Some remarks on locally representable algebraic weak factorisation systems

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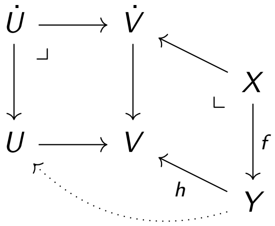
Theorem (Cohen, Coquand, Huber and Mörtberg)

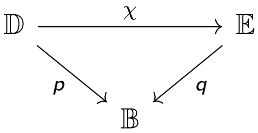
Homotopy type theory can be modelled (constructively) in cubical sets.

Dependent types appear in the model as maps with a certain algebraic structure called a *Kan filling operation*.

The universe of small types $\dot{U} \to U$ classifies Kan filling operations in the following sense. U is a pullback of the Hofmann-Streicher universe $\dot{V} \to V$ of maps with small fibres. If a map f is a pullback of $\dot{V} \to V$ along a fixed map h, then Kan filling operations on f correspond precisely to factorisations of h through U:



Suppose we are given a functor χ between Grothendieck fibrations that preserves cartesian maps:



χ restricts to a functor between the wide subcategories of cartesian maps, $\mathsf{Cart}(\chi) : \mathsf{Cart}(\mathbb{D}) \to \mathsf{Cart}(\mathbb{E})$.

Definition

We say χ *creates cartesian lifts* if $\mathsf{Cart}(\chi)$ is a discrete fibration.

We refer to a pair $(\mathbb{D}, \chi)$ where χ creates cartesian lifts as a *notion of fibered structure* on the fibration $q : \mathbb{E} \to \mathbb{B}$.

Suppose we are given a functor $\chi : \mathbb{D} \to \mathbb{E}$ that creates cartesian lifts. For each $X \in \mathbb{E}$, we can define a presheaf $\bar{\chi}_X$ on $\mathbb{B}/q(X)$ as follows. Given a map $\sigma : I \to q(X)$, we take $\bar{\chi}_X(\sigma)$ to be the set of objects of $\chi^{-1}(\sigma^*(X))$.

Definition (Shulman)

We say χ is *locally representable* if it satisfies any of the equivalent conditions below.

1. For every $X \in \mathbb{E}$ the presheaf $\bar{\chi}_X$ is representable.
2. $\mathrm{Cart}(\chi)$ has a right adjoint as a functor $\mathrm{Cart}(\mathbb{D}) \to \mathrm{Cart}(\mathbb{E})$ in **Cat**.
3. $\mathrm{Cart}(\chi)$ is comonadic as a functor $\mathrm{Cart}(\mathbb{D}) \to \mathrm{Cart}(\mathbb{E})$ in **Cat**.

Well known definitions such as Bénabou's definable classes, many instances of Johnstone's comprehension schemes, and local smallness of a fibration can all be phrased in terms of local representability.

Definition

Let $q : \mathbb{E} \to \mathbb{B}$ be a fibration. A *monad over* $\mathbb{B}$ is a functor $T : \mathbb{E} \to \mathbb{E}$ such that $T \circ q = q$, together with pointwise vertical natural transformations $\eta : 1_{\mathbb{E}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ satisfying the usual monad laws.

Theorem

If T is a monad, then the forgetful functor on T-algebras, $\chi : T\text{-}\mathrm{Alg} \to \mathbb{E}$ is a notion of fibred struture. We refer to such notions of fibred structure as monadic.

If $q : \mathbb{E} \to \mathbb{B}$ is locally small and T preserves cartesian maps, then χ is locally representable.

Algebraic weak factorisation systems (awfs's) are usually formulated in terms of functorial factorisations with extra structure. We can alternatively view them as certain monadic notions of fibred structure by the following result.

Theorem (Bourke-Garner)

An algebraic weak factorisation system (awfs) on $\mathbb{B}$ is precisely a monadic notion of fibred structure on $\mathsf{cod} : \mathbb{B}^{\rightarrow} \to \mathbb{B}$, *say* $\chi : T\text{-}\mathsf{Alg} \to \mathbb{B}^{\rightarrow}$, *together with a "composition" operator on* $T\text{-}\mathsf{Alg}$ *that makes* $T\text{-}\mathsf{Alg}$ *into a double category and* χ *a double functor.*

For many naturally occuring examples of awfs's the monad does not preserve cartesian maps.

Kan fibrations in simplical sets are usually defined as maps with the right lifting property against horn inclusions. However, the following alternative definitions are sometimes used with the same underlying wfs but different awfs's.

1. Right lifting property against pushout product of monomorphism and interval endpoint inclusion[1]
2. Right lifting property against pushout product of boundary inclusion and interval endpoint inclusion[2]
3. Monoidal lifting property against horn inclusions

Theorem (S)

None of the awfs's on $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ *listed above are locally representable.*

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[1] Definition of Kan fibration in cubical sets, B_3 in Gabriel and Zisman

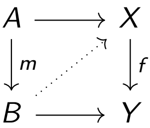
[2] B_2 in Gabriel and Zisman

Key ideas in the proof:

1. If $T\text{-}\mathrm{Alg} \to \mathbf{Set}^{\Delta^{\mathrm{op}}\to}$ is locally representable, then $\mathrm{Cart}(T\text{-}\mathrm{Alg}) \to \mathrm{Cart}(\mathbf{Set}^{\Delta^{\mathrm{op}}\to})$ is comonadic, and so creates colimits.
2. The interval object $1 \rightrightarrows \Delta_1$ admits a linear order in the internal language of simplicial sets (in fact simplicial sets is the classifying topos for linear orders with endpoints).
3. The interval object has disjoint endpoints and is connected (so the map $1 + 1 \to \Delta_1$ is a monomorphism but not an isomorphism).
4. We construct in the internal language of $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ two different Kan fibration structures on a certain pushout that agree when restricted to each pushout inclusion.

Given a vertical map $m : A \to B$ in a locally small Grothendieck fibration $q : \mathbb{E} \to \mathbb{B}$ over $I \in \mathbb{B}$, we define a notion of fibred structure on the fibration of vertical maps $V(\mathbb{E}) \overset{\mathrm{cod}}{\to} \mathbb{E} \to \mathbb{B}$ as follows.

A *lifting structure* on a vertical map $f : X \to Y$ is a section of the canonical map $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)$.



If the notion of fibred structure given by lifting structures is monadic, then it determines an awfs on each fibre category $\mathbb{E}_J$. We call this the *fibred awfs cofibrantly generated by m*.

Definition

We say an object B of $\mathbb{E}$ is *tiny relative to* J if the functor $\text{Hom}(B, -) : \mathbb{E}_J \to \mathbb{B}/(I \times J)$ has a right adjoint.

Example

An object in a set indexed family fibration $(B_i)_{i \in I}$ is tiny if the hom set functor $\text{Hom}(B_i, -)$ has a right adjoint for each i in the set I.

Example

For a codomain fibration $\text{cod} : \mathbb{B}^{\to} \to \mathbb{B}$, we can view an object B of $\mathbb{B}$ as a family over 1, $B \to 1$. It is tiny relative to 1 when the exponential $(-)^B$ has a right adjoint.

Theorem (S)

Suppose we are given a map $m : A \to B$ *in a locally small fibration* $q : \mathbb{E} \to \mathbb{B}$*. If* B *is tiny relative to* J *then lifting structures on* $\mathrm{cod} : \mathbb{E}_J^{\to} \to \mathbb{E}_J$ *are locally representable.*

In particular if lifting structures are monadic, then the resulting awfs's cofibrantly generated by m are locally representable. Applying to codomain fibrations, we recover results due to Licata-Orton-Pitts-Spitters, and a later version by Awodey.

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Suppose we are given a map $m : A \to B$ in a locally small fibration $q : \mathbb{E} \to \mathbb{B}$. If B is tiny relative to J then lifting structures on $\mathrm{cod} : \mathbb{E}_J^{\to} \to \mathbb{E}_J$ *are locally representable.*

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A paper will appear soon, *Locally representable algebraic weak factorisation systems*.
The definition of cofibrantly generated awfs appears in Swan, *Lifting problems in Grothendieck fibrations*, arXiv:1802.06718

Thank you for your attention!