The \((\infty,1)\)-category of Types

Eric Finster

CMU-HoTT Seminar

February 17, 2023
The Problem

- Type theory in its current form does not appear to allow us to describe infinitely coherent algebraic structures on non-truncated types.
  (e.g. $(\infty,1)$-categories, $\mathbb{A}_\infty$-monoids, $\mathbb{E}_\infty$-monoids...)

- What kind of extension would enable these types of definitions?
Two-level Type Theory

- Add a second universe of "pre-types" which have a "strict" equality.
  (In particular, pre-types are 0-truncated)

- Use this meta-level theory to repeat classical constructions by induction on "external" natural numbers.

- Can define semi-simplicial types, etc...
Advantages

1) Leverage existing techniques
2) Clear implementation and meta-theory
3) Completeness

Disadvantages

1) Preferential treatment for sets
2) Duplication: \( \mathbb{N}, \mathbb{N}_0 \)
3) What are pre-types?
4) Univalence is restricted.
Presentations of Types

- In addition to basic type formers \((\Pi, \Sigma)\), modern type theories provide us with a means of presenting types with inductive/coinductive definitions.

- Moreover, we have an internal description of what the data of such a presentation consists of. (Polynomial/Indexed container).

- The collection of such presentations is itself a model of type theory.
Higher Presentations

- There is not yet any clear picture of what the data of a Higher Inductive Type should be.
- This suffers from a coherence problem!
- I regard the search for a definition of (semi-simplicial/opetopic/cubical)-types as the search for a Theory of higher presentations.
- Higher presentations $\Rightarrow$ Higher structures
A Two-Level Theory of Presentations

- Directly provide type theory with a theory of higher presentations.
- Explain the 2\textsuperscript{nd} level in terms of the 1\textsuperscript{st}.
  (the non-fibrant types have a definition).
- Equations which make higher presentations well-defined should be regarded as part of the theory.
- Precedent: "Levitation"
Advantages

1) Univalence preserved
2) Clear explanation of non-fibrant types
3) Direct access to higher structures

Disadvantages

1) Meta-theory undeveloped
2) Requires new techniques
3) Completeness?
Opetopic Types (Signature)

\( \text{OType} : \mathbb{N} \rightarrow \text{Type} \)

\( \text{Frm} : \{ n : \mathbb{N} \} \rightarrow \text{OType} \ n \rightarrow \text{Type} \)

\( \text{Pd} : \{ n : \mathbb{N} \} \ (X : \text{OType} \ n) \rightarrow (\ P : \text{Frm} X \rightarrow \text{Type}) \)

\( \rightarrow (\ P : \text{Frm} X \rightarrow \text{Type}) \)
Opetopic Types (Def'n)

\( \text{OType}^{n+1} = \sum_{X: \text{OType}} \text{Frm}(X) \rightarrow \text{Type} \)

\( \text{Frm}(X, P) = \sum_{f: \text{Frm}(X)} (Pd P f) \times (P f) \)

\( \text{Pd}(X, P) = \text{An inductively defined type of trees} \)
Opetopic Types (Egny’s)

- The type constructor $Pd$ is a polynomial monad.
- In order for the definition to be well-formed, we must strictify the laws for the monadic operators: $\mu, \eta$.
- We regard these operations as primitive operators of the theory with prescribed definitional behavior.
The Type Theory of Opetopic Types

- We can think of opetopic types as our universe of presentations.

- But we quickly find that to make progress we need more: morphisms of opetopic types, for example, also require equations.

- It becomes natural therefore to axiomatize the CWF structure these types have (i.e., introduce dependent opetopic types/terms).
What can we do with this?

- Definitions
  - $\infty$-groupoid, $(\infty,1)$-category
  - $\infty$-planar operad
  - $\infty$-monoid group
  - $(\infty,n)$-category

- Constructions
  - $\Sigma_0$, $\Pi_0$, $\text{Grp}(X)$
  - $X \star_0 Y$, $1 \times 1$, $\infty$-limits/colimits

- Theorems
  - Type $\Rightarrow \infty$-groupoid
  - 1-cakes $\Rightarrow$ truncated $(\infty,1)$-cat

And...
Dependent Operadic Types

\( \text{OType} \downarrow : \forall n : \text{IN3} \to \text{OType}_n \to \text{Type} \)

\( \text{Frm} \downarrow : \forall n : \text{IN3} \ (X : \text{OType}_n) \ (x : \text{OType}_n X) \to \text{Frm} X \to \text{Type} \)

\( \text{Pd} \downarrow : \forall n : \text{IN3} \ (X : \text{OType}_n) \ (x : \text{OType}_n X) \to (P : \text{Frm} X \to \text{Type}) \to (Pd : \forall f : \text{Frm} X \exists f : \text{Frm} x \to \text{Frm} x \exists f \to \text{Frm} x \exists f \to \text{Type}) \to \forall f : \text{Frm} X \exists f : \text{Frm} x \to \text{Frm} x \exists f \to \text{Pd} P f \to \text{Type} \)
The Opuspic Universe

\[ U_0 : (n : \mathbb{N}) \rightarrow \text{OType}_n \]

\[ V_0 : (n : \mathbb{N}) \rightarrow \text{OType}_n (U_0^n) \]

\[ U_0 (n+1) = (U_0^n, \lambda f. \text{Form} f \rightarrow \text{Type}) \]

\[ V_0 (n+1) = (V_0^n, \lambda f \ni p. \text{Pf} f) \]
\( x : X \quad f : F \times y \quad a : A \times y \times z \times f \times g \times h \)

\( y : Y \quad g : G \times y \times z \)

\( z : Z \quad h : H \times z \)

\( X : \text{Type} \quad F : X \rightarrow Y \rightarrow \text{Type} \)

\( Y : \text{Type} \quad G : Y \rightarrow Z \rightarrow \text{Type} \)

\( Z : \text{Type} \quad H : X \rightarrow Z \rightarrow \text{Type} \)

\( A : \left( (x : X) \times (y : Y) \times (z : Z) \right) \rightarrow F \times y \rightarrow G \times y \times z \rightarrow H \times z \rightarrow \text{Type} \)
Fibrant Relations

A : (x: X)(y: Y)(z: Z) → P = y → G y z → H x z → Type

A is a fibrant relation if given x, y, z, f, g we have:

\[ \text{is-contr} \left( \sum_{h: H x z} A \ldots h \right) \]
The \((\infty, 1)\)-category of Types

**Def** \( \Upsilon \) := the subobject of \( \mathcal{U}_0 \) consisting of the fibrant relations

**Thm** \( \Upsilon \) is an \((\infty, 1)\)-category

\[
\text{若 } F \text{ is fibrant } \iff (x : x) \to \text{co-cone} \sum_{y : F y} \sim x - \Upsilon
\]
Thanks!