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Joyal's representation theorem for Heyting categories

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Abstract

An elegant result by André Joyal, believed to have been formulated around 1970, extends the Stone representation theorem for Boolean algebras to Heyting categories. Namely :

Every small Heyting category $\mathcal{H}$ can be embedded in one of the form $\mathbf{Set}^{\mathbb{C}}$ via a Heyting, conservative functor.

Representation theorems can be regarded as categorical equivalents to completeness theorems, specifically, they imply the latter through the construction of syntactic categories encapsulating first-order theories. Through this bridge, Joyal's representation theorem provides completeness results for intuitionistic propositional calculus, coherent logic, classical first-order logic, and notably, intuitionistic first-order logic.

In this work, we introduce a new, categorical approach to the model-theoretic proof of Joyal's theorem originally presented by Makkai and Reyes in [MR77].

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Contents

Introduction

The result at the core of this work, often referred to as *Joyal's Completeness Theorem*, *Joyal's Representation Theorem* or *"canonical version" of Kripke completeness*, has been discussed in different places. In 1977, Makkai and Reyes provided a model-theoretic proof ([MR77]). A refined version of the theorem has been presented as *Kripke-Joyal Completeness for Heyting Fibrations* by Makkai in 1993 [Mak93], with a proof following a similar approach. In their compelling dissertation on the history of categorical logic [MR], Marquis and Reyes analyzed and contextualized Joyal's theorem. Finally, Makkai and Reyes demonstrated how to modify the embedding to achieve completeness with respect to Kripke models in 1995, see [MR95].

The proof due to Makkai and Reyes relies on the modification of the theory $\mathbb{T}_{\mathcal{H}}$ associated to the category $\mathcal{H}$ to a specific new theory $\mathbb{T}^*_{\mathcal{H}}$, of which a model should witness the conditions that make the embedding preserve the Heyting structure. For that construction, the model-theoretic "method of diagrams" is used. To establish the existence of a model for this new theory, both a compactness theorem for finitary logic and Gödel completeness theorem for first-order logic are employed.

Our proof will be a categorical translation of this logic-oriented proof, i.e all these constructions will be done in terms of categories (for the theories), and of functors (for the models). We will introduce our work by giving an overview of the concepts that allow these correspondences, in 1.1.

Joyal's result is a representation theorem that generalizes the Stone representation theorem for Boolean algebras. The latter asserts that every Boolean algebra is isomorphic to the dual algebra of its associated Stone space. An equivalent formulation is the following :

For any Boolean algebra $\mathcal{B}$, there is a set X and an injective Boolean homomorphism $\mathcal{B} \hookrightarrow 2^X$

where $2^X \cong \mathcal{P}(X)$ is the set of all functions from X to a two-valued set. It is possible to consider Heyting algebras instead of Boolean ones :

For any Heyting algebra $\mathcal{H}$, there is an injective homomorphism of Heyting algebras $\mathcal{H} \hookrightarrow \mathbb{2}^X$

where $\mathbb{2}^X$ is the Heyting algebra of monotone maps from the poset $\mathbb{2} = \{0 \leq 1\}$ to a poset X. This latter result gives completeness of intuitionistic propositional calculus with respect to Kripke semantics and is merely a special case of Joyal's representation theorem, when the Heyting category $\mathcal{H}$ is a poset.

To advance the result, we will have to transition from a power of $\mathbb{2}$ to a power of **Set**, specifically to the category of covariant functors $\mathbf{Set}^{\mathbb{C}}$. We will see that the embedding is under the form of a double dual, in fact, $\mathbb{C}$ can be taken to be a full subcategory of the category of coherent models from $\mathcal{H}$ to **Set**, denoted as $\mathrm{Coh}(\mathcal{H}, \mathbf{Set})$. This is reminiscent of the Heyting algebra case, where X is taken as the poset of distributive lattice homomorphisms $\mathrm{DLat}(\mathcal{H}, \mathbb{2})$ (and not the Heyting homomorphisms !).

Another analogy is the use of the Prime Ideal Theorem in the context of distributive lattices compared to the application of Deligne's theorem in our case. The Prime Ideal Theorem says that we can always separate a point from a certain ideal with another prime ideal. In our framework, this corresponds to separating (morphisms in) $\mathcal{H}$ by the points $\mathrm{Coh}(\mathcal{H}, \mathbf{Set})$. This property of separation equate Deligne's theorem, which states that every coherent topos has enough points.

Interestingly, Deligne's result will be used twice in our proof. First, it provides the conservativity of the functor $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$ - a functor is conservative if it reflects isomorphisms. Second, it will be utilized in the final step of the proof in 2.3 to demonstrate the existence of the desired coherent model, echoing the classical Gödel completeness theorem used by Makkai, for the same purpose.

1 Background

1.1 Tastes of categorical logic

This section overviews several concepts and constructions for categorical first-order logic, all of which are detailed in [Awo24]. This particularly motivates our subsequent interpretation of theories as categories and functors as models. It also provides the context needed to derive completeness results categorically, as will be developed in Section 3.

We begin by focusing on the coherent fragment of first-order logic, which involves coherent theories :

Definition 1.1.1. [Awo24] A (many-sorted) *coherent theory* $\mathbb{T}$ is a (many-sorted) type theory together with a set of axioms expressed in the fragment of logic built from $=, \top, \wedge, \exists, \bot$, and $\vee$.

A first-order theory consists in fact of a type theory and of axioms, which are built from formulas. These formulas are themselves constructed using typed relation symbols and function symbols, along with operations from a specific fragment of logic. There are rules governing typing judgments and the formation of formulas according to the relevant fragment of logic. Inference rules then specify how new sequents can be derived from given sequents and contexts.

Categories provide semantics for such theories : there is a specific way to *interpret* different theories within suitable categories, in terms of subobjects and additional structure. For example, regular and coherent theories can be interpreted in regular and coherent categories, respectively (see the definitions below). An example of a regular theory is the theory of categories; the theory of local rings or the theory of fields are coherent theories.

Moreover, there is a notion of satisfaction of a sequent within a categorical interpretation. An interpretation is considered a *model* of a theory if each axiom of the theory is satisfied within that interpretation. The models of a theory $\mathbb{T}$ in a category $\mathcal{C}$, and the homomorphisms between them, form a category denoted $\mathrm{Mod}(\mathbb{T}, \mathcal{C})$.

Definition 1.1.2. A finitely complete category $\mathcal{C}$ is *regular* if :

1. coequalizers of kernel pairs exist in $\mathcal{C}$;
2. regular epimorphisms are pullback-stable in $\mathcal{C}$.

In any category $\mathcal{C}$ with finite limits, for any object $x \in \mathcal{C}$, the poset of subobjects $\mathrm{Sub}(x)$ is a meet semi-lattice. In coherent categories, this poset has coproducts and is a distributive lattice.

Definition 1.1.3. A *coherent category* is a regular category in which posets of subobjects $\mathrm{Sub}(a)$ have finite unions (i.e coproducts) and where each pullback functor $f^{-1} : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ preserves them.

Here is an useful equivalent definition of a regular category :

Theorem 1.1.4. *Let $\mathcal{C}$ be a finitely complete category. Then $\mathcal{C}$ is a regular category if and only if :*

1. *any arrow in $\mathcal{C}$ factorizes as a regular epimorphism followed by a monomorphism;*
2. *these factorizations are pullback-stable.*

Proof. See [Gra20, 1.14] □

Remark 1.1.5. For a morphism $f : a \to b$, we will denote its reg-epi/mono factorization as $\mathrm{im}(f)$ (it is unique up to isomorphism). It has in fact the property that whenever f factors through a monic h, so does $\mathrm{im}(f)$. Moreover it gives a right adjoint to the pullback functor $f^{-1} : \mathrm{Sub}(b) \longrightarrow \mathrm{Sub}(a)$, $\exists_f : \mathrm{Sub}(a) \longrightarrow \mathrm{Sub}(b)$, $\exists_f(x : u \rightarrowtail a) = \mathrm{im}(f \circ x)$.

The approach used to prove logical results in categorical ways is called *functorial semantics.* As suggested by the name, functors play a central role.

Definition 1.1.6. A *regular functor* $F : \mathcal{C} \to \mathcal{D}$ is a functor between regular categories which preserves finite limits and regular epimorphisms.

Definition 1.1.7. A *coherent functor* is a regular functor between coherent categories that preserves finite unions of subobjects.

Notation 1.1.8. We denote the category of coherent (resp. regular) functors between coherent (resp. regular) categories $\mathcal{C}$ and $\mathcal{D}$, and natural transformations between them, as $\mathrm{Coh}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Reg}(\mathcal{C}, \mathcal{D})$. In the case of categories with finite limits, we denote the category of finite limit preserving functors $\mathrm{Lex}(\mathcal{C}, \mathcal{D})$.

In the context of coherent logic, functorial semantics rely on the ability to construct a *classifying category* $\mathcal{C}_{\mathbb{T}}$ for any coherent theory $\mathbb{T}$. This category has the following universal property :

Theorem 1.1.9. *For any coherent theory $\mathbb{T}$, the syntactic category $\mathcal{C}_{\mathbb{T}}$ classifies $\mathbb{T}$-models in coherent categories. Specifically, for any coherent category $\mathcal{C}$, there is an equivalence of categories, natural in $\mathcal{C}$,*

$$\mathrm{Coh}\left(\mathcal{C}_{\mathbb{T}}, \mathcal{C}\right) \simeq \mathrm{Mod}(\mathbb{T}, \mathcal{C})$$

where $\mathrm{Coh}\left(\mathcal{C}_{\mathbb{T}}, \mathcal{C}\right)$ *is the category of coherent functors and natural transformations. In particular, there is a universal model U in $\mathcal{C}_{\mathbb{T}}$.*

Proof. See [Awo24, 3.2.40]. □

In other words, models of a coherent theory $\mathbb{T}$ in a category $\mathcal{C}$ can be identified to coherent functors from this classifying category $\mathcal{C}_{\mathbb{T}}$ to $\mathcal{C}$. Moreover, the homomorphisms of models are the natural transformations between the corresponding functors.

In this work, we will therefore assume the liberty to treat any coherent category $\mathcal{H}$ as the syntactic category of a certain coherent theory $\mathbb{T}$, and coherent functors as models, without explicitly stating it.

An *universal model* U is a model in the syntactic category $\mathcal{C}_{\mathbb{T}}$ such that any model of $\mathbb{T}$ in a category $\mathcal{C}$ is a functorial image of U via a unique functor. This model has the property that provability of a sequent $\varphi \vdash \psi$ from the given rules of inference and the axioms of $\mathbb{T}$ is equivalent to satisfaction in U :

$$U \models (\Gamma \mid \varphi \vdash \psi) \quad \text{iff} \quad \mathbb{T} \text{ proves } (\Gamma \mid \varphi \vdash \psi) \tag{1.1.1}$$

Finally, the rules for formation of formulas and the rules of inference for coherent logic can be extended to the universal quantifier $\forall$. The categories that support an interpretation of this logical operator are the Heyting categories. This fragment of logic is called *Heyting logic* or more commonly *intuitionistic first-order logic.*

Definition 1.1.10. A *Heyting category* is a coherent category in which for each map $f : a \to b$, the pullback functor $f^{-1} : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ has a right adjoint $\forall_f$:

$$\begin{tikzcd} \mathrm{Sub}(b) \arrow[r, bend left, "f^{-1}"] \arrow[r, phantom, "\perp"] & \mathrm{Sub}(a) \arrow[l, bend left, "\forall_f"] \end{tikzcd}$$

Notice that this fragment has the logical operators $\Rightarrow$ and $\neg$: in a Heyting category, it is possible to define the exponentiation $\Rightarrow$ in any lattice $\mathrm{Sub}(a)$ in terms of the universal quantifier. Then the negation is obtained via $u \Rightarrow 0$, where $u \in \mathrm{Sub}(a)$ and 0 is the initial subobject, corresponding to the logical operator $\perp$.

1.2 Ind-objects

Ind-objects of a category $\mathcal{C}$ (or inductive systems of $\mathcal{C}$) have been introduced in [Ver+72]. They will allow us to play with different properties of lex (i.e finite limits preserving) functors $\mathcal{H} \to \mathbf{Set}$, which are (equivalent to) ind-objects of $\mathcal{H}^{op}$ when $\mathcal{H}$ has finite limits. Therefore, among different equivalent definitions, we consider the following one :

Definition 1.2.1. An *Ind-object* of a small category $\mathcal{H}^{op}$ is a presheaf $F : \mathcal{H} \to \mathbf{Set}$ that is a filtered colimit of representables, i.e such that there is a natural isomorphism :

$$F \cong \operatorname*{colim}_{i \in I} \overline{F_i}$$

with $F_i \in \mathcal{H}$ for all $i \in I$, and I a filtered category.

Notation 1.2.2. We use the following notation for the Yoneda embedding

$$(\overline{\ \ }) : \mathcal{H}^{op} \to \mathbf{Set}^{\mathcal{H}}$$

$$h \mapsto \overline{h} := \mathrm{Hom}_{\mathcal{H}^{op}}(-, h) = \mathrm{Hom}_{\mathcal{H}}(h, -)$$

Definition 1.2.3. For $\mathcal{H}^{op}$ a small category, the category $\mathrm{Ind}(\mathcal{H}^{op})$ is the full subcategory of $\mathbf{Set}^{\mathcal{H}}$ which objects are Ind-objects of $\mathcal{H}^{op}$.

We are interested in the setting where $\mathcal{H}$ has finite limits, in which case the three following propositions hold :

Lemma 1.2.4. *Let $\mathcal{H}$ be a small category with finite limits. Then* $\mathrm{Ind}(\mathcal{H}^{op}) \simeq \mathrm{Lex}(\mathcal{H}, \mathbf{Set})$.

Proof. See [Joh82, VI.1.3] □

Proposition 1.2.5. *Let $\mathcal{H}$ be a small category with finite limits. Then $\mathrm{Ind}(\mathcal{H}^{op})$ is cocomplete and the embedding $\mathcal{H}^{op} \hookrightarrow \mathrm{Ind}(\mathcal{H}^{op})$ preserves finite colimits.*

Proof. See [Joh82, VI.1.6] □

Proposition 1.2.6. *Let $\mathcal{H}$ be a small category with finite limits. Then $\mathrm{Ind}(\mathcal{H}^{op})$ is complete.*

Proof. See [Joh82, VI.1.7] □

Definition 1.2.7. If $\mathcal{C}$ is a (locally small) category with filtered colimits, we say that an object X of $\mathcal{C}$ is *finitely-presentable* if the functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \text{ Set}$$

preserves filtered colimits.

The following result characterizes the category of Ind-objects of a small category $\mathcal{C}$ as the finite colimit-preserving cocompletion of $\mathcal{C}$.

Theorem 1.2.8. *Let $\mathcal{C}$ be a small category with finite colimits, and $F : \mathcal{C} \to \mathcal{E}$ a fully faithful functor, with $\mathcal{E}$ a cocomplete category.*

1. *If the objects in the image of F are finitely-presentable in $\mathcal{E}$, F extends to a fully faithful functor $\tilde{F} : \mathrm{Ind}\,\mathcal{C} \to \mathcal{E}$;*

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "F"] \arrow[d, "(\overline{\ \ })"'] & \mathcal{E} \\ \mathrm{Ind}(\mathcal{C}) \arrow[ur, dashed, "\tilde{F}"'] & \end{tikzcd}$$

2. *Assume additionally that any object of $\mathcal{E}$ is a colimit of objects in the image of F and that F preserves finite colimits. Then $\tilde{F}$ is an equivalence.*

Proof. See [Joh82, VI.1.8]. We just recall here the constructions of $\tilde{F}$ and its inverse. Let $D \in \mathrm{Ind}(\mathcal{C})$, $D \cong \mathrm{colim}_{i \in I} \overline{D_i}$ where $D_i \in \mathcal{C}$ for all i.

1. $$\tilde{F}(D) := \operatorname*{colim}_{i \in I} F(D_i)$$

   Notice that $\tilde{F}$ preserves filtered colimits.

2. Let $P \in \mathcal{E}$, then the proof shows that the comma category $F \downarrow P$ is filtered and that

   $$P \cong \operatorname*{colim}_{j \in F \downarrow P} F(C_j)$$

   where J is the canonical functor : $F \downarrow P \to \mathcal{C}$. Then $\mathrm{colim}_{j \in F\downarrow P} \overline{C_j}$ is an object of $\mathrm{Ind}\,\mathcal{C}$ such that its image under $\tilde{F}$ is isomorphic to P. This represents a functorial construction $\tilde{F}^{-1} : \mathcal{E} \to \mathrm{Ind}(\mathcal{C})$ that is an equivalence inverse to $\tilde{F}$.

□

Lemma 1.2.9. *If $\mathcal{H}$ is a small category, then any object $\overline{F}$ in the image of the embedding $\mathcal{H}^{op} \to \mathrm{Ind}(\mathcal{H}^{op})$ is finitely-presentable. In other words, there is a bijection :*

$$\mathrm{Hom}(\overline{F}, \operatorname*{colim}_{i \in I} X_i) \cong \operatorname*{colim}_{i \in I} \mathrm{Hom}(\overline{F}, X_i)$$

for any filtered colimit $I \to \mathrm{Ind}(\mathcal{H}^{op})$.

Proof. See [Joh82, VI.1.8] □

1.3 Additional results

We present here a variety of supplementary results that will be particularly useful in the subsequent proof.

Remark 1.3.1. Recall that, by Proposition 1.2.5, when $\mathcal{H}$ has finite limits, the embedding

$$(^{-}) : \mathcal{H}^{op} \to \mathrm{Lex}(\mathcal{H}, \mathbf{Set}) \simeq \mathrm{Ind}(\mathcal{H}^{op})$$

$$h \mapsto \overline{h} := \mathrm{Hom}_{\mathcal{H}}(h, -)$$

preserves finite colimits, i.e it sends finite limits in $\mathcal{H}$ to finite colimits in $\mathrm{Ind}(\mathcal{H}^{op})$.

Lemma 1.3.2. *Let $u \rightarrowtail a$ be a monomorphism in a small category $\mathcal{H}$, $N \in \mathrm{Lex}(\mathcal{H}, \mathbf{Set})$, and $y \in N(a)$. Then*

$$y \in N(u)$$

$$\Leftrightarrow y \textit{ factorizes as}$$

$$\begin{tikzcd} \overline{a} \arrow[r, "y"] \arrow[d, two heads] & N \\ \overline{u} \arrow[ur, dotted] & \end{tikzcd}$$

Proof. This is a consequence of the Yoneda lemma. □

Lemma 1.3.3. *Let $\mathbb{C}$ be a small category, the category of presheaves $\mathbf{Set}^{\mathbb{C}^{op}}$ is Heyting. If $f : P \to Q$ is a natural transformation between presheaves, the inverse image of $V \in \mathrm{Sub}(Q)$ is given by*

$$f^{-1}(V)(c) = f_c^{-1}(Vc) = \{x \in Pc \mid f_c(x) \in Vc\}\,.$$

for $c \in \mathbb{C}$.

For $U \in \mathrm{Sub}(P)$, the universal quantification is given by :

$$\forall_f(U)(c) = \{x \in Qc \mid \text{ for all } h : d \to c \text{ in } \mathbb{C}, \text{ for all } y \in Pd \text{ such that } f_d(y) = Qh(x),\ y \in Ud\}$$

Proof. See [Awo24, 3.3.28] $\square$

Lemma 1.3.4. *A slice of a regular category is regular.*

Proof. The forgetful functor $\mathcal{H}/b \to \mathcal{H}$ preserves and creates pullbacks and equalizers (see [BB04, 2.1.3]. This means that pullbacks and equalizers are computed in the underlying category; monomorphisms as well because they are pullback diagrams.

Binary products in the slice are computed as pullbacks in the base category. Consequently, the slice category $\mathcal{H}/b$ has finite products and equalizers provided that $\mathcal{H}$ does. Thus, $\mathcal{H}/b$ has finite limits if $\mathcal{H}$ has them.

The forgetful functor also preserves and creates colimits that exist in $\mathcal{H}/b$. Thus regular epimorphisms are taken in the base. We deduce that the reg-epi/mono factorization of a morphism over b always exists and is given as its factorization in the underlying category. Moreover, this factorization is stable under pullbacks in $\mathcal{H}/b$ if it is in $\mathcal{H}$. This justifies that $\mathcal{H}/b$ is regular using Theorem 1.1.4. $\square$

Lemma 1.3.5. *A slice of a coherent category is coherent.*

Proof. Suppose that for any object x in $\mathcal{H}$ the meet semi-lattice $\mathrm{Sub}_{\mathcal{H}}(x)$ has coproducts (i.e unions of subobjects), and that they are stable under pullbacks.

Recall that a monomorphism in $\mathcal{H}/b$ is a morphism which image under the forgetful functor is a mono in $\mathcal{H}$. Then, for any object over $b \in \mathcal{H}$, say $\boldsymbol{x} : x \to b$, it is easy to show that the coproducts in $\mathrm{Sub}_{\mathcal{H}/b}(\boldsymbol{x})$ are given by the corresponding coproducts in $\mathrm{Sub}_{\mathcal{H}}(x)$ post-composed with $\boldsymbol{x}$.

Now these union of subobjects are preserved under pullbacks in the slice as well, since pullbacks are computed in the underlying category as explained in 1.3.4. $\square$

Lemma 1.3.6. *For a small category $\mathcal{H}$ with finite limits and an object $b \in \mathcal{H}$ there is an equivalence of categories :*

$$\bar{b}/\operatorname{Ind}(\mathcal{H}^{op}) \simeq \operatorname{Ind}(b/\mathcal{H}^{op}) \simeq \operatorname{Ind}((\mathcal{H}/b)^{op})$$

Proof. We are going to use the equivalence in 1.2.4, namely $\operatorname{Ind}(\mathcal{H}^{\mathrm{op}}) \simeq \operatorname{Lex}(\mathcal{H}, \mathbf{Set})$, to show that we have functors $\kappa : \bar{b}/\operatorname{Lex}(\mathcal{H}, \mathbf{Set}) \to \operatorname{Lex}(\mathcal{H}/b, \mathbf{Set})$ and $\varepsilon : \operatorname{Lex}(\mathcal{H}/b, \mathbf{Set}) \to \bar{b}/\operatorname{Lex}(\mathcal{H}, \mathbf{Set})$ and that their composition is naturally isomorphic to the identity.

First, for any lex functor $F : \mathcal{H}/b \to \mathbf{Set}$, let us define $\varepsilon(F) = N$ as in the following diagram :

$$\begin{tikzcd} \mathcal{H} \arrow[r, "N"] \arrow[d, "\boldsymbol{b}^*"'] & \mathbf{Set} \\ \mathcal{H}/b \arrow[ur, "F"'] & \end{tikzcd}$$

where $\boldsymbol{b} : b \to 1$ in $\mathcal{H}$ and $\boldsymbol{b}^* : h \mapsto (h \times b \to b)$ is the right adjoint to the forgetful functor $\boldsymbol{b}_! : \mathcal{H}/b \to \mathcal{H}$. Moreover, there is a natural transformation $\bar{b} \to N$, i.e an element $x \in N(b)$ given by

$$F(\delta_b) : \{*\} \to F(b \times b) = N(b)$$

where δ_b is the diagonal morphism over b. N preserves finite limits because F and $\boldsymbol{b}^*$ do.

On the other hand, for a lex functor $N : \mathcal{H} \to \mathbf{Set}$ and a natural transformation $x : \bar{b} \to N$ we define $\kappa(N) = F$ as the following pushout :

$$\begin{tikzcd}
F(h) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & N(h) \arrow[d, "N(h\to b)"] \\
\{*\} \arrow[r, "x"] & N(b)
\end{tikzcd} \tag{1.3.1}$$

Then recall that binary products in $\mathcal{H}/b$ are computed as pullbacks in $\mathcal{H}$ and that equalizers are computed as equalizers in $\mathcal{H}$. N preserves both pullbacks and equalizers, and pullbacks commute with all limits. Therefore, F preserves finite limits in $\mathcal{H}/b$.

Now, for any $h \in \mathcal{H}$, observe that $\varepsilon\kappa(N)(h)$ is given by the pushout below :

$$\begin{tikzcd}
\varepsilon\kappa(N)(h) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & N(h \times b) \arrow[d, "N(\pi)"] \\
\{*\} \arrow[r, "x"] & N(b)
\end{tikzcd}$$

But $N(h\times b) \cong N(h) \times N(b)$ and therefore $\varepsilon\kappa(N)(h) \cong N(h)$. Moreover, the natural transformation associated to $\varepsilon\kappa(N)(h)$ is given by the dashed arrow $\kappa(N)(\delta_b)$ below :

$$\begin{tikzcd}
\{*\} \arrow[r, "x"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[drr, dotted] & N(b) \arrow[d, Rightarrow] \arrow[drr, "N(\delta_b)"] & & \\
\{*\} \arrow[r, "x"] \arrow[drr, Rightarrow] & N(b) \arrow[drr, Rightarrow] & N(b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & N(b \times b) \arrow[d] \\
 & & \{*\} \arrow[r, "x"] & N(b)
\end{tikzcd}$$

This map is universal from the pullback in the front thus makes the top square commute and is given by x (up to composition with an isomorphism), as wanted.

Now consider $\kappa\varepsilon(F)(h)$, for any lex functor $F : \mathcal{H}/b \to \mathbf{Set}$ as before, and any object h over b.

$$\begin{tikzcd}
\kappa\varepsilon(F)(h) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(h \times b) \arrow[d, "F(h\times \mathrm{Id}_b)"] \\
\{*\} \arrow[r, "F(\delta_b)"] & F(b \times b)
\end{tikzcd}$$

Because F preserve finite limits we have $F(h \times b) \cong F(h)$ and $F(b \times b) \cong \{*\}$, from which follows $\kappa\varepsilon(F)(h) \cong F(h)$. □

Remark 1.3.7. The equivalence stated in 1.3.6 holds even if the small category $\mathcal{H}$ lacks finite limits. This can be proved directly from the definition of an Ind-object together with the isomorphism given in 1.2.9 and commutation of filtered colimits with finite limits.

Lemma 1.3.8. *For a small coherent category $\mathcal{H}$ and an object $b \in \mathcal{H}$ there is an equivalence*

$$\overline{b}/\operatorname{Coh}(\mathcal{H}, \mathbf{Set}) \simeq \operatorname{Coh}(\mathcal{H}/b, \mathbf{Set})$$

Proof. Note that a slice $\mathcal{H}/b$ is coherent if $\mathcal{H}$ is coherent (see 1.3.4). We refer to the proof of 1.3.6, which shows $\overline{b}/\operatorname{Lex}(\mathcal{H}, \mathbf{Set}) \simeq \operatorname{Lex}(\mathcal{H}/b, \mathbf{Set})$. It remains to show that the finite limit-preserving functors we constructed preserve finite union of subobjects and regular epimorphisms.

First, for any coherent functor $F : \mathcal{H}/b \to \mathbf{Set}$ we defined a functor $N \in \bar{b}/\operatorname{Coh}(\mathcal{H}, \mathbf{Set})$ as the following composition :

$$\begin{tikzcd} \mathcal{H} \arrow[r, "N"] \arrow[d, "\boldsymbol{b}^*"'] & \mathbf{Set} \\ \mathcal{H}/b \arrow[ur, "F"'] & \end{tikzcd}$$

where $\boldsymbol{b} : b \to 1$ in $\mathcal{H}$ and $\boldsymbol{b}^* : h \mapsto (h \times b \to b)$. But the latter preserves finite union of subobjects and regular-epis since it is the pullback functor along $\boldsymbol{b}$. Therefore $N = F \circ b^*$ is coherent.

Next, for a coherent functor $N : \mathcal{H} \to \mathbf{Set}$ along with a natural transformation $1 \to N(b)$, the functor defined as the pullback (1.3.1) in 1.3.4 preserves finite unions of subobjects and regular-epis because N is coherent and they are stable under pullbacks.

$$\begin{tikzcd} F(h \cup h') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & N(h \cup h') = N(h) \cup N(h') \arrow[d, "N(h \cup h' \to b)"] \\ \{*\} \arrow[r, "x"] & N(b) \end{tikzcd}$$

□

Note that some authors refer to *final* functors when using the following definition, but we use the terminology introduced in [Ver+72] :

Definition 1.3.9. A functor $K : \mathcal{C} \to \mathcal{D}$ is *cofinal* if for any functor $F : \mathcal{D} \to \mathcal{M}$, the canonical map

$$\operatorname*{colim}_{\mathcal{C}} FK \xrightarrow{\cong} \operatorname*{colim}_{\mathcal{D}} F$$

is an isomorphism, when the colimits exist.

Lemma 1.3.10. *A functor $K : \mathcal{C} \to \mathcal{D}$ is cofinal if and only if for each $d \in \mathcal{D}$, the comma category $d \downarrow K$ is non-empty and connected.*

Proof. See [Rie14, II.8.3]. □

Let us recall

Definition 1.3.11. A category $\mathcal{J}$ is *filtered* when

1. it is not empty,
2. for every two objects j and j' in J, there exists an object k and two arrows $f : j \to k$ and $f' : j' \to k$ in J,
3. for every two parallel arrows $u, v : i \to j$ in J, there exists an object k and an arrow $w : j \to k$ such that $wu = wv$.

Corollary 1.3.12. *Let I be a filtered category, k any object of I. Then for any cocomplete category $\mathcal{C}$ and any functor $F : I \to \mathcal{C}$;*

$$\operatorname*{colim}_{i \in I} F(i) \cong \operatorname*{colim}_{l \in k/I} F(l)$$

Proof. It suffices to show that the forgetful functor $U : k/I \hookrightarrow I$ is cofinal via Lemma 1.3.10, whenever I is filtered. Let $i \in I$, consider $i \downarrow U$.
It is non-empty because there exists $l \in I$ and morphisms $k \to l \leftarrow i$ by 2. of Definition 1.3.11 of a filtered category.

It is connected because for any two objects $i \to l$, $i \to l'$ in $i \downarrow U$ there exist $l \to m \leftarrow l'$ such that the composites

$$\begin{tikzcd} & l \arrow[dr] & \\ i \arrow[ur] \arrow[dr] & & m \\ & l' \arrow[ur] & \end{tikzcd}$$

are equal, by points 2. and 3. of Definition 1.3.11. Our two objects are therefore connected by the morphisms $l \to m$ and $l' \to m$. ☐

A definition of a (pseudo)-colimit in a 2-category is given in [Lac07, 6.10]. The colimit of a pseudo-diagram in **Cat** can usually be computed by localizing the Grothendieck construction for that diagram, inverting the cocartesians morphisms. When the colimit is filtered, the computation is made easier by the following formula.

Lemma 1.3.13. *Let I be a small filtered category, and $F : I \to$ **Cat** a pseudo-functor, where $\mathcal{C}_i := F(i)$. Then*

$$\mathrm{Ob}(\operatorname*{colim}_{i\in I} \mathcal{C}_i) = \bigsqcup_{i\in I} \mathrm{Ob}(\mathcal{C}_i)\big/\!\sim$$

where $\sim$ is the equivalence relation generated by setting $c_i \sim c_j$ iff there exists $k \in I$, $\varphi_i : i \to k$, $\varphi_j : j \to k$ such that $F(\varphi_i)(c_i) = F(\varphi_j)(c_j)$.

For two objects $[c_i]$, $[c_j]$ of $\operatorname{colim}_{i\in I} \mathcal{C}_i$ with representants c_i, c_j the morphisms between them are given by :

$$\mathrm{Mor}([c_i],[c_j]) = \operatorname*{colim}_{l\in k/I} \mathrm{Hom}_{\mathcal{C}_l}(F(\varphi_i)(c_i), F(\varphi_j)(c_j))$$

where :
- k is such that there exists two morphisms $i \to k$, $j \to k$ (such k exists because I is filtered)
- k/I is the coslice category
- $\varphi_i : i \to l$, $\varphi_j : j \to l$ are morphisms in I

Proof. [Bre+14, 2.3.2] ☐

Lemma 1.3.14. *A filtered colimit $F : I \to$ **Cat** of coherent categories and coherent functors $F(i \to j)$ between them is coherent.*

Proof. The proof is essentially the same as in the case of pretopos, see [Bre+14, 2.3.3]. Notice that the functors $F(i) \to \operatorname{colim}_{i\in I} F(i)$ are also coherent.

We outline the proof here : the coherent structure, which includes finite limits, regular epimorphisms, and finite unions of subobjects, is built from finite diagrams and equations within these diagrams. These are finite collections of objects and morphisms as in Lemma 1.3.13. Each component are lifted individually to categories $F(j)$'s, and then brought together to a single category $F(k)$ due to the filtered nature of I. For example, for an equalizer diagram in the colimit, 5 "lifting" steps are involved, allowing to retrieve two parallel morphisms in a single category $F(k)$. The desired structure can then be obtained from this category $F(k)$. The universal properties hold in the colimit and are independent of the choice of this category $F(k)$ because they hold in each category of the diagram and the functors between them preserve them.

☐

2 Joyal's theorem

Theorem 2.0.1. *Let $\mathcal{H}$ be a small coherent category. There is a small category $\mathbb{C}$ and a coherent conservative functor $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{C}}$ preserving the Heyting structure that exists in $\mathcal{H}$.*

Proof. We show that $\mathbb{C}$ can be taken to be the category of coherent models of $\mathcal{H}$, i.e the category of coherent functors $\mathrm{Coh}(\mathcal{H}, \mathbf{Set})$. Notice that this category is not always small, we show how to reduce to a small category at the end of this proof.

Then, the functor is given by :

$$ev : \mathcal{H} \longhookrightarrow \mathbf{Set}^{\mathbb{C}}$$

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, "f"] & (F \mapsto F(a)) \arrow[d, "ev(f)_F = Ff"] \\
b \arrow[r, mapsto] & (F \mapsto F(b))
\end{tikzcd}$$

We refer to Appendix A for the conservativity of the functor, which is just Deligne's theorem.

The proof that it is a coherent functor relies on the fact that limits, colimits and image factorizations are taken pointwise in presheaves categories (in a topos, the binary union of two subobjects is the image factorization of the coproduct of the subobjects).

However, the universal quantification is not pointwise (see Lemma 1.3.3), which is why the proof of the conservation of the Heyting structure requires more effort. The following part of this work is devoted to that matter.

Suppose that $f : a \to b$ is a morphism in $\mathcal{H}$ such that the pullback functor $f^{-1} : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ has a right adjoint $\forall_f$. We need to show that the following diagram commutes :

$$\begin{tikzcd}
\mathrm{Sub}(a) \arrow[r, "ev"] \arrow[d, "\forall_f"] & \mathrm{Sub}(ev(a)) \arrow[d, "\forall_{ev(f)}"] \\
\mathrm{Sub}(b) \arrow[r, "ev"] & \mathrm{Sub}(ev(b))
\end{tikzcd}$$

where ev restricts to subobjects because it preserves finite limits. Therefore we want to show that for all $u \in \mathrm{Sub}(a)$, $\forall_{ev(f)(ev(u))} = ev(\forall_f(u))$. This amounts to show that for any coherent model $M \in \mathbb{C}$, $\forall_{ev(f)(ev(u))}(M) = M(\forall_f(u))$.

Notice that $\forall_{ev(f)(ev(u))}(M) = \forall_{ev(f)_M}(ev(u)(M)) = \forall_{Mf} M(u)$.

The inclusion $M(\forall_f(u)) \subseteq \forall_{Mf} M(u)$ follows easily, applying the finite limit preserving functor M to the following commuting diagram :

$$\begin{tikzcd}
u \arrow[dr, tail] & \\
f^{-1}(\forall_f u) \arrow[u, tail] \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & a \arrow[d, "f"] \\
\forall_f u \arrow[r, tail] & b
\end{tikzcd}$$

gives a map $M(f^{-1}(\forall_f u)) \cong (Mf)^{-1}(M(\forall_f(u))) \rightarrowtail M(u)$ in $\mathrm{Sub}(M(a))$. Therefore, there is an inclusion $M(\forall_f(u)) \subseteq \forall_{Mf} M(u)$ given by the adjunction $(Mf)^{-1} \dashv \forall_{Mf}$.

For the inclusion $\forall_{Mf}M(u) \subseteq M(\forall_f(u))$, let us use the definition of the universal quantification in presheaves, Lemma 1.3.3 :

$$\forall_{Mf}M(u) = \{x \in ev(b)(M) \mid \text{ for all } h : M \to N \text{ in } \mathrm{Coh}(\mathcal{H}, \mathbf{Set}),$$

$$\text{for all } y \in ev(a)(N) \text{ such that } ev(f)_N(y) = ev(b)(h)(x) \text{ then } y \in ev(u)(N)\}$$

Along with the definition of ev, this gives

$$\forall_{Mf}M(u) = \{x \in M(b) \mid \text{ for all } h : M \to N \text{ in } \mathrm{Coh}(\mathcal{H}, \mathbf{Set}),$$

$$\text{for all } y \in N(a) \text{ such that } Nf(y) = h_b(x) \text{ then } y \in N(u)\}$$

Therefore, let us assume that $x \in M(b)$, $x \notin M(\forall_f u)$. We have to prove the following :

$$\text{There exists a coherent model } N, \text{ a natural transformation } h : M \to N, \text{ and an element } y \in N(a) \text{ such that } Nf(y) = h_b(x) \text{ but } y \notin N(u). \tag{2.0.1}$$

2.1 Left exact models

In this section, we prove that we can find a finite-limit preserving model M^* and an element $w \in M^*(a)$ such that (2.0.1) holds. In other words, we could chose $\mathbb{C} = \mathrm{Lex}(\mathcal{H}, \mathbf{Set})$ and ev would preserve the Heyting structure !

Notice that $M \in \mathrm{Coh}(\mathcal{H}, \mathbf{Set}) \subseteq \mathrm{Lex}(\mathcal{H}, \mathbf{Set})$. Since $\mathcal{H}$ has all finite limits, we know that $\mathrm{Ind}(\mathcal{H}^{\mathrm{op}}) \simeq \mathrm{Lex}(\mathcal{H}, \mathbf{Set})$ and moreover it is cocomplete, by Lemma 1.2.4 and Proposition 1.2.5. We construct the following pushout in $\mathrm{Ind}(\mathcal{H}^{op})$:

$$\begin{tikzcd} \overline{b} \arrow[r, "x"] \arrow[d, "\overline{f}"'] & M \arrow[d, "\tilde{h}"] \\ \overline{a} \arrow[r, "w"] \arrow[ur, phantom, "\ulcorner"] & M^* \end{tikzcd}$$

First, the element $w \in M^*(a)$ verifies $M^*f(w) = \tilde{h}_b(x)$: this follows directly from commutativity of the above square at $b \in \mathcal{H}$ and for $\mathrm{Id}_b \in \overline{b}_b = \mathrm{Hom}(b, b)$.

Furthermore, we claim that $w \in M^*(a)$ but $w \notin M^*(u)$.

For our lex functor M^*, a consequence of Yoneda lemma is that $w \in M^*(u)$ if and only if there exists a factorization of w as in the diagram below, and similarly for the condition $x \in M(\forall_f u)$ (see Lemma 1.3.2).

$$\begin{tikzcd} \overline{\forall_f u} \arrow[dr, dashed] & \\ \overline{b} \arrow[u] \arrow[r, "x"] \arrow[d, "\overline{f}"'] & M \arrow[d, "\tilde{h}"] \\ \overline{a} \arrow[r, "w"] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & M^* \\ \overline{u} \arrow[ur, dashed] & \end{tikzcd}$$

Since we assumed that $x \notin M(\forall_f u)$ and we want to show that $w \notin M^*(u)$, it is sufficient to show that $\mathrm{Hom}_{\overline{a}/\mathrm{Ind}(\mathcal{H}^{op})}(\overline{u}, M^*) \cong \mathrm{Hom}_{\overline{b}/\mathrm{Ind}(\mathcal{H}^{op})}(\overline{\forall_f(u)}, M)$.

By Lemma 1.3.6, $\overline{b}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}}) \cong \operatorname{Ind}((\mathcal{H}/b)^{op})$, so we can write M as a filtered colimit $M = \operatorname{colim}_{i\in I} \overline{m_i}$, with each m_i in $\mathcal{H}/b$. Let us write $\overline{a} \sqcup_{\overline{b}} \overline{m_i}$ for the pushout of $\overline{m_i}$ along $\overline{f}$. Since colimits commute with colimits, we have :

$$M^* = \overline{a} \sqcup_{\overline{b}} M \cong \operatorname*{colim}_{i\in I} \overline{a} \sqcup_{\overline{b}} \overline{m_i}$$

By Remark 1.3.1,

$$\overline{a} \sqcup_{\overline{b}} \overline{m_i} \cong \overline{f^*(m_i)}$$

where $f^* : \mathcal{H}/b \to \mathcal{H}/a$ is the pullback functor.

Now

$$\begin{aligned}
\operatorname{Hom}_{\overline{a}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}})}(\overline{u}, \overline{f^*(m_i)}) &\cong \operatorname{Hom}_{\mathcal{H}/a}(f^*(m_i), u) && (2.1.1)\\
&\cong \operatorname{Sub}(a)(\operatorname{im}(f^*(m_i)), u) && (2.1.2)\\
&\cong \operatorname{Sub}(a)(f^{-1}(\operatorname{im} m_i), u) && (2.1.3)\\
&\cong \operatorname{Sub}(b)(\operatorname{im} m_i, \forall_f u) && (2.1.4)\\
&\cong \operatorname{Hom}_{\mathcal{H}/b}(m_i, \forall_f u) && (2.1.5)\\
&\cong \operatorname{Hom}_{\overline{b}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}})}(\overline{\forall_f u}, \overline{m_i}) && (2.1.6)
\end{aligned}$$

where (2.1.1) and (2.1.6) are Yoneda, (2.1.2) and (2.1.5) are given by the property of the image, see Remark 1.1.5. (2.1.3) is commutation of the image with pullbacks, and (2.1.4) is our adjoint property.

$$\begin{tikzcd}
\operatorname{Sub}(b) \arrow[r, bend left, "f^{-1}"] \arrow[r, phantom, "\perp"] & \operatorname{Sub}(a) \arrow[l, bend left, "\forall_f"]
\end{tikzcd}$$

Therefore, by Lemma 1.2.9,

$$\begin{aligned}
\operatorname{Hom}_{\overline{a}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}})}(\overline{u}, M^*) &\cong \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\overline{a}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}})}(\overline{u}, \overline{f^*(m_i)})\\
&\cong \operatorname*{colim}_{i\in I} \operatorname{Hom}_{\overline{b}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}})}(\overline{\forall_f u}, \overline{m_i})\\
&\cong \operatorname{Hom}_{\overline{b}/\operatorname{Ind}(\mathcal{H}^{\mathrm{op}})}(\overline{\forall_f u}, M)
\end{aligned}$$

and we have therefore proved that we have a finite limit preserving functor M^*, a natural transformation $\tilde{h} : M \to M^*$ and $w \in M^*(a)$ such that (2.0.1) holds.

2.2 Coherent models

Now we would like to level up the lex functor $M^* \in \operatorname{Lex}(\mathcal{H}, \mathbf{Set})$ and the natural transformation $\tilde{h} : M \to M^*$ to a functor $N \in \operatorname{Coh}(\mathcal{H}, \mathbf{Set})$ and a natural transformation $h : M \to N$ such that we again have an element $y \in N(a)$, $Nf(y) = h_b(x)$, but $y \notin N(u)$.

Let us write $M^* \in \operatorname{Ind}(a/\mathcal{H}^{op})$ as a filtered colimit of representables, $M^* = \operatorname{colim}_{i\in I} \overline{M_i}$.

We construct here a new theory $\mathcal{H}(M^*)$ and show it is coherent. We add new constants to the language of the theory $\mathcal{H}$, one for each element of each **Set**-model $\overline{M_i}$. Then, the theory consists of all formulas that are build of these constants, and are true in M^* when the constants are interpreted as the elements of the $\overline{M_i}$. This construction is done in [MR77] and referred to as the "positive diagram" of a model. Categorically, this corresponds to slicing $\mathcal{H}$ by each M_i and taking the filtered 2-colimit of the categories $\mathcal{H}/M_i$.

First, we will show that the new theory $\mathcal{H}(M^*)$ is simply described in terms of pushouts in $\mathrm{Ind}(\mathcal{H}^{\mathrm{op}})$ (or equivalently pullbacks in $\mathrm{Pro}(\mathcal{H}^{\mathrm{op}})$).

Then, we will establish the important property of this theory : any coherent model of $\mathcal{H}(M^*)$ corresponds to a coherent model N of $\mathcal{H}$ with a natural transformation $M^* \to N$.

2.2.1 The new theory $\mathcal{H}(M^*)$

Let define the category $\mathcal{H}(M^*)^{op}$ as the full subcategory of $M^*/\operatorname{Ind}(\mathcal{H}^{op})$ where the objects are pushouts of maps in $\mathcal{H}$, as in :

$$\begin{tikzcd}
\bar{d} \arrow[r] \arrow[d, "\bar{g}"] & M^* \arrow[d] \\
\bar{c} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y
\end{tikzcd}$$

$g : c \to d$ is a map in $\mathcal{H}$.

We claim that

$$\mathcal{H}(M^*) \simeq \operatorname*{colim}_{i \in I} \mathcal{H}/M_i$$

where the diagram $F : I \to \mathbf{Cat}$ is given by $F(i) = \mathcal{H}/M_i$, $F(\varphi : j \to i) = u^*$ is the pullback along $u : M_i \to M_j$, whenever $\overline{u} = \overline{M_j} \to \overline{M_i}$ is a morphism in the diagram of $\operatorname{colim}_{i\in I} \overline{M_i}$. Notice that F is a pseudo-functor. By Lemma 1.3.13, we describe the objects and morphisms of $\operatorname{colim}_{i\in I} \mathcal{H}/M_i$ as follows :

- An object of $\operatorname{colim}_{i\in I} \mathcal{H}/M_i$ is an object x_i of $\mathcal{H}/M_i$, for a certain i, under the equivalence relation
$$x_i \sim x_j$$
if and only if there is an indice $k \in I$ and two morphisms $u : M_k \to M_i$, $v : M_k \to M_j$ such that
$$u^*(x_i) = v^*(x_j)$$
- For two objects $[x_i]$ and $[y_j]$ with representants $x_i \in \mathcal{H}/M_i$ and $y_j \in \mathcal{H}/M_j$, there exists a $k \in I$ such that there are morphisms $u^* : \mathcal{H}/M_i \to \mathcal{H}/M_k \leftarrow \mathcal{H}/M_j : v^*$, and thus;
$$\operatorname{Hom}([x_i],[y_j]) = \operatorname*{colim}_{l \in k/I} \operatorname{Hom}_{\mathcal{H}/M_l}(x_l, y_l)$$
where we abuse notation and write x_l and y_l for the images of $u^*(x_i)$ and $v^*(y_j)$ under a pullback functor $\mathcal{H}/M_k \to \mathcal{H}/M_l$ in the diagram.

We construct a functor :

$$\operatorname*{colim}_{i \in I} \mathcal{H}/M_i \to \mathcal{H}(M^*)$$

$$[M_i' \xrightarrow{x_i} M_i] \mapsto \begin{tikzcd}
\overline{M_i} \arrow[r] \arrow[d] & M^* \arrow[d] \\
\overline{M_i'} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y
\end{tikzcd}$$

and we will show that it is essentially surjective on objects and fully faithful.

Remark 2.2.1. Notice that objects of $\mathcal{H}(M^*)$ are formally the same as objects of $\mathcal{H}(M^*)^{op}$, even though they are Pro- and Ind-objects respectively. Here we choose to work in $\mathrm{Ind}(\mathcal{H}^{op})$ to be able to directly use results from the literature, that are more commonly available for Ind-objects (see section 1.2). Additionally, this sets the stage for the next subsection, where we will be interested in $\mathrm{Ind}(\mathcal{H}(M^*)^{op})$.
Therefore, we regard objects of $\mathcal{H}(M^*)$ as those Ind-objects that are pushouts of maps in $\mathcal{H}$, even though it would be more rigourous to consider them as those Pro-objects that are pullbacks of maps in $\mathcal{H}$.

Verifying that this functor is well-defined on objects and morphisms is a matter of using the pasting lemma for pullbacks in $\mathcal{H}/M_i$ and in $\mathcal{H}(M^*)$.

Now let $M^* \to Y$ be an object of $\mathcal{H}(M^*)$:

$$\begin{tikzcd} \overline{d} \arrow[r] \arrow[d, "\overline{g}"] & M^* \arrow[d] \\ \overline{c} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

where $g : c \to d$ is a map in $\mathcal{H}$. By Lemma 1.2.9, $\overline{d}$ is finitely-presentable, i.e.,

$$\mathrm{Hom}(\overline{d}, M^*) \cong \operatorname*{colim}_{i \in I} \mathrm{Hom}(\overline{d}, \overline{M_i})$$

This says that there exists $i \in I$ such that the morphism $\overline{d} \to M^*$ factorizes through $\overline{M_i}$. By Lemma 1.2.5, the pushout of the left square below is in the image of the embedding $\mathcal{H}^{op} \to \mathrm{Ind}(\mathcal{H}^{op})$, so that Y is the pushout of an object of $\mathcal{H}/M_i$. The functor we defined previously is essentially surjective on objects.

$$\begin{tikzcd} \bar{d} \arrow[r] \arrow[d, "\overline{g}"] & \overline{M_i} \arrow[r] \arrow[d] & M^* \arrow[d] \\ \bar{c} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \overline{Z} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

Now for two objects $[x_j]$, $[y_i]$, let us have the following pushouts (from the definition of the morphisms in $\mathrm{colim}_{i\in I}\, \mathcal{H}/M_i$ we can assume $i = j$) :

$$\begin{tikzcd} \overline{M_j} \arrow[r] \arrow[d, "\overline{x_j}"] & M^* \arrow[d] \\ \overline{M_j'} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \end{tikzcd} \quad \begin{tikzcd} \overline{M_j} \arrow[r] \arrow[d, "\overline{y_j}"] & M^* \arrow[d] \\ \overline{M_j''} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

To establish that our functor is full and faithful we want to show :

$$\mathrm{Hom}_{\mathcal{H}(M^*)}(X, Y) \cong \mathrm{Hom}_{M^*/\,\mathrm{Ind}(\mathcal{H}^{op})}(Y, X) \cong \mathrm{Hom}([y_j], [x_j]) = \operatorname*{colim}_{l \in j/I} \mathrm{Hom}_{\mathcal{H}/M_l}(y_l, x_l) \qquad (2.2.1)$$

Observe that the first isomorphism results from reversing the direction of the morphisms when transitioning from $\mathcal{H}(M^*)$ to $\mathcal{H}(M^*)^{op}$; refer to Remark 2.2.1.

First, by the pushout property of Y we have :

$$\mathrm{Hom}_{M^*/\,\mathrm{Ind}(\mathcal{H}^{op})}(Y, X) \cong \mathrm{Hom}_{\overline{M_j}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M_j''}, X) \qquad (2.2.2)$$

By Corollary 1.3.12 we have $M^* \cong \mathrm{colim}_{l \in j/I}\, \overline{M_l}$. For each $j \to l$, define $\overline{M_l'}$ (and analogously $\overline{M_l''}$) as :

$$\begin{tikzcd}
\overline{M_j} \arrow[r] \arrow[d, "\overline{x_j}"] \arrow[dr, phantom, "\ulcorner", very near end] & \overline{M_l} \arrow[d] & \overline{M_j} \arrow[r] \arrow[d, "\overline{y_j}"] \arrow[dr, phantom, "\ulcorner", very near end] & \overline{M_l} \arrow[d] \\
\overline{M'_j} \arrow[r] & \overline{M'_l} & \overline{M''_j} \arrow[r] & \overline{M''_l}
\end{tikzcd}$$

Since colimits commute with colimits, we have

$$X \cong \operatorname*{colim}_{l \in j/I} \overline{M'_l}$$

Using Lemma 1.2.9 again, it follows :

$$\mathrm{Hom}_{\overline{M_j}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_j}, X) \cong \operatorname*{colim}_{l \in j/I} \mathrm{Hom}_{\overline{M_j}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_j}, \overline{M'_l}) \tag{2.2.3}$$

Now observe that $\mathrm{Hom}_{\overline{M_l}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_l}, \overline{M'_l})$ and $\mathrm{Hom}(\overline{M''_l}, \overline{M'_l})$ are the pullbacks below :

$$\begin{tikzcd}
\mathrm{Hom}_{\overline{M_l}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_l}, \overline{M'_l}) \arrow[r] \arrow[d] & \{*\} \arrow[d] \\
\mathrm{Hom}(\overline{M''_l}, \overline{M'_l}) \arrow[r] \arrow[d] & \mathrm{Hom}(\overline{M_l}, \overline{M'_l}) \arrow[d] \\
\mathrm{Hom}(\overline{M''_j}, \overline{M'_l}) \arrow[r] & \mathrm{Hom}(\overline{M_j}, \overline{M'_l})
\end{tikzcd}$$

Therefore the outer rectangle is a pullback as well, but this is $\mathrm{Hom}_{\overline{M_j}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_j}, \overline{M'_l})$. In other words, we just showed

$$\operatorname*{colim}_{l \in j/I} \mathrm{Hom}_{\overline{M_j}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_j}, \overline{M'_l}) \cong \operatorname*{colim}_{l \in j/I} \mathrm{Hom}_{\overline{M_l}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_l}, \overline{M'_l}) \tag{2.2.4}$$

Using successively equations (2.2.2), (2.2.3) and (2.2.4), we showed eq. (2.2.1) :

$$\mathrm{Hom}_{M^*/\,\mathrm{Ind}(\mathcal{H}^{op})}(Y, X) \cong \operatorname*{colim}_{l \in j/I} \mathrm{Hom}_{\overline{M_l}/\,\mathrm{Ind}(\mathcal{H}^{op})}(\overline{M''_l}, \overline{M'_l}) = \operatorname*{colim}_{l \in j/I} \mathrm{Hom}_{\mathcal{H}/M_l}(y_l, x_l)$$

where the last isomorphism results from the Yoneda lemma and the definition of x_l and y_l.

$\mathcal{H}(M^*)$ is a coherent theory :

Since $\mathcal{H}$ is coherent, each slice category $\mathcal{H}/M_i$ is coherent, see Lemma 1.3.4. The pullback functors $u^* : \mathcal{H}/M_j \to \mathcal{H}/M_i$ involved in the colimit are coherent (because $\mathcal{H}$ is coherent). I is filtered, so by Lemma 1.3.14, $\mathrm{colim}_{i \in I}\, \mathcal{H}/M_i$ is a coherent category, and the equivalence $\mathrm{colim}_{i \in I}\, \mathcal{H}/M_i \simeq \mathcal{H}(M^*)$ is an equivalence of coherent categories.

2.2.2 Models of $\mathcal{H}(M^*)$

We are now about to prove that

$$\mathrm{Ind}(\mathcal{H}(M^*)^{op}) \simeq M^*/\,\mathrm{Ind}(\mathcal{H}^{op}) \tag{2.2.5}$$

Notice that $F : \mathcal{H}(M^*)^{op} \hookrightarrow M^*/\,\mathrm{Ind}(\mathcal{H}^{op})$ is fully faithful, and $M^*/\,\mathrm{Ind}(\mathcal{H}^{op})$ is cocomplete as the coslice of a cocomplete category. Thus, we are in the setting of Theorem 1.2.8, let us prove both points 1. and 2. This will imply the existence of an extension $\tilde{F}$ as below, that is an equivalence :

$$\begin{tikzcd}
\mathcal{H}(M^*)^{op} \arrow[r, hook, "F"] \arrow[d, "\overline{(\,)}"'] & M^*/\,\mathrm{Ind}(\mathcal{H}^{op}) \\
\mathrm{Ind}(\mathcal{H}(M^*)^{op}) \arrow[ur, dashed, "\tilde{F}"'] &
\end{tikzcd}$$

1. Let $\begin{tikzcd} \bar{d} \arrow[r] \arrow[d, "\bar{g}"'] & M^* \arrow[d] \\ \bar{c} \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$ be any object in the image of F, we want to show that is is finitely-presentable. Consider any filtered colimit $P = \mathrm{colim}_{k\in K} P_k$ in $M^*/\operatorname{Ind}(\mathcal{H}^{op})$. For any $k \in K$, notice that

$$\operatorname{Hom}_{M^*/\operatorname{Ind}(\mathcal{H}^{op})}(Y, P_k) \cong \operatorname{Hom}_{\overline{d}/\operatorname{Ind}(\mathcal{H}^{op})}(\overline{c}, P_k)$$

Moreover, $\operatorname{Hom}_{\overline{d}/\operatorname{Ind}(\mathcal{H}^{op})}(\overline{c}, P)$ and $\mathrm{colim}_{k\in K} \operatorname{Hom}_{\overline{d}/\operatorname{Ind}(\mathcal{H}^{op})}(\overline{c}, P_k)$ are the pullbacks in the diagram below (the latter because filtered colimits commute with finite limits), in **Set**.

$$\begin{tikzcd}
\mathrm{colim}_{k\in K} \operatorname{Hom}_{\overline{d}/\operatorname{Ind}(\mathcal{H}^{op})}(\overline{c}, P_k) \arrow[rr] \arrow[dd] \arrow[dr, "\cong"] & & \mathrm{colim}_{k\in K} \operatorname{Hom}(\overline{c}, P_k) \arrow[dd] \arrow[dr, "\cong"] & \\
& \operatorname{Hom}_{\overline{d}/\operatorname{Ind}(\mathcal{H}^{op})}(\overline{c}, P) \arrow[rr] \arrow[dd] & & \operatorname{Hom}(\overline{c}, P) \arrow[dd] \\
\{*\} \arrow[rr] \arrow[dr, equal] & & \mathrm{colim}_{k\in K} \operatorname{Hom}(\overline{d}, P_k) \arrow[dr, "\cong"] & \\
& \{*\} \arrow[rr] & & \operatorname{Hom}(\overline{d}, P)
\end{tikzcd}$$

All transversal maps are bijections (Lemma 1.2.9), therefore the map between the pullbacks is so and we have

$$\operatorname{Hom}(Y, P) \cong \operatorname*{colim}_{k\in K} \operatorname{Hom}_{M^*/\operatorname{Ind}(\mathcal{H}^{op})}(Y, P_k)$$

2. Let $M^* \to P$ be an object in $M^*/\operatorname{Ind}(\mathcal{H}^{op})$. We want to write it as the (filtered) colimit of objects of $\mathcal{H}(M^*)^{op}$.
Notice that $M^* \to P$ is an object of the arrow category $\operatorname{Ind}(\mathcal{H}^{op})^{\to}$. Under the equivalence $\operatorname{Ind}((\mathcal{H}^{op})^{\to}) \xrightarrow{\simeq} \operatorname{Ind}(\mathcal{H}^{op})^{\to}$ (see [KS06, 6.4.2]), we can write $M^* \to P$ as a filtered colimit of Ind-objects $\overline{A_i} \to \overline{B_i}$, where A_i, $B_i \in \mathcal{H}$. For all $i \in I$, consider the following pushout in $\operatorname{Ind}(\mathcal{H}^{op})$:

$$\begin{tikzcd} \overline{A_i} \arrow[r] \arrow[d] & M^* \arrow[d] \\ \overline{B_i} \arrow[r] & P_i \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Then in $M^*/\operatorname{Ind}(\mathcal{H}^{op})$, $P \cong \mathrm{colim}_{i\in I} P_i$. Indeed, it is easy to show that the maps from the pushouts P_i to P realize $M^* \to P$ as a cocone under the diagram I. The universal property of $\mathrm{colim}_{i\in I}(\overline{A_i} \to \overline{B_i})$ in the arrow category is a universal property for $\mathrm{colim}_{i\in I} P_i$ in $M^*/\operatorname{Ind}(\mathcal{H}^{op})$.

It is left to show that F preserves finite colimits. Since $\mathcal{H}(M^*)^{op}$ has finite colimits, it is sufficient to show that they are "coming from" colimits in $M^*/\operatorname{Ind}(\mathcal{H}^{op})$, i.e that the full subcategory inclusion $\mathcal{H}(M^*)^{op} \subseteq M^*/\operatorname{Ind}(\mathcal{H}^{op})$ is closed under finite colimits. But recall from Lemma 1.3.14, and under the equivalence $\mathcal{H}(M^*)^{op} \simeq \mathrm{colim}_{i\in I} M_i/\mathcal{H}^{op}$, that finite colimits of $\mathcal{H}(M^*)^{op}$ (i.e finite limits in $\mathcal{H}(M^*)$) are computed in the slices.

In other words, let $M^* \to Y$, $M^* \to Z$ in $\mathcal{H}(M^*)^{op}$. Then there exists $i \in I$, and objects $M_i \to M_i'$, $M_i \to M_i''$ such that the coproduct $Y + Z$ is isomorphic to the image of $M_i' + M_i''$ (or more precisely of its representant $[M_i' + M_i'']$) through the functor $\mathrm{colim}_{i\in I} M_i/\mathcal{H}^{op} \to \mathcal{H}(M^*)^{op}$. But this functor is "pushout along $\overline{M_i} \to M^*$", i.e the functor :

$$M^* \sqcup_{\overline{M_i}} (-) : \overline{M_i}/\operatorname{Ind}(\mathcal{H}^{op}) \to M^*/\operatorname{Ind}(\mathcal{H}^{op})$$

which is a colimit and therefore preserves colimits. Therefore the coproduct of $Y + Z$ in $M^*/\operatorname{Ind}(\mathcal{H}^{op})$ is the coproduct in $\mathcal{H}(M^*)^{op}$. The reasoning is the same for coequalizers.

To proceed with the crucial step of this proof, which involves Deligne's theorem and the coherence of $\mathcal{H}(M^*)$, we must show that under the equivalence $\operatorname{Ind}(\mathcal{H}^{\mathrm{op}}) \simeq \operatorname{Lex}(\mathcal{H}, \mathbf{Set})$, the equivalence (2.2.5) "restricts" to coherent functors. In other words, we need to demonstrate that any coherent functor $N' : \mathcal{H}(M^*) \to \mathbf{Set}$ corresponds to a coherent functor $N : \mathcal{H} \to \mathbf{Set}$ with a natural transformation $M^* \to N$, and vice-versa.

We can deduce from the construction of $\tilde{F}^{-1}$ in the Proof of Theorem 1.2.8, that in our case lex functors $N : \mathcal{H} \to \mathbf{Set}$ (along with $M^* \to N$) are identified to lex functors $N' : \mathcal{H}(M^*) \to \mathbf{Set}$ via :

$$N(Y) \cong \operatorname{Hom}_{M^*/\operatorname{Lex}(\mathcal{H},\mathbf{Set})}(Y, N') \tag{2.2.6}$$

where $Y \in \mathcal{H}(M^*)$, but again we consider its corresponding Ind-object in $\mathcal{H}(M^*)^{op}$, see Remark 2.2.1.

- Suppose that N' preserves finite union of subobjects. Let u, $v \in \operatorname{Sub}(y)$ in $\mathcal{H}$, we have :

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Lex}(\mathcal{H},\mathbf{Set})}(\overline{u \cup v}, N) &\cong \operatorname{Hom}_{M^*/\operatorname{Lex}(\mathcal{H},\mathbf{Set})}(\overline{u \cup v} + M^*, N) \\
&\cong N'(\overline{u \cup v} + M^*) \\
&\cong N'(\overline{u} + M^* \cup \overline{v} + M^*) \\
&\cong \operatorname{Hom}_{\operatorname{Lex}(\mathcal{H},\mathbf{Set})}(\overline{u}, N) \cup \operatorname{Hom}_{\operatorname{Lex}(\mathcal{H},\mathbf{Set})}(\overline{v}, N)
\end{aligned}$$

  and therefore N preserves finite union of subobjects too (Yoneda).

  Notice that $M^* + \overline{u}$, $M^* + \overline{v}$ are coproducts in $\operatorname{Lex}(\mathcal{H}, \mathbf{Set})$. Let us explain $N'(\overline{u \cup v} + M^*) \cong N'(\overline{u} + M^* \cup \overline{v} + M^*)$. In $\mathcal{H}(M^*)^{op}$, $\overline{u} + M^*$, $\overline{v} + M^*$ are the pushouts of $0 \to M^*$ along $0 \to \overline{u}$, $0 \to \overline{v}$ respectively. Because I is filtered and by Lemmas 1.2.9 and 1.2.5, we can assume there exists $i \in I$ such that they are in fact the following pushouts :

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \overline{M_i} \arrow[r] \arrow[d] & M^* \arrow[d] \\
\overline{u} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \overline{u \times M_i} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \overline{u} + M^*
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \overline{M_i} \arrow[r] \arrow[d] & M^* \arrow[d] \\
\overline{v} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \overline{v \times M_i} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \overline{v} + M^*
\end{tikzcd}$$

  But the operation of constructing these successive pushouts for a $u \in \mathcal{H}$ corresponds exactly to the composite $\mathcal{H} \xrightarrow{- \times M_i} \mathcal{H}/M_i \to \operatorname{colim}_{i \in I} \mathcal{H}/M_i \cong \mathcal{H}(M^*)$. The first functor is coherent as the "slicing" of a coherent category and the second one is coherent by Lemma 1.3.14. In other words $\overline{u \cup v} + M^* \cong \overline{u} + M^* \cup \overline{v} + M^*$, as subobjects of the Pro-object corresponding to $\overline{y} + M^*$.

- Suppose that N preserves finite union of subobjects. Let U, $V \in \operatorname{Sub}(Y)$ in $\mathcal{H}(M^*)$. Again, by Lemma 1.3.14, we know that all the coherent structure of $\mathcal{H}(M^*)$ (limits, regular epimorphisms, union of subobjects) can be computed in the slices. This means that there exists an $i \in I$ such that $U \cup V$ is the image through $\mathcal{H}/M_i \to \mathcal{H}(M^*)$ of the union of two subobjects u_i, v_i of an object y_i (sent to Y).

  In other words, the following is a pushout in $\operatorname{Lex}(\mathcal{H}, \mathbf{Set})$:

$$\begin{tikzcd}
\overline{M_i} \arrow[r] \arrow[d] & M^* \arrow[d] \\
\overline{u_i \cup v_i} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & U \cup V
\end{tikzcd}$$

Therefore :

$$\begin{aligned} N'(U \cup V) &\cong \mathrm{Hom}_{M^*/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(U \cup V, N) \\ &= \mathrm{Hom}_{\overline{M_i}/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(\overline{u_i \cup v_i}, N) \\ &\cong \mathrm{Hom}_{\overline{M_i}/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(\overline{u_i}, N) \cup \mathrm{Hom}_{\overline{M_i}/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(\overline{v_i}, N) \\ &\cong N'(U) \cup N'(V) \end{aligned}$$

Remark 2.2.2. For conciseness, we omit verifying that the functors preserve regular epimorphisms. The proof is analogous to that for finite unions of subobjects; using that if $f : Y \to Y'$ is a coequalizer in $\mathcal{H}(M^*)$, it can be lifted to a coequalizer $f_i : M_i' \to M_i''$ in a slice $\mathcal{H}/M_i$, and that the functor $\mathcal{H} \to \mathcal{H}(M^*)$ is coherent.

2.3 Deligne's theorem, again

To conclude the proof, notice that in $\mathcal{H}(M^*)^{op}$, there is an object Y that is the following pushout :

$$\begin{tikzcd} \overline{a} \arrow[r, "w"] \arrow[d, two heads] & M^* \arrow[d, two heads] \\ \overline{u} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \end{tikzcd}$$

such that $M^* \ncong Y$. Indeed, if this was an isomorphism then we would have a factorization of w through $\overline{u}$ but it is not the case under the assumption $x \notin M(\forall_f u)$, see 2.1.

Therefore, we can use Deligne's Theorem A.0.5 : there exists a coherent model $N' : \mathcal{H}(M^*) \to \mathbf{Set}$ such that $N'(Y) \ncong N'(M^*)$. This is a coherent model $N : \mathcal{H} \to \mathbf{Set}$ along with $h' : M^* \to N$, such that

$$\mathrm{Hom}_{M^*/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(Y, N) \ncong \mathrm{Hom}_{M^*/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(M^*, N)$$

Notice that $M^* \twoheadrightarrow Y$ is an epimorphism because $u \in \mathrm{Sub}(a)$ in $\mathcal{H}$, therefore $\overline{a} \twoheadrightarrow \overline{u}$ in $\mathrm{Ind}(\mathcal{H}^{op})$ is epi and they are preserved under pushouts. Therefore the left hand side can only be a singleton or the empty set. But the right hand side is a singleton, therefore :

$$\mathrm{Hom}_{M^*/\,\mathrm{Lex}(\mathcal{H},\mathbf{Set})}(Y, N) = \varnothing$$

i.e h' does not factorizes through Y.

Coming back to our original diagram :

$$\begin{tikzcd} \overline{\forall_f u} \arrow[dr, dashed] & & \\ \overline{b} \arrow[u] \arrow[r, "x"] \arrow[d, "\overline{f}"'] & M \arrow[d, "\tilde{h}"] & \\ \overline{a} \arrow[r, "w"] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & M^* \arrow[r, "h'"] \arrow[d] & N \\ \overline{u} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y & \end{tikzcd}$$

It is now easy to show that the coherent model N, along with $h = h' \circ \tilde{h}$ and the element $y = h_a'(w) \in N(a)$ are such that

$$N(f)(h_a'(w)) = (h' \circ \tilde{h})_b(x) \text{ but } h_a'(w) \notin N(u)$$

(see Lemma 1.3.2).

In other words we have proved 2.0.1, i.e we have a coherent model N with $h : M \to N$ and $y \in N(a)$ s.t :

$$N(f)(y) = (h)_b(x) \text{ but } y \notin N(u)$$

$\square$

Remark 2.3.1. Coh($\mathcal{H}$, Set) may be large. However, smallness of $\mathcal{H}$ along with the Strong Collection axiom from CZF Set Theory allow us to find a small full subcategory $\mathbb{S} \subseteq$ Coh($\mathcal{H}$, Set) that is sufficient to make the embedding conservative. Beforehand, let us establish a sufficient condition for a regular functor to be conservative.

Lemma 2.3.2. *A regular and conservative functor between regular categories is faithful.*

Proof. Consider the equalizer of a parallel pair of morphisms. $\square$

Corollary 2.3.3. *A regular functor F between regular categories is conservative if and only if it reflects monomorphisms, i.e if for any $a \rightarrowtail c$ in $\mathcal{C}$ such that $F(a) \cong F(c)$ then $a \cong c$.*

Proof. This follows from the fact that faithful functors reflect monomorphisms and from Lemma 2.3.2. $\square$

Now, let φ be a formula and A a set. The Strong Collection axiom can be written in the following terms (see [AR01]) :

$$\forall \cdots \forall A[\forall x \in A \exists y \varphi(x, y, \ldots) \to \exists B \mathbb{B}(x \in A, y \in B)\varphi(x, y, \ldots)]$$

where $\mathbb{B}(x \in A, y \in B)\varphi$ means

$$\forall x \in A \exists y \in B \varphi \wedge \forall y \in B \exists x \in A \varphi$$

This amounts to say that if every element in A is related to an element y by some relation φ, then there exists a set B of those elements y that are related to some x in A.

Now we can use $A = \{(C, S) \in Ob(\mathcal{H}) \times Ob(\mathcal{H}) \mid S \subsetneq C\}$ where we write $S \subsetneq C$ for a proper subobject S of C. Let X be the class of objects of $\mathbb{C}$, i.e the coherent functors $\mathcal{H} \to$ Set. Let φ be a relation on $X \times A$ such that $(F, (C, S)) \in \varphi$ if and only $F(S) \subsetneq F(C)$. Since the functor ev is conservative, there exists a model $F : \mathcal{H} \to$ Set in X such that $F(S) \subsetneq F(C)$, for all $(C, S) \in$ A.

Therefore, this collection axiom gives us the existence of a set B of models F that are such that $F(S) \subsetneq F(C)$ for some $S \subsetneq C$ in $\mathcal{H}$. We can take $\mathbb{S}$ to be the full subcategory of $\mathbb{C}$ whose objects are the functors in B. Then the embedding $\mathcal{H} \hookrightarrow \mathbf{Sets}^{\mathbb{S}}$ is conservative and Heyting. The proof that this embedding is Heyting is still under development and is not included here.

3 Completeness results

Joyal's theorem is fundamental because it allows to retrieve completeness results for intuitionistic first-order logic (IFOL), as we shall see now.

Let us state that for coherent or Heyting theories, typing contexts Γ are interpreted as objects $[\![\Gamma]\!]$ of the category, and formulas in context $\Gamma \mid \varphi$ are interpreted as subobjects of their typing context, $[\![\Gamma \mid \varphi]\!] \rightarrowtail [\![\Gamma]\!]$.

Additionally, a model M of a first-order theory $\mathbb{T}$ satisfies a formula in context $(\Gamma \mid \varphi)$, denoted as $M \models (\Gamma \mid \varphi)$, if the subobject $[\![\Gamma \mid \varphi]\!]_M \rightarrowtail [\![\Gamma]\!]_M$ is the maximal one, $\mathrm{Id}_{[\![\Gamma]\!]_M}$.

The Kripke-Joyal forcing relation $\Vdash$ is defined on formulas and generalized elements of a Heyting category. It allows reformulating the condition of satisfaction of a formula in an interpretation in terms of objects and morphisms of the category. While we do not provide the details here, the result for a first-order logic theory $\mathbb{T}$ is as follows :

Lemma 3.0.1. *[Awo24] For any model $\mathbb{T}$-model M in a Heyting category $\mathcal{C}$, denote A_M the interpretation of a type symbol A in M, we have:*

$$M \models (x : A \mid \varphi) \quad \textit{iff} \quad \textit{for all } z : Z \to A_M \textit{ in } \mathcal{C},\ Z \Vdash \varphi(z).$$

These semantics behave well with respect to particular models. In the case of models in presheaves categories $\mathbf{Set}^{\mathbb{C}^{op}}$, the following theorem holds (y denotes the Yoneda embedding):

Theorem 3.0.2. *(Kripke-Joyal Semantics). [Awo24] For any presheaf category $\mathbf{Set}^{\mathbb{C}^{op}}$ and model M of a first-order theory $\mathbb{T}$, let $(x : A \mid \varphi), (x : A \mid \psi)$, and $(x : A, y : B \mid \vartheta)$ be formulas (in context) in the language of $\mathbb{T}$, and let $C \in \mathbb{C}$ and $c, c_1, c_2 : \mathrm{y}C \to A_M$ be any maps. Then we have*

1. $\mathrm{y}C \Vdash \top(c)$ *always.*
2. $\mathrm{y}C \Vdash \bot(c)$ *never.*
3. $\mathrm{y}C \Vdash c_1 = c_2$ *iff* $c_1 = c_2$ *as arrows* $\mathrm{y}C \to A_M$
4. $\mathrm{y}C \Vdash \varphi(c) \wedge \psi(c)$ *iff* $\mathrm{y}C \Vdash \varphi(c)$ *and* $\mathrm{y}C \Vdash \psi(c)$
5. $\mathrm{y}C \Vdash \varphi(c) \vee \psi(c)$ *iff* $\mathrm{y}C \Vdash \varphi(c)$ *or* $\mathrm{y}C \Vdash \psi(c)$
6. $\mathrm{y}C \Vdash \varphi(c) \Rightarrow \psi(c)$ *iff for all* $d : D \to C,\ \mathrm{y}D \Vdash \varphi(c.d)$ *implies* $\mathrm{y}D \Vdash \psi(c.d)$
7. $\mathrm{y}C \Vdash \neg\varphi(c)$ *iff for no* $d : D \to C,\ \mathrm{y}D \Vdash \varphi(c.d)$
8. $\mathrm{y}C \Vdash \exists y : B.\vartheta(c, y)$ *iff for some* $c' : C \to B_M,\ \mathrm{y}C \Vdash \vartheta(c, c')$
9. $\mathrm{y}C \Vdash \forall y : B.\vartheta(c, y)$ *iff for all* $d : D \to C$*, for all* $d' : D \to B_M,\ \mathrm{y}D \Vdash \vartheta(c.d, d')$

When the category $\mathbb{C}^{op}$ is a poset, the result is simplified as :

Theorem 3.0.3. *(Kripke Semantics) [Awo24] For any first-order theory $\mathbb{T}$ and poset K and model M in the functor category Set^K, let $(x : A \mid \varphi), (x : A \mid \psi)$, and $(x : A, y : B \mid \vartheta)$ be formulas in context in the language of $\mathbb{T}$, and let $i \in K$ and $a, a_1, a_2 : yi \to A_M$ be any maps (respectively elements a, $a_1, a_2 \in A_M(i)$). For each $i \in K$ we write $i \Vdash \varphi(a)$ for the relation $a \in ([\![x : A \mid \varphi]\!]_M)(i)$. Then we have :*

1. $i \Vdash \top(a)$ *always.*
2. $i \Vdash \bot(a)$ *never.*
3. $i \Vdash a_1 = a_2$ *iff* $a_1 = a_2$ *as elements of the set* $A_M(i)$.

4. $i \Vdash \varphi(a) \wedge \psi(a)$ *iff* $i \Vdash \varphi(a)$ *and* $i \Vdash \psi(a)$.

5. $i \Vdash \varphi(a) \vee \psi(a)$ *iff* $i \Vdash \varphi(a)$ *or* $i \Vdash \psi(a)$.

6. $i \Vdash \varphi(a) \Rightarrow \psi(a)$ *iff for all* $j \geq i, j \Vdash \varphi(a_j)$ *implies* $j \Vdash \psi(a_j)$.

7. $i \Vdash \neg\varphi(a)$ *iff for no* $j \geq i,\ j \Vdash \varphi(a_j)$.

8. $i \Vdash \exists y : B.\vartheta(a, y)$ *iff for some* $b : \mathrm{y}i \to B_M,\ i \Vdash \vartheta(a, b)$.

9. $i \Vdash \forall y : B.\vartheta(a, y)$ *iff for all* $j \geq i$, *for all* $b : \mathrm{y}j \to B_M,\ j \Vdash \vartheta(a_j, b)$.

It is possible to show that the rules for Heyting logic are sound with respect to their interpretation in Heyting categories. Moreover, for any first-order theory $\mathbb{T}$, there exists a syntactic category $\mathcal{C}_{\mathbb{T}}$, that is Heyting and contains an universal model U (see (1.1.1)). Subsequently, for any intuitionistic first-order theory, one can show soundness and completeness of the rules of IFOL with respect to semantics in Heyting categories :

Proposition 3.0.4. *An intuitionistic first-order theory* $\mathbb{T}$ *proves a formula* $\Gamma \mid \varphi$ *if and only if every model of* $\mathbb{T}$ *satisfies it.*

Joyal's representation theorem enhances this result by providing completeness with respect to restricted classes of models, that allow for the above Kripke-Joyal semantics : models in presheave categories and Kripke models. Indeed, it says that there is a small category $\mathbb{C}^{op}$ and a conservative Heyting functor :

$$F : \mathcal{C}_{\mathbb{T}} \hookrightarrow \mathbf{Set}^{\mathbb{C}^{op}}$$

A key feature of functorial semantics can then be applied : since F is a Heyting functor, the image of the universal model U in $\mathcal{C}_{\mathbb{T}}$ under F is again a model FU of $\mathbb{T}$ in the category $\mathbf{Set}^{\mathbb{C}^{op}}$. Thus, for a formula $\Gamma \mid \varphi$ to be provable in $\mathbb{T}$, it suffices for it to be satisfied in the model FU. Indeed, by conservativity of F :

$$\begin{aligned}
&FU \models (\Gamma \mid \varphi) \Leftrightarrow \mathrm{Id}_{[\![\Gamma \mid \varphi]\!]_{FU}} = [\![\Gamma \mid \varphi]\!]_{FU} \\
&\Leftrightarrow F(\mathrm{Id}_{[\![\Gamma \mid \varphi]\!]_U}) = F\left([\![\Gamma \mid \varphi]\!]_U\right) \\
&\Leftrightarrow \mathrm{Id}_{[\![\Gamma \mid \varphi]\!]_U} = [\![\Gamma \mid \varphi]\!]_U \\
&\Leftrightarrow U \models (\Gamma \mid \varphi). \\
&\Leftrightarrow \mathbb{T} \vdash (\Gamma \mid \varphi)
\end{aligned}$$

and the following result is obtained :

Theorem 3.0.5. *There is a model* M *in a presheaf category* $\mathbf{Set}^{\mathbb{C}^{op}}$ *such that for any intuitionistic first-order theory* $\mathbb{T}$, *and every closed formula* φ,

$$\mathbb{T} \vdash \varphi \quad \textit{iff} \quad M \models \varphi$$

For a model in a category $\mathbf{Set}^K$, where K is a particular type of poset, namely, a "tree", we speak of *Kripke models.* To obtain the completeness theorem with respect to the semantics of Theorem 3.0.3, it is necessary to further reduce the small category $\mathbb{S} \subseteq \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$ in the embedding $\mathcal{H} \hookrightarrow \mathbf{Set}^{\mathbb{S}}$ to a tree K and show that $\mathcal{H} \hookrightarrow \mathbf{Set}^K$ remains conservative and Heyting. This is done in [MR95, 3], see also [God13]. Joyal's theorem consequently establishes Kripke completeness :

Theorem 3.0.6. *(Kripke Completeness for Intuitionistic First-Order Logic) [Kri65] If* A *is valid, then it is provable in Heyting predicate calculus.*

where A is a formula of IFOL, and where valid means that A is valid in every Kripke-model.

4 Conclusion

It is noteworthy that our proof could be enhanced by uniformizing the perspectives between Pro- and Ind-objects, as well as by transitioning from colimits over filtered categories to directed posets. Indeed, it is always possible to make this reduction and obtain equivalent colimits, as shown in [AN82] : in our context, it would allow for an easier manipulation of the colimit we took in **Cat**.

Nonetheless, the project has successfully demonstrated Joyal's theorem by employing various tools from category theory, drawing on categorical logic and Deligne's theorem, which is fundamental for first-order logic due to its equivalence to Gödel's completeness theorem. Ultimately, the proof is structured around three main stages, focusing on well-known properties of left exact functors - which have been treated as Ind-objects in order to leverage their advantageous properties and those of filtered colimits.

Appendix A Deligne's theorem

Definition A.0.1. Given (Grothendieck) toposes $\mathcal{E}$ and $\mathcal{F}$, a geometric morphism $p : \mathcal{F} \to \mathcal{E}$ is a pair of adjoint functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "p^*"] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, bend left, "p_*"] \end{tikzcd}$$

such that the left adjoint p^* preserves finite limits.

Remark A.0.2. A Grothendieck topos verifies the hypotheses of Freyd's adjoint functor theorem. Therefore, a geometric morphism can also be defined as a functor $p^* : \mathcal{E} \to \mathcal{F}$ preserving all small colimits and finite limits.

Definition A.0.3. A *point* p of a (Grothendieck) topos $\mathcal{E}$ is a geometric morphism $p : \mathbf{Set} \to \mathcal{E}$.

Definition A.0.4. A topos $\mathcal{E}$ is said to have *enough points* if any morphism $f : a \to b$ in $\mathcal{E}$ is an isomorphism whenever $p^*(f)$ is an isomorphism for every point p of $\mathcal{E}$.

Theorem A.0.5. ***(Deligne's theorem)*** *A coherent topos has enough points.*

Proof. See e.g in [MM94, IX.11]. □

Let $\mathcal{H}$ be the classifying category of a coherent (any geometric in fact) theory $\mathbb{T}$, i.e there is an equivalence

$$\mathrm{Mod}(\mathbb{T}) \simeq \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$$

It is shown in [MM94, X.6] that there exists a Grothendieck topology $J_{\mathcal{H}}$ such that the topos $X = \mathrm{Sh}(\mathcal{H}, J_{\mathcal{H}})$ of sheaves on $\mathcal{H}$ with respect to this topology $J_{\mathcal{H}}$ is the classifying topos for the theory $\mathbb{T}$, namely that :

$$\mathrm{pt}(X) := \mathrm{Geom}(\mathbf{Set}, X) \simeq \mathrm{Coh}(\mathcal{H}, \mathbf{Set})$$

Moreover this topos is coherent because the Grothendieck topology $J_{\mathcal{H}}$ is given by a basis consisting of finite covering families.

Therefore, an equivalent statement of Deligne's theorem is the following :

For any morphism $f : a \to b$ in $\mathcal{H}$, if $F(a) \cong F(b)$ for all coherent functor $F : \mathcal{H} \to \mathbf{Set}$, then f is an isomorphism.

This precisely states that the functor from Theorem 2.0.1 is conservative.

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