Plan for the three talks:

1. Basic syntax of H.O.T.T.
2. Symmetries and semicartesian cubes
3. Semantics of univalent universes
Outline

1. A calculus of telescopes
2. Some problems revealed by cubes
3. Symmetry solves all problems
4. Semicartesian cubes
5. Semantic identity types
• Last week I described the “Book” version of H.O.T.T., starting with simple ideas, and introducing complexity only as necessary.

• By way of review, let’s reformulate the resulting theory more concisely and cleanly.

In particular, we eventually ended up with $n$-variable $ap$ (and $Id$) that bind a finite list of variables:

\[
\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash t : B \quad \cdots \\
\Gamma \vdash ap_{x_1, \ldots, x_n}(p_1, \ldots, p_n) : Id_B(\cdots)
\]

Such a “context suffix” is also called a **telescope**. We now reify these into a “telescope calculus”.
Telescopes

Telescopes are defined inductively as finite lists of types:

\[ \Gamma \vdash \epsilon \text{ tel} \]

\[ \Gamma \vdash \Delta \text{ tel} \quad \Gamma, \Delta \vdash A : U \]

\[ \Gamma \vdash (\Delta, x : A) \text{ tel} \]

The “elements” of a telescope are substitutions:

\[ (\epsilon, \delta) : \Gamma \vdash A : U \quad \delta : \Delta \]

\[ a : A \quad a[\delta] : A[\delta] \]

These are defined mutually with their action on terms (and types):
Now we can define identity telescopes from identity types:

\[
\Delta \text{ tel } \delta : \Delta \quad \delta' : \Delta \\
\frac{}{\text{Id}_\Delta(\delta, \delta') \text{ tel}}
\]

\[
\text{Id}_\epsilon((\epsilon, \epsilon)) \equiv \epsilon
\]

These are defined mutually with \(n\)-ary \(\text{Id}\), which depends on them:

\[
\rho : \text{Id}_\Delta(\delta, \delta') \quad \Delta \vdash A : U \\
\frac{}{\text{Id}_\Delta^\to_A(a, a') : U}
\]

We write \(\text{Id}_A(a, a') \equiv \text{Id}^{(\epsilon)}_{\epsilon, A}(a, a')\) in the non-dependent case.

(Last time I defined dependent \(\text{Id}\) in terms of \(\text{ap}\); here we postulate it separately and then make them coincide later.)
As we saw last time, \( \text{Id} \) computes on all type formers:

\[
\text{Id}_A \times B(s, t) \equiv \text{Id}_{A \times B}(\pi_1 s, \pi_1 t) \times \text{Id}_{A}(\pi_2 s, \pi_2 t)
\]

\[
\text{Id}_{\Sigma x:A B}(s, t) \equiv \sum_{q:\text{Id}_A(\pi_1 s, \pi_1 t)} \text{Id}_{\Sigma(A,B)}(\Delta, x:A).B(\pi_2 s, \pi_2 t)
\]

\[
\text{Id}_{A \rightarrow B}(f, g) \equiv \prod_{u:A} \prod_{v:A} \prod_{q:\text{Id}_A(u, v)} \text{Id}_{\Delta \rightarrow B}(f u, g v)
\]

\[
\text{Id}_{\Pi(x:A) B}(f, g) \equiv \prod_{u:A} \prod_{v:A} \prod_{q:\text{Id}_A(u, v)} \text{Id}_{\Pi(A,B)}(\Delta, x:A).B(f u, g v)
\]
Id is a 1-1 correspondence

All identity types are 1-1 correspondences:

\[ \rho : \text{Id}_\Delta(\delta, \delta') \quad \Delta \vdash A : U \quad a : A[\delta] \]
\[ \overline{\text{corr}}^\rho_{\Delta.A}(a) : \text{isContr(} \sum_{(a' : A[\delta'])} \text{Id}^\rho_{\Delta.A}(a, a')) \]

\[ \rho : \text{Id}_\Delta(\delta, \delta') \quad \Delta \vdash A : U \quad a' : A[\delta'] \]
\[ \overline{\text{corr}}^\rho_{\Delta.A}(a') : \text{isContr(} \sum_{(a : A[\delta])} \text{Id}^\rho_{\Delta.A}(a, a')) \]

The centers of contraction constitute transport:

\[ \rho : \text{Id}_\Delta(\delta, \delta') \quad \Delta \vdash A : U \quad a : A[\delta] \]
\[ \overrightarrow{\text{tr}}^\rho_{\Delta.A}(a) : A[\delta'] \quad \overrightarrow{\text{lift}}^\rho_{\Delta.A}(a) : \text{Id}^\rho_{\Delta.A}(a, \overrightarrow{\text{tr}}^\rho_{\Delta.A}(a)) \]

These witnesses compute on type formers:

\[ \overline{\text{corr}}^\rho_{\Delta.A \times B}(a) \equiv \ldots \]

hence also

\[ \overrightarrow{\text{tr}}^\rho_{\Delta.A \times B}(a) \equiv \ldots \]
Computing ap

A term can be applied to \( \text{Id} \) of any telescope it depends on:

\[
\varrho : \text{Id}_\Delta(\delta, \delta') \quad \Delta \vdash t : B \\
\frac{}{\text{ap}_\Delta t(\varrho) : \text{Id}^\varrho_{\Delta B}(t[\delta], t[\delta'])}
\]

This higher-dimensional explicit substitution computes on all* terms:

\[
\text{ap}_\Delta (s,t)(\varrho) \equiv (\text{ap}_\Delta s(\varrho), \text{ap}_\Delta t(\varrho))
\]

\[
\text{ap}_\Delta .\pi_1s(\varrho) \equiv \pi_1 \text{ap}_\Delta s(\varrho) \\
\text{ap}_\Delta .\pi_2s(\varrho) \equiv \pi_2 \text{ap}_\Delta s(\varrho)
\]

\[
\text{ap}_\Delta .fb(\varrho) \equiv \text{ap}_\Delta .f(p)(b[a/x], b[a'/x], \text{ap}_\Delta .b(\varrho)).
\]

\[
\text{ap}_\Delta .(\lambda y.t)(\varrho) \equiv \lambda u.\lambda v.\lambda q.\text{ap}_\Delta .y.t(\varrho, q).
\]

We define reflexivity as the 0-ary ap: \( \text{refl}_a \equiv \text{ap}_{\epsilon a}() \).
Univalence

\( \text{Id}_U(A, B) \) contains as a retract the type of 1-1 correspondences:

\[
1\text{-1-Corr}(A, B) \equiv \sum_{(R: A \to B \to U)} \left( \prod_{(a:A)} \text{isContr}(\sum_{(b:B)} R(a, b)) \right) \\
\times \left( \prod_{(b:B)} \text{isContr}(\sum_{(a:A)} R(a, b)) \right).
\]

\[
1\text{-1-Corr}(A, B) \uparrow_{\sim} \text{Id}_U(A, B) \downarrow_{\sim} 1\text{-1-Corr}(A, B) \quad p_{\uparrow \downarrow} \equiv p
\]

We identify dependent \( \text{Id} \) with \( \text{ap} \) into the universe:

\[
\text{Id}^\omega_{\Delta, B}(b, b') \equiv \pi_1(\text{ap}_{\Delta, B}(\varrho)\downarrow)(b, b')
\]

\[
\overline{\text{corr}}^\omega_{\Delta, B}(b, b') \equiv \pi_1\pi_2(\text{ap}_{\Delta, B}(\varrho)\downarrow)(b, b')
\]

\[
\underline{\text{corr}}^\omega_{\Delta, B}(b, b') \equiv \pi_2\pi_2(\text{ap}_{\Delta, B}(\varrho)\downarrow)(b, b')
\]

(Last time, we defined the LHS as the RHS. Separating them is more natural for Tarski universes, and permits types not lying in any universe.)
I claimed that \( \text{ap} \) is never a normal form, but there's one exception:

**When \( y \) is a variable, \( \text{refl}_y \) is neutral** (hence normal).

Since \( \text{refl} \) is nullary \( \text{ap} \), the rule that would apply is

\[
\text{ap}_{x_1 \ldots x_n}(p_1, \ldots, p_n) \equiv \text{refl}_y \quad \text{(if \( y \) is a variable } \notin \{x_1, \ldots, x_n\}\text{)}
\]

where \( n = 0 \), but this just reduces \( \text{refl}_y \equiv \text{ap}_{()}(y) \) to itself!

This includes other terms that obviously must also be neutral:

- \( \text{ap}_{x.f}(p) \equiv \text{refl}_f(a_0, a_1, p) \) for a variable \( f : A \to B \).
- \( \text{Id}_A(a_0, a_1) \equiv (\pi_1 \text{refl}_A)(a_0, a_1) \) for a variable \( A : U \).

Similarly, \( \text{refl}_{\text{refl}_x} \), \( \text{refl}_{\text{refl}_{\text{refl}_x}} \), etc., are also neutral.
1. A calculus of telescopes
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H.O.T.T. is **not** a “cubical type theory”: there are no explicit cubes in the syntax. But like any other type theory with dependent identity types (including Book HoTT!), it has an **emergent** notion of cube:

\[ a_{02} : \text{Id}_A(a_{00}, a_{01}) \quad a_{12} : \text{Id}_A(a_{10}, a_{11}) \quad a_{20} : \text{Id}_A(a_{00}, a_{10}) \]

\[ a_{21} : \text{Id}_A(a_{01}, a_{11}) \quad a_{22} : \text{Id}^{a_{02}, a_{12}}_{x,y} \text{Id}_A(x,y)(a_{20}, a_{21}) \]

![Diagram of squares and cubes](image)

Similarly, \( \text{Id}_{\text{Id}_{\text{Id}_A}} \) is a type of 3-dimensional cubes, etc.

**Very important point**

The roles of \( a_{02}, a_{12} \) and \( a_{20}, a_{21} \) are asymmetrical!
Given \( a_{02}, a_{12}, a_{20} \), we have fillers of left-to-right cubical horns:

\[
\begin{align*}
\text{tr}^{a_{02}, a_{12}}_{x. y. \text{Id}_A(x, y)}(a_{20}) : \text{Id}_A(a_{01}, a_{11}) \\
\text{lift}^{a_{02}, a_{12}}_{x. y. \text{Id}_A(x, y)}(a_{20}) : \text{Id}^{a_{02}, a_{12}}_{x. y. \text{Id}_A(x, y)}(a_{20}, \text{tr}^{a_{02}, a_{12}}_{x. y. \text{Id}_A(x, y)}(a_{20}))
\end{align*}
\]

Similarly, \( \text{tr} \) and \( \text{lift} \) fill right-to-left cubical horns. And \( \text{tr}_{\text{Id}_A} \), etc. fill higher-dimensional left-right horns.

**Problem #1**

We don’t seem to have top-to-bottom or bottom-to-top fillers.
Degenerate cubes

Given \( a_2 : \text{Id}_A(a_0, a_1) \), there are two degenerate squares:

\[
\begin{align*}
\text{refl}_{a_2} & : \text{Id}_{\text{Id}_A(a_0,a_1)}(a_2, a_2) \equiv \text{Id}_{x,y.\text{Id}_A(x,y)}(a_2, a_2) \\
ap_{x.\text{refl}_x}(a_2) & : \text{Id}_{x.\text{Id}_A(x,x)}(\text{refl}_{a_0}, \text{refl}_{a_1}) \equiv \text{Id}_{x,y.\text{Id}_A(x,y)}(\text{refl}_{a_0}, \text{refl}_{a_1})
\end{align*}
\]
Degenerate cubes

Given $a_2 : \text{Id}_A(a_0, a_1)$, there are two degenerate squares:

For $a : A$, the two doubly-degenerate squares seem to be definitionally unrelated.
Problem #3

Our rules so far compute $\text{refl}_{a_2}$ based on the structure of $a_2$, but $\text{ap}_x.\text{refl}_x(a_2)$ is stuck, even if $a_2$ is very concrete.

- $\text{refl}_x$ doesn’t reduce when $x$ is a variable.
- $\text{ap}$ doesn’t inspect its identification argument.
Problem #3

Our rules so far compute \( \text{refl}_{a_2} \) based on the structure of \( a_2 \), but \( \text{ap}_{x, \text{refl}_x}(a_2) \) is stuck, even if \( a_2 \) is very concrete.

- \( \text{refl}_x \) doesn’t reduce when \( x \) is a variable.
- \( \text{ap} \) doesn’t inspect its identification argument.

A bit nonobviously, this also breaks canonicity for \( \mathbb{N} \).

Intuitive homotopy-theoretic reason

For a type \( A : U \), the square \( \text{ap}_{x, \text{refl}_x}(\text{refl}_A) \) in \( U \) is essentially a self-homotopy of the identity equivalence of \( A \), i.e. \( \prod_{(a : A)} \text{Id}_A(a, a) \).

Taking \( A = S^1 \) we get a stuck loop in \( \text{Id}_{S^1}(\text{base}, \text{base}) \), hence in \( \mathbb{Z} \).

(There’s also an explicit argument using two universes instead of \( S^1 \).)
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To solve these problems, we introduce a symmetry operation that transposes squares:

$$a_{10} \xrightarrow{a_{12}} a_{11}$$

$$\overrightarrow{a_{20}} \overrightarrow{a_{22}} \overrightarrow{a_{21}}$$

$$a_{00} \xrightarrow{a_{02}} a_{01}$$

$$a_{01} \xrightarrow{a_{21}} a_{11}$$

$$a_{02} \xrightarrow{\text{sym}_A(a_{22})} a_{12}$$

$$\text{sym}_A(a_{22}) : \text{Id}_{x,y} \cdot \text{Id}_A(x,y)(a_{02}, a_{12})$$
The other Kan operations

Now we can fill other cubical horns, solving problem #1:

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{01} & \rightarrow & a_{11} \\
  a_{02} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{10} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{01} & \rightarrow & a_{11} \\
  a_{02} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{10} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]

\[
\begin{array}{ccc}
  a_{10} & \rightarrow & a_{11} \\
  a_{20} & \uparrow & \downarrow a_{21} \\
  a_{00} & \rightarrow & a_{01} \\
  a_{02} & \downarrow & \uparrow a_{02} \\
  a_{00} & \rightarrow & a_{10} \\
\end{array}
\]
To solve problem #3, we define

$$\text{ap}_x.\text{refl}_x(a_2) \equiv \text{sym}_A(\text{refl}_{a_2})$$

This computes based on $a_2$... if sym also computes!
Computing symmetry

To solve problem #3, we define

\[ \text{ap}_x \text{refl}_x(a_2) \equiv \text{sym}_A(\text{refl}_{a_2}). \]

This computes based on \( a_2 \) ... if \( \text{sym} \) also computes!

For the most part, computing symmetry is straightforward, e.g.:

\[
\text{Id}_{s_0^2, s_1^2}^{u \cdot v} \text{Id}_{A \times B}(u, v)(s_{20}, s_{21}) \\
\equiv \text{Id}_{s_0^2, s_1^2}^{u \cdot v} \text{Id}_A(\pi_1 u, \pi_1 v) \times \text{Id}_B(\pi_2 u, \pi_2 v)(s_{20}, s_{21}) \\
\equiv \text{Id}_{s_0^2, s_1^2}^{u \cdot v} \text{Id}_A(\pi_1 s_{20}, \pi_1 s_{21}) \times \text{Id}_{s_0^2, s_1^2}^{u \cdot v} \text{Id}_B(\pi_2 s_{20}, \pi_2 s_{21}) \\
\equiv \text{Id}_{s_0^2, s_1^2}^{x \cdot w} \text{Id}_A(x, w)(\pi_1 s_{20}, \pi_1 s_{21}) \times \text{Id}_{s_0^2, s_1^2}^{y \cdot z} \text{Id}_B(y, z)(\pi_2 s_{20}, \pi_2 s_{21}).
\]

So we can define

\[ \text{sym}_{A \times B}((p, q)) \equiv (\text{sym}_A(p), \text{sym}_B(q)) \]
To generalize this to \( \Sigma \)-types, we need dependent symmetry over a square in a telescope (don’t worry too much about the syntax):

\[
\begin{align*}
\delta_{22} &: \text{Id}^\delta_{02, \delta_2} (\delta, \delta') (\delta_{20}, \delta_{21}) \\
a_{22} &: \text{Id}^\delta_{02, \delta_2, \delta_{22}, a_{02}, a_{12}} (\delta, \delta', \rho, u, v) (a_{20}, a_{21}) \\
\text{sym}^\delta_{22} (a_{22}) &: \text{Id}^\delta_{02, \delta_2, \rho, u, v} (\delta, \delta', \rho, u, v) (a_{20}, a_{22}, a_{21})
\end{align*}
\]

Then we can define

\[
\text{sym}^\delta_{22, \Sigma(x:A)B} ((p, q)) \equiv (\text{sym}^\delta_{22, A} (p), \text{sym}^\delta_{22, p} (\Delta, x:A).B (q))
\]
Symmetry for functions

\[ \text{Id}_{f \cdot g \cdot \text{Id}}(f, g)(f_{20}, f_{21}) \equiv \text{Id}_{f \cdot g \cdot \Pi(x_0: A) \Pi(x_1: A) \Pi(x_2: \text{Id}_A(x_0, x_1)) \text{Id}_B(f x_0, g x_1)}(f_{20}, f_{21}) \]

\[ \equiv \Pi(x_{00}: A) \Pi(x_{01}: A) \Pi(x_{02}: \text{Id}_A(x_{00}, x_{01})) \]
\[ \Pi(x_{10}: A) \Pi(x_{11}: A) \Pi(x_{12}: \text{Id}_A(x_{10}, x_{11})) \]
\[ \Pi(x_{20}: \text{Id}_A(x_{00}, x_{10})) \Pi(x_{21}: \text{Id}_A(x_{01}, x_{11})) \Pi(x_{22}: \text{Id}_x^x \cdot \text{Id}_y^x \cdot \text{Id}_z^x \cdot \text{Id}_t^x \cdot \text{Id}_u^x \cdot \text{Id}_v^x)(f_{20} x_{20}, f_{21} x_{21}) \]

So \( f_{22} : \text{Id}_{f \cdot g \cdot \text{Id}}(f, g)(f_{20}, f_{21}) \) is a function from squares in \( A \), with arbitrary boundary, to squares in \( B \) with specified boundary. Thus we define \( \text{sym}_{A \rightarrow B} \) by transposing both input and output:

\[ \text{sym}_{A \rightarrow B}(f_{22})(x_{00}, x_{10}, x_{20}, x_{01}, x_{11}, x_{21}, x_{02}, x_{12}, x_{22}) \equiv \text{sym}(f_{22}(x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, \text{sym}(x_{22}))) \]

Symmetry for \( \Pi \)-types is similar, using dependent symmetry.
Some obvious rules for symmetry are that it should be an involution:

\[ \text{sym}_A(\text{sym}_A(a_{22})) \equiv a_{22} \]

and it should commute with iterated \text{ap} on squares:

\[ \text{sym}_B(\text{ap}_\text{ap}_f(a_{22})) \equiv \text{ap}_\text{ap}_f(\text{sym}_A(a_{22})) \]

The nullary case of the latter is \( \text{sym}(\text{refl}_{\text{refl}_a}) \equiv \text{refl}_{\text{refl}_a} \).

This solves problem #2:

\[ \text{ap}_{x.\text{refl}_x}(\text{refl}_a) \equiv \text{sym}(\text{refl}_{\text{refl}_a}) \equiv \text{refl}_{\text{refl}_a} \]
Higher-dimensional symmetry

For $n$-dimensional cubes (i.e. $n$-fold iterated Id-types):

- We would expect symmetries to permute all $n$ dimensions. The symmetric group $S_n$ should act on $n$-cubes.

- We have transpositions of adjacent dimensions, from our sym.
  (E.g. $\text{sym}_{\text{Id}_A} : \text{Id}_{\text{Id}_{\text{Id}_A}} \to \text{Id}_{\text{Id}_{\text{Id}_A}}$ and $\text{ap}_{\text{sym}_A} : \text{Id}_{\text{Id}_{\text{Id}_A}} \to \text{Id}_{\text{Id}_{\text{Id}_A}}$.)

Fortunately, $S_n$ is generated by adjacent transpositions!

$$S_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \begin{array}{l}
\sigma_k\sigma_k = 1 \\
\sigma_j\sigma_k = \sigma_k\sigma_j \ (j + 1 < k) \\
\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}
\end{array} \right\rangle$$

The first two relations follow from the equations on the last slide. To obtain the third, we assert

$$\text{sym}_{\text{Id}_A}(\text{ap}_{\text{sym}_A}(\text{sym}_{\text{Id}_A}(a_{222}))) \equiv \text{ap}_{\text{sym}_A}(\text{sym}_{\text{Id}_A}(\text{ap}_{\text{sym}_A}(a_{222}))).$$
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Symmetry computes the previously stuck term $\text{ap}_{x.\text{refl}_x}(a_2)$. But how do we know there aren’t other stuck terms?

Obviously, by proving canonicity/normalization.

We haven’t done this yet, but the first step (from a modern perspective) is constructing a set-based semantic model to be the codomain for Artin gluing.
Identity contexts

**Question**

What categorical structure corresponds to our identity types?

- The objects of a category $\mathcal{C}$ correspond to syntactic contexts.
- The fundamental operation on contexts takes $\Delta$ to

$$\text{ID}_\Delta \equiv (\delta : \Delta, \delta' : \Delta, \varrho : \text{Id}_\Delta(\delta, \delta'))$$

which factors the diagonal (i.e. is a path object):

$$\Delta \xrightarrow{\text{refl}} \text{ID}_\Delta \rightarrow (\delta : \Delta, \delta' : \Delta) \simeq \Delta \times \Delta.$$

- This operation is functorial (via ap).
- We have natural symmetries $\text{ID}_{\text{ID}_\Delta} \simeq \text{ID}_{\text{ID}_\Delta}$, yielding an $S_n$-action on $n$-fold identity contexts.
Cubical actions

Thus, an ID-structure on $\mathcal{C}$ is the same as

- A functor $\text{ID} : \mathcal{C} \to \mathcal{C}$
- Nat. trans. $r : 1_\mathcal{C} \to \text{ID}$ and $s, t : \text{ID} \Rightarrow 1_\mathcal{C}$ with $sr = tr = 1_{1_\mathcal{C}}$
- Natural symmetries $\text{ID} \circ \text{ID} \cong \text{ID} \circ \text{ID}$ satisfying $S_n$ relations.

**Definition**

Let $\Box^{\text{op}}$ be the monoidal category freely generated by an object $\blacksquare$, morphisms $r : 1 \to \blacksquare$ and $s, t : \blacksquare \to 1$ with $sr = tr = 1_1$, where 1 is the unit, and symmetries $\blacksquare \otimes \blacksquare \cong \blacksquare \otimes \blacksquare$ satisfying $S_n$ relations.

Then an ID-structure on $\mathcal{C}$ is also equivalently

- A monoidal functor $\Box^{\text{op}} \to [\mathcal{C}, \mathcal{C}]$

and therefore also equivalently

- A coherent action $\Box^{\text{op}} \times \mathcal{C} \to \mathcal{C}$. 
The semicartesian cube category

• □ is a semicartesian monoidal category: symmetric monoidal and its unit 1 is terminal. Projections, but no diagonals.
• It is also the semicartesian monoidal category freely generated by an object I and morphisms s, t : 1 → I.

We call □ the semicartesian cube category.

This is the category used by:
• Bernardy–Coquand–Moulin, for internal parametricity (actually they used a unary version, this would be the binary one)
• Bezem–Coquand–Huber, for the original cubical model
• Cavallo–Harper, for the parametricity direction of parametric cubical type theory
The presheaf category $\hat{\square} = \text{Set}^{\square^{\text{op}}}$ inherits a Day convolution monoidal structure (also semicartesian):

$$(X \otimes Y)_n = \int^{k,\ell} X_k \times Y_\ell \times \square(n, k \oplus \ell).$$

We write $\square^n$ for the representable $\square(-, I^{\otimes n})$. Note $\square^0$ is terminal.

**Theorem**

An action $\triangleright : \square^{\text{op}} \times C \to C$ is the same as an enrichment of $C$ over $\hat{\square}$ that has powers by representables (write $\square^n \triangleright X \equiv I^{\otimes n} \triangleright X$).

$$\text{Map}(A, B)_n := C(A, \square^n \triangleright B)$$

$$\hat{\square}(X, \text{Map}(A, \square^n \triangleright B)) \cong \hat{\square}(X \otimes \square^n, \text{Map}(A, B))$$

$\hat{\square}$-enriched categories are the natural home for H.O.T.T. semantics.
Cubical objects

Of course, $\square$ is enriched over itself.

Similarly, any category $\mathcal{E}^{\square^{\text{op}}}$ of cubical objects is $\square$-enriched, with powers and copowers if $\mathcal{E}$ is complete and cocomplete:

\[(A \odot X)_n = \int_{k, \ell} (A_k \times \square(n, k \oplus \ell)) \cdot X_\ell\]

\[(A \pitchfork X)_n = \int_{k, \ell} (X_k \times \square(k, n \oplus \ell)) A_\ell\]

\[(\square^m \pitchfork X)_n = X_{n \oplus m}\]

\[\text{Map}(X, Y)_n = \mathcal{E}^{\square^{\text{op}}}(X, \square^n \pitchfork Y)\]
More about the cube category

Up to equivalence:

- The objects of $p$ are finite sets.
- A morphism $\phi \in p(m, n)$ is a function $\phi : n \to m \sqcup \{-, +\}$ that is injective on the preimage of $m$.
- The monoidal structure $m \oplus n$ is disjoint union.

Sometimes use a skeletal version with objects $\underline{n} = \{0, 1, \ldots, n - 1\}$, but often the non-skeletal version with all finite sets is better.

- The coface $\delta_k, \pm \in p(n \setminus \{k\}, n)$ is the identity on $n \setminus \{k\}$ and sends $k$ to $\pm$.
- The codegeneracy $\sigma_k \in p(n, n \setminus \{k\})$ is the inclusion.
- The endomorphism monoid $p(n, n)$ is the symmetric group $S_n$.
The monoidal structure of \( \square \) is “almost” cartesian; only the injectivity requirement spoils it. If it were cartesian we would have

\[
\square(n, k \oplus \ell) \cong \square(n, k) \times \square(n, \ell).
\]

Instead, we have

\[
\square(n, k \oplus \ell) \cong \sum_{\phi: \square(n, k)} \square(n \setminus \phi(k), \ell).
\]

Removing \( \phi(k) \) from the second domain ensures the copaired function \( k \sqcup \ell \to n \sqcup \{-, +\} \) is still injective on the preimage of \( n \).

But in some ways this is even better!
Copowers by representables

For $A \in \hat{\square}$ and $X \in \mathcal{E}^{\square^{\text{op}}}$, we have

$$(A \odot X)_n = \int^{k, \ell} (A_k \times \square(n, k \oplus \ell)) \cdot X_\ell$$

$$(\square^m \odot X)_n = \int^{k, \ell} (\square(k, m) \times \square(n, k \oplus \ell)) \cdot X_\ell$$

$$= \int^\ell \square(n, m \oplus \ell) \cdot X_\ell$$

$$= \int^\ell \left( \sum_{\phi \in \square(n, m)} \square(n \setminus \phi(m), \ell) \right) \cdot X_\ell$$

$$= \sum_{\phi \in \square(n, m)} \int^\ell \square(n \setminus \phi(m), \ell) \cdot X_\ell$$

$$= \sum_{\phi \in \square(n, m)} X_{n \setminus \phi(m)}.$$
Taking $m = 1$, we get

$$(\Box^1 \circ X)_n = \sum_{\phi \in \Box(n, 1)} X_{n \setminus \phi(1)}.$$

A morphism $\phi \in \Box(n, 1)$ is a function $1 \to n \sqcup \{-, +\}$, so either:

- some $k \in n$, in which case $n \setminus \phi(1) = n \setminus \{k\}$, or
- $+$ or $-$, in which case $n \setminus \phi(1) = n$. Thus:

$$(\Box^1 \circ X)_n = X_n + X_n + \sum_{k \in n} X_{n \setminus \{k\}}.$$

An $n$-cube in $\Box^1 \circ X$ is either an $n$-cube in the left-hand copy of $X$, an $n$-cube in the right-hand copy of $X$, or an $(n - 1)$-cube in $X$ stretched out in some dimension along the cylinder.

There is almost no other cube category for which this holds.
Outline

1. A calculus of telescopes
2. Some problems revealed by cubes
3. Symmetry solves all problems
4. Semicartesian cubes
5. Semantic identity types
In a $\hat{\mathcal{C}}$-enriched category with representable powers, we also need:

1. Coherence theorems.
2. Transport and lifting ("fibrancy").
3. Categorical computation rules for $\text{Id}$, up to isomorphism.

It's tempting to think that, at least in $\hat{\mathcal{C}}$, we can just define $\text{Id}_{A \times B}$, $\text{Id}_{A \to B}$, etc., to be whatever we want. But we can't: $\text{Id}_X$ must be defined as $\square^1_X$. What we can define is the individual sets of $n$-cubes in a particular $X \in \hat{\mathcal{C}}$. But:

- It can be non-obvious how these lead to a categorical characterization of the entire cubical set $\text{Id}_X$.
- For type formers like $A \times B$, $A \to B$, we don't even have this much choice: they are determined by their universal properties.

The computation rules for $\text{Id}$ are non-trivial theorems about $\mathcal{E} \hat{\mathcal{C}}^{\text{op}}$. 
Semantic identity types

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Identity types of products

Note $x : A, y : A \vdash \text{Id}_A(x, y) : U$ is represented semantically by the projection from the representable power $\square^1 \pitchfork A \to A \times A$.

Since $(\square^1 \pitchfork -)$ is a right adjoint, it preserves products:

$$
\begin{align*}
\square^1 \pitchfork (A \times B) & \cong (\square^1 \pitchfork A) \times (\square^1 \pitchfork B) \\
\downarrow & \\
(A \times B) \times (A \times B) & \cong (A \times A) \times (B \times B)
\end{align*}
$$

Syntactically, this gives

$$
\text{Id}_{A \times B}(u, v) \cong \text{Id}_A(\pi_1 u, \pi_1 v) \times \text{Id}_B(\pi_2 u, \pi_2 v).
$$

Same idea works for $\Sigma$-types. A coherence theorem will improve $\cong$ to $=$.
Plan for the three talks:

1. Basic syntax of H.O.T.T.
2. Symmetries and semicartesian cubes
3. Univalent universes