Tutorial on
Polynomial Functors and Type Theory

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Workshop on Polynomial Functors
Topos Institute
March 2022
Outline

Part I

1 Polynomials
2 Type theory
3 Natural models of type theory

Part II

4 Universes in presheaves
5 A polynomial monad
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1. Polynomials

Let $\mathcal{E}$ be a locally cartesian closed category. Thus for every map $f : B \to A$ we have adjoint functors on the slice categories,

$$
\begin{array}{c}
B \\
\downarrow f \\
A
\end{array} \quad \begin{array}{c}
\Sigma_f \\
\downarrow f^* \\
\Pi_f \\
\downarrow \quad \downarrow \\
\mathcal{E}/B \\
\mathcal{E}/A
\end{array}
$$

When $A = 1$ we write

$$
\Sigma_B \dashv B^* \dashv \Pi_B
$$

for the corresponding functors determined by $B \to 1$. 
1. Polynomials

Definition

The polynomial endofunctor $P_f : \mathcal{E} \to \mathcal{E}$ determined by a map

$$f : B \to A$$

is the composite

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{P_f} & \mathcal{E} \\
\downarrow \downarrow & \searrow & \uparrow \uparrow \\
\mathcal{E}/B & \longrightarrow & \mathcal{E}/A \\
\downarrow \pi_f & & \downarrow \Sigma_A \\
\mathcal{E}/B & \longrightarrow & \mathcal{E}/A
\end{array}
\]

which we may write in the internal language of $\mathcal{E}$ as

$$P_f X = \Sigma_A \Pi_f B^* X = \Sigma_A \Pi_f f^* A^* X$$

$$= \Sigma_A \Pi_f f^* A^* X = \Sigma_A (A^* X)^f = \Sigma_{x:A} X^{B(x)}.$$
1. Polynomials

The construction of $P_fX$ can be visualized as follows:

$$
\begin{array}{c}
\mathcal{E} \\
\downarrow B^* \\
\mathcal{E}/B \\
\downarrow \Pi_f \\
\mathcal{E}/A \\
\downarrow \Sigma_A \\
\mathcal{E}
\end{array}
$$

$$
\begin{array}{c}
X \\
\downarrow \\
X \times B \\
\downarrow \\
B \\
\downarrow _f \\
A
\end{array}
$$
1. Polynomials

Lemma (UMP of $P_f X$)

*Maps* $p : Z \to P_f X$ *correspond naturally to pairs* $(a, b)$ *where*

$$a : Z \to A \quad b : a^* B \to X.$$ 

**Proof.**
1. Polynomials

Now suppose we have a pullback square

\[
\begin{array}{ccc}
D & \rightarrow & B \\
\downarrow g & & \downarrow f \\
C & \rightarrow & A \\
\end{array}
\]
Then for each $X$ we get a map $t_X : P_g X \to P_f X$ as follows:

\[
\begin{array}{c}
D 	imes X \\[0.5cm]
\downarrow g \\[0.5cm]
\downarrow f \\
D \\[0.5cm]
\downarrow t \\
C \\
\downarrow t_X \\
P_g X \\
\end{array} \\
\begin{array}{c}
B 	imes X \\[0.5cm]
\downarrow B \\
A \\
\downarrow A \\
P_f X, \\
\end{array}
\]

because the lower square is a pullback by Beck-Chavalley,

\[P_g X \cong t^* P_f X.\]
1. Polynomials

Then for each \( X \) we get map \( t_X : P_g X \to P_f X \) as follows:

\[
\begin{array}{c}
D \times X \quad \xrightarrow{t_X} \quad B \times X \\
\downarrow{D} \quad \quad \downarrow{B} \\
D \quad \xrightarrow{t} \quad B \\
\quad \downarrow{t^* f = g} \quad \downarrow{f} \\
C \quad \xrightarrow{t} \quad A \\
\downarrow{C} \quad \quad \quad \quad \quad \quad \downarrow{A} \\
P_{t^* f} X \quad \xrightarrow{t_X} \quad P_f X,
\end{array}
\]

because the lower square is a pullback by Beck-Chavvalley,

\[ P_g X \cong t^* P_f X. \]

Indeed, since \( g = t^* f \), we have

\[ P_{t^* f} X \cong P_g X \cong t^* P_f X. \]
1. Polynomials

Then for each $h : Y \to X$ we have the pullback square below.

\[
\begin{array}{c}
D \times Y \ar[r]^{t \times Y} \ar[d]_{D \times h} & B \times Y \ar[d]_{B \times h} \\
D \times X \ar[r]_{t \times X} & B \times X \\
D \ar[r] \ar[d]_{g} & B \ar[d]_{f} \\
C \ar[r]_{t} & A \\
P_{g}X \ar[r]_{t_{X}} \ar[u]_{P_{g}h} & P_{f}X \ar[u]_{P_{f}h} \\
P_{g}Y \ar[r]_{t_{Y}} & P_{f}Y \ar[u]_{P_{f}h}
\end{array}
\]
1. Polynomials

Proposition

Taking the polynomial functor \( P_f : \mathcal{E} \to \mathcal{E} \) of a map \( f : B \to A \) determines a functor

\[
P : \mathcal{E}^{\text{cart}} \to \text{End}(\mathcal{E}).
\]

The cartesian squares in \( \mathcal{E}^{\rightarrow} \) are taken to cartesian natural transformations between endofunctors on \( \mathcal{E} \). Moreover, the polynomials are closed under composition.

Proof.

It remains only to show that polynomial functors compose: given any \( f : B \to A \) and \( g : D \to C \), there is a map \( h : F \to E \) such that

\[
P_g \circ P_f = P_h : \mathcal{E} \to \mathcal{E}.
\]

See Spivak (2022) for the definition of \( h = g \circ f \).\qed
2. Dependent type theory

Types:

\[ A, B, \ldots \]

Terms:

\[ x : A, \ b : B, \ldots \]

Dependent types: ("type-indexed families of types")

\[ x : A \vdash B(x) \]
\[ x : A, y : B(x) \vdash C(x, y) \]
\[ \ldots \]

Type forming operations:

\[ \sum_{x : A} B(x), \ \prod_{x : A} B(x), \ \ldots \]

Term forming operations:

\[ \langle a, b \rangle, \ \lambda x . b(x), \ \ldots \]

Equations:

\[ s = t : A \]
2. Dependent type theory: Rules

**Contexts:**

\[
\frac{x : A \vdash B(x)}{x : A, \ y : B(x) \vdash}
\]

Writing \( \Gamma \) for any context, we have:

\[
\frac{\Gamma \vdash C}{\Gamma, \ z : C \vdash}
\]
2. Dependent type theory: Rules

Sums:

\[
\Gamma, x : A \vdash B(x) \quad \frac{\Gamma \vdash \sum_{x : A} B(x)}{\Gamma \vdash \sum_{x : A} B(x)}
\]

\[
\frac{\Gamma \vdash a : A, \Gamma \vdash b : B(a)}{\Gamma \vdash \langle a, b \rangle : \sum_{x : A} B(x)}
\]

\[
\Gamma \vdash c : \sum_{x : A} B(x) \quad \frac{\Gamma \vdash c}{} \Gamma \vdash \text{fst} \ c : A
\]

\[
\frac{\Gamma \vdash c : \sum_{x : A} B(x)}{} \Gamma \vdash \text{snd} \ c : B(\text{fst} \ c)
\]

\[
\Gamma \vdash \text{fst} \langle a, b \rangle = a : A
\]

\[
\Gamma \vdash \text{snd} \langle a, b \rangle = b : B
\]

\[
\Gamma \vdash \langle \text{fst} \ c, \text{snd} \ c \rangle = c : \sum_{x : A} B(x)
\]
2. Dependent type theory: Rules

Sums:

\[
\begin{align*}
\Gamma, x : A & \vdash B(x) & \Gamma \vdash \sum_{x:A} B(x) \\
\Gamma & \vdash \sum_{x:A} B(x) & \Gamma, a : A, \Gamma & \vdash b : B(a) \\
\Gamma \vdash c : \sum_{x:A} B(x) & \Gamma & \vdash \sum_{x:A} B(x) \\
\Gamma & \vdash \text{fst } c : A & \Gamma & \vdash \text{snd } c : B(\text{fst } c) \\
\Gamma & \vdash \text{fst} \langle a, b \rangle = a : A & \Gamma & \vdash \text{snd} \langle a, b \rangle = b : B \\
\Gamma & \vdash \langle \text{fst } c, \text{snd } c \rangle = c : \sum_{x:A} B(x)
\end{align*}
\]
2. Dependent type theory: Rules

Sums:

\[
\begin{align*}
  & x : A \vdash B(x) \\
  & \sum_{x : A} B(x) \\
  & c : \sum_{x : A} B(x) \\
  & \text{fst } c : A \\
  & \text{snd } c : B(\text{fst } c) \\
  & \langle a, b \rangle = a : A \\
  & \langle \text{fst } c, \text{snd } c \rangle = c : \sum_{x : A} B(x)
\end{align*}
\]
2. Dependent type theory: Rules

**Products:**

\[ x : A \vdash B(x) \]

\[
\prod_{x : A} B(x)
\]

\[ x : A \vdash b(x) : B(x) \]

\[
\lambda x.b(x) : \prod_{x : A} B(x)
\]

\[ a : A \]

\[
f : \prod_{x : A} B(x)
\]

\[
fa : B(a)
\]

\[ x : A \vdash (\lambda x.b)x = b : B(x) \]

\[ \lambda x.fx = f : \prod_{x : A} B(x) \]
2. Dependent type theory: Substitution

A tuple of terms in context $\sigma : \Delta \rightarrow \Gamma$ induces an operation

$$\sigma : \Delta \rightarrow \Gamma \quad \Gamma \vdash a : A$$

$$\Delta \vdash a[\sigma] : A[\sigma]$$

which preserves everything.

For example given $y : Y \vdash s : Z$ and $z : Z, x : A(z) \vdash B(z, x)$ we can do

$$y : Y \vdash s : Z \quad \frac{z : Z, x : A(z) \vdash B(z, x)}{z : Z \vdash \prod_{x : A(z)} B(z, x)}$$

$$\Downarrow$$

$$y \vdash (\prod_{x : A(z)} B(z, x))[s/z]$$

or

$$\frac{y : Y \vdash s : Z \quad z : Z, x : A(z) \vdash B(z, x)}{y, x : A(s) \vdash B(s, x)}$$

$$\Downarrow$$

$$y : Y \vdash \prod_{x : A(s)} B(s, x)$$

and syntactically the results are the same,

$$\left( \prod_{x : A(z)} B(z, x) \right)[s/z] = \prod_{x : A(s)} B(s, x).$$

This suggests a reformulation as an indexed algebraic structure.
3. Natural models

Definition
A natural transformation \( f : Y \to X \) of presheaves on a category \( C \) is called *representable* if its pullback along any \( y : C \to X \) is representable:

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
y D & \longrightarrow & Y \\
\downarrow \downarrow & & \searrow f \\
y C & \longrightarrow & X \\
\end{array}
\]

Proposition (A, Fiore)
A representable natural transformation is the same thing as a **Category with Families** in the sense of Dybjer.
3. Natural models

Definition
A natural transformation $f : Y \to X$ of presheaves on a category $\mathcal{C}$ is called representable if its pullback along any $y_C \to X$ is representable: for all $C \in \mathcal{C}$ and $x \in X(C)$ there is given $p : D \to C$ and $y \in Y(D)$ such that the following is a pullback:

![Diagram]

Proposition (A, Fiore)
A representable natural transformation equipped with a choice of such pullbacks is the same thing as a Category with Families in the sense of Dybjer.
3. Natural models

Write the objects and arrows of $\mathbb{C}$ as $\sigma: \Delta \to \Gamma$, thinking of a *category of contexts and substitutions*.

Let $p: \hat{U} \to U$ be a representable map of presheaves on $\mathbb{C}$.

Think of $U$ as the *presheaf of types*, $\hat{U}$ as the *presheaf of terms*, and then $p$ gives the type of a term.

$$\Gamma \vdash A \approx A \in U(\Gamma)$$

$$\Gamma \vdash a : A \approx a \in \hat{U}(\Gamma)$$

where $A = p \circ a$. 

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{A} & U \\
\downarrow & & \downarrow p \\
\hat{U} & \xrightarrow{a} & U
\end{array}
\]
Naturality of $p : \hat{U} \to U$ means that for any substitution $\sigma : \Delta \to \Gamma$, we have the required action on types and terms:

\[
\begin{align*}
\Gamma \vdash A & \Rightarrow \Delta \vdash A[\sigma] \\
\Gamma \vdash a : A & \Rightarrow \Delta \vdash a[\sigma] : A[\sigma]
\end{align*}
\]
Given any further $\tau : \Delta' \to \Delta$ we clearly have

\[
A[\sigma][\tau] = A[\sigma \circ \tau] \quad a[\sigma][\tau] = a[\sigma \circ \tau]
\]

and for the identity substitution $1 : \Gamma \to \Gamma$

\[
\]

This is the basic structure of a CwF.
The remaining operation of **context extension**

\[
\Gamma \vdash A \\
\frac{}{\Gamma, x : A \vdash}
\]

is modeled by the representability of \( p : \hat{U} \to U \) as follows.
3. Natural models, context extension

Given $\Gamma \vdash A$ we need a new context $\Gamma . A$ together with a substitution $p_A : \Gamma . A \to A$ and a term

$$\Gamma . A \vdash q_A : A[p_A] .$$

Let $p_A : \Gamma . A \to \Gamma$ be the pullback of $p$ along $A$.

\[
\begin{array}{c}
\Gamma . A \xrightarrow{\quad q_A \quad} \hat{U} \\
p_A \downarrow \quad \downarrow p \\
\Gamma \quad \xrightarrow{\quad A \quad} \quad U
\end{array}
\]

The map $q_A : \Gamma . A \to \hat{U}$ gives the required term $\Gamma . A \vdash q_A : A[p_A]$. Syntactically, this is just the term

$$\Gamma, x : A \vdash x : A .$$
3. Natural models, context extension

\[
\begin{array}{c}
\Delta \\
\downarrow \sigma \\
\downarrow (\sigma, a) \\
(\sigma, a) \\
\downarrow \\
\Gamma.A \\
\downarrow q_A \\
\downarrow \\
\Gamma \\
\downarrow A \\
\downarrow p_A \\
\downarrow \\
\Gamma \\
\downarrow A \\
\downarrow p \\
\downarrow \\
\U \\
\end{array}
\]

The pullback means that given any substitution \( \sigma : \Delta \rightarrow \Gamma \) and term \( \Delta \vdash a : A[\sigma] \) there is a map

\[(\sigma, a) : \Delta \rightarrow \Gamma.A \]

satisfying

\[p_A(\sigma, a) = \sigma\]
\[q_A[\sigma, a] = a.\]
3. Natural models, context extension

By the uniqueness of \((\sigma, a)\), we also have

\[(\sigma, a) \circ \tau = (\sigma \circ \tau, a[\tau])\]

for any \(\tau : \Delta' \to \Delta\)

and

\[(p_A, q_A) = 1.\]

These are all the laws for a CwF.
3. Natural models, algebraic formulation

Natural models can be presented as an essentially algebraic theory, with several sorts, partial operations, and equations between terms.

We have four basic sorts:

\[ C_0, C_1, A, B \]

and the following operations and equations:

**category:** the usual domain, codomain, identity and composition operations for the index category \( C \):

\[
\begin{array}{c}
C_1 \times C_0 \xrightarrow{\circ} C_1 \\
\xrightarrow{\text{dom}} \quad \xleftarrow{\text{cod}} \quad \xrightarrow{id} \quad C_0,
\end{array}
\]

Together with the familiar equations for a category.
3. Natural models, algebraic formulation

**presheaf:** the indexing and action operations for the presheaves $A, B : \mathbb{C}^{\text{op}} \to \text{Set}$:

$$
\begin{align*}
C_1 \times C_0 A \xrightarrow{\alpha} A \\
\downarrow \quad p_A \\
C_0
\end{align*}
$$

$$
\begin{align*}
C_1 \times C_0 B \xrightarrow{\beta} B \\
\downarrow \quad p_B \\
C_0
\end{align*}
$$

together with the equations making $\alpha$ an action:

\[
p_A(\alpha(u, a)) = \text{dom}(u),
\]

\[
\alpha(u \circ v, a) = \alpha(v, \alpha(u, a)),
\]

\[
\alpha(1_{p_A(a)}, a) = a,
\]

and similarly for $\beta$. 
3. Natural models, algebraic formulation

natural transformation: an operation

\[ f : A \rightarrow B \]

satisfying the naturality equations:

\[ p_B \circ f = p_A, \quad f \circ \alpha = \beta \circ (C_1 \times C_0 f) \]

representable: a natural transformation \( f : A \rightarrow B \) is representable just if the associated functor,

\[ \int_C f : \int_C A \rightarrow \int_C B \]

on the categories of elements has a right adjoint

\[ f^* : \int_C B \rightarrow \int_C A \]

(an algebraic condition, see Newstead (2018)).
3. Natural models and initiality

- The notion of a natural model is thus *essentially algebraic*.
- The algebraic homomorphisms correspond exactly to syntactic translations.
- There is an *initial algebra* as well as a *free algebra* over any signature of basic types and terms.
- The rules of dependent type theory specify a procedure for generating the free algebra.
3. Natural models and tribes

Let \( p : \hat{U} \to U \) be a natural model.

The fibration
\[
\int_C U \to C
\]
of all \textit{display maps} \( p_A : \Gamma.A \to \Gamma \), for all \( A : \Gamma \to U \), determines a \textit{clan} in the sense of Joyal (2017).

Conversely, given a clan \( \mathcal{D} \hookrightarrow \mathcal{C} \), there is a natural model in \( \hat{\mathcal{C}} \),
\[
\bigsqcup_{f \in \mathcal{D}} yf : \bigsqcup_{f \in \mathcal{D}} y\text{dom}(f) \longrightarrow \bigsqcup_{f \in \mathcal{D}} y\text{cod}(f).
\]

This natural model \( p_\mathcal{D} : \hat{\mathcal{U}}_\mathcal{D} \to U_\mathcal{D} \) determines a \textit{splitting} of the associated fibration \( \mathcal{D} \to \mathcal{C} \).
3. Natural models and tribes

**Theorem (ish)**

*There is an adjunction between the categories of clans and of natural models, which specializes to a biequivalence between (certain) tribes and natural models with (certain) type-forming operations.*

Part II
4. Universes in presheaves

Recall the notion of a Hofmann-Streicher universe

\[ \hat{V} \to V \]

in a category of presheaves \( \hat{C} = \text{Set}^{\text{op}} \).

1. Let \( \text{set} \to \text{Set} \) be the full subcategory of small sets \( s < \kappa \).
2. Let \( \hat{\text{set}} = 1/\text{set} \) be the category of small pointed sets.
3. Then for \( c \in C \) let:

\[ V(c) = \text{Cat}(C/c^{\text{op}}, \text{set}) \]

the set of small presheaves on \( C/c \),

\[ \hat{V}(c) = \text{Cat}(C/c^{\text{op}}, \hat{\text{set}}) \]

... small pointed presheaves on \( C/c \).

4. The action on \( d \to c \) is given by precomposition with postcomposition \( C/d \to C/c \).

5. There is a natural transformation \( \hat{V} \to V \) determined by composing with the forgetful functor \( \hat{\text{set}} \to \text{set} \).
4. Universes in presheaves

**Definition**

In a category $\hat{\mathcal{C}} = \text{Set}^{\mathcal{C}^{\text{op}}}$ of presheaves,

- an object $A$ is *small* if its values $A(c)$ are small, for all $c \in \mathcal{C}$,
- a map $A \to X$ is *small* if its fibers $A_x = x^*A$ are small, for all $x : yc \to X$.

![Diagram]

\[ \begin{array}{ccc}
A_x & \to & A \\
\downarrow & & \downarrow \\
yc & \to & X.
\end{array} \]

Note that small maps are stable under pullback. And that the map $\hat{V} \to V$ is small, since the fiber $\hat{V}_S$ over $S : yc \to V$ has as elements pointed presheaves $\hat{S} : \mathcal{C}/c \to \text{set}$. 
4. Universes in presheaves

Proposition

For every small map $A \to X$ there is a canonical classifying map $\alpha : X \to V$ fitting into a pullback diagram of the form

$$
\begin{array}{ccc}
A & \to & \hat{V} \\
\downarrow & & \downarrow \\
X & \to & V.
\end{array}
$$

Proof.

Do it first for the small maps $A_x \to yc$, for all $x : yc \to X$, for which there is a canonical choice of $\alpha_x : yc \to V$. Then use the presentation of $X$ as a colimit over its category of elements $(c, x) \in \int_C X$ to get $\alpha : X \to U$. \qed
4. Universes in presheaves

Remark
For large enough \( \kappa \) the small maps are closed under the adjoints \( \Sigma_A \vdash A^* \vdash \Pi_A \) to pullback along small maps \( A \to X \).

This fact gives rise to natural operations on the universe \( \hat{V} \to V \) that can be used to (coherently!) model the corresponding type-forming operations, as follows:

- a universe \( \hat{V} \to V \) is a natural model on the category of contexts \( \hat{C} \),

- a universe \( \hat{V} \to V \) generates a polynomial endofunctor

\[
P : \hat{C} \to \hat{C}.
\]

- The type forming operations in the natural model will correspond to algebraic structure on the polynomial endofunctor.
5. Polynomial monad and type formers

Let \( p : \hat{U} \to U \) be a natural model on an arbitrary category \( \mathbb{C} \), and consider the associated polynomial endofunctor,

\[
P = U! \circ p_* \circ \hat{U}^* : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}},
\]

which we can write as,

\[
P(X) = \sum_{A : U} X^{[A]},
\]

where \([A] = p^{-1}(A)\) is the fiber of \( p : \hat{U} \to U \) at \( A : U \).

Lemma

Maps \( \Gamma \to P(X) \) correspond naturally to pairs \((A, B)\) where

\[
\begin{array}{c}
X \xleftarrow{B} \Gamma \cdot A \xrightarrow{} \hat{U} \\
\downarrow \quad \downarrow \\
\Gamma \xrightarrow{A} U
\end{array}
\]
Applying $P$ to $U$ \textit{itself} therefore gives the object

$$PU = \sum_{A:U} U^A,$$

for which maps $\Gamma \to PU$ correspond naturally to pairs $(A, B)$ of the form,

$$U \leftarrow^B \Gamma.A \rightarrow^A U$$

Since maps $\Gamma \to U$ correspond naturally to types in context $\Gamma \vdash A$, we see that maps $\Gamma \to PU$ correspond naturally to types in the extended context $\Gamma.A \vdash B$. 
5. Polynomial monad and type formers

Proposition

For a natural model \( \dot{U} \rightarrow U \), the polynomial object

\[
P_U = \sum_{A:U} U^A
\]

classifies types in context. Specifically, there is a natural isomorphism between maps \( \Gamma \rightarrow PU \) and pairs \((A, B)\) where

\[
\Gamma . A \vdash B.
\]

Similarly, the object

\[
P_{\dot{U}} = \sum_{A:U} \dot{U}^A
\]

classifies terms in context: pairs \((A, b : B)\) where \(\Gamma . A \vdash b : B\), for \((A, B)\) the composite with \(Pp : P_{\dot{U}} \rightarrow PU\).
5. Polynomial monad and type formers: $\Pi$

Proposition

*The natural model* $p : \hat{U} \rightarrow U$ *models the rules for products just if there are maps* $\lambda, \Pi$ *making the following a pullback.*

\[
\begin{array}{ccc}
PU & \xrightarrow{\lambda} & \hat{U} \\
\downarrow Pp & & \downarrow p \\
PU & \xrightarrow{\Pi} & U
\end{array}
\]
Proposition

The map \( p : \hat{U} \to U \) models the rules for products just if there are maps \( \lambda, \Pi \) making the following a pullback.

Proof:
Proposition

The map $p : \hat{U} \to U$ models the rules for products just if there are maps $\lambda, \Pi$ making the following a pullback.

Proof:
The map $p : \hat{U} \to U$ models the rules for products just if there are maps $\lambda, \Pi$ making the following a pullback.

Proof:

\[
\begin{aligned}
A \vdash b : B & \quad \lambda_{A}b \\
\sum_{A : U} \hat{U}^A & \quad P(\hat{U}) \xrightarrow{\lambda} \hat{U} \\
\sum_{A : U} U^A & \quad P(U) \xrightarrow{\Pi} U \\
A \vdash B & \quad \Pi_{AB}
\end{aligned}
\]
5. Polynomial monad and type formers: Π

Proposition

The map $\rho : \hat{U} \to U$ models the rules for products just if there are maps $\lambda, \Pi$ making the following a pullback.

Proof:

$$
\begin{array}{c}
\sum_{A : U} \hat{U}[A] \\
\sum_{A : U} U[A] \\
A \vdash B \\
\end{array}
\begin{array}{c}
P(\hat{U}) \\
P(U) \\
\Pi_A B \\
\end{array}
\begin{array}{c}
\lambda \\
\Pi \\
\end{array}
\begin{array}{c}
\hat{U} \\
U \\
\end{array}
\begin{array}{c}
f \\
\end{array}
$$
5. Polynomial monad and type formers: $\Pi$

**Proposition**

The map $p : \hat{U} \to U$ models the rules for products just if there are maps $\lambda, \Pi$ making the following a pullback.

**Proof:**

\[
\begin{align*}
A \vdash fx : B & \quad \lambda_A fx = f \\
\sum_{A : U} \hat{U}^A & \quad P(\hat{U}) \xrightarrow{\lambda} \hat{U} \\
\sum_{A : U} U^A & \quad P(U) \xrightarrow{\Pi} U \\
A \vdash B & \quad \Pi AB
\end{align*}
\]
5. Polynomial monad and type formers: $\Sigma$

Proposition

The map $p : U \to U$ models the rules for sums just if there are maps $(\text{pair}, \Sigma)$ making the following a pullback

\[
\begin{array}{ccc}
Q & \xrightarrow{\text{pair}} & U \\
\downarrow q & & \downarrow p \\
P(U) & \xrightarrow{\Sigma} & U
\end{array}
\]

where $q = p \triangleleft p : Q \to P(U)$ is the generating map of the composite $P_q = P_{p \triangleleft p} = P_p \circ P_p$.

Explicitly:

$$Q = \sum_{A : U} \sum_{B : U^A} \sum_{x : A} B(x)$$
5. Polynomial monad and type formers: \( T \)

Rules for a terminal type \( T \)

\[
\begin{align*}
\Gamma & \vdash T \\
\Gamma & \vdash \ast : T \\
x : T & \vdash x = \ast : T
\end{align*}
\]

Proposition

The map \( p : \hat{U} \to U \) models the rules for a terminal type just if there are maps \((\ast, T)\) making the following a pullback.

\[
\begin{array}{ccc}
1 & \xrightarrow{*} & \hat{U} \\
\downarrow & & \downarrow p \\
1 & & U \\
\downarrow T & & \downarrow \end{array}
\]
5. Polynomial monad

Consider the pullback squares for $T$ and $\Sigma$.

\[
\begin{array}{ccc}
1 & \xrightarrow{*} & \bar{U} \\
\downarrow & & \downarrow p \\
1 & \xrightarrow{T} & U
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Q & \xrightarrow{\text{pair}} & \bar{U} \\
\downarrow p \triangleright p & & \downarrow p \\
P(U) & \xrightarrow{\Sigma} & U
\end{array}
\]

These determine cartesian natural transformations between the corresponding polynomial endofunctors.

\[
\tau : 1 \Rightarrow P \quad \quad \sigma : P \circ P \Rightarrow P
\]
5. Polynomial monad

Theorem (A-Newstead)

A natural model \( p : \mathbb{U} \rightarrow \mathbb{U} \) models the \( T \) and \( \Sigma \) type formers just if the associated polynomial endofunctor \( P \) has the structure maps of a cartesian monad.

\[
\tau : 1 \Rightarrow P \quad \sigma : P \circ P \Rightarrow P
\]

What about the monad laws?
5. Polynomial monad

The monad laws correspond to the following type isomorphisms.

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \circ P\sigma = \sigma \circ \sigma_P$</td>
<td>$\sum_a \sum_b C(a, b) \cong \sum_{(a,b)} B(a)$</td>
</tr>
<tr>
<td>$\sigma \circ P\tau = 1$</td>
<td>$\sum_a 1 \cong A$</td>
</tr>
<tr>
<td>$\sigma \circ \tau_P = 1$</td>
<td>$\sum_x A \cong A$</td>
</tr>
</tbody>
</table>
The pullback square for $\Pi$

\[
\begin{array}{c}
P\dot{U} \xrightarrow{\lambda} \dot{U} \\
\downarrow Pp \quad \downarrow p \\
PU \xrightarrow{\Pi} U
\end{array}
\]

determines a cartesian natural transformation

$$\pi : P^2 p \Rightarrow p$$

where $P^2 : \hat{C}^2 \rightarrow \hat{C}^2$ is the extension of $P$ to the arrow category.
5. Polynomial monad

Theorem (A-Newstead)

A natural model \( p : \mathbb{U} \to \mathbb{U} \) models the \( \prod \) type former just if it has an algebra structure for the extended endofunctor \( P^2 \),

\[
\pi : P^2 p \Rightarrow p.
\]
5. Polynomial monad

The algebra laws correspond to the following type isomorphisms.

\[
\begin{array}{|c|c|}
\hline
\pi \circ P \pi & = & \pi \circ \sigma \\
\hline
\prod_{a:A} \prod_{b:B(a)} C(a, b) & \cong & \prod_{(a,b):\sum \! A} C(a, b) \\
\hline
\pi \circ \tau & = & 1 \\
\prod_{x:1} A & \cong & A \\
\hline
\end{array}
\]
6. Propositions and types

We can compare these operations on types

\[ \Sigma, \Pi : PU \rightarrow U \]

with those on subobjects of objects \( A \) in the topos \( \hat{C} \),

\[ \exists_A, \forall_A : \Omega^A \rightarrow \Omega. \]

Consider

\[ P\Omega = \sum_{A:U} \Omega^A \]

for the polynomial endofunctor of \( \dot{U} \rightarrow U \). We then have the comparable maps

\[ \exists, \forall : P\Omega \rightarrow \Omega. \]
Proposition

There is a retraction \( i : \Omega \leftrightarrow U \), \( s : U \rightarrow \Omega \) such that the following squares commute.

\[
\begin{array}{ccc}
P\Omega & \xrightarrow{\exists} & \Omega \\
Pi & \downarrow & s \\
PU & \xrightarrow{\Sigma} & U \\
\end{array}
\]

\[
\begin{array}{ccc}
P\Omega & \xrightarrow{\forall} & \Omega \\
Pi & \downarrow & s \\
PU & \xrightarrow{\prod} & U \\
\end{array}
\]
6. Propositions and types

For the proof, factor the natural model \( p : \dot{U} \to U \) as on the right below.

\[
\begin{align*}
\Gamma.A & \to \dot{U} \\
\Gamma.\llbracket A \rrbracket & \to \llbracket \dot{U} \rrbracket \\
\Gamma & \to U
\end{align*}
\]

So \( \llbracket \dot{U} \rrbracket \to U \) is a universal family of small propositions.
Let $s : U \to \Omega$ classify the mono $\||\dot{U}|| \hookrightarrow U$. 

\[
\begin{array}{ccc}
\Gamma.A & \rightarrow & \dot{U} \\
\downarrow & & \downarrow \\
\Gamma.||A|| & \rightarrow & ||\dot{U}|| \\
\downarrow & & \downarrow \\
\Gamma & \rightarrow & U \\
\downarrow A & & \downarrow s \\
\Gamma & \rightarrow & \Omega \\
\end{array}
\]
Let $s : U \to \Omega$ classify the mono $\|\dot{U}\| \hookrightarrow U$.

Let $i : \Omega \to U$ classify the family of small propositions $1 \rightarrow \Omega$. 
Let

\[ || \cdot || := i \circ s : U \to U. \]

We have

\[ s \circ i = 1 : \Omega \to \Omega. \]

So

\[ \Omega = \text{im}(|| \cdot ||). \]
The following diagrams then commute, as required.
References


Appendix: Natural models of HoTT

Theorem

A category $\mathcal{C}$ with a terminal object $1$ admits a natural model of Homotopy type theory if it has a class of maps $\mathcal{D}$ satisfying the following conditions:

- **total:** every $\mathcal{C} \to 1$ is in $\mathcal{D}$,
- **stable:** $\mathcal{D}$ is closed under pullbacks along all maps in $\mathcal{C}$,
- **closed:** $\mathcal{D}$ is closed under composition and under dependent products along all maps in $\mathcal{D}$,
- **factorizing:** every map $f : A \to B$ in $\mathcal{C}$ factors as $f = d \circ a$ with $a \in \mathcal{D}$ and $d \in \mathcal{D}$.

Proof.

Uses the main idea of the Lumsdaine-Warren coherence theorem: a left-adjoint splitting of the fibration of $\mathcal{D}$-maps.
Examples of categories satisfying the conditions of the theorem:

- Kan complexes with the fibration wfs on sSets.
- Any right-proper Cisinski model category (restricted to the fibrant objects).
- Groupoids, $n$-Groupoids, $\infty$-Groupoids.
- Joyal’s $\pi h$-tribes.
- The syntactic category of contexts of type theory itself.