Three-dimensional cosmography of the high redshift Universe using intergalactic absorption

Collin Politsch

@capolitsch

Carnegie Mellon University

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Collaborators (PhD advisors)

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Jessi Cisewski-Kehe (UW-Madison)

Rupert Croft (CMU)

Overview of Thesis Work

- **Extrasolar planets.**
  - Modeling phase-folded transit light curves with higher-order changepoint methods
- **Eclipsing binary stars.**
  - Nonparametric modeling of phase-folded light curves
- **Spectroscopic classification and redshift estimation.**
  - Efficient spectral template generation with observational spectra
- **Supernovae.**
  - Light-curve template generation
  - Nonparametric estimation of observable parameters
- **Intergalactic medium (via the Lyman-α forest).**
  - Nonparametric estimation of quasar continua
  - One-dimensional reconstruction of the intergalactic medium
  - Three-dimensional reconstruction of the intergalactic medium

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Chapters 2-3 of Thesis
Politsch et al. (2020a,b)
MNRAS

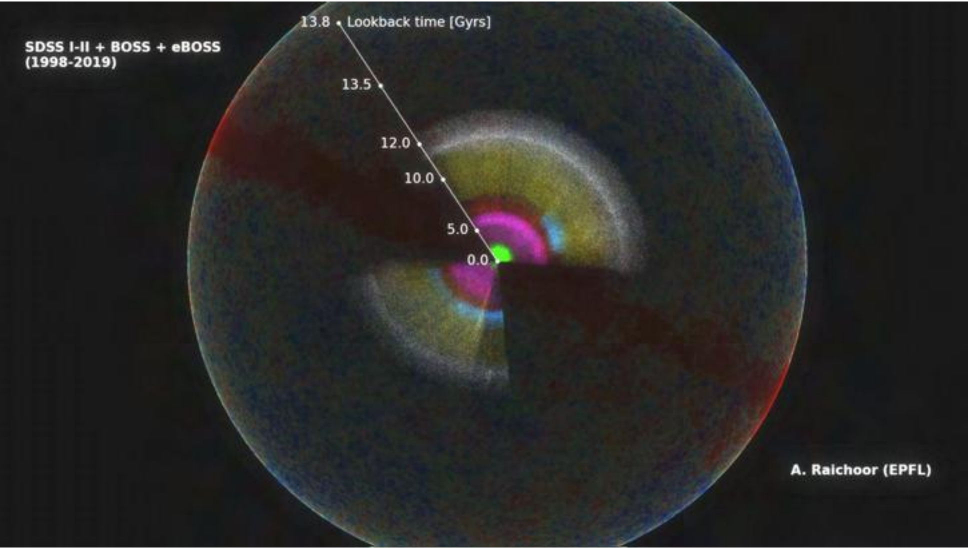
- **Intergalactic medium (via the Lyman-α forest).**
  - Nonparametric estimation of quasar continua
  - One-dimensional reconstruction of the intergalactic medium
  - Three-dimensional reconstruction of the intergalactic medium

Focus of this talk

Talk outline

1. The Lyman-α forest
   - Background
   - The Sloan Digital Sky Survey
   - Summary of work & results
2. One-dimensional mapping the intergalactic medium
   - Denoising observational spectra
   - Estimating the underlying density field
   - Uncertainty quantification
3. Three-dimensional mapping the intergalactic medium
   - Spatial model
   - Distributed computing
   - Optimized absorption field reconstruction
   - Uncertainty quantification
   - High-significance candidates for galaxy protoclusters and voids

The Sloan Digital Sky Survey



The cosmological matter distribution

The **intergalactic medium** is a highly diffuse gaseous medium that pervades the volume of intergalactic space and contains a majority of the baryonic matter in the Universe.

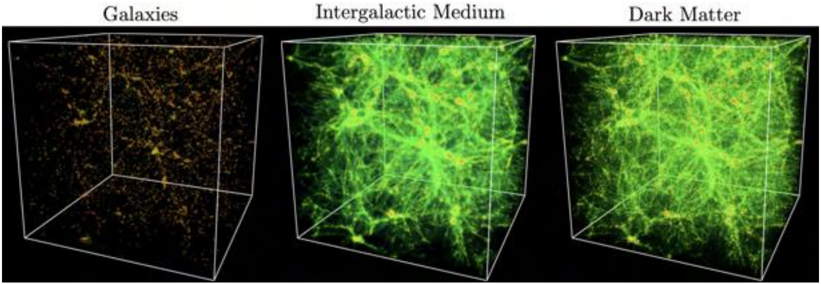
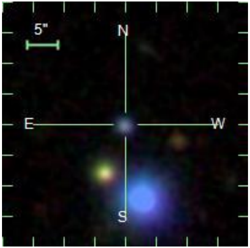


Image credit: R. Cen and J. P. Ostriker

Absorption spectroscopy with quasar backlights

Absorption spectroscopy: Spectroscopic techniques that measure the absorption of light, as a function of wavelength, due to its interaction with a medium.



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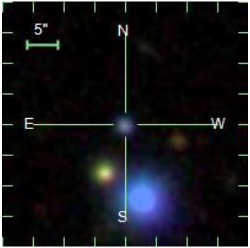

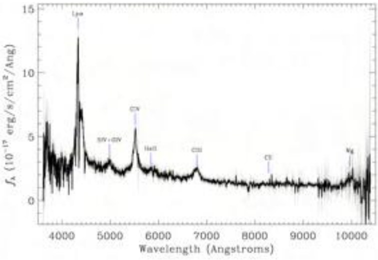

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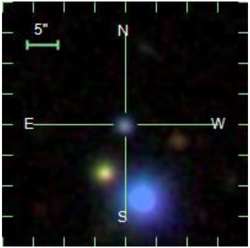

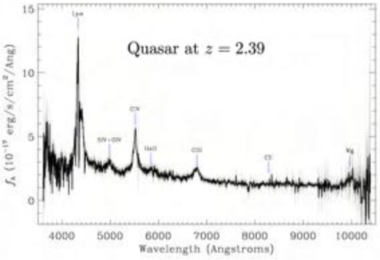

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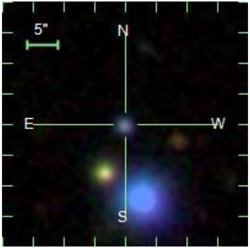
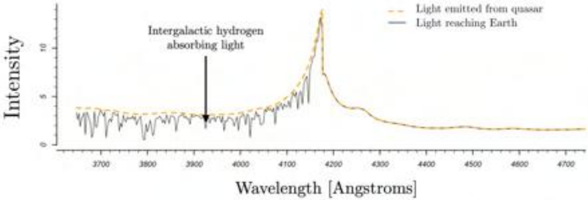
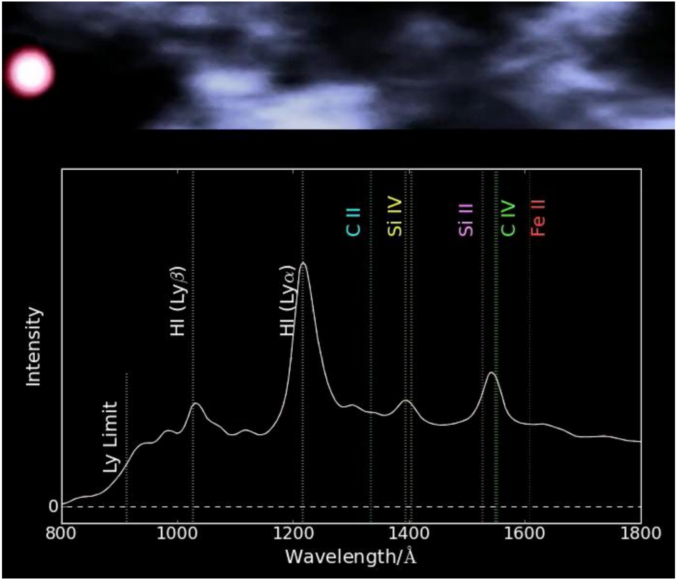


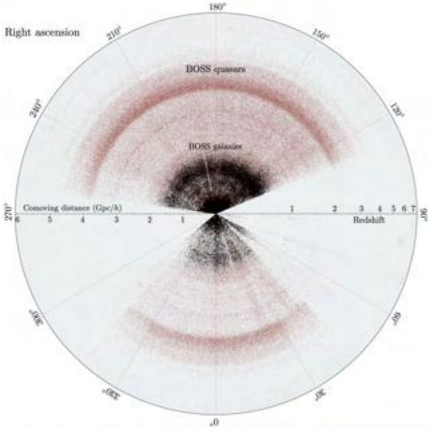


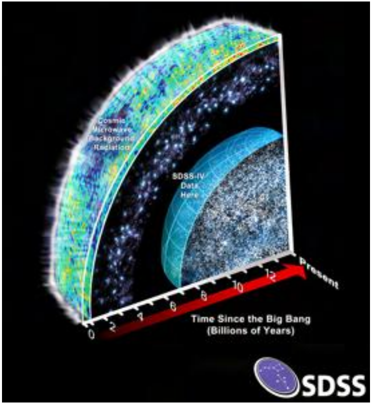
The Lyman-α forest absorption process



Video credit: Andrew Pontzen

The Baryon Oscillation Spectroscopic Survey





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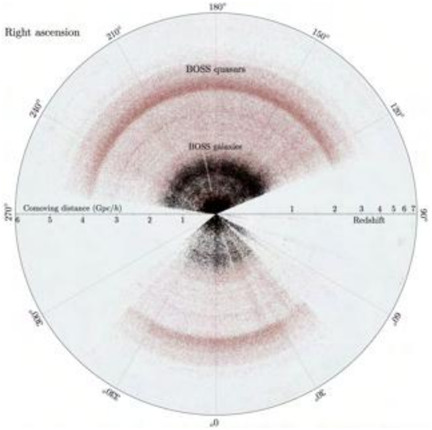
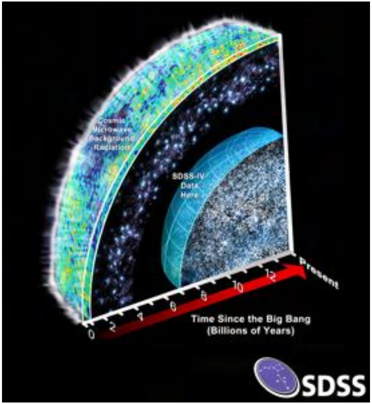
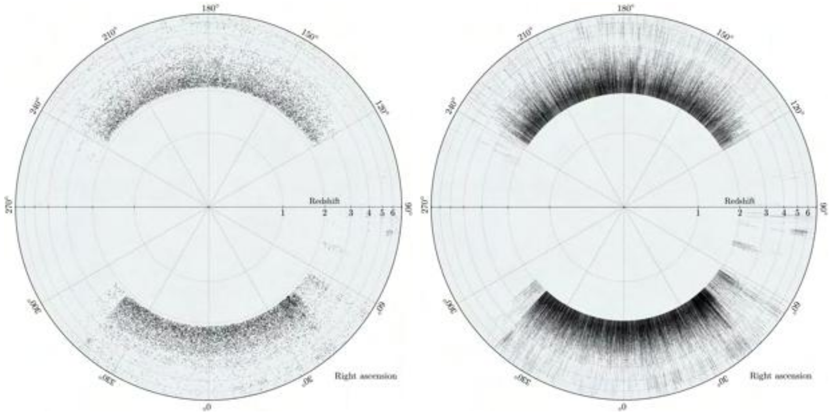




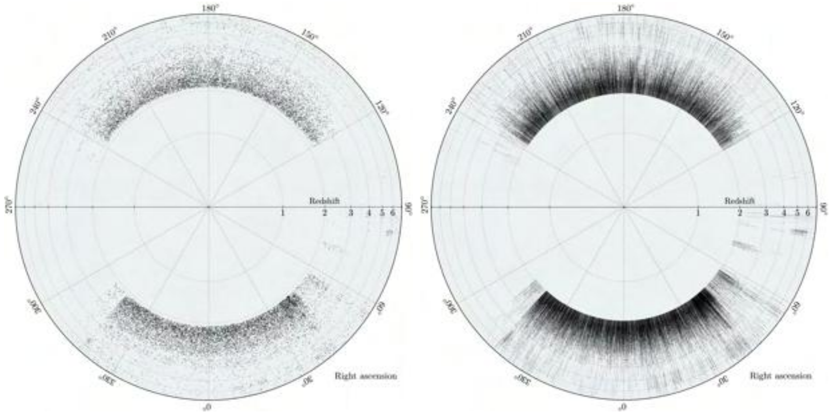
~160,000 usable
Lyman-α quasar spectra

High-level summary of work



High-level summary of work

1. Infer the relative density of the IGM along each 1D quasar sightline



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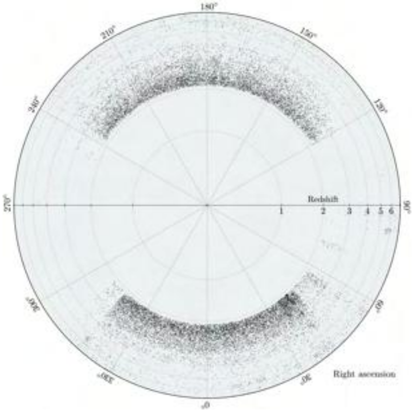
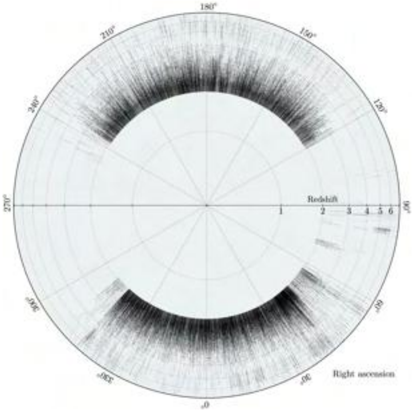
2. Pool the sightlines and reconstruct a full 3D large-scale structure map





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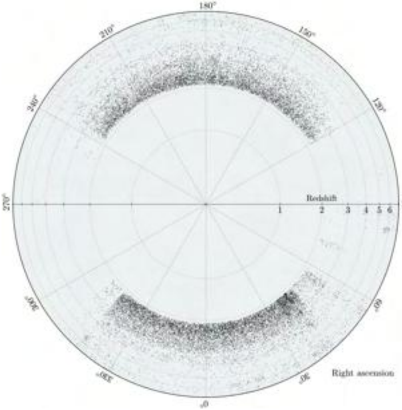
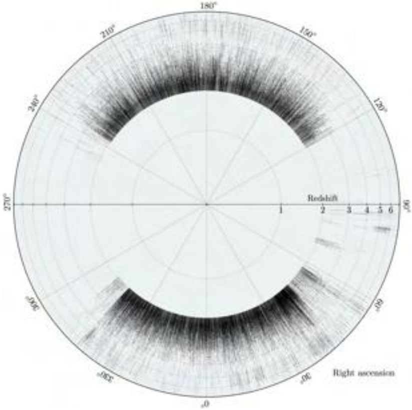
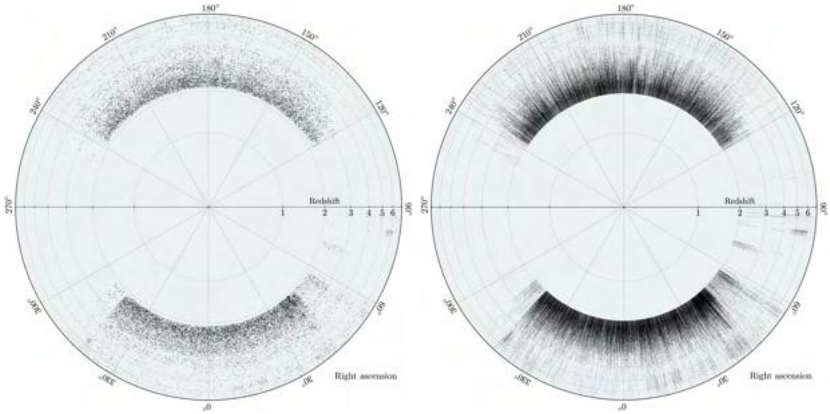




High-level summary of work

1. Infer the relative density of the IGM along each 1D quasar sightline
2. Pool the sightlines and reconstruct a full 3D large-scale structure map
3. Quantify the total statistical uncertainty in the full 3D map
4. Identify statistically significant candidates for galaxy protoclusters and cosmic voids



Summary of results: Largest volume LSS map of the Universe to date

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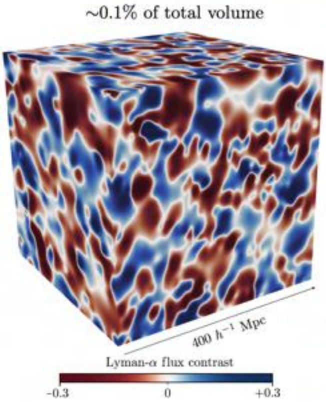
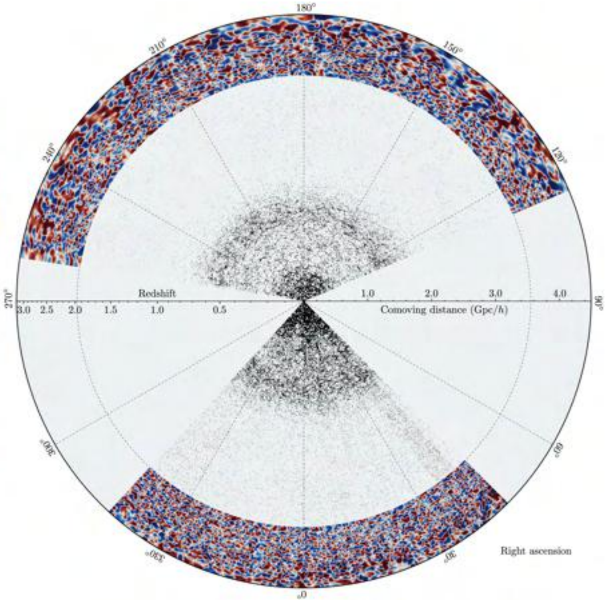
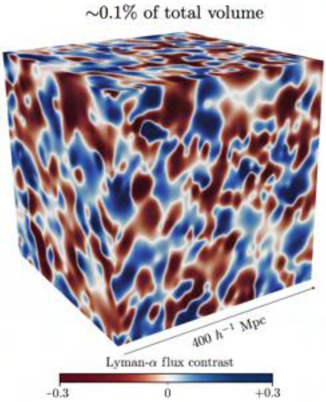
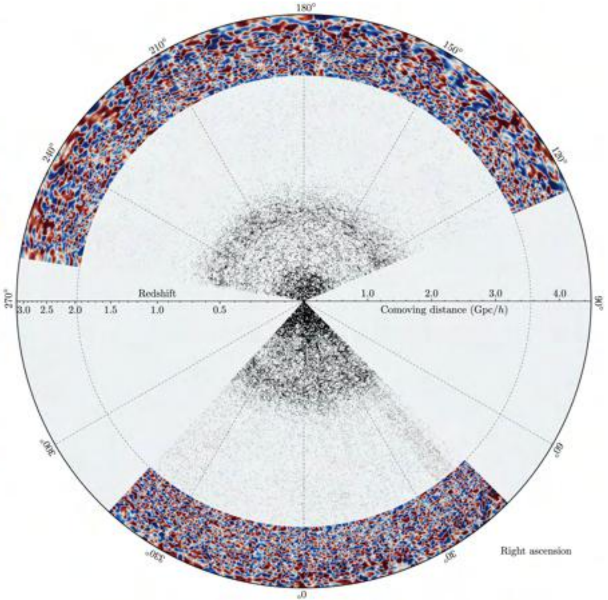




Summary of results: Largest volume LSS map of the Universe to date

- 47 h^{-3} Gpc^3 = 154 Gpc^3 (Planck CMB)





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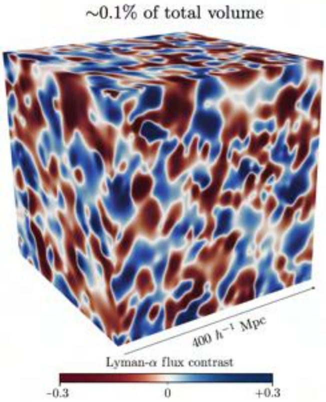
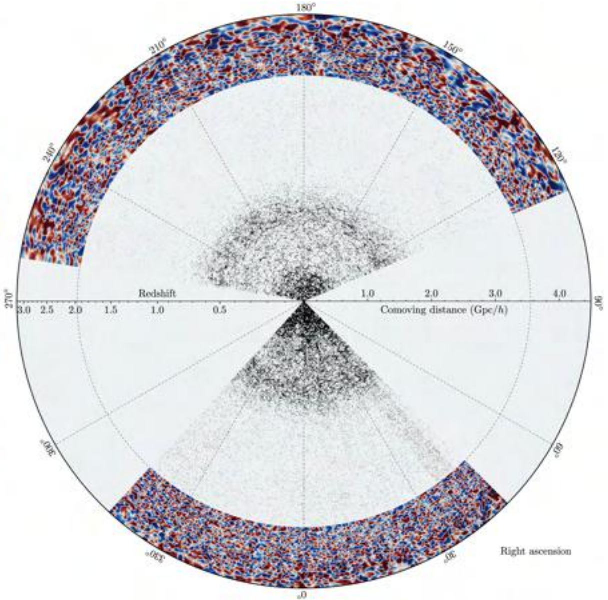
- 47 h^{-3} Gpc^3 = 154 Gpc^3 (Planck CMB)
- 10,332 deg^2 footprint (25% sky coverage)





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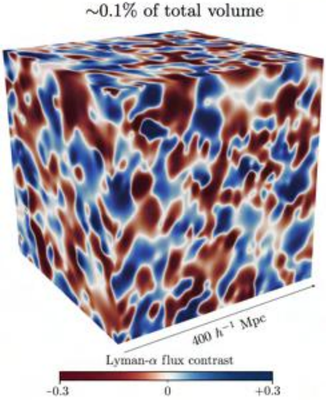

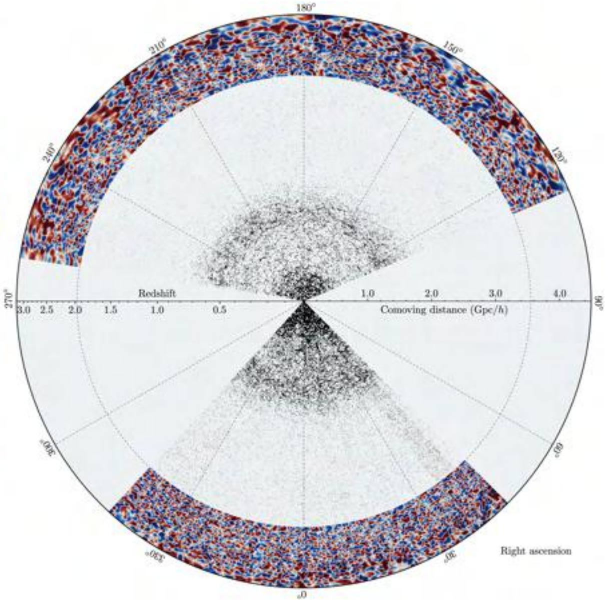

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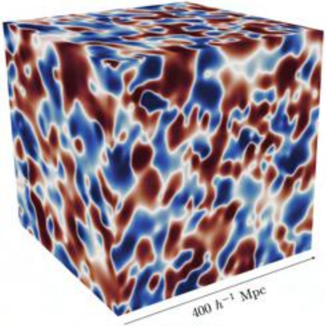
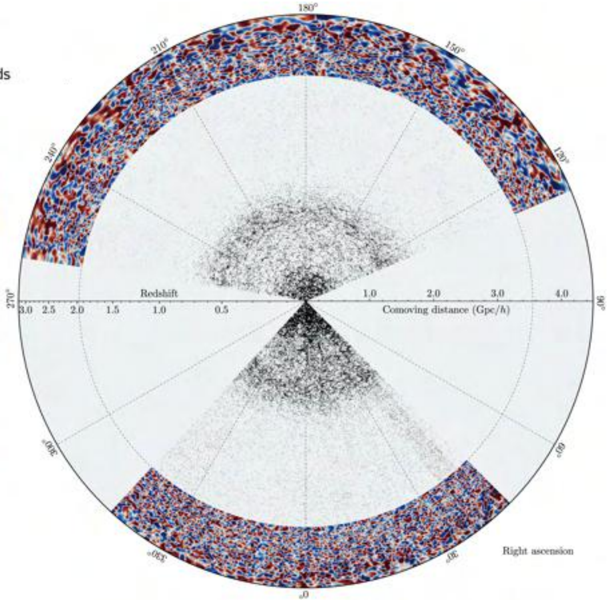
- ~180,000 candidates for protoclusters/voids

~0.1% of total volume





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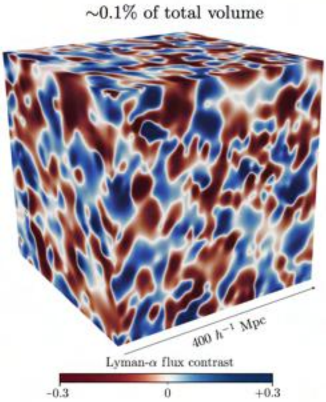

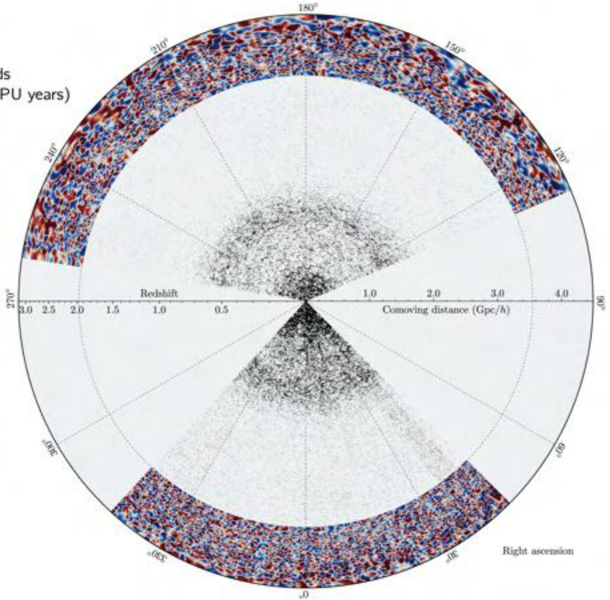

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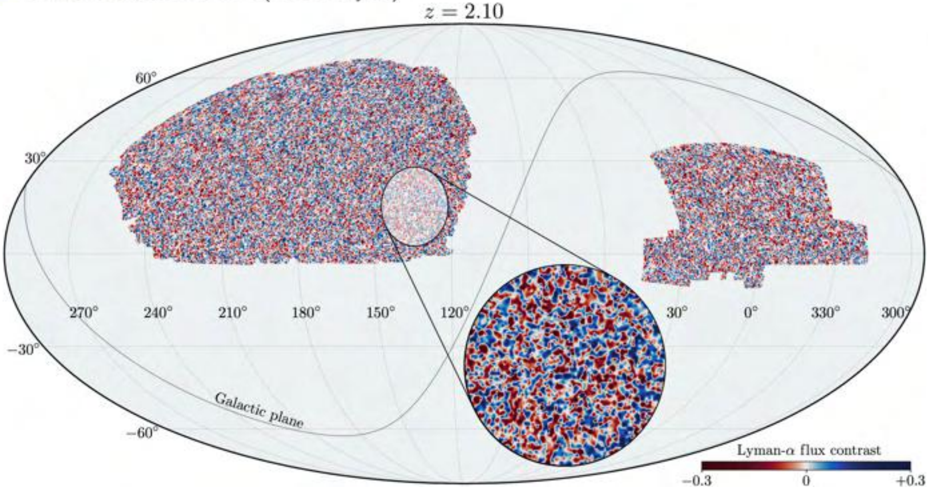
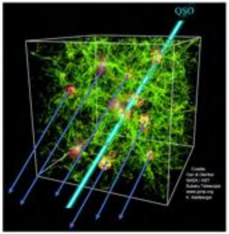
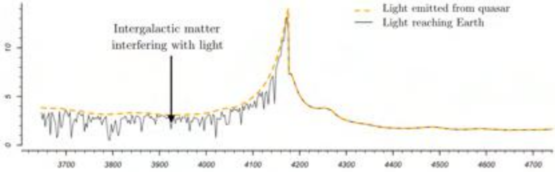


Outline

One-dimensional mapping the intergalactic medium





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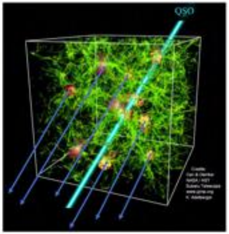
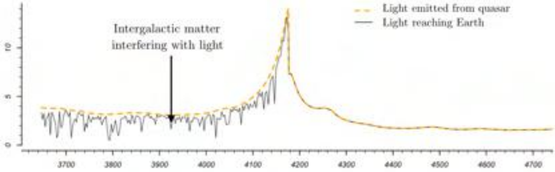
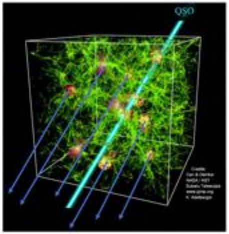
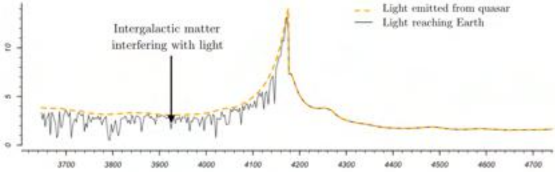




$$f(\lambda_i) = f_0(\lambda_i) + \epsilon_i$$

One-dimensional mapping the intergalactic medium





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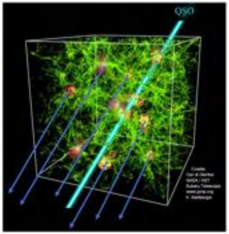

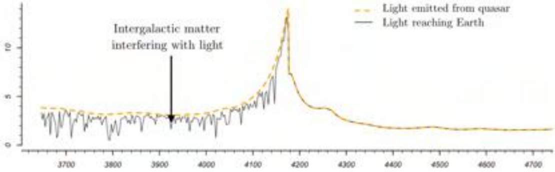

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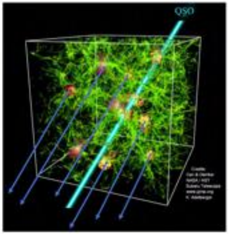
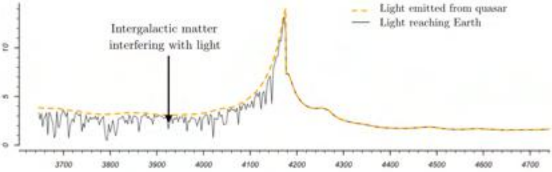
One-dimensional mapping the intergalactic medium





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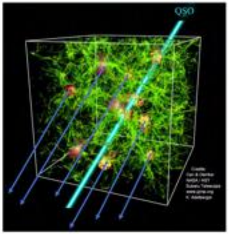
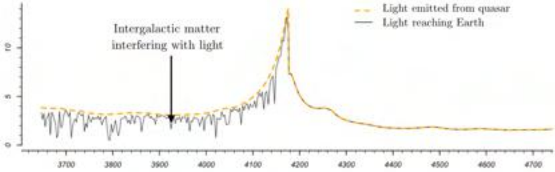
Lyman-α flux contrast:

$$\delta_F(z) = \frac{F(z) - \overline{F}(z)}{\overline{F}(z)}$$

One-dimensional mapping the intergalactic medium





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Estimator:

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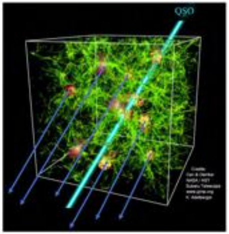
where $\quad m(z) = C(z) \cdot \overline{F}(z)$

One-dimensional mapping the intergalactic medium



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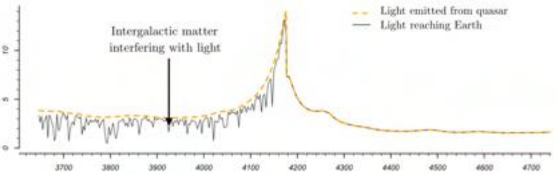

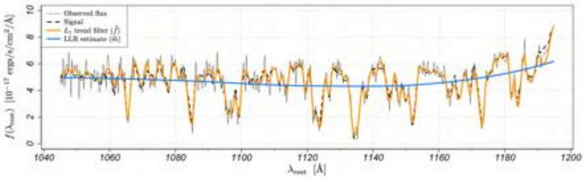

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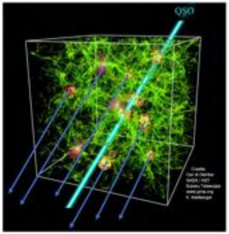

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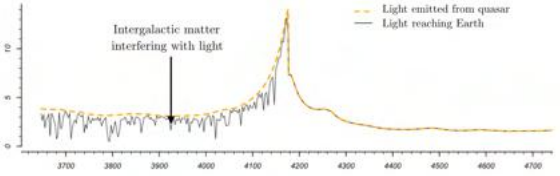
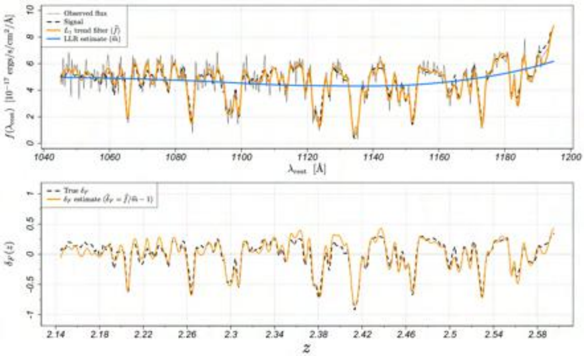
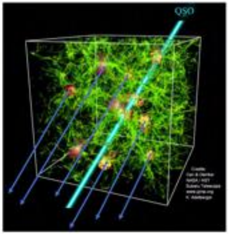




Trend filtering: Tibshirani (2014)

One-dimensional mapping the intergalactic medium



$$\begin{aligned} f(\lambda_i) &= f_0(\lambda_i) + \epsilon_i \\ &= C(\lambda_i) \cdot F(\lambda_i) + \epsilon_i \\ &= C(\lambda_i) \cdot \overline{F}(\lambda_i) \cdot (1 + \delta_F(\lambda_i)) + \epsilon_i \end{aligned}$$

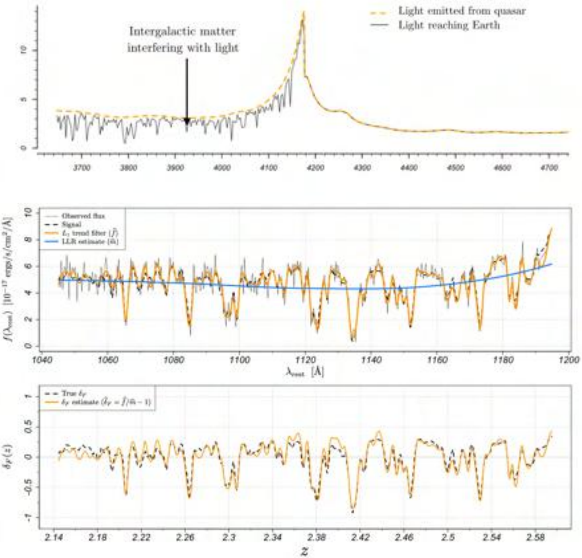
Lyman-α flux contrast:

$$\delta_F(z) = \frac{F(z) - \overline{F}(z)}{\overline{F}(z)}$$

Estimator:

$$\widehat{\delta}_F(z) = \frac{\widehat{f}_0(z) - \widehat{m}(z)}{\widehat{m}(z)}$$

where $\quad m(z) = C(z) \cdot \overline{F}(z)$



Trend filtering: Tibshirani (2014), Politsch et al. (2020 a,b)

Parametric bootstrapping the flux contrast

Uncertainty quantification?

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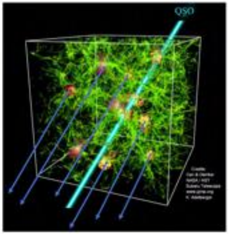
4. Contruct the pointwise $1-\alpha$ percentile band:

$$V_{1-\alpha}(z_i) = \left(\widehat{\delta}_{F,\alpha/2}^*(z_i),\ \ \widehat{\delta}_{F,1-\alpha/2}^*(z_i)\right)$$

One-dimensional absorption field reconstruction



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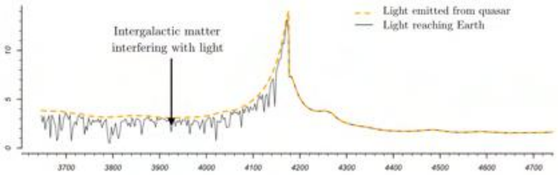

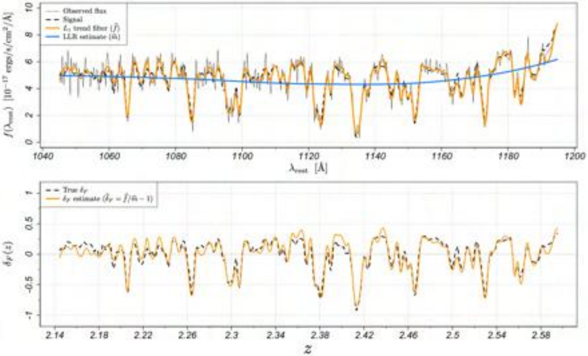

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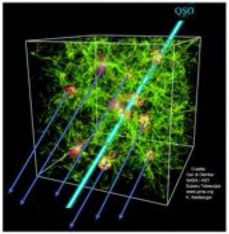

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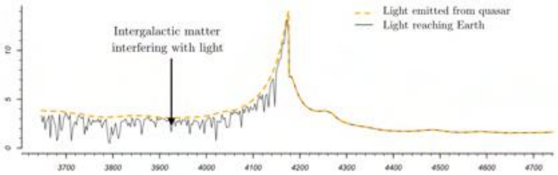

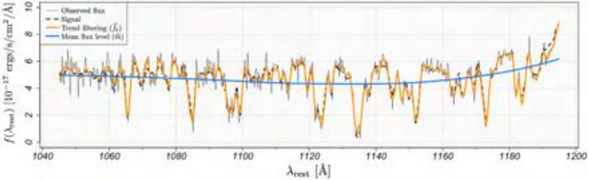

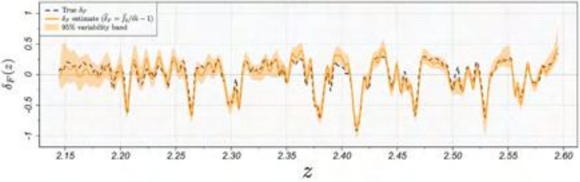

Outline

1. The Lyman-α forest
   - Background
   - The Sloan Digital Sky Survey
   - Summary of work & results
2. One-dimensional mapping the intergalactic medium
   - Denoising observational spectra
   - Estimating the underlying density field
   - Uncertainty quantification
3. Three-dimensional mapping the intergalactic medium
   - Spatial model
   - Distributed computing
   - Optimized absorption field reconstruction
   - Uncertainty quantification
   - High-significance candidates for galaxy protoclusters and voids

Lyman-α forest tomography

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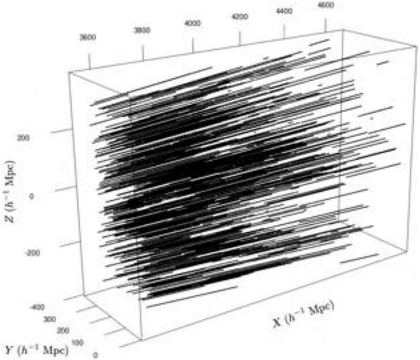

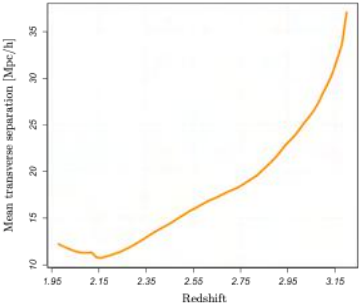

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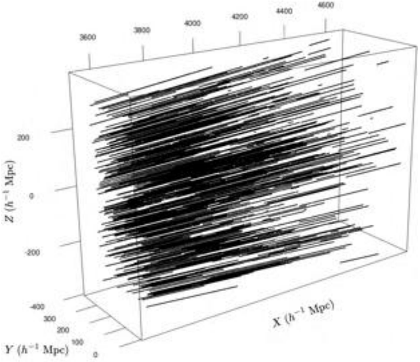

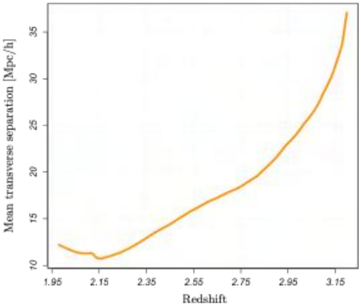

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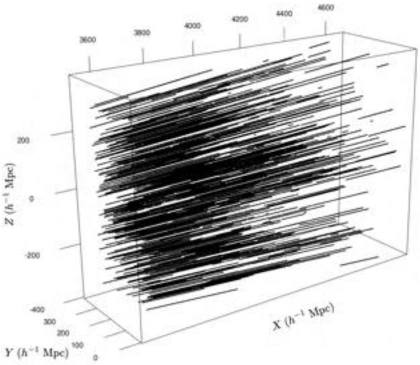

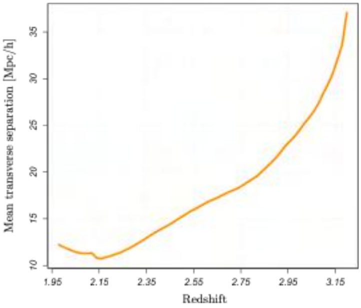

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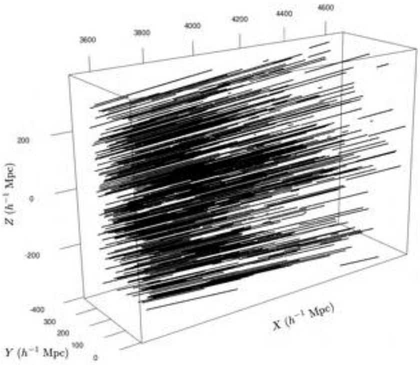

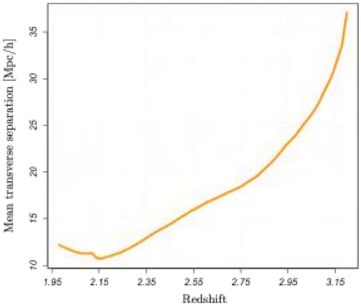

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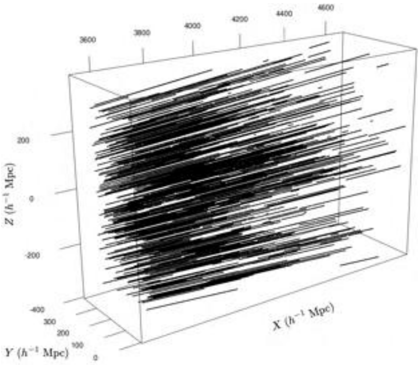

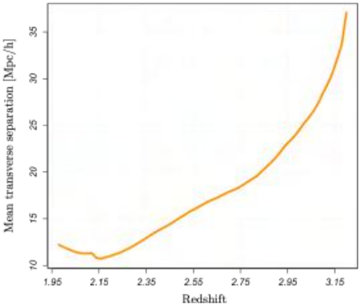

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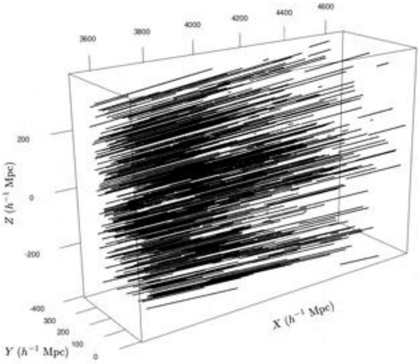
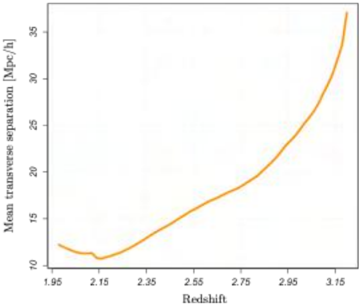
5. **Computational efficiency and scalability.** Sparsity and distributed computing





Multi-resolution spatial kernel ridge regression

Cost functional:

$$\min_{\beta \in \mathbb{R}^m} (\widehat{\delta}_F - \Phi\beta)^T W (\widehat{\delta}_F - \Phi\beta) + \|D\beta\|_2^2$$

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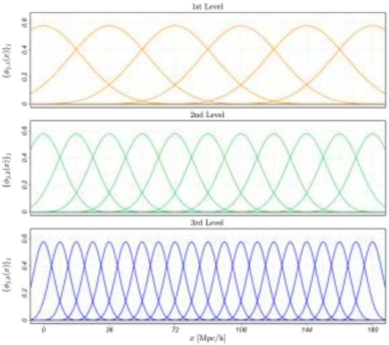

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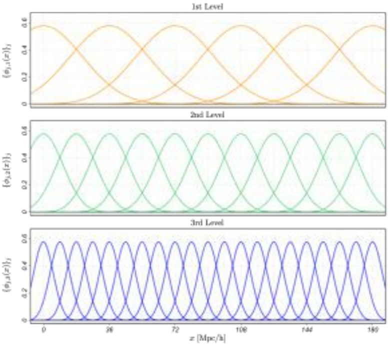

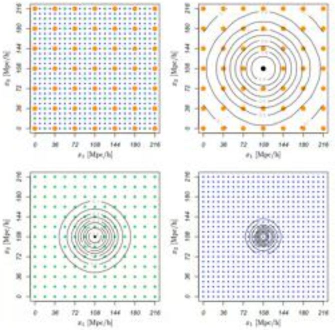

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$$D = \begin{bmatrix} \sqrt{\gamma_1}(\Delta_1 + \alpha_1 I) & 0 & \cdots & 0 \\ 0 & \sqrt{\gamma_2}(\Delta_2 + \alpha_2 I) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sqrt{\gamma_d}(\Delta_d + \alpha_d I) \end{bmatrix}.$$

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$$\Delta_{\ell_{ij}} = \begin{cases} 6 & i = j, \\ -1 & j \in \mathcal{N}_i, \\ 0 & \text{otherwise,} \end{cases}$$

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Expanded penalty:

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$n \approx$ **72 million** **40 million** $\lesssim p \lesssim$ **50 million**

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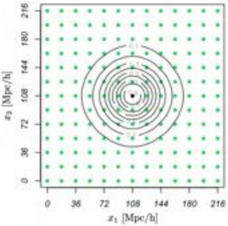

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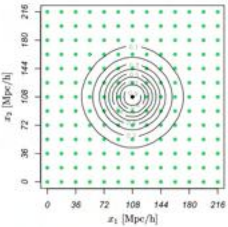

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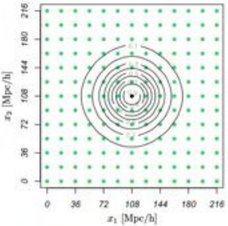

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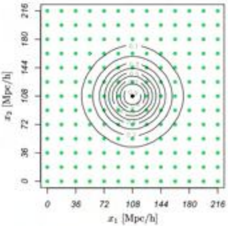

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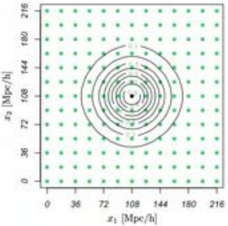

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Kendall tau ranking distance:

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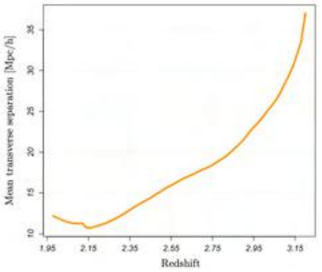

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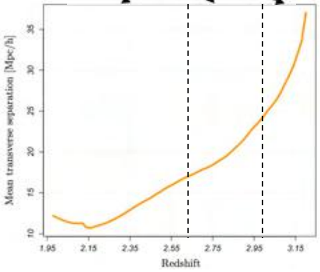
Optimize separately



Cross-sectional sky map of the intergalactic medium

$$\widehat{\delta}_F^{L\|}(x) = \sum_{j=1}^{3}\sum_{k=1}^{r_j} \mathbb{1}(x \in A_{j,k}) \cdot \widehat{\delta}_F^{L,j,k}(x; \widehat{\eta}_j)$$

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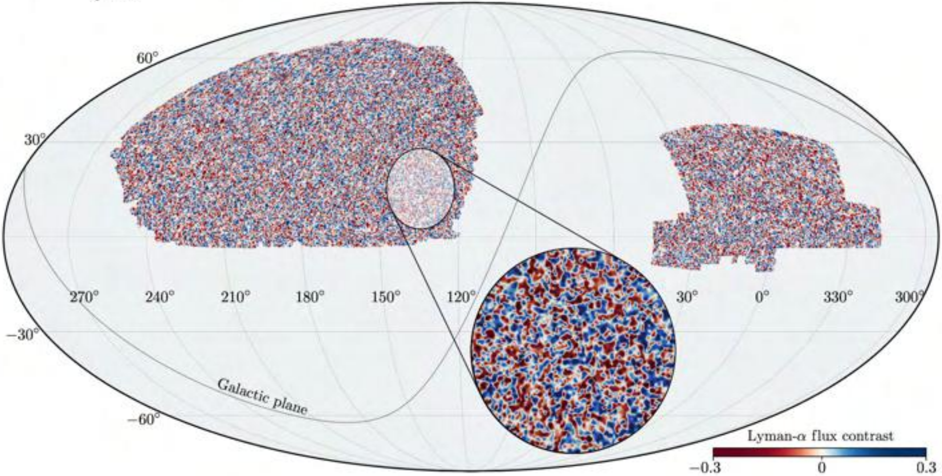
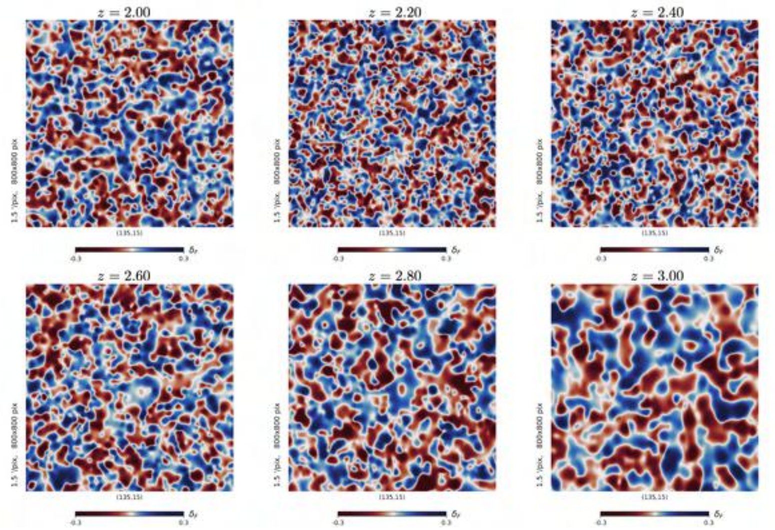


Slice of intergalactic medium at various redshifts



Uncertainty quantification

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1. Construct a bootstrap sample of quasar spectra

$$f^*_{q,b}(\lambda_{q_i}) \sim N(\widehat{f}_{q,0}(\lambda_{q_i}), \widehat{\sigma}^2_{q_i}), \quad i = 1, \ldots, n, \quad q = 1, \ldots, 159,581$$

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4. Compute the distributed-SKRR 3D reconstruction $\widehat{\delta}^{L\|*}_F(x)$ from the pooled sightline flux contrast estimates.

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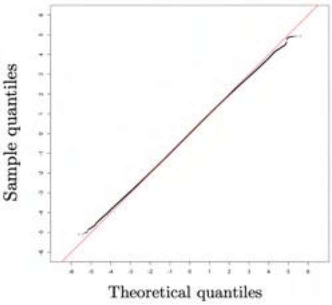

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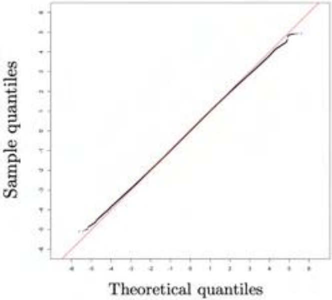

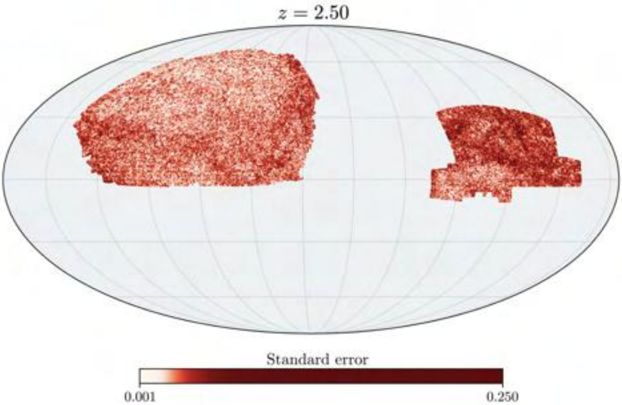

Candidates for galaxy protoclusters and voids

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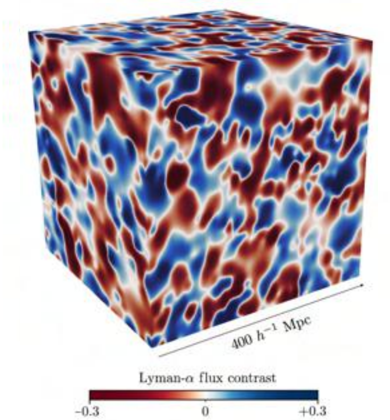

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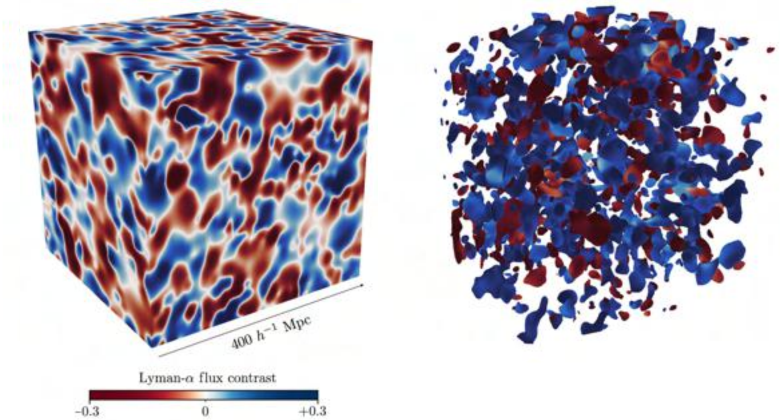

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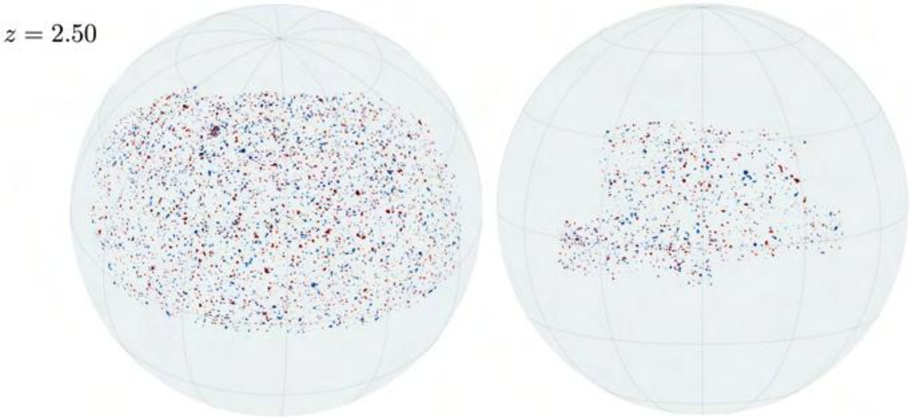

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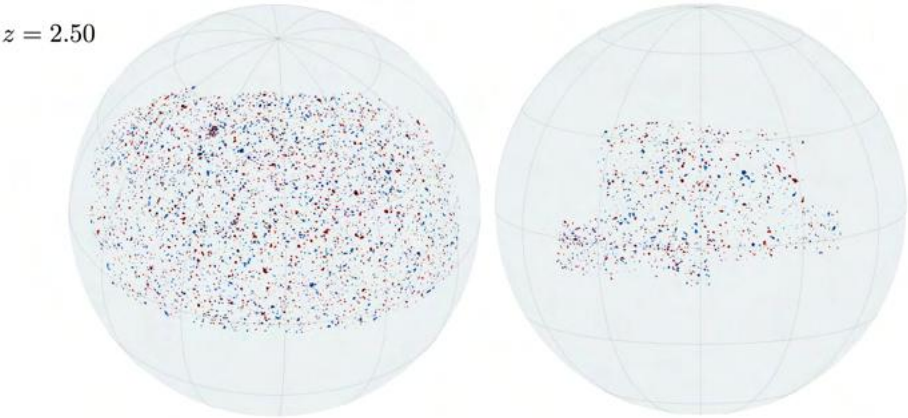

Census of candidates (rough estimates based on partial catalog)

	Significance level 3σ	4σ	5σ	6σ	7σ	8σ
Galaxy Protoclusters	184,250	130,535	82,209	50,029	29,493	18,272
Cosmic Voids	173,882	127,098	82,016	54,507	34,948	22,266

Dissemination of data products (in progress)

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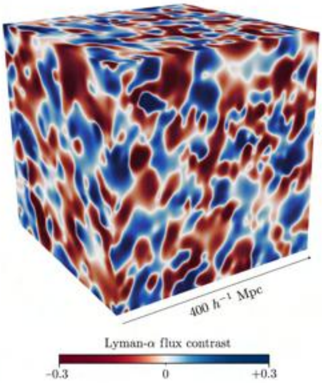
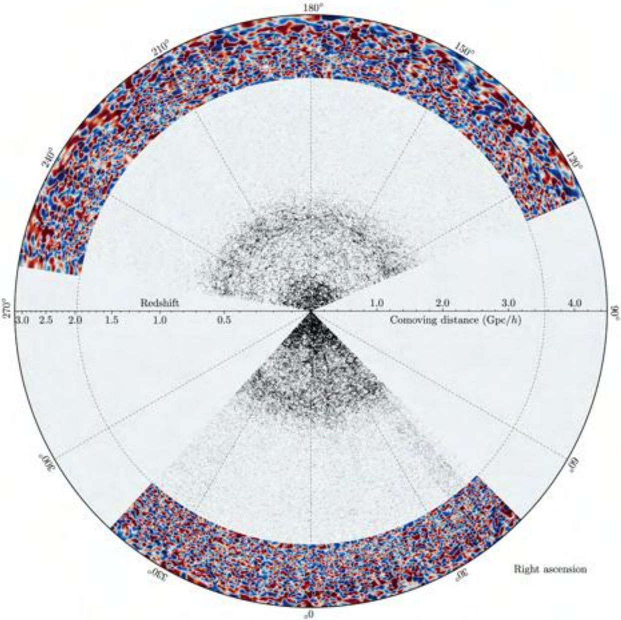
Thank you!

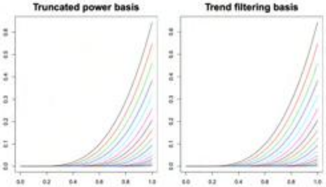

400 h^{-1} Mpc
Lyman-α flux contrast
−0.3
0
+0.3


180°
210°
150°
240°
120°
270°
90°
300°
60°
330°
30°
0°
Redshift
3.0 2.5 2.0 1.5 1.0 0.5
1.0 2.0 3.0 4.0
Comoving distance (Gpc/h)
Right ascension

Overview of trend filtering

$$f(t_i) = f_0(t_i) + \epsilon_i \qquad i = 1, \ldots, n$$

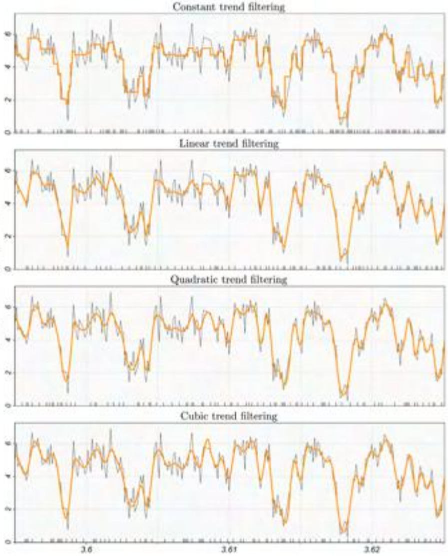
$$f_0(t) = \sum_j \beta_j h_j(t)$$



$$\min_{\{\beta_j\}} \sum_{i=1}^{n} \left(f(t_i) - \sum_{j=1}^{n} \beta_j h_j(t_i) \right)^2 + \gamma \cdot k! \cdot \Delta t^k \sum_{j=k+2}^{n} |\beta_j|$$

$$\widehat{f}_0(t;\gamma) = \sum_{j=1}^{n} \widehat{\beta}_j h_j(t)$$

References: Tibshirani & Taylor (2011), Tibshirani (2014), Ramdas & Tibshirani (2015), Politsch et al. (2020a,b)



ℓ_p-penalized linear models

	High-dimensional	1D splines*
$p = 0$	**Subset selection** $\min_{\{\beta_j\}} \sum_{i=1}^{n} \left(y_i - \sum_j \beta_j x_{ji} \right)^2$ s.t. $\sum_{j=1}^{d} \mathbb{1}\{\beta_j \neq 0\} = s$	**Variable-knot regression spline** $\min_{\{\beta_j\}} \sum_{i=1}^{n} \left(f(t_i) - \sum_j \beta_j \eta_j(t_i) \right)^2$ s.t. $\sum_{j \geq k+2} \mathbb{1}\{\beta_j \neq 0\} = s$
$p = 1$	**Lasso** $\min_{\{\beta_j\}} \sum_{i=1}^{n} \left(y_i - \sum_j \beta_j x_{ji} \right)^2$ s.t. $\sum_{j=1}^{d} \lvert\beta_j\rvert \leq \tau$	
$p = 2$	**Ridge regression** $\min_{\{\beta_j\}} \sum_{i=1}^{n} \left(y_i - \sum_j \beta_j x_{ji} \right)^2$ s.t. $\sum_{j=1}^{d} \beta_j^2 \leq \tau$	**Smoothing spline** $\min_{\{\beta_j\}} \sum_{i=1}^{n} \left(f(t_i) - \sum_j \beta_j \eta_j(t_i) \right)^2$ s.t. $\sum_{j,k=1}^{n} \beta_j \beta_k \omega_{jk} \leq \tau$

ℓ_p-penalized linear models

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