The $(\infty,1)$-category of Types

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The Problem

- Type theory in its current form does not appear to allow us to describe infinitely coherent algebraic structures on non-truncated types.

  (e.g. $(\infty,1)$-categories, A_∞-monoids, E_∞-monoids...)

- What kind of extension would enable these types of definitions?

Two-level Type Theory

- Add a second universe of "pre-types" which have a "strict" equality.
  (in particular, pre-types are 0-truncated)
- Use this meta-level theory to repeat classical constructions by induction on "external" natural numbers.
- Can define semi-simplicial types, etc...

Advantages	Disadvantages
1) Leverage existing techniques	1) Preferential treatment for sets
2) Clear implementation and meta-theory	2) Duplication : $\mathbb{N}, \mathbb{N}_S$
3) Completeness	3) What **are** pre-types?
	4) Univalence is restricted.

Presentations of Types

- In addition to basic type formers (Π, Σ) modern type theories provide us with a means of <u>presenting</u> types with inductive/coinductive definitions.
- Moreover, we have an internal description of what the data of such a presentation consists of. (Polynomial / Indexed container).
- The collection of such presentations is itself a model of type theory.

Higher Presentations

- There is not yet any clear picture of what the data of a Higher Inductive Type should be.
- This suffers from a coherence problem!
- I regard the search for a definition of (semi-simplicial / opetopic / cubical)-types as the search for a theory of higher presentations.
- Higher presentations ⇒ Higher structures

A Two-Level Theory of Presentations

- Directly provide type theory with a theory of higher presentations
- Explain the 2nd level in terms of the 1st. (the non-fibrant types have a definition).
- Equations which make higher presentations well-defined should be regarded as part of the theory.
- Precedent: "Levitation"

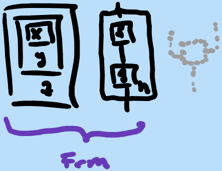
Advantages	Disadvantages
1) Univalence preserved	1) Meta-theory undeveloped
2) Clear explanation of non-fibrant types	2) Requires new techniques
3) Direct access to higher structures	3) Completeness?

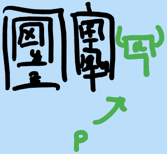
Opetopic Types (Signature)

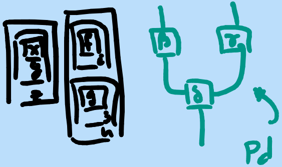
$$\mathbb{O}\mathrm{Type} : \mathbb{N} \to \mathrm{Type}$$

$$\mathrm{Frm} : \{n : \mathbb{N}\} \to \mathbb{O}\mathrm{Type}\ n \to \mathrm{Type}$$

$$\mathrm{Pd} : \{n : \mathbb{N}\}\ (X : \mathbb{O}\mathrm{Type}\ n) \to (P : \mathrm{Frm}\ X \to \mathrm{Type}) \to (\mathrm{Frm}\ X \to \mathrm{Type})$$







Opetopic Types (Def'n)

$$\mathrm{OType}\,(n+1) = \sum_{X\,:\,\mathrm{OType}\,n} : \mathrm{Frm}(X) \to \mathrm{Type}$$

$$\mathrm{Frm}\,(X, P) = \sum_{f\,:\,\mathrm{Frm}(X)} (\mathrm{Pd}\; P\; f) \times (P\; f)$$

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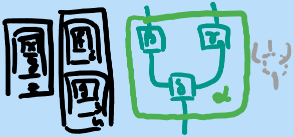


$$\mathrm{Pd}\,(X, P) =$$ An inductively defined type of trees

Opetopic Types (Eqn's)

- The type constructor Pd is a polynomial monad.
- In order for the definition to be well-formed, we must strictify the laws for the monadic operators: μ, η.
- We regard these operations as primitive operators of the theory with prescribed definitional behavior.

The Type Theory of Opetopic Types

- We can think of opetopic types as our universe of presentations.
- But we quickly find that to make progress we need more: morphisms of opetopic types, for example, also require equations.
- It becomes natural therefor to axiomatize the CWF structure these types have

  (i.e. introduce dependent opetopic types/terms)

What can we do with this?

- Definitions

  ∞-groupoid, $(\infty,1)$-category

  ∞-planar operad

  A_∞-monoid group

  (∞,n)-category

- Constructions

  Σ_0, Π_0, $Grp(X)$

  $X *_0 Y$, $|X|$, ∞-limits/colimits

- Theorems

  Type $\simeq$ ∞-groupoid

  1-cat $\simeq$ truncated $(\infty,1)$-cat

And ....

Dependent Opetopic Types

$$\mathsf{OType}{\downarrow} : \{n : \mathbb{N}\} \to \mathsf{OType}\ n \to \mathsf{Type}$$

$$\mathsf{Frm}{\downarrow} : \{n : \mathbb{N}\}(X : \mathsf{OType}\ n)(X{\downarrow} : \mathsf{OType}{\downarrow}\ X) \to \mathsf{Frm}\ X \to \mathsf{Type}$$

$$\begin{aligned}\mathsf{Pd}{\downarrow} : \{n : \mathbb{N}\}&(X : \mathsf{OType}\ n)(X{\downarrow} : \mathsf{OType}{\downarrow}\ X)\\ &\to (P : \mathsf{Frm}\ X \to \mathsf{Type})\\ &\to (P{\downarrow} : \{f : \mathsf{Frm}\ X\} \to \mathsf{Frm}{\downarrow}\ X{\downarrow}\ f \to P\ f \to \mathsf{Type})\\ &\to \{f : \mathsf{Frm}\ X\} \to \mathsf{Frm}{\downarrow}\ f \to \mathsf{Pd}\ P\ f \to \mathsf{Type}\end{aligned}$$

$$\mathsf{OType}{\downarrow}\ (n+1)\ (X, P) = \sum_{X{\downarrow} : \mathsf{OType}{\downarrow}\ X} \{f : \mathsf{Frm}\ X\} \to \mathsf{Frm}{\downarrow}\ X{\downarrow}\ f \to P\ f \to \mathsf{Type}$$

The Opetopic Universe

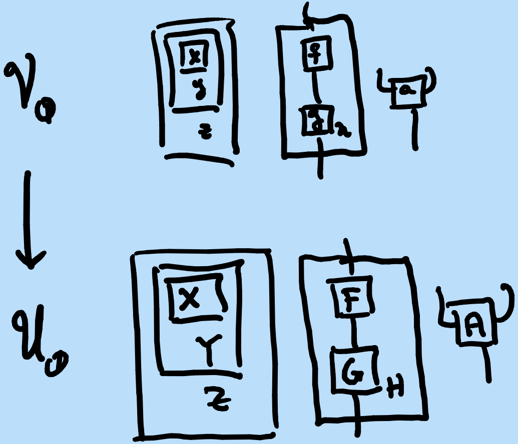
$$\mathcal{U}_{O} : (n : \mathbb{N}) \to \mathrm{OType}\ n$$

$$\mathcal{V}_{O} : (n : \mathbb{N}) \to \mathrm{OType}{\downarrow}\ (\mathcal{U}_{O}\ n)$$

$$\mathcal{V}_{O} \to \mathcal{U}_{O}$$

$$\mathcal{U}_{O}\ (n+1) = (\mathcal{U}_{O}\ n\ ,\ \lambda f .\ \mathrm{Frm}{\downarrow}\ f \to \mathrm{Type})$$

$$\mathcal{V}_{O}\ (n+1) = \left(\mathcal{V}_{O}\ n\ ,\ \lambda\, f\ f{\downarrow}\ P .\ P\ f{\downarrow}\right)$$



$x : X \quad f : F\,x\,y \quad a : A\,x\,y\,z\,f\,g\,h$

$y : Y \quad g : G\,y\,z$

$z : Z \quad h : H\,x\,z$

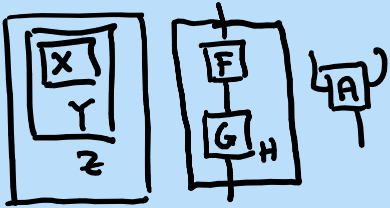
$X : \mathrm{Type} \quad F : X \to Y \to \mathrm{Type}$

$Y : \mathrm{Type} \quad G : Y \to Z \to \mathrm{Type}$

$Z : \mathrm{Type} \quad H : X \to Z \to \mathrm{Type}$

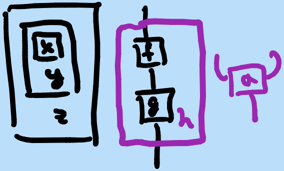
$A : (x : X)(y : Y)(z : Z) \to F\,x\,y \to G\,y\,z \to H\,x\,z \to \mathrm{Type}$

Fibrant Relations



$A : (x : X)(y : Y)(z : Z) \to F\,x\,y \to G\,y\,z \to H\,x\,z \to \text{Type}$

A is a fibrant relation if given x, y, z, f, g we have:

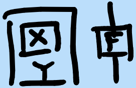


$$\text{is-contr}\left(\sum_{h : H\,x\,z} A \ldots h \right)$$

The $(\infty,1)$-category of Types

Def $\mathcal{T}$:= the subobject of $\mathcal{U}_0$ consisting of the fibrant relations

Thm $\mathcal{T}$ is an $(\infty,1)$-category



$\rightsquigarrow$ F is fibrant $\Longleftrightarrow$ $(x:X) \to \text{is-contr} \sum_{y:Y} F\,x\,y$

$\simeq X \to Y$

Thanks!