Every Elementary Higher Topos has a Natural Number Object

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Pre-talk Comments

Three Comments:

- This is not a type theory talk (but has some type theory words)!
- Will assume basic category theory, but no ∞-category theory.
- For more details see the paper *"Every Elementary Higher Topos has a Natural Number Object"* arXiv:1809.01734 or Theory and Applications of Categories, Vol. 37, 2021, No. 13, pp 337-377.

1-Categories vs. Type Theories

There are various 1-categories which are models of various type theories (table by Shulman):

propositional logic	Boolean/Heyting algebras
first-order logic	Boolean/Heyting categories
simply typed λ-calculus	cartesian closed categories
higher-order type theory	elementary toposes

1-Categories vs. Type Theories

There are various 1-categories which are models of various type theories (table by Shulman):

propositional logic	Boolean/Heyting algebras
first-order logic	Boolean/Heyting categories
simply typed λ-calculus	cartesian closed categories
higher-order type theory	elementary toposes

Focus on the last line!

Elementary Toposes

Definition (Elementary Topos)

An elementary topos $\mathcal{E}$ is a **locally Cartesian closed** category with **subobject classifier**.

- **LCCC:** For all $f : X \to Y$ we have an adjunction

$$\mathcal{E}_{/Y} \underset{f_*}{\overset{f^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}_{/X} \ .$$

- **SOC:** Natural isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(X, \Omega) \cong \mathcal{S}\mathrm{ub}(X)$$

where $\mathcal{S}\mathrm{ub}(X)$ are iso classes of subobjects.

Examples of Elementary Toposes

- The category of sets $\mathcal{S}\mathrm{et}$, where $\Omega = \{0, 1\}$.
- The category of finite sets $\mathcal{S}\mathrm{et}^{fin}$ (more generally category of sets smaller than a certain inaccessible cardinal).
- Presheaves on set $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}\mathrm{et})$.
- Left-exact localizations thereof (Grothendieck toposes).
- Filter products

Constructions in an Elementary Topos

Using the axioms of an elementary topos we can already prove a lot of cool stuff:

1. Existence of finite colimits
2. Giraud axiom's
3. Classification of left exact localizations
4. epi-mono factorization
5. ...

Non-Constructions in an Elementary Topos

Some constructions cannot be carried out in every elementary topos. For example constructing *free monoids*.

Example

The category of finite sets does not have free monoids. Similarly, presheaves of finite sets and filter products on finite sets.

We need an additional axiom.

An infinite object

- Key difference between sets and finite sets: existence of **infinite sets**.
- Need to add an axiom that gives us infinite sets.
- One way would be to simply assume the existence of *small colimits*. But the definition of small colimits would not be elementary!
- Want to do better!

Freyd natural number object

One elementary way to indicate an object is infinite is to state it has a non-trivial self-injection.

Definition (Freyd NNO)

A triple $(\mathbb{N}, o : 1 \to \mathbb{N}, s : \mathbb{N} \to \mathbb{N})$ is a *Freyd natural number object* if the following are colimits:

$$\mathbb{N} \underset{s}{\overset{id}{\rightrightarrows}} \mathbb{N} \longrightarrow 1 \;, \quad \begin{array}{ccc} \emptyset & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow s \\ 1 & \xrightarrow{o} & \mathbb{N} \end{array} .$$

Peano natural number object

Another elementary way to define an infinite object is to focus on the set of natural numbers and axiomatize the Peano axioms.

Definition (Peano NNO)

A triple $(\mathbb{N}, o : 1 \to \mathbb{N}, s : \mathbb{N} \to \mathbb{N})$ is a *Peano natural number object* if s is monic, o and s are disjoint subobjects of $\mathbb{N}$, and for every subobject $\mathbb{N}' \hookrightarrow \mathbb{N}$ that is closed under the maps o and s, meaning we have a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{N}' & \xrightarrow{s} & \mathbb{N}' \\ & \overset{o}{\nearrow} & \downarrow & & \downarrow \\ 1 & & & & \\ & \underset{o}{\searrow} & & & \\ & & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \end{array}$$

the inclusion $\mathbb{N}' \overset{\cong}{\hookrightarrow} \mathbb{N}$ is an isomorphism.

Lawvere natural number object

Finally, Lawvere found another categorical way to think about NNOs, namely via a universal property.

Definition (Lawvere NNO)

A triple $(\mathbb{N}, o: 1 \to \mathbb{N}, s: \mathbb{N} \to \mathbb{N})$ is a *Lawvere natural number object* if for any other triple $(X, b: 1 \to X, u: X \to X)$ there is a unique map $f: \mathbb{N} \to X$ making the following diagram commute

$$\begin{array}{ccccc} & & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \overset{o}{\nearrow} & \Big\downarrow{\scriptstyle \exists ! f} & & \Big\downarrow{\scriptstyle \exists ! f} \\ 1 & & & & \\ & \underset{b}{\searrow} & & & \\ & & X & \xrightarrow{u} & X \end{array}.$$

A Zoo of NNOs

How are all these notions related? We have in fact the best possible result.

Theorem (Elephant D5.1.2)

Let $(\mathbb{N}, o, s)$ be a triple in an elementary topos. Then the following are equivalent.

1. *It is a Freyd natural number object.*
2. *It is a Peano natural number object.*
3. *It is a Lawvere natural number object.*

Hence, we can simply call such an object a natural number object.

Natural Number Objects are Useful

Using natural number objects we now have nice extra results.

Proposition (Elephant D5.3.3)

Let $\mathcal{E}$ be an elementary topos with NNO. Then forgetful functor $\mathcal{M}\mathrm{on}(\mathcal{E}) \to \mathcal{E}$ has a left adjoint given explicitly by

$$F(X) = \mathcal{E}_{/\mathbb{N}}(\mathbb{N}_1, X \times \mathbb{N}),$$

$$\coprod_n X^n \to \mathbb{N}$$

where $\mathbb{N}_1 \to \mathbb{N}$ is the universal finite cardinal.

Recall, this condition cannot be relaxed and assuming NNOs is in fact necessary!

What about higher dimensions?

Let's summarize:

1. We have several notions of natural number objects that all coincide.
2. The existence does not follow from the axioms of elementary toposes.
3. Assuming its existence we can prove cool things.

How about ∞-categories? Can we generalize the results we just reviewed? Are there are any differences?

What is an ∞-Category?

For the purposes of this talk we will take an ∞-category to be the following data:

1. Objects X, Y, ... in $\mathcal{C}$.
2. Mapping **space** $\mathrm{Map}_{\mathcal{C}}(X, Y)$ with a composition operation defined up to contractible ambiguity.
3. All classical categorical terms (limits, adjunctions, Cartesian closure, ...) still hold, although some need to be adjusted.

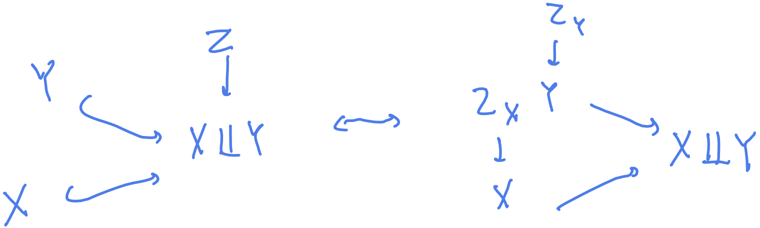
We will review two concepts in more detail: **descent** and **subobject classifier**.

Weak Descent

If $\mathcal{E}$ is an elementary topos then Giraud's axioms imply that there is an equivalence of categories

$$\mathcal{E}_{/X \coprod Y} \simeq \mathcal{E}_{/X} \times \mathcal{E}_{/Y}$$

which takes an object $Z \to X \coprod Y$ to the pullback $((\iota_X)^* Z, (\iota_Y)^* Z)$.



In ∞-categories this can be generalized to all colimits

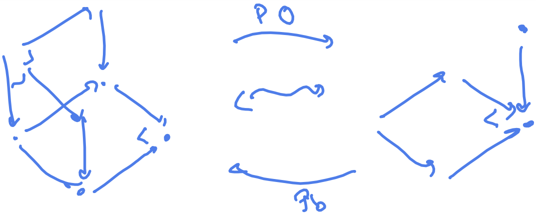
Descent

Definition (Descent)

Let $\mathcal{E}$ be a category with finite limits and colimits. Then $\mathcal{E}$ satisfies *descent* for a diagram $F : I \to \mathcal{E}$ if there is an equivalence

$$\mathcal{E}_{/\mathrm{colim}_I F} \xrightarrow{\simeq} (\mathcal{E}^I_{/F})^{Cart},$$

where $(\mathcal{E}^I_{/F})^{Cart}$ consists of natural transformations over F where all naturality squares are pullbacks.



Subobject classifier

Subobject classifiers for ∞-categories are defined analogously.

Definition (Subobject classifier for ∞-Categories)

Let $\mathcal{E}$ be an ∞-category with finite limits. A subobject classifier Ω represents

$$\mathcal{S}\mathrm{ub} : \mathcal{E}^{op} \to \mathcal{S}\mathrm{et}$$

taking an object X to the set of isomorphism classes of subobjects $\mathcal{S}\mathrm{ub}(X)$ in the ∞-category $\mathcal{E}$.

Note $f : X \to Y$ is a subobject if $\Delta : X \to X \times_Y X$ is an equivalence in $\mathcal{C}$. $\mathcal{E}$

Which ∞-Categories?

We want to study various constructions internal to ∞-categories. Here we would like to use a similar table relating **intensional type theories** and ∞-**categories**, but that table is not complete yet.

Which ∞-Categories?

We want to study various constructions internal to ∞-categories. Here we would like to use a similar table relating **intensional type theories** and ∞-**categories**, but that table is not complete yet.

For the remainder fix an ∞-category $\mathcal{E}$ that satisfies following conditions:

1. It has finite limits and colimits.
2. It is locally Cartesian closed.
3. It has a subobject classifier.
4. It satisfies descent for all colimit diagrams that exist.

Note these are very reasonable conditions and are satisfied by any commonly discussed definition of elementary ∞-topos.

Do they exist?

Examples include:

- The ∞-category of spaces.
- Presheaves $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$
- Grothendieck ∞-toposes
- Filter product ∞-toposes.
- ...

Natural Number Objects in ∞-Categories

Given that we have finite colimits, we can adjust all three definitions for Freyd, Peano and Lawvere natural number objects.

- **Freyd:** We have colimits $\mathbb{N} \underset{s}{\overset{id}{\rightrightarrows}} \mathbb{N} \longrightarrow 1$, $1 \coprod \mathbb{N} \xrightarrow[\cong]{o+s} \mathbb{N}$.
- **Peano:** s is monic, o and s are disjoint subobjects of $\mathbb{N}$, and every subobject $\mathbb{N}' \hookrightarrow \mathbb{N}$ that is closed under the maps o and s is isomorphic to $\mathbb{N}$.
- **Lawvere:** The **space** of maps

$$\begin{array}{ccccc} & \overset{o}{\nearrow} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ 1 & & \downarrow f & & \downarrow f \\ & \underset{b}{\searrow} & X & \xrightarrow{u} & X \end{array} .$$

if X 0-truncated

is contractible.

Main Theorem

Theorem (R)

Let $\mathcal{E}$ be a locally Cartesian closed finitely bicomplete ∞-category which satisfies descent and has a subobject classifier. Then all three notions of natural number object coincide and exist!

For the remainder of this talk we focus on the proof and various complications, and (maybe) some implications.

Main Steps of the Proof

The proof neatly breaks down in three major steps each using a different type of mathematics:

1. **Algebraic Topology:** Constructing an object that includes the NNO
2. **Elementary Topos Theory:** Constructing the desired object and proving it is a Freyd and Peano NNO in $\mathcal{E}$
3. **Homotopy Type Theory:** Proving it is a Lawvere NNO in $\mathcal{E}$

The Circle and its Loops

The first step uses standard algebraic topology.

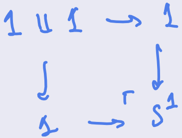
Definition (Circle)

We define the circle in $\mathcal{E}$ as $1 \underset{id}{\overset{id}{\rightrightarrows}} 1 \xrightarrow{i} S^1$.

Definition (Loop Object of Circle)

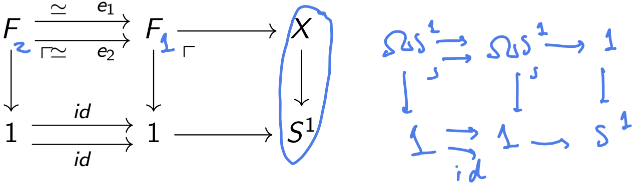
The loop object ΩS^1 is defined as the pullback

$$\begin{array}{ccc} \Omega S^1 & \longrightarrow & 1 \\ \downarrow & \ulcorner & \downarrow i \\ 1 & \xrightarrow{i} & S^1 \end{array} .$$



The Circle and Descent

Using descent we have an equivalence between maps $X \to S^1$ and objects F along with a choice of self equivalence.



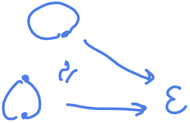
Under this equivalence, $1 \to S^1$ corresponds to $(\Omega S^1, s)$.

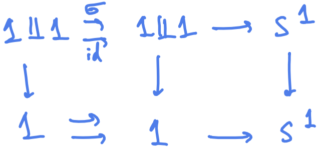
The Loop Object is Nice

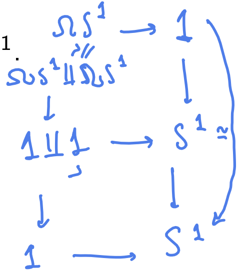
Using descent we can prove following nice properties about ΩS^1:

1. **Universality of Pullback:** It is 0-truncated.
2. **Covering Spaces:** We have $\Omega S^1 \cong \Omega S^1 \coprod \Omega S^1$.
3. **Loop Spaces:** ΩS^1 is a group object.

This ends the first part!







The Underlying Elementary Topos

Definition (0-Truncated object)

An object X in $\mathcal{E}$ is 0-truncated if for all Y, $\mathrm{Map}_{\mathcal{E}}(Y, X)$ is (equivalent to) a set. We denote the full subcategory of $\mathcal{E}$ consisting of 0-truncated objects by $\tau_0\mathcal{E}$.

$\mathcal{E}$ has finite limits, is locally Cartesian closed and has a subobject classifier. Hence $\tau_0\mathcal{E}$ is an elementary topos, which we call the **underlying elementary topos**. The previous result implies that ΩS^1 is an object in $\tau_0\mathcal{E}$.

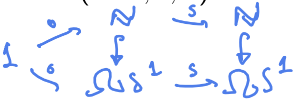
Using the Underlying Elementary Topos

We can use our understanding of elementary topos theory and the object ΩS^1 to construct natural number objects.

1. **Non-Canonical:** $\Omega S^1 \cong \Omega S^1 \coprod \Omega S^1$ implies that $\tau_0\mathcal{E}$ has an NNO by Corollary 5.1.3 in the Elephant. The constructed NNO depends on the choice of isomorphism.

$\mathbb{Z} \cong \mathbb{Z}^{even} \amalg \mathbb{Z}^{odd} \rightsquigarrow \mathbb{N} = \{2^n\}$

2. **Canonical:** We can prove the minimal subobject of $(\Omega S^1, s, o)$ closed under s, o is an NNO.



So, $\tau_0\mathcal{E}$ has all kinds of NNOs. Can we move it back up?

Moving up Peano NNOs

Recall the axioms of a Peano natural number object are about subobjects and disjointness. However, the inclusion $\tau_0\mathcal{E} \to \mathcal{E}$ is limit preserving and so we immediately get following result:

Proposition

$\mathcal{E}$ has a Peano natural number object.

Moving Up Freyd NNOs

Moving up Freyd is a little more subtle. The map $1 \coprod \mathbb{N} \xrightarrow[\cong]{o+s} \mathbb{N}$ is still formally an equivalence in $\mathcal{E}$.
The equivalence

$$\mathbb{N} \underset{s}{\overset{id}{\rightrightarrows}} \mathbb{N} \longrightarrow 1$$

can be computed. Hence:

Proposition

$\mathcal{E}$ has a Freyd natural number object.

How about Lawvere NNOs?

The Lawvere NNO condition does not generalize, so we need a different strategy. We want to prove being a Peano NNO implies Lawvere NNO. However, the proof in the elementary topos setting doesn't generalize:

How about Lawvere NNOs?

The Lawvere NNO condition does not generalize, so we need a different strategy. We want to prove being a Peano NNO implies Lawvere NNO. However, the proof in the elementary topos setting doesn't generalize:

- It uses type theoretical language to construct object in the topos!
- Even if we manually translated the construction, it involves identity types, which behave very differently.
- Even if we took care of that it relies on a lemma from a different construction, which does not give us the desired object in an ∞-category (in fact, even in spaces).

A Counter-Example in Spaces

Enter Homotopy Type Theory

Here we use our knowledge of homotopy type theory to prove a Peano natural number object in $\mathcal{E}$ is a Lawvere natural number object. The key is to use the wisdom of Shulman, who does a similar construction in homotopy type theory.

Constructing Lawvere NNOs in HoTT

Let $(X, b : 1 \to X, u : X \to X)$ be a triple:

- There is a map that takes $f : \mathbb{N} \to X$ to the product of the restriction maps $(f|_{[n]} : [n] \to X)_{n \in \mathbb{N}}$
- Being indexed over $\mathbb{N}$ in type theory allows us to restrict to the diagonal $(f|_{[n]}(n))_{n \in \mathbb{N}} : \mathbb{N} \to X$.
- As the appearance suggests this map is the identity (i.e. $f|_{[n]}(n) = f(n)$) and so we have a retract diagram.
- Finally, the type of maps $f_{[n]} : [n] \to X$ with assumption $(X, b : 1 \to X, u : X \to X)$ is contractible and the product of contractible types is contractible.

Transforming HoTT into Category Theory

We want to make this into a categorical argument. The correct argument would be to realize homotopy type theory as the internal language of our $(\infty, 1)$-category and be done with it...

We don't have that, so we will manually translate some ideas and add some other ideas to complete the proof! In particular, there are two key tricks we use to make the manual translation work.

Parametrizing over the Natural Numbers

In order to be able to restrict the collection of partial maps to a global map out of $\mathbb{N}$ it needs to be parameterized over $\mathbb{N}$ by definition. So, for a triple (X, b, u) we define the space $Part(X, b, u)$ as

$$Part(\pi_2 : \mathbb{N}_1 \to \mathbb{N}, (\pi_2 : X \times \mathbb{N} \to \mathbb{N}, (b, id_{\mathbb{N}}) : \mathbb{N} \to X \times \mathbb{N}, u \times id_{\mathbb{N}} : X \times \mathbb{N} \to X \times \mathbb{N}))$$

which is the space of maps f that fit into the diagram

$$\begin{array}{ccccc} & \overset{o \times \mathrm{id}}{\nearrow} & \mathbb{N}_1 & \xrightarrow{\iota_2} & \mathbb{N} \coprod \mathbb{N}_1 \\ \mathbb{N} & & \Big\downarrow{\scriptstyle f \circ inc} & & \Big\downarrow{\scriptstyle f} \\ & \underset{b}{\searrow} & X \times \mathbb{N} & \xrightarrow{u \times \mathrm{id}} & X \times \mathbb{N} \end{array} .$$

The Object of Contractibility

The second trick is to use the object of contractibility $isContr_B : (\mathcal{E}_{/B})^{\simeq} \to (\mathcal{E}_{/B})^{\simeq}$, which has following properties:

- $isContr_B(p : E \to B) \to B$ is final if and only if p is an equivalence.
- $isContr_B(p : E \to B) \to B$ is always (-1)-truncated.

Combining this with the axioms of a Peano NNO we have following useful lemma.

Lemma

A map $X \to \mathbb{N}$ is an equivalence if and only if $isContr_{\mathbb{N}}(X \to \mathbb{N})$ is closed under (o, s).

Building the maps

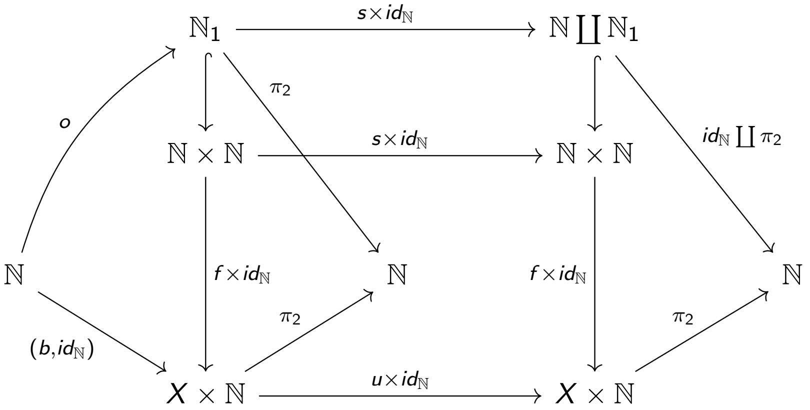
We now use these observations to build the maps! Let $Ind(X, b, u)$ be the space of diagrams of the form

$$\begin{array}{ccccc} & & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ & \overset{o}{\nearrow} & \Big\downarrow f & & \Big\downarrow f \\ 1 & & & & \\ & \underset{b}{\searrow} & & & \\ & & X & \xrightarrow{u} & X \end{array}$$

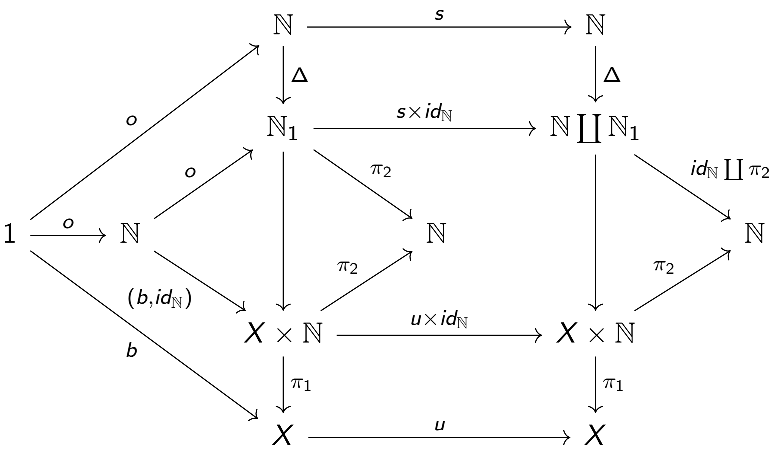
and recall $Part(X, b, u)$ is the space of diagrams of the form

$$\begin{array}{ccccc} & & \mathbb{N}_1 & \xrightarrow{\iota_2} & \mathbb{N} \coprod \mathbb{N}_1 \\ & \overset{o \times \mathrm{id}}{\nearrow} & \Big\downarrow f \circ inc & & \Big\downarrow f \\ \mathbb{N} & & & & \\ & \underset{b}{\searrow} & & & \\ & & X \times \mathbb{N} & \xrightarrow{u \times \mathrm{id}} & X \times \mathbb{N} \end{array} .$$

From Global to Local



From Local to Global



Combining the Steps!

We can now combine everything to finish the proof:

1. Show the two maps compose to the identity i.e. we have a retract diagram

$$Ind(X, b, u) \to Part(X, b, u) \to Ind(X, b, u).$$

2. Prove $Part(X, b, u) \to \mathbb{N}$ is an equivalence using $isContr_{\mathbb{N}}(Part(X, b, u))$ and induction.
3. Deduce $Ind(X, b, u)$ is contractible, proving $\mathbb{N}$ is a Lawvere natural number object.

Who cares?

We want to end this talk with some implications:

1. Relation to enveloping ∞-toposes
2. External and Internal Colimits
3. Connections to Truncations
4. Future Directions

Not All Elementary Toposes Lift

Here is a cool implication of these results relating 1-categories and ∞-categories.

Corollary

Let $\mathcal{E}$ be an elementary topos without natural number object (such as finite sets), then there does not exist a locally Cartesian closed finitely bicomplete ∞-category $\hat{\mathcal{E}}$ satisfying descent such that $\tau_0(\hat{\mathcal{E}}) \simeq \mathcal{E}$.

This is in stark contrast to Grothendieck toposes, which always have *enveloping ∞-toposes*.

Implications for Examples of ∞-Toposes

Note this also means that the ∞-category of finite spaces cannot give us ∞-toposes (unlike finite sets). Rather we have to take κ-small spaces for κ large enough. This has also been studied by Lo Monaco.

Universes

For further applications of NNOs we need universes.

Definition

A map $p : \mathcal{U}_* \to \mathcal{U}$ in $\mathcal{E}$ is a **universe** if the induced pullback map of spaces

$$p^*(-) : \mathrm{Map}_{\mathcal{E}}(X, \mathcal{U}) \to (\mathcal{E}_{/X})^{\simeq}$$

is fully faithful and the essential image is closed under finite limits and colimits.

Here $(\mathcal{E}_{/X})^{\simeq}$ is the maximal sub-space of the ∞-category $\mathcal{E}_{/X}$.

External Countable Colimits

Using NNOs we can give a nice criterion for the existence of countable colimits.

Proposition

Let $\mathcal{E}$ be a locally Cartesian closed finitely bicomplete ∞-category with subobject classifier satisfying descent. Let $\mathcal{U}$ be a universe and denote the full subcategory of $\mathcal{E}$ in the image of $\mathcal{U}$ by $\mathcal{E}^S$. Then $\mathcal{E}^S$ is countably bicomplete if and only if $\mathbb{N}$ is in $\mathcal{E}^S$ and is the countable colimit of 1.

The key is to recover general (co)products from morphism $\coprod_{\mathbb{N}} 1 \to \mathcal{U}$.

Internal Coproducts

Using NNOs we can study colimits internally! Fix $\mathcal{E}$ as before with a universe $\mathcal{U}$.

Definition

A sequence is a morphism $\{A_n\}_n : \mathbb{N} \to \mathcal{U}$. The **internal coproduct** $\sum A_n$ is the pullback along $p : \mathcal{U}_* \to \mathcal{U}$.

For example, the internal coproduct of a constant diagram $\{X\} : \mathbb{N} \to \mathcal{U}$ can be evaluated to be $X \times \mathbb{N}$.

Internal Sequential Colimits

For a sequence $\{A_n\}_n : \mathbb{N} \to \mathcal{U}$ a sequential diagram is a map

$$\{f_n\}_n : \sum_{n:\mathbb{N}} A_n \to \sum_{n:\mathbb{N}} A_{n+1}$$

and the sequential colimit A_∞ is defined as the coequalizer:

$$\sum_{n:\mathbb{N}} A_n \underset{id}{\overset{f}{\rightrightarrows}} \sum_{n:\mathbb{N}} A_n \longrightarrow A_\infty \ .$$

We can prove cofinality results using this definition.

Theorem

Let $\{f_n : A_n \to A_{n+1}\}_{n:\mathbb{N}}$ be a sequential diagram. Then $\{f_n\}_{n:\mathbb{N}}$ has the same sequential colimit as $\{f_{n+1}\}_{n:\mathbb{N}}$.

Truncations

The existence of natural number objects and sequential colimits can be used in a variety of ways.

- Using natural number objects we can define internal truncation levels (which can differ from the classical ones in spaces).
- We can translate other homotopy type theory results, such as the join construction (due to Rijke) to get (-1)-truncations.

For more details see *"An Elementary Approach to Truncations"* arXiv:1812.10527.

Further Questions

Can we use natural number objects to construct free A_∞-monoids?
We can use the same construction to get the object we expect to be the free A_∞-monoid, however:

1. In the ∞-setting we need operads to define A_∞-monoids.
2. Even more generally the definition of operads could depend on the natural number object.
3. Developing A_∞-objects in homotopy type theory has been challenging.
4. ...

The End

Thank You!

Questions?