# Sheaf Universes via Coalgebra

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# Motivation: Extending the Model in Presheaves

There is a well-known construction of a model of type theory in $Set^{C^{op}}$. It is based on the equivalence

$$Set^{C^{op}}/P \simeq Set^{(\int P)^{op}} \quad ,$$

for each $P \in Set^{C^{op}}$.

- The (large) presheaf of types on $Set^{C^{op}}$ is given by $\mathrm{Ty}(P) :\equiv Set^{(\int P)^{op}}$.
- The context extension is given by the functor $Set^{(\int P)^{op}} \to Set^{C^{op}}/P$.

Furthermore, when $C$ is small (for some Grothendieck universe *set*), there is a universe that classifies small types.

- The universe $\mathcal{U} \in Set^{C^{op}}$ is given on $c \in C$ by $\mathcal{U}(c) :\equiv set^{(C/c)^{op}}$.

# Problem (Hofmann and Streicher 1999)

We would like to extend this picture to the category of sheaves $\mathrm{Sh}(C, J)$ on a site $(C, J)$. However, when $(C, J)$ is small, the putative universe $\mathcal{U}$ given on $c \in C$ by

$$\mathcal{U}(c) :\equiv \mathrm{sh}(C/c, J/c) \quad ,$$

the category of small sheaves on a suitably restricted site, is not a sheaf!

Rather it forms a stack (2-sheaf), in which amalgamations are unique only up to coherent isomorphism.

# Options

How to get around this issue?

- We can use the **stack semantics** described by Thierry Coquand in an earlier seminar (Coquand *et al.* 2020).
- We can **strictify** to get a sheaf of sheafs, as described by Denis-Charles Cisinski in an earlier seminar.
- ...

However, we may want to maintain a 1-categorical approach, at least where viable.

# Current Approach

Zwanziger (2020) gives a 1-categorical approach to constructing universes that applies to certain sheaf toposes, namely those with enough points.[1] In those cases, a suitable presheaf of sheaves is a sheaf (not a stack).

We will focus here on the case of sheaves on a topological space, which can be described in relatively familiar terms.

I will work with the natural model formulation of CwFs (Awodey 2012, 2018, Fiore 2012).

[1] Thank you to Thierry for suggesting this application.

# Outline

# Outline

# Natural Models

**Definition**

*A* natural model *consists of:*

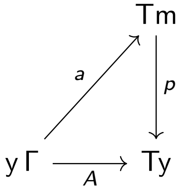
- *a category* $\mathcal{E}$ *with a terminal object,*
- *presheaves* $\mathrm{Ty}, \mathrm{Tm} : \mathcal{E}^{op} \to Set$,
- *a representable natural transformation* $p : \mathrm{Tm} \to \mathrm{Ty}$.

# Conventions

## Conventions

- *An object* $\Gamma \in \mathcal{E}$ *is a "context".*
- *An element* $A \in \mathrm{Ty}(\Gamma)$ *is a "type in context* $\Gamma$*".*
- *An element* $a \in \mathrm{Tm}(\Gamma)$ *such that* $p_\Gamma(a) = A$ *is a "term of type* $A$ *in context* $\Gamma$*".*

This last is represented by the following commutative diagram:




Below, as here, we will freely use the Yoneda lemma to identify presheaf elements $x \in P(\Gamma)$ with the corresponding map $x : \mathrm{y}\,\Gamma \to P$.

# Comprehension as Representability

Representability of $p : \mathrm{Tm} \to \mathrm{Ty}$ means the following:

**Definition**

*Given a context* $\Gamma \in \mathcal{E}$ *and a type* $A \in \mathrm{Ty}(\Gamma)$ *in the context* $\Gamma$*, there is* $\Gamma.A \in \mathcal{E}$, $p_A : \Gamma.A \to \Gamma$*, and* $v_A : \mathrm{y}(\Gamma.A) \to \mathrm{Tm}$ *such that the following diagram is a pullback:*

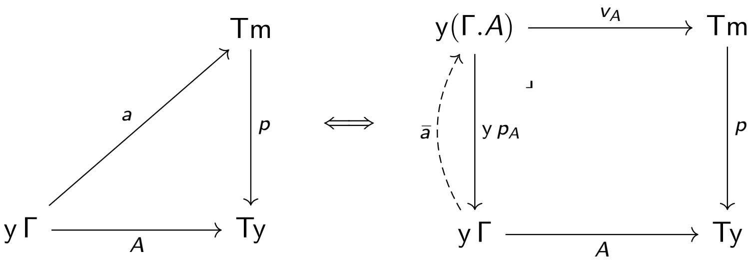
$$
\begin{array}{ccc}
\mathrm{y}(\Gamma.A) & \xrightarrow{v_A} & \mathrm{Tm} \\
{\scriptstyle \mathrm{y}\,p_A}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\
\mathrm{y}\,\Gamma & \xrightarrow[A]{} & \mathrm{Ty}
\end{array}
$$

*These* $\Gamma.A, p_A, v_A$ *together constitute the* comprehension *of* $A$*.*

# Terms *vs.* Sections

### Remark

*Terms are interchangeable with a "comprehension" as sections, as depicted by the following:*




See Awodey (2018) for more on the natural model formulation.

# Outline

# Sheaves

We will use multiple equivalent characterizations for the category of sheaves.

### Convention

*We reserve the term "sheaf" for the (objects of the) usual reflective subcategory of* $\mathrm{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$*, which we denote by*

$$a \dashv i : \mathrm{Sh}(X) \hookrightarrow \mathrm{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \qquad .$$

*Recall that this adjunction is a geometric inclusion, which we denote*

$$i : \mathrm{Sh}(X) \hookrightarrow \mathrm{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \qquad .$$

# Local Homeomorphisms

The category of sheaves on $X$ is equivalent to the category of local homeomorphisms over $X$.

**Convention**

*Let $LH$ denote the full subcategory of $Top^{\rightarrow}$ on the local homeomorphisms. Let $LH_X$ denote the fiber of $\mathrm{cod} : LH \rightarrow Top$ at $X$.*

**Proposition**

$\mathrm{Sh}(X) \simeq LH_X$.

**Proposition**

*The forgetful functor $|-|_X : LH_X \rightarrow Set/|X|$ is strictly comonadic.*

**Proof.**

Elephant A4.2.4(e). □

# Inverse Image

### Convention

*Let*

$$f : X \to Y$$

*be a continuous function. Recall that $f$ induces, by pullback in* Top*, a functor*

$$f^* : LH_Y \to LH_X \quad ,$$

*the* inverse image *of local homeomorphisms.*

It will be crucial for us to have an inverse image operation that is strictly coherent, but the inverse image of local homeomorphisms (and sheaves, for that matter,) is only coherent up to isomorphism.

Solution: Use precomposition, rather than pullback.

# Outline

# Brooms

Consider the composite of geometric morphisms

$$Set^{|X|} \simeq Set/|X| \twoheadrightarrow LH_X \quad .$$

This is a geometric surjection, and thus induces a comonad

$$\flat : Set^{|X|} \to Set^{|X|}$$

such that the category of coalgebras is equivalent to $LH_X$, *i.e.*

$$(Set^{|X|})^\flat \simeq LH_X \quad .$$

We thus have yet another equivalent characterization of $\mathrm{Sh}(X)$, which, for the purposes of this talk, we need a name for:

**Definition**

*A* broom *on $X$ is a coalgebra for $\flat : Set^{|X|} \to Set^{|X|}$. We denote by* $\mathrm{Br}(X) :\equiv (Set^{|X|})^\flat$ *the category of brooms on $X$.*

# Intuition for Terminology

The term "sheaf" evokes a resemblence of local homeomorphisms to sheaves of grain. Here, "broom" evokes an upside-down sheaf of grain:

Figure: A sheaf of wheat

Figure: A wheat broom

# Intuitive Characterization of Brooms

Since brooms are less familiar than local homeomorphisms, we will use the following characterization when reasoning about brooms:

## Proposition

*The category* $\mathrm{Br}(X)$ *is a 1-pullback*

$$\begin{array}{ccc} \mathrm{Br}(X) & \xrightarrow[p_{(-)}]{\sim} & LH_X \\ {\scriptstyle |-|_X}\downarrow & \lrcorner & \downarrow{\scriptstyle |-|_X} \\ Set^{|X|} & \xrightarrow[\Phi]{\sim} & Set/|X| \end{array}$$

## Proof.

From the fact that $LH_X$ is strictly comonadic with comonad $\flat' : Set/|X| \to Set/|X|$, and $\Phi : Set^{|X|} \simeq Set/|X|$ extends to an embedding (indeed, an equivalence) of comonads $(Set^{|X|}, \flat) \simeq (Set/|X|, \flat')$. □

# Intuitive Characterization of Brooms (Cont’d)

We can thus reason about brooms $B$ as if they were pairs $(|B|, p_B) \in Set^{|X|} \times LH_X$, subject to the condition suggested by

$$
\begin{array}{ccc}
E_B & & \\
\Big\downarrow{\scriptstyle p_B} & \quad\mapsto\quad & \\
X & &
\end{array}
\qquad
\begin{array}{ccc}
|X|.|B| & \longrightarrow & Set_\bullet \\
\Big\downarrow{\scriptstyle p_{|B|}} & \lrcorner & \Big\downarrow \\
|X| & \xrightarrow[|B|]{} & Set
\end{array}
\quad ,
$$

where (and henceforth) $\mapsto$ indicates the action of $|-| : Top \to Set$.

# Broom Sections

We have also the composite of adjunctions

$$Set_\bullet^{|X|} \simeq \mathrm{sect}(Set/|X|) \twoheadrightarrow \mathrm{sect}(LH_X) \quad ,$$

which induces a comonad

$$\flat_\bullet : Set_\bullet^{|X|} \to Set_\bullet^{|X|}$$

such that the category of coalgebras is equivalent to $\mathrm{sect}(LH_X)$, *i.e.* $(Set_\bullet^{|X|})^{\flat_\bullet} \simeq \mathrm{sect}(LH_X)$.

## Convention

*We will write* $\mathrm{Br}_\bullet(X)$ *for* $(Set_\bullet^{|X|})^{\flat_\bullet}$ *and call an object of* $\mathrm{Br}_\bullet(X)$ *a* broom section *on* $X$*.*

# Intuitive Characterization of Broom Sections

Since broom sections are less familiar than sections of local homeomorphisms, we will use the following characterization when reasoning about broom sections:

**Proposition**

*The category* $\mathrm{Br}_\bullet(X)$ *is a 1-pullback*

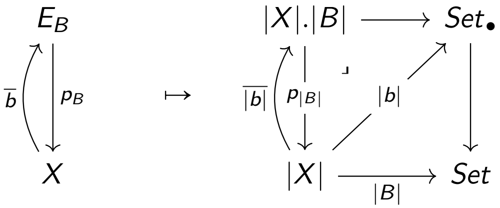
$$\begin{array}{ccc}
\mathrm{Br}_\bullet(X) & \xrightarrow[\overline{(-)}]{\sim} & \mathrm{sect}(LH_X) \\
{\scriptstyle |-|_X}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle |-|_X} \\
Set_\bullet^{|X|} & \xrightarrow[\Phi]{\sim} & \mathrm{sect}(Set/|X|)
\end{array}$$

**Proof.**

Analogous. □

# Intuitive Characterization of Brooms (Cont'd)

We can thus reason about broom sections $b$ as if they were pairs $(|b|, \overline{b}) \in Set_{\bullet}^{|X|} \times \mathrm{sect}(LH_X)$, subject to the condition suggested by

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# Strict Inverse Image

**Proposition (Marmolejo 1998)**

*Let*

$$f : X \to Y$$

*be a continuous function. Then f induces an* inverse image *functor*

$$f^* : \mathrm{Br}(Y) \to \mathrm{Br}(X) \quad ,$$

*equivalent to the usual inverse image of local homeomorphisms or sheaves. Furthermore, this operation is strictly coherent in the sense of yielding a strict functor*

$$\mathrm{Br}(-) : \mathit{Top}^{op} \to \mathit{Topos}_{\mathit{Cart}} \quad .$$

# Strict Inverse Image (Cont'd)

**Proof.**

Let

$$f : X \to Y$$

be a continuous function. Then, for each broom $B \in Br(Y)$ (associated to the pair $(|B|, p_B)$), $f$ induces a canonical pullback square at left:

$$\begin{array}{ccccc}
|X|.|B|[|f|] & \overset{|f|.|B|}{\dashrightarrow} & |Y|.|B| & \longrightarrow & Set_{\bullet} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
|X| & \underset{|f|}{\longrightarrow} & |Y| & \underset{|B|}{\longrightarrow} & Set
\end{array}$$

.

# Strict Inverse Image (Cont'd)

Proof (cont'd).

Because the forgetful functor $|-| : Top \to Set$ creates pullbacks, there is a unique pullback in $Top$, at left, over that in $Set$, at right:

$$
\begin{array}{ccc}
E_{B[f]} & \xrightarrow{f.B} & E_B \\
{\scriptstyle p_{B[f]}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_B} \\
X & \xrightarrow[f]{} & Y
\end{array}
\quad \mapsto \quad
\begin{array}{ccc}
|X|.|B|[|f|] & \xrightarrow{|f|.|B|} & |Y|.|B| \\
\downarrow & \lrcorner & \downarrow \\
|X| & \xrightarrow[|f|]{} & |Y|
\end{array}
\quad .
$$

Since $p_B$ is a local homeomorphism, so is $p_{B[f]}$. We thus let $B[f]$ be the broom associated to the pair $(|B|[|f|], p_{B[f]})$, and define

$$f^*B :\equiv B[f] \quad .$$

# Strict Inverse Image (Cont'd)

Proof (cont'd).

The strict coherence of the inverse image is then inherited from that of the canonical natural model on *Set*. □

Remark

*We have constructed a natural model on Top, with* $\mathrm{Ty}(X) :\equiv \mathrm{Br}(X)$.

# Strict Restriction of Brooms

**Definition**

*Let* $B \in \mathrm{Br}(X)$ *and* $V \in \mathcal{O}(X)$. *Then we write* $B|_V$ *for* $B[i] \in \mathrm{Br}(V)$, *where* $i : V \hookrightarrow X$.

**Lemma**

*Let* $B \in \mathrm{Br}(X)$ *and* $V \in \mathcal{O}(X)$. *Then* $U(B|_V) = (|B|)|_{|V|}$ *and, furthermore, we have*

$$\begin{array}{ccc} E_{B|_V} & \overset{i.B}{\hookrightarrow} & E_B \\ {\scriptstyle p_{B|_V}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_B} \\ V & \underset{i}{\hookrightarrow} & X \end{array} .$$

# Strict Restriction of Broom Sections

**Definition**

*Let* $b \in \mathrm{Br}_\bullet(X)$ *and* $V \in \mathcal{O}(X)$. *Then we write* $b|_V$ *for* $b[i] \in \mathrm{Br}(V)$, *where* $i : V \hookrightarrow X$.

**Lemma**

*Let* $b \in \mathrm{Br}_\bullet(X)$ *and* $V \in \mathcal{O}(X)$. *Then* $U(b|_V) = (|b|)|_{|V|}$ *and, furthermore, we have*

$$\begin{array}{ccc} E_{B|_V} & \xhookrightarrow{\;i.B\;} & E_B \\ {\scriptstyle \overline{b|V}}\uparrow\downarrow{\scriptstyle p_{B|_V}} & \lrcorner & {\scriptstyle \overline{b}}\uparrow\downarrow{\scriptstyle p_B} \\ V & \xhookrightarrow[\;i\;]{} & X \end{array}.$$

# Small Brooms

**Definition**

*We say a broom* $B \in \mathrm{Br}(X)$ *is* small *if* $|B|(x)$ *is small, for all* $x \in X$.

**Definition**

*We write* $\mathrm{br}(X)$ *for the category of small brooms on* $X$.

**Definition**

*We say a broom section* $b \in \mathrm{Br}_\bullet(X)$ *is* small *if* $|b|(x)$ *is small, for all* $x \in X$.

**Definition**

*We write* $\mathrm{br}_\bullet(X)$ *for the category of small broom sections on* $X$.

# Outline

# The Model Sh($X$)

Intuition: just as we exploited the equivalence

$$Set^{C^{op}}/P \simeq Set^{(\int P)^{op}}$$

for the natural model on $Set^{C^{op}}$, we essentially exploit the equivalence

$$\mathrm{Sh}(X)/S \simeq \mathrm{Sh}(E_S)$$

for the natural model on $Sh(X)$. More precisely, we will exploit the equivalence

$$\mathrm{Br}(X)/B \simeq \mathrm{Br}(E_B) \quad .$$

# The Natural Model Sh($X$)

### Theorem

*The category* $\mathsf{Sh}(X)$ *extends to a natural model via the following:*

- *The presheaf* $\mathsf{Ty}$ *is given on* $S \in \mathsf{Sh}(X)$ *by*

$$\mathsf{Ty}(S) :\equiv \mathsf{Br}(E_S)_0 \quad .$$

- *The presheaf* $\mathsf{Tm}$ *is given on* $S \in \mathsf{Sh}(X)$ *by*

$$\mathsf{Tm}(S) :\equiv \mathsf{Br}_\bullet(E_S)_0 \quad .$$

- *The natural transformation* $p : \mathsf{Tm} \to \mathsf{Ty}$ *is given on* $S \in \mathsf{Sh}(X)$ *by the obvious projection*

$$\mathsf{Br}_\bullet(E_S)_0 \to \mathsf{Br}(E_S)_0 \quad .$$

# The Presheaf $\mathcal{U}$

**Definition**

*For any small $X$, let the presheaf $\mathcal{U} : \mathcal{O}(X)^{op} \to \mathbf{Set}$ be given on $U \in \mathcal{O}(X)$ by*

$$\mathcal{U}(U) :\equiv \mathrm{br}(U)_0 \quad ,$$

*and on*

$$i : V \hookrightarrow U$$

*by*

$$\mathcal{U}(i)(B) :\equiv B|V \in \mathrm{br}(V)_0 \quad ,$$

*for any $B \in \mathrm{br}(U)$.*

# $\mathcal{U}$ is a Sheaf

**Theorem**

*The presheaf $\mathcal{U}$ is a sheaf.*

**Proof.**

Let $\bigcup_{i \in \mathcal{I}} V_i = V \in \mathcal{O}(X)$, and $(B_i \in \text{br}(V_i))_{i \in \mathcal{I}}$ a matching family for $\mathcal{U}$ at $V$. Then $(|B_i| \in set^{|V_i|})_{i \in \mathcal{I}}$ is a matching family for the sheaf $set^{|-|}$ at at $V$, so we have a unique amalgamation $|B| \in Set^{|V|}$.

Furthermore, we construct the space $E_B$ on the set $|V|.|B|$ by taking the final topology for $(|E_{B_i}| \hookrightarrow |V|.|B|)_{i \in \mathcal{I}}$. Then $p_{|B|}$ clearly lifts to a local homeomorphism $p_B : E_B \to V$ and the broom $B$ associated to $(|B|, p_B)$ is an amalgamation for $(B_i \in \text{br}(V_i))_{i \in \mathcal{I}}$.

Any amalgamation $B'$ must have $|E_{B'}| = |V|.|B|$ with the final topology on $E_{B'}$, or else we would not have open inclusions $E_{B_i} \hookrightarrow E_{B'}$ for all $i \in \mathcal{I}$. Thus, $B$ is the unique amalgamation. □

# $\mathcal{U}$ Classifies Small Types

**Remark**

*For all* $S \in \mathrm{Sh}(X)$, $\mathrm{ty}(S) \cong \mathrm{Hom}_{\mathrm{Sh}(X)}(S, \mathcal{U})$.

The proof involves slightly more machinery.

# Natural Models of Coalgebras

This model emerges from much more general considerations:

**Theorem**

*The strict 2-category NM of natural models is closed under the construction of coalgebras for comonads.*

**Lemma**

*The comonad* $\flat : \mathit{Set}^{|X|} \to \mathit{Set}^{|X|}$ *extends to a comonad of natural models.*

This recovers the natural model on Sh$(X)$.

# Outline

# Future Work

- Can the model in $\mathrm{Sh}(X)$ be obtained more formally from the model in *Top*? Can the model in *Top* be obtained more formally?
- Extend to more sheaf toposes.
  - Use that for any *ionad* $X$ and $S \in \mathrm{Sh}(X)$,

$$\mathrm{Sh}(X)/S \simeq \mathrm{Sh}(E_S) \quad ,$$

    in a suitable sense.