Polynomial Functors and Natural Models of Type Theory

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Workshop on Polynomial Functors
Topos Institute
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Outline

1. Dependent type theory
2. Natural models
3. Type formers
4. Polynomial monad
5. Propositions as types

1. Dependent type theory

Types:

$$A, B, \ \ldots, \ A \times B, \ A \to B, \ \ldots$$

Terms:

$$x : A, \ b : B, \ \langle a, b \rangle, \ \lambda x. b(x), \ \ldots$$

Dependent Types:

$x : A \vdash B(x)$ "indexed families of types"

Type Forming Operations:

$$\sum_{x:A} B(x), \quad \prod_{x:A} B(x), \ \ldots$$

Equations:

$$s = t : A$$

1. Dependent type theory: Rules

Contexts:

$$\frac{x:A \vdash B(x)}{x:A,\ y:B(x) \vdash} \qquad \frac{\Gamma \vdash C}{\Gamma, z:C \vdash}$$

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Sums:

$$\frac{\Gamma, x : A \vdash B(x)}{\Gamma \vdash \sum_{x:A} B(x)} \qquad \frac{\Gamma \vdash a : A, \quad \Gamma \vdash b : B(a)}{\Gamma \vdash \langle a, b\rangle : \sum_{x:A} B(x)}$$

$$\frac{\Gamma \vdash c : \sum_{x:A} B(x)}{\Gamma \vdash \mathsf{fst}\, c : A} \qquad \frac{\Gamma \vdash c : \sum_{x:A} B(x)}{\Gamma \vdash \mathsf{snd}\, c : B(\mathsf{fst}\, c)}$$

$$\Gamma \vdash \mathsf{fst}\langle a, b\rangle = a : A \qquad \Gamma \vdash \mathsf{snd}\langle a, b\rangle = b : B$$

$$\Gamma \vdash \langle \mathsf{fst}\, c, \mathsf{snd}\, c\rangle = c : \sum_{x:A} B(x)$$

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Products:

$$\frac{x : A \vdash B(x)}{\prod_{x:A} B(x)} \qquad \frac{x : A \vdash b : B(x)}{\lambda x.b : \prod_{x:A} B(x)}$$

$$\frac{a : A \qquad f : \prod_{x:A} B(x)}{fa : B(a)}$$

$$x : A \vdash (\lambda x.b)x = b : B(x)$$

$$\lambda x.fx = f : \prod_{x:A} B(x)$$

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Substitution:

$$\frac{\sigma : \Delta \to \Gamma \qquad \Gamma \vdash a : A}{\Delta \vdash a[\sigma] : A[\sigma]}$$

2. Natural models

Definition

A natural transformation $f : Y \to X$ of presheaves on a category $\mathbb{C}$ is called *representable* if its pullback along any $\mathrm{y}C \to X$ is representable:

$$\begin{array}{ccc} \mathrm{y}D & \longrightarrow & Y \\ \big\downarrow & \lrcorner & \big\downarrow f \\ \mathrm{y}C & \longrightarrow & X \end{array}$$

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A natural transformation $f : Y \to X$ of presheaves on a category $\mathbb{C}$ is called *representable* if its pullback along any $\mathrm{y}C \to X$ is representable: for all $C \in \mathbb{C}$ and $x \in X(C)$ there is given $p : D \to C$ and $y \in Y(D)$ such that the following is a pullback:

$$\begin{array}{ccc} \mathrm{y}D & \xrightarrow{\;y\;} & Y \\ {\scriptstyle \mathrm{y}p}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathrm{y}C & \xrightarrow[\;x\;]{} & X \end{array}$$

Proposition (A, Fiore)

A representable natural transformation equipped with a choice of such pullbacks is the same thing as a **Category with Families** *in the sense of Dybjer.*

2. Natural models

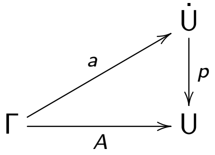
Write the objects and arrows of $\mathbb{C}$ as $\sigma : \Delta \to \Gamma$, thinking of a *category of contexts and substitutions*.

Let $p : \dot{\mathsf{U}} \to \mathsf{U}$ be a representable map of presheaves on $\mathbb{C}$.

Think of U as the *presheaf of types*, $\dot{\mathsf{U}}$ as the *presheaf of terms*, and then p gives the type of a term.

$$\begin{aligned} \Gamma \vdash A \quad &\approx \quad A \in \mathsf{U}(\Gamma) \\ \Gamma \vdash a : A \quad &\approx \quad a \in \dot{\mathsf{U}}(\Gamma) \end{aligned}$$

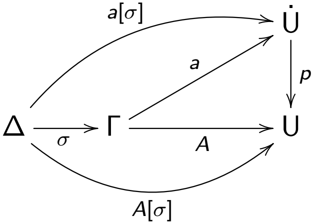
where $A = p \circ a$.



2. Natural models

Naturality of $p : \dot{\mathsf{U}} \to \mathsf{U}$ means that for any *substitution* $\sigma : \Delta \to \Gamma$, we have the required action on types and terms:

$$\begin{array}{rcl} \Gamma \vdash A & \Rightarrow & \Delta \vdash A[\sigma] \\ \Gamma \vdash a : A & \Rightarrow & \Delta \vdash a[\sigma] : A[\sigma] \end{array}$$



2. Natural models

Given any further $\tau : \Delta' \to \Delta$ we clearly have

$$A[\sigma][\tau] = A[\sigma \circ \tau] \qquad\qquad a[\sigma][\tau] = a[\sigma \circ \tau]$$

and for the identity substitution $1 : \Gamma \to \Gamma$

$$A[1] = A \qquad\qquad a[1] = a.$$

This is the basic structure of a CwF.

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The remaining operation of **context extension**

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash}$$

is given by the representability of $p : \dot{\mathrm{U}} \to \mathrm{U}$ as follows.

2. Natural models, context extension

Given $\Gamma \vdash A$ we need a new context $\Gamma.A$ together with a substitution $p_A : \Gamma.A \to A$ and a term

$$\Gamma.A \vdash q_A : A[p_A]\,.$$

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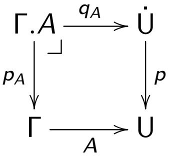
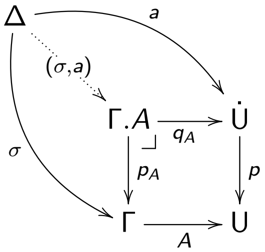
Let $p_A : \Gamma.A \to \Gamma$ be the pullback of p along A.



The map $q_A : \Gamma.A \to \dot{\mathrm{U}}$ gives the required term $\Gamma.A \vdash q_A : A[p_A]$.

2. Natural models, context extension



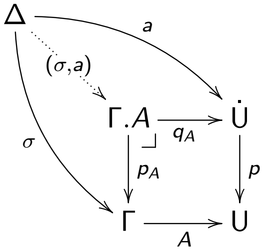
The pullback means that given any substitution $\sigma : \Delta \to \Gamma$ and term $\Delta \vdash a : A[\sigma]$ there is a map

$$(\sigma, a) : \Delta \to \Gamma.A$$

satisfying

$$p_A(\sigma, a) = \sigma$$
$$q_A[\sigma, a] = a.$$

2. Natural models, context extension



By the uniqueness of (σ, a), we also have

$$(\sigma, a) \circ \tau \ = \ (\sigma \circ \tau, a[\tau]) \qquad \text{for any } \tau : \Delta' \to \Delta$$

and

$$(p_A, q_A) = 1.$$

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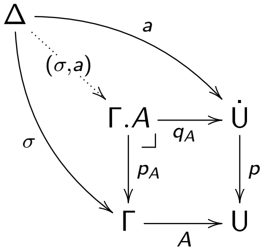

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These are all the laws for a CwF.

2. Natural models and initiality

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- The algebraic homomorphisms correspond to syntactic translations.
- There are *initial algebras* as well as *free algebras* over basic types and terms.
- The rules of type theory are a procedure for generating the free algebras.

2. Natural models and tribes

Let $p : \dot{\mathrm{U}} \to \mathrm{U}$ be a natural model.

The fibration $\mathcal{F} \to \mathbb{C}$ of all *display maps*

$$p_A : \Gamma.A \to \Gamma \qquad \text{for all } A : \Gamma \to \mathrm{U}$$

form a *clan* in the sense of Joyal.

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Let $p : \dot{\mathsf{U}} \to \mathsf{U}$ be a natural model.

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Conversely, given a clan $(\mathbb{C}, \mathcal{F})$, there is a natural model in $\hat{\mathbb{C}}$,

$$\coprod_{f \in \mathcal{F}} \mathsf{y}(f) : \coprod_{f \in \mathcal{F}} \mathsf{y}(\mathrm{dom}(f)) \to \coprod_{f \in \mathcal{F}} \mathsf{y}(\mathrm{cod}(f)).$$

The natural model determines a *splitting* of the fibration $\mathcal{F} \to \mathbb{C}$.

3. Modeling the type formers

Consider the *polynomial endofunctor* $P = \mathsf{U}_! p_* \dot{\mathsf{U}}^* : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ determined by $p : \dot{\mathsf{U}} \to \mathsf{U}$,

$$P(X) \; = \; \sum_{A : \mathsf{U}} X^{[A]}$$

where $[A] = p^{-1}(A)$ is the fiber of $p : \dot{\mathsf{U}} \to \mathsf{U}$ at $A : \mathsf{U}$.

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Lemma

Maps $\Gamma \to P(X)$ *correspond naturally to pairs* (A, B) *where*

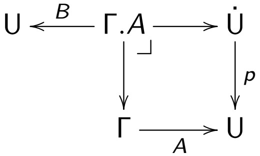
$$\begin{array}{ccccc}
X & \xleftarrow{\;B\;} & \Gamma.A & \longrightarrow & \dot{\mathsf{U}} \\
 & & \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\
 & & \Gamma & \xrightarrow[\;A\;]{} & \mathsf{U}
\end{array} \; .$$

3. Modeling the type formers

Applying P to U itself therefore gives an object

$$P(\mathsf{U}) = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

maps $\Gamma \to P(\mathsf{U})$ into which correspond naturally to types in an extended context $\Gamma.A \vdash B$



3. Modeling the type formers: Π

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ *models the rules for products just if there are maps* λ, Π *making the following a pullback.*

$$\begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ {\scriptstyle P(p)}\downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

3. Modeling the type formers: Π

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Proof:

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The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for products just if there are maps λ, Π making the following a pullback.

Proof:

$A \vdash f(x) : B$ $\qquad\qquad$ $\lambda_A f(x) = f$

$$\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} \qquad \begin{array}{ccc} P(\dot{\mathsf{U}}) & \xrightarrow{\lambda} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\Pi]{} & \mathsf{U} \end{array}$$

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$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

3. Modeling the type formers: Σ

Proposition

The map $p : \dot{\mathsf{U}} \to \mathsf{U}$ *models the rules for sums just if there are maps* (pair, Σ) *making the following a pullback*

$$\begin{array}{ccc} Q & \xrightarrow{\ \mathsf{pair}\ } & \dot{\mathsf{U}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\ \Sigma\ ]{} & \mathsf{U} \end{array}$$

where $q : Q \to P(\mathsf{U})$ *is the polynomial composition* $P_q = P \circ P$.

Explicitly:

$$Q \;=\; \sum_{A:\mathsf{U}} \sum_{B:\mathsf{U}^A} \sum_{x:A} B(x)$$

3. Modeling the type formers: T

Rules for a terminal type T

$$\frac{}{\vdash \mathrm{T}} \qquad\qquad \frac{}{\vdash * : \mathrm{T}} \qquad\qquad \frac{}{x : \mathrm{T} \vdash x = * : \mathrm{T}}$$

Proposition

The map $p : \dot{\mathrm{U}} \to \mathrm{U}$ *models the rules for a terminal type just if there are maps* $(*, \mathrm{T})$ *making the following a pullback.*

$$\begin{array}{ccc} 1 & \xrightarrow{\;\;*\;\;} & \dot{\mathrm{U}} \\ \big\downarrow & & \big\downarrow p \\ 1 & \xrightarrow[\;\;\mathrm{T}\;\;]{} & \mathrm{U} \end{array}$$

4. Polynomial monad

Consider the pullback squares for T and Σ.

$$\begin{array}{ccc} 1 & \xrightarrow{\;*\;} & \dot{\mathsf{U}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ 1 & \xrightarrow[\;\mathsf{T}\;]{} & \mathsf{U} \end{array} \qquad\qquad \begin{array}{ccc} Q & \xrightarrow{\;\mathsf{pair}\;} & \dot{\mathsf{U}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ P(\mathsf{U}) & \xrightarrow[\;\Sigma\;]{} & \mathsf{U} \end{array}$$

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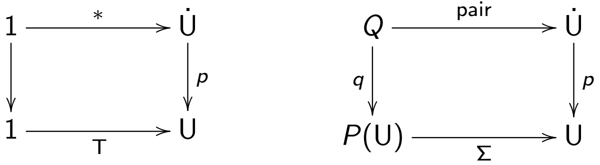

These determine cartesian natural transformations between the corresponding polynomial endofunctors.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

4. Polynomial monad

Theorem (A-Newstead)

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ *models the* T *and* Σ *type formers iff the associated polynomial endofunctor* P *has the structure maps of a cartesian monad.*

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

4. Polynomial monad

The monad laws correspond to the following type isomorphisms.

$\sigma \circ P\sigma = \sigma \circ \sigma_P$	$\sum_{a:A} \sum_{b:B(a)} C(a,b) \;\cong\; \sum_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\sigma \circ P\tau = 1$	$\sum_{a:A} 1 \;\cong\; A$
$\sigma \circ \tau_P = 1$	$\sum_{x:1} A \;\cong\; A$

4. Polynomial monad

The pullback square for Π

$$\begin{array}{ccc} P(\dot{\mathrm{U}}) & \xrightarrow{\lambda} & \dot{\mathrm{U}} \\ {\scriptstyle P(p)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ P(\mathrm{U}) & \xrightarrow[\Pi]{} & \mathrm{U} \end{array}$$

determines a cartesian natural transformation

$$\pi : P^{\mathbf{2}} p \Rightarrow p$$

where $P^{\mathbf{2}} : \hat{\mathbb{C}}^{\mathbf{2}} \to \hat{\mathbb{C}}^{\mathbf{2}}$ is the extension of P to the arrow category.

4. Polynomial monad

Theorem (A-Newstead)

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ *models the* Π *type former iff it has an algebra structure for the lifted endofunctor* $P^{\mathbf{2}}$.

$$\pi : P^{\mathbf{2}} p \Rightarrow p$$

4. Polynomial monad

The algebra laws correspond to the following type isomorphisms.

$\pi \circ P\pi \;=\; \pi \circ \sigma$	$\prod_{a:A} \prod_{b:B(a)} C(a,b) \;\cong\; \prod_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\pi \circ \tau \;=\; 1$	$\prod_{x:1} A \;\cong\; A$

5. Propositions as types

Let $p : \dot{\mathsf{U}} \to \mathsf{U}$ be a universe of *small* objects in $\mathcal{E} = \hat{\mathbb{C}}$.

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Though p is not representable in $\mathcal{E}$ it is still a natural model in $\hat{\mathcal{E}}$.

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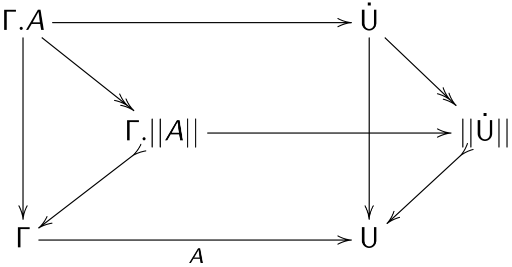
Factor p as on the right below.



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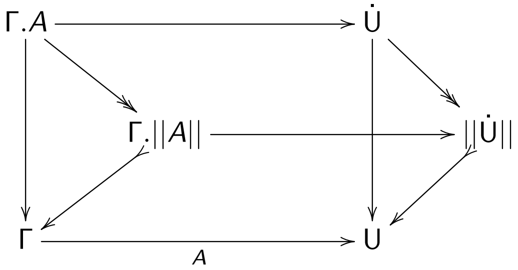
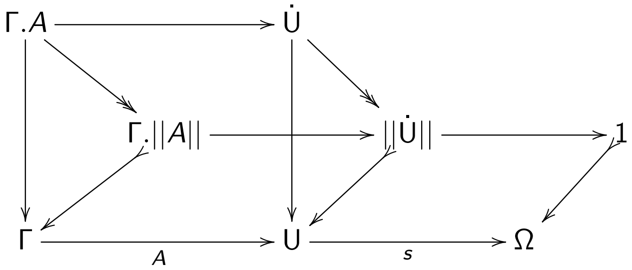


So $||\dot{\mathsf{U}}|| \rightarrowtail \mathsf{U}$ is a universal family of small propositions.

5. Propositions as types

Let $s : \mathsf{U} \to \Omega$ classify the mono $||\dot{\mathsf{U}}|| \rightarrowtail \mathsf{U}$.



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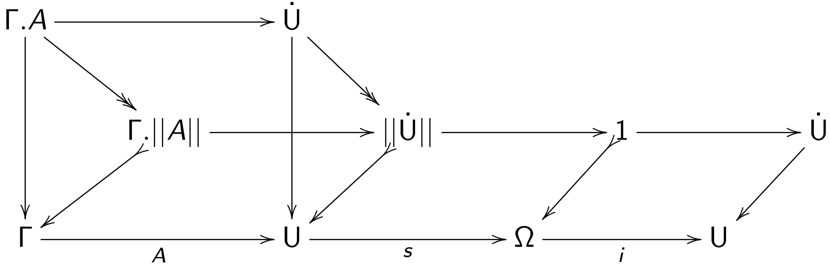
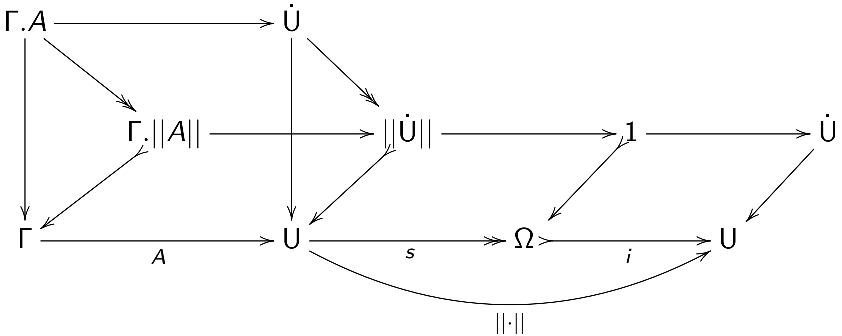


Let $i : \Omega \to \mathsf{U}$ classify the family of small propositions $1 \rightarrowtail \Omega$.

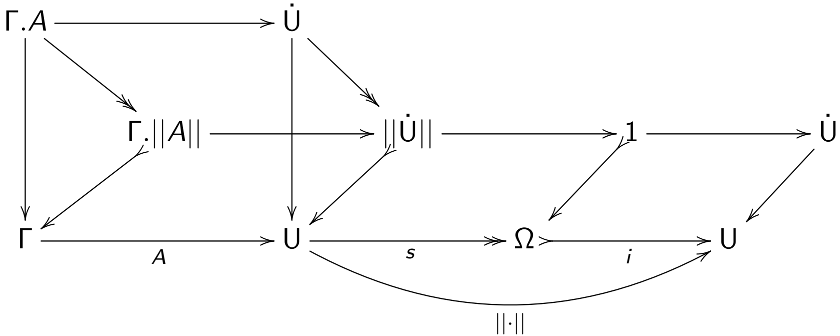
5. Propositions as types



Let

$$||\cdot|| := i \circ s : \mathsf{U} \to \mathsf{U}.$$

5. Propositions as types



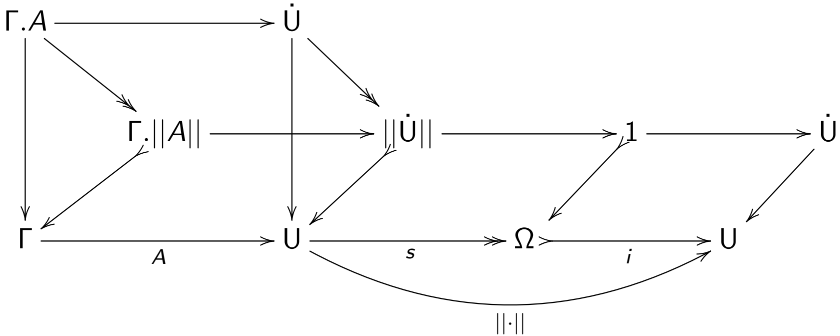
Let

$$||\cdot|| := i \circ s : \mathsf{U} \to \mathsf{U}.$$

We have

$$s \circ i = 1 : \Omega \to \Omega.$$

5. Propositions as types



Let

$$||\cdot|| := i \circ s : \mathrm{U} \to \mathrm{U}.$$

We have

$$s \circ i = 1 : \Omega \to \Omega.$$

So

$$\Omega = \mathrm{im}(||\cdot||).$$

5. Propositions as types

The following commute.

$$
\begin{array}{ccc}
\sum_{A:\mathrm{U}} \Omega^A & \xrightarrow{\exists} & \Omega \\
{\scriptstyle Pi}\downarrow & & \uparrow{\scriptstyle s} \\
\sum_{A:\mathrm{U}} \mathrm{U}^A & \xrightarrow[\Sigma]{} & \mathrm{U}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\sum_{A:\mathrm{U}} \Omega^A & \xrightarrow{\forall} & \Omega \\
{\scriptstyle Pi}\downarrow & & \uparrow{\scriptstyle s} \\
\sum_{A:\mathrm{U}} \mathrm{U}^A & \xrightarrow[\Pi]{} & \mathrm{U}
\end{array}
$$

Where, recall,

$$PX \;=\; \sum_{A:\mathrm{U}} X^A$$

is the polynomial functor of the natural model $p : \dot{\mathrm{U}} \to \mathrm{U}$.

References

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