Great Ideas from Optimal Transport and Riemannian Geometry

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For full proofs please refer to this QR code.

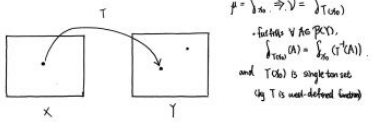
From Monge Problem (MP) to Kantorovich Problem (KP)

Monge Problem

- given probability measures μ on X and ν on Y, and cost $c : X \times Y \to \mathbb{R}$, minimize

$$\int_X c(x, T(x))\, \mathrm{d}\mu(x)$$

- over measurable maps $T : X \to Y$ with $T_{\#}\mu = \nu$.



$\mu = \int_{x_0} \Rightarrow \nu = \int_{T(x_0)}$
- for this $\forall$ Borel $B \subseteq Y$,
$\int_{T(x_0)}(A) = \int_{x_0}(T^{-1}(A))$
and $T(x_0)$ is singleton set
(by T is well-defined function)

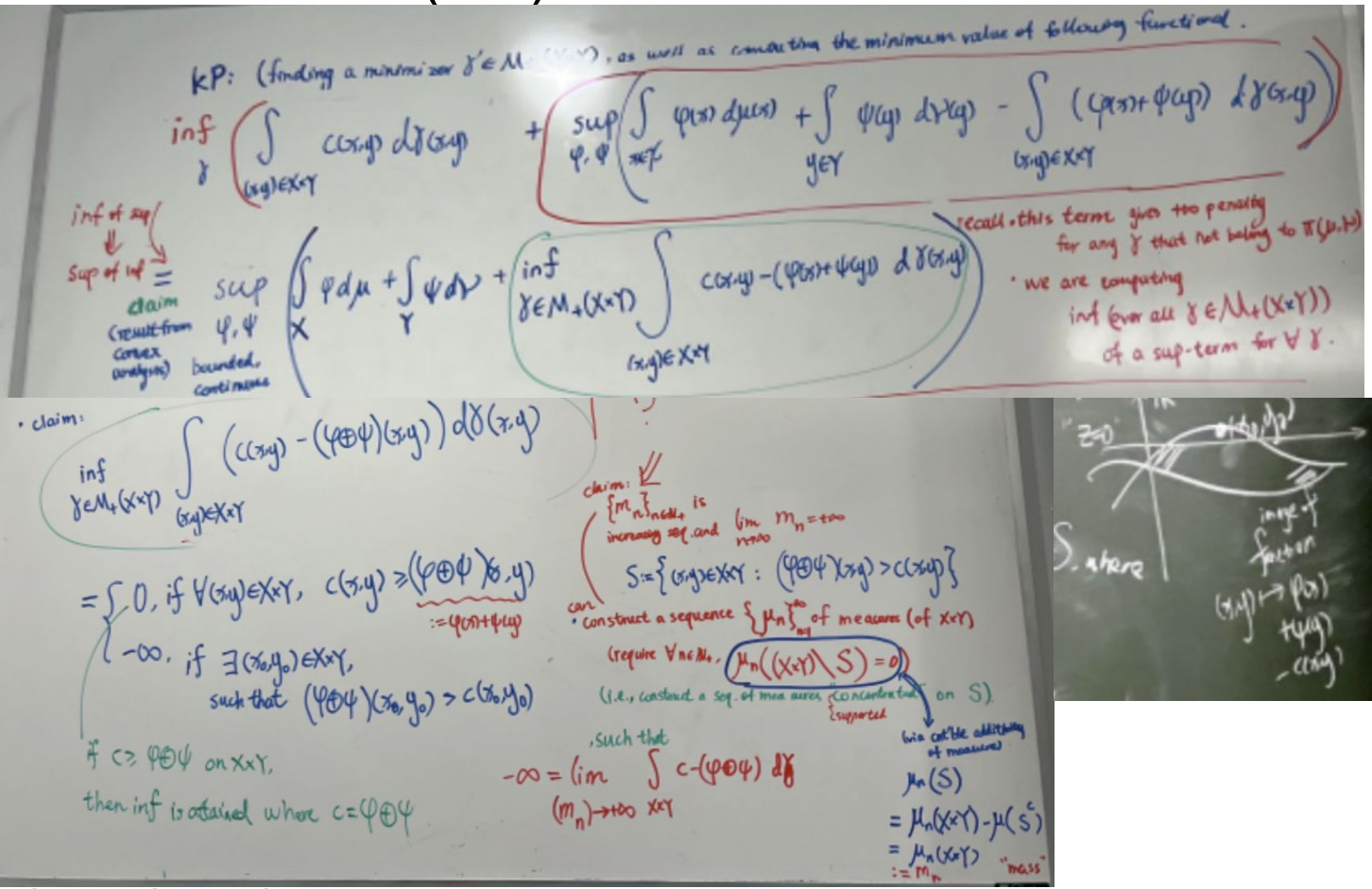
Relaxation viewpoint: MP vs KP

- MP: optimization over maps T.
- KP: optimization over measures γ.
- KP is a relaxation of MP, and typically

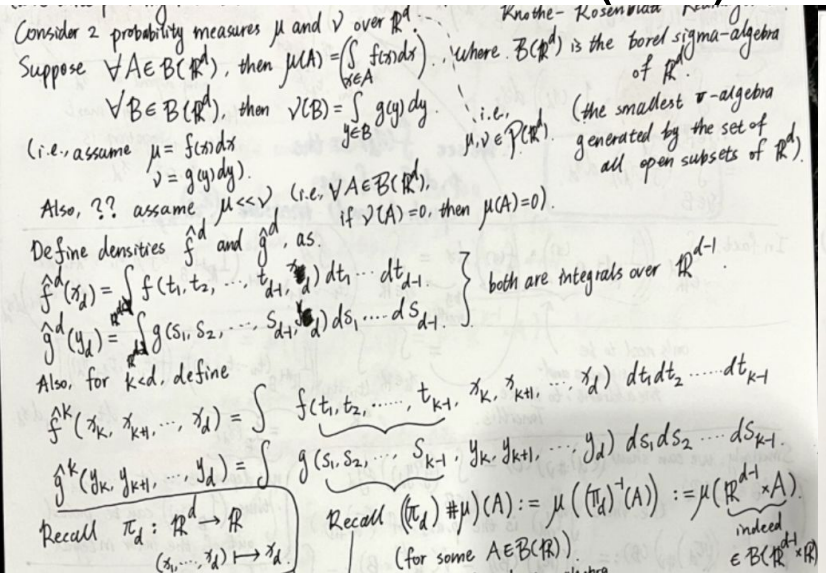
$$\mathrm{Opt(KP)} \leq \mathrm{Opt(MP)}$$

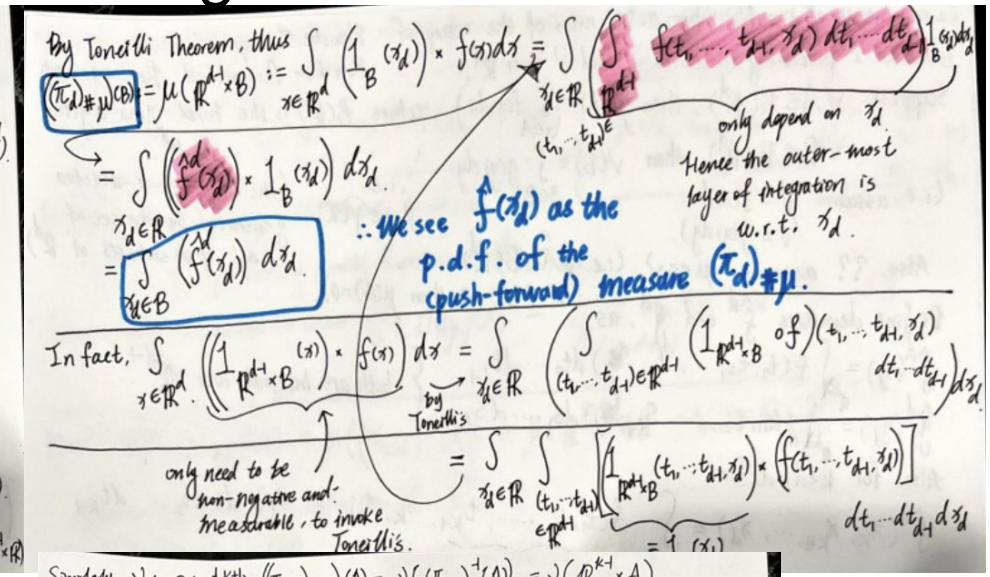
$\exists$ deterministic transport plan
$\gamma := (\mathrm{id}, T)_{\#}\mu$
encoding Monge problem.
i.e., $\forall A \subseteq \mathcal{B}(X \times Y)$,
$\gamma(A) := \mu\left((\mathrm{id}, T)^{-1}(A)\right)$
$= \mu(\{x \in X : (x, T(x)) \in A\})$

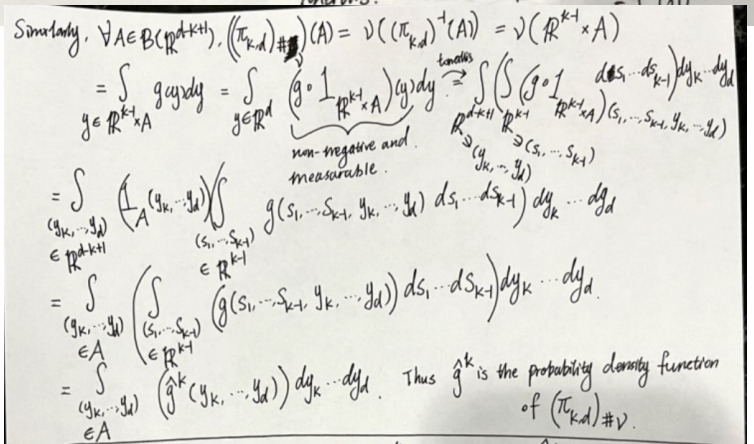
Dual Problem (DP)



Knothe-Rosenblatt (KR) Rearrangement

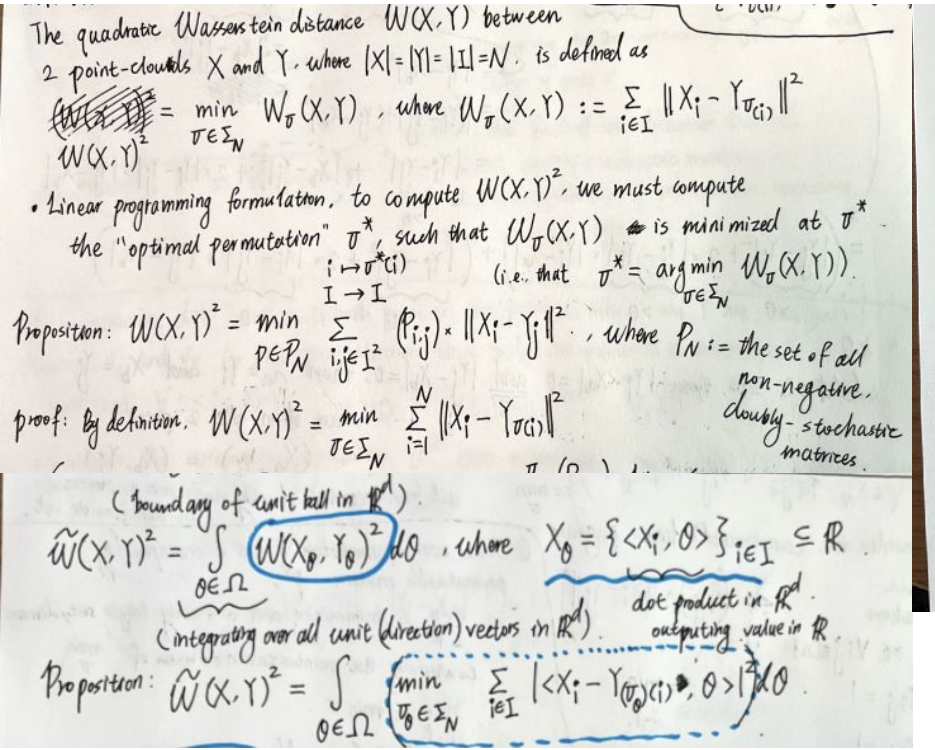


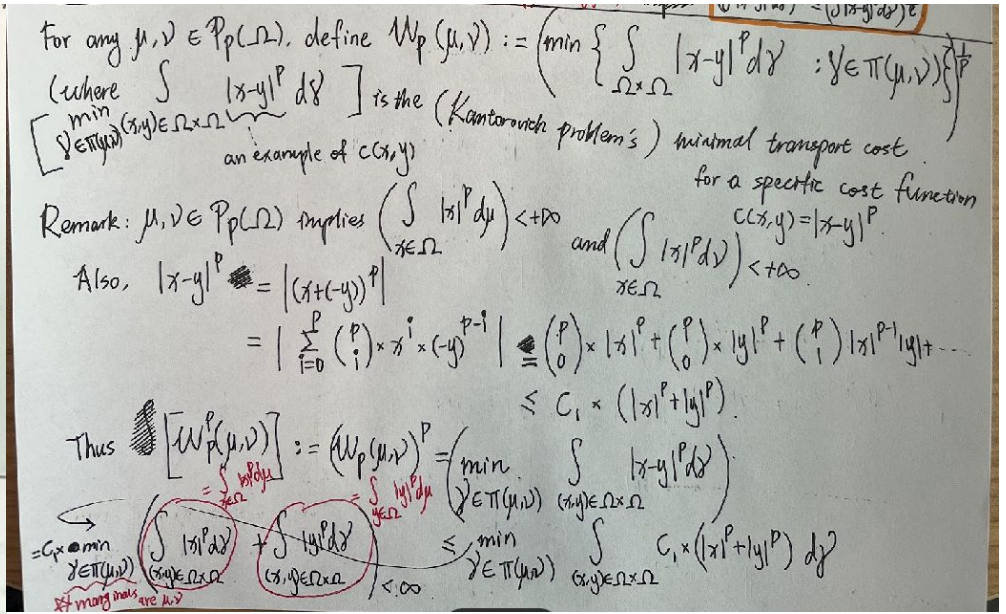




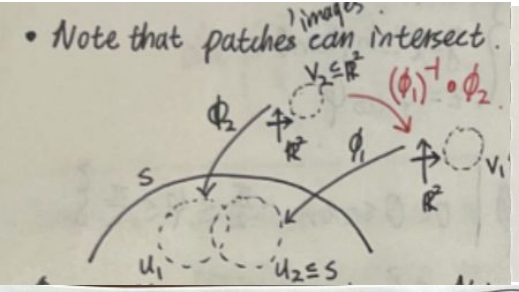
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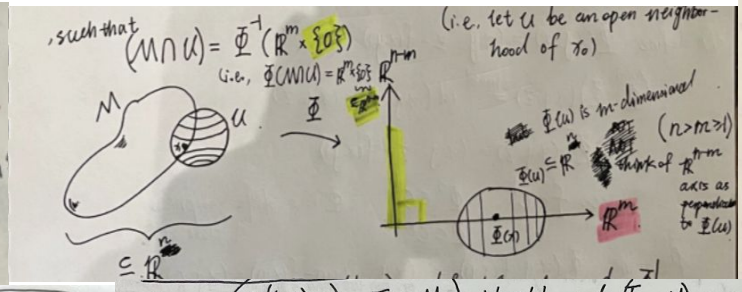
(Sliced) Wasserstein Distance

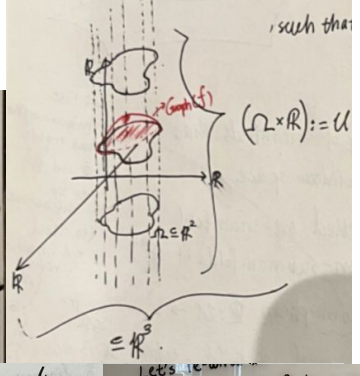




Geometric Intuitions of Manifolds



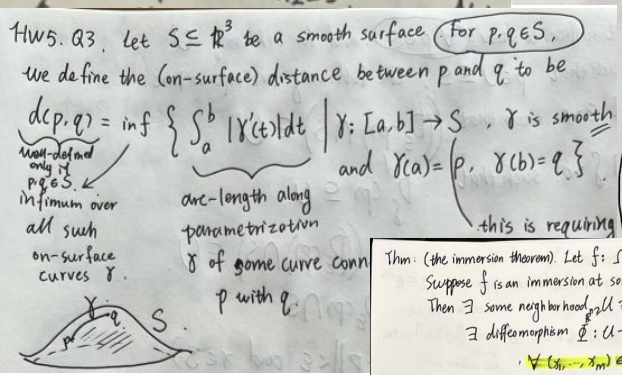




HW5. Q3. Let $S \subseteq \mathbb{R}^3$ be a smooth surface (for $p, q \in S$,)
we define the (on-surface) distance between p and q to be

$$d(p,q) = \inf \left\{ \int_a^b |\gamma'(t)| dt \;\middle|\; \gamma: [a,b] \to S,\ \gamma \text{ is smooth and } \gamma(a) = p,\ \gamma(b) = q \right\}$$

- well-defined only if $p, q \in S$.
- infimum over all such on-surface curves γ.
- arc-length along parametrization γ of some curve connecting p with q.
- this is requiring

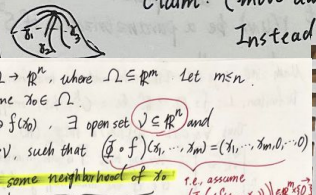


Intuition: $\mathrm{Im}((Df)(U)) = T_{f(U)}M$, $\forall u \in U$, and $(T_{f(U)}M) \subseteq \mathbb{R}^n$

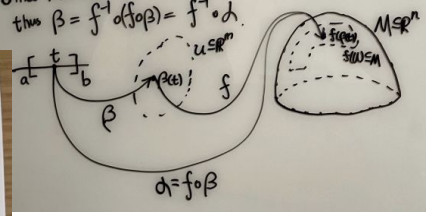
$(T_{f(U)}M)$:= the set of velocities of curves in M through $f(U)$

$= \{\gamma'(0) : \gamma(t) \subseteq M,\ \gamma(0) = f(U)\}$

Claim: (move away from the perspective of $(Df)(U)$ as a
Instead view $(Df)(U)$ as a linear transforma



Let's
Set-up:
Let $M \subseteq \mathbb{R}^n$ be a m-dimensional man
Let $f: U \to f(U)$ be a local chart.
$\subseteq \mathbb{R}^m$ $\subseteq M \subseteq \mathbb{R}^n$.
If $\alpha: [a,b] \to f(U)$ is a curve, then $\exists$ curve $\beta: [a,b] \to U$
such that $f \circ \beta = \alpha$. (i.e. such that $\forall t \in [a,b]$ we have $f(\beta(t)) = \alpha(t)$)
Since local chart $f: U \to f(U)$ is a homeomorphism
thus $\beta = f^{-1} \circ \alpha = f^{-1} \circ \alpha$.



Thm: (the immersion theorem). Let $f: \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^m$. Let $m \leq n$.
Suppose f is an immersion at some $\gamma_0 \in \Omega$.
Then $\exists$ some neighborhood $U \ni f(\gamma_0)$, $\exists$ open set $V \subseteq \mathbb{R}^n$ and
$\exists$ diffeomorphism $\Phi: U \to V$, such that $(\Phi \circ f)(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$
$\forall (x_1, \cdots, x_m) \in$ some neighborhood of γ_0
$(V_\delta(\gamma_0)) \subseteq \mathbb{R}^m$
i.e. assume $(\Phi(f(x_1, \cdots, x_m))) \in \mathbb{R}^m \times \{\vec{0}\}$
(despite $V = \mathrm{Codomain}(\Phi)$ is such that $\mathrm{image}/\mathrm{image\ set} \subseteq \mathbb{R}^m \times \{\vec{0}\}$)

Theorem: $[\widetilde{g}_{ij}(\tilde{u})] = (D\Phi(\tilde{u}))^T [g_{ij}(u)] (D\Phi(\tilde{u}))$, suppose $u = \Phi(\tilde{u})$
$1 \leq i \leq m$, $1 \leq j \leq m$
$\in \mathbb{R}^{m \times m}$ $\in \mathbb{R}^{m_1 \times m}$ $\in \mathbb{R}^{m \times m}$ $\in \mathbb{R}^{m_2 \times m_1}$

In order for LHS, RHS 's dimension to check, we must have $m_2 = m$ and $m_1 = m$

$M \subseteq \mathbb{R}^n$
$f(U) = \tilde{f}(\tilde{U})$
$f = \tilde{f} \circ \Phi$
$\tilde{U} \subseteq \mathbb{R}^{m_1}$
$U \subseteq \mathbb{R}^{m_2}$
Suppose Φ is a diffeomorphism
$m_1 < n$, $m_2 < n$, and $m_1 = m_2$

Suppose M is a Riemannian manifold with the metric $\langle \cdot, \cdot \rangle_x := G_x(\cdot, \cdot)$, $\forall x \in M$
(Recall each g_{ij} coefficient corresponds to a parametrization/local chart f of M)
Denote g_{ij} and $\widetilde{g}_{ij}$ to be the coefficients of Riemannian metric on M, corresponding respectively to local charts f and $\tilde{f}$.

$g_{ij}: U \to \mathbb{R}$, $x \mapsto G_{f(x)}\left((Df)(x)\vec{e}_i, (Df)(x)\vec{e}_j\right)$, $\forall i,j \in [m]$, $\forall x \in U$.
(thus $f(x) \in M$)
$\widetilde{g}_{ij}: \tilde{U} \to \mathbb{R}$, $x \mapsto G_{\tilde{f}(x)}\left((D\tilde{f})(x)\vec{e}_i, (D\tilde{f})(x)\vec{e}_j\right)$
$\therefore G_{f(x)}: T_{f(x)} \times T_{f(x)} \to \mathbb{R}$

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