Formal Model Theory & Higher Topology

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This talk is based on three preprints.

1. **General facts on the Scott Adjunction**, ArXiv:2009.14023.
2. **Towards Higher Topology**, ArXiv:2009.14145.
3. **Formal Model Theory & Higher Topology**, ArXiv:2010.00319.

Which were estracted from my PhD thesis.

4. **The Scott Adjunction**, ArXiv:2009.07320.

Sketches of an elephant

These cover three different aspects of the same story.

1. Category Theory;
2. (Higher) Topology;
3. Logic.

We will start our tour from the crispiest one: (Higher) Topology.

The topological picture

$$\begin{tikzcd}
& \mathsf{Loc} \arrow[dl, bend left, "\mathsf{pt}"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{Top} \arrow[ur, bend left, "\mathcal{O}"] & & \mathsf{Pos}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left, "\mathsf{S}"]
\end{tikzcd}$$

Top is the category of topological spaces and continuous mappings between them.

Pos_ω is the category of posets with directed suprema and functions preserving directed suprema.

The topological picture

$$\begin{tikzcd}
 & \mathsf{Loc} \arrow[dl, bend left, "\mathsf{pt}"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{Top} \arrow[ur, bend left, "\mathcal{O}"] & & \mathsf{Pos}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left, "\mathsf{S}"]
\end{tikzcd}$$

Loc is the category of Locales. It is defined to be the opposite category of frames, where objects are frames and morphisms are morphisms of frames. A frame is a poset with infinitary joins ($\bigvee$) and finite meets ($\wedge$), verifying the infinitary distributivity rule,

$$(\bigvee x_i) \wedge y = \bigvee (x_i \wedge y)$$

The poset of open sets $\mathcal{O}(X)$ of a topological space X is the archetypal example of a locale.

The topological picture

$$
\begin{tikzcd}
 & \mathsf{Loc} \arrow[dl, bend left=15, "\mathsf{pt}"'] \arrow[dr, bend left=15, "\mathsf{pt}"] & \\
\mathsf{Top} \arrow[ur, bend left=15, "\mathcal{O}"] & & \mathsf{Pos}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left=15, "\mathsf{S}"]
\end{tikzcd}
$$

The diagram is relating three different approaches to *geometry*.

Top is the *classical* approach.

Loc is the *pointfree/constructive* approach.

Pos_ω was approached from a geometric perspective by Scott, motivated by *domain theory* and *λ-calculus*.

The topological picture

$$
\begin{tikzcd}
 & \mathsf{Loc} \arrow[dl, bend left, "\mathsf{pt}"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{Top} \arrow[ur, bend left, "\mathcal{O}"] & & \mathsf{Pos}_\omega \arrow[ul, bend left, "\mathsf{S}"] \arrow[ll, "\mathsf{ST}"]
\end{tikzcd}
$$

pt maps in both cases a locale to its set of *formal* points. A formal point of a locale $\mathcal{L}$ is a morphism of locales $\mathbb{T} \to \mathcal{L}$. This set admits a topology, but also a partial order.

S maps a poset with directed colimits to the frame $\mathsf{Pos}_\omega(P, \mathbb{T})$.

ST equips a poset P with its Scott topology, which can be essentially identified with the frame above.

The topological picture

$$
\begin{tikzcd}
 & \mathsf{Loc} \arrow[dl, dashed, bend left, "\mathsf{pt}"] \arrow[dr, dashed, bend left, "\mathsf{pt}"] & \\
\mathsf{Top} \arrow[ur, bend left, "\mathcal{O}"] & & \mathsf{Pos}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, "\mathsf{S}"]
\end{tikzcd}
$$

1. $\mathcal{O} \dashv \mathsf{pt}$, is sometimes called **Isbell adjunction**.
2. $\mathsf{S} \dashv \mathsf{pt}$, might be called **Scott adjunction**.
3. The solid diagram above commutes.
4. This is all very classical. What did I do? **Categorify!**

The project of Categorification

$$
\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, bend left, "pt"] \arrow[dr, bend left, "pt"] & \\
\mathsf{LBlon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_{\omega} \arrow[ul, bend left, "\mathsf{S}"] \arrow[ll, "\mathsf{ST}"]
\end{tikzcd}
$$

1. Topoi is the 2-category of Grothendieck topoi. A Grothendieck **topos is** precisely **a cocomplete category with lex colimits**, an analog of the infinitary distributivity rule, **and a generating set**. The latter is just a smallness assumption which is secretly hidden and even stronger in locales, indeed a locale is a set.

The project of Categorification

$$
\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, bend left, "pt"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{LBIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_{\omega} \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left, "\mathsf{S}"]
\end{tikzcd}
$$

1. Acc_{ω} is the 2-category of accessible categories with directed colimits and functors preserving them. **An accessible category with directed colimits is a category with directed colimits** (notice the analogy with directed suprema) **and a (suitable) generating set**. As in the case of topoi, the request of a (nice enough) generating set makes constructions more tractable.

Ionads

The 2-category of Ionads was introduced by Garner. A **ionad** $\mathcal{X} = (X, \mathrm{Int})$ is a set X together with a comonad $\mathrm{Int} : \mathrm{Set}^X \to \mathrm{Set}^X$ preserving finite limits. While topoi are the categorification of locales, Ionads are the categorification of the notion of topological space, to be more precise, Int categorifies the interior operator of a topological space.

Thm. (Garner)

The category of coalgebras for a ionad is indicated with $\mathbb{O}(\mathcal{X})$ and is a cocomplete elementary topos. A ionad is bounded if $\mathbb{O}(\mathcal{X})$ is a Grothendieck topos. Thus one should look at the functor

$$\mathbb{O} : \mathrm{BIon} \to \mathrm{Topoi},$$

as the categorification of the functor that associates to a space its frame of open sets.

$$
\begin{tikzcd}
& \mathsf{Topoi} \arrow[dl, dashed, bend right] \arrow[dr, bend left, "\mathrm{pt}"] & \\
\mathsf{BIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_\omega \arrow[ul, dashed, bend right] \arrow[ll, dashed]
\end{tikzcd}
$$

1. The functor pt was also known to the literature. For every topos $\mathcal{E}$ one can define its category of points to be $\mathsf{Topoi}(\mathsf{Set}, \mathcal{E})$, and it is a classical result that this category is accessible and has directed colimits.
2. My task was to provide all the dashed arrows in this diagram, to show that they form adjunctions and to describe their properties.

The Scott Adjunction (Henry, DL)

There is an 2-adjunction

$$\mathsf{S} : \mathsf{Acc}_\omega \leftrightarrows \mathsf{Topoi} : \mathsf{pt}.$$

1. Acc_ω is the 2-category of accessible categories with directed colimits, a 1-cell is a functor preserving directed colimits, 2-cells are invertible natural transformations.
2. Topoi is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints.

The Scott construction

Let $\mathcal{A}$ be a 0-cell in Acc_ω. $\mathsf{S}(\mathcal{A})$ is defined as the category $\mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$. Let $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω.

$$\begin{tikzcd} \mathcal{A} \arrow[d, "f"] \\ \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{S}\mathcal{A} \arrow[d, bend left] \arrow[d, phantom, "f^* \dashv f_*"] \\ \mathsf{S}\mathcal{B} \arrow[u, bend left] \end{tikzcd}$$

$\mathsf{S}f = (f^* \dashv f_*)$ is defined as follows: f^* is the precomposition functor $f^*(g) = g \circ f$. This is well defined because f preserve directed colimits. f^* preserve all colimits and thus has a right adjoint, that we indicate with f_*. Observe that f^* preserve finite limits because finite limits commute with directed colimits in Set.

$$\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, dashed, bend right] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{BIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_\omega \arrow[ll, dashed] \arrow[ul, dashed, bend right, "\mathsf{S}"']
\end{tikzcd}$$

1. Unfortunately the definition of Garner does not allow to find a right adjoint for $\mathbb{O}$.
   In order to fix this problem, one needs to stretch Garner's definition and introduce **generalized (bounded) Ionads**.

Generalized Ionads

A generalized ionad $\mathcal{X} = (X, \mathrm{Int})$ is a locally small (but possibly large) pre-finitely cocomplete category X together with a lex comonad $\mathrm{Int} : \P(X) \to \P(X)$.

Why isn't it just the data of a locally small category X together with a lex comonad on Set^X?

- By $\P(X)$ we mean the full subcategory of Set^X made by small copresheaves over X, namely those functors $X \to \mathrm{Set}$ that are small colimits of corepresentables (in Set^X). This is a locally small category, as opposed to Set^X which might be locally large.
- Obviously, when X is small, every presheaf is small.
- $\P(X)$ is the free completion of X° under colimits.
- The category of small presheaves $\mathcal{P}(X)$ over a (locally small) large category X is a bit pathological. In full generality $\mathcal{P}(X)$ is not complete.

Generalized Ionads

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- By $¶(X)$ we mean the full subcategory of Set^X made by small copresheaves over X, namely those functors $X \to \mathrm{Set}$ that are small colimits of corepresentables (in Set^X). This is a locally small category, as opposed to Set^X which might be locally large.

Prop.

If X is finitely pre-cocomplete, then $¶(X)$ has finite limits.

Prop.

If X is small or it is accessible, then $¶(X)$ is complete.

Generalized Ionads

A generalized ionad $\mathcal{X} = (X, \mathrm{Int})$ is a locally small (but possibly large) pre-finitely cocomplete category X together with a lex comonad $\mathrm{Int} : \P(X) \to \P(X)$.

Why isn't it just the data of a locally small category X together with a lex comonad on Set^X?

Prop.

If $f^* : \mathcal{G} \to \P(X)$ is a cocontinuous functor from a total category, then it has a right adjoint f_*.

The result above allows to produce comonads on $\P(X)$ (just compose $f^* f_*$) and follows from the general theory of total categories, but needs $\P(X)$ to be locally small to stay in place. Thus the choice of Set^X would have generated size issues. A similar issue would arise with Kan extensions.

$$
\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, dashed, bend left, "pt"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{LBIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_\omega \arrow[ul, bend left, "\mathsf{S}"] \arrow[ll, dashed, "\mathsf{ST}"]
\end{tikzcd}
$$

Every topos induces a generalized bounded ionad over its points

- For a topos $\mathcal{E}$, there exists a natural evaluation pairing

$$\mathrm{ev} : \mathcal{E} \times \mathrm{pt}(\mathcal{E}) \to \mathrm{Set},$$

  mapping the couple (e, p) to its evaluation $p^*(e)$.
- Its mate functor $ev^* : \mathcal{E} \to \mathrm{Set}^{\mathrm{pt}(\mathcal{E})}$, preserves colimits and finite limits.
- ev^* takes values in $¶(\mathrm{pt}(\mathcal{E}))$. Since a topos is a total category, ev^* must have a right adjoint ev_*, and we get an adjunction,

$$ev^* : \mathcal{E} \leftrightarrows ¶(\mathrm{pt}(\mathcal{E})) : ev_*.$$

- The comonad ev^*ev_* is lex and thus induces a ionad over $\mathrm{pt}(\mathcal{E})$.

$$
\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, bend left, "pt"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{LBIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_{\omega} \arrow[ul, bend left, "\mathsf{S}"] \arrow[ll, dashed, "\mathsf{ST}"]
\end{tikzcd}
$$

Every accessible category with directed colimits is a ionad.

- The Scott topos $\mathsf{S}\mathcal{A} = \mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$ of $\mathcal{A}$ sits naturally in $\P(\mathcal{A})$.
- The inclusion $\iota_\mathcal{A}$ of $\mathsf{S}\mathcal{A}$ in $\P(\mathcal{A})$ has a right adjoint $r_\mathcal{A}$,

$$\iota : \mathsf{S}(\mathcal{A}) \leftrightarrows \P(\mathcal{A}) : r.$$

- The comonad is lex and induces a ionad over $\mathcal{A}$.

Thm. (DL)

Replacing bounded Ionads with generalized bounded Ionads, there exists a right adjoint for $\mathbb{O}$ and a Scott topology-construction ST such that $\mathsf{S} = \mathbb{O} \circ \mathsf{ST}$, in complete analogy to the posetal case.

$$
\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, bend left, "pt"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{LBIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left, "\mathsf{S}"]
\end{tikzcd}
$$

The generalized Isbell adjunction (DL)

There is a 2-adjunction

$$\mathbb{O} : \mathsf{LBIon} \leftrightarrows \mathsf{Topoi} : \mathsf{pt}.$$

Thm. (DL)

The adjunction is idempotent and restrict to a bi-equivelence between sober bounded ionads and topoi with enough points.

Our geometric picture is completed. We now move to a **categorical understanding of the Scott adjunction**.

Thm.

$\mathrm{Acc}_\omega(\mathcal{A}, \mathcal{B})$ is an accessible category with directed colimits. Thus Acc_ω has an internal hom.

Thm. (DL)

Acc_ω is monoidal closed $(\otimes, \mathrm{Acc}_\omega(-,-))$ with respect to this internal hom.

Thm. (DL)

The 2-category of topoi is enriched over the bicategory Acc_ω. Moreover it has tensors.

$$\mathcal{A} \boxtimes \mathcal{E} := \mathrm{Acc}_\omega(\mathcal{A}, \mathcal{E}).$$

Cor.

As a corollary of the fact that Topoi is tensored over Acc_ω, the Scott adjunction re-emerges.

$$\mathrm{Topoi}(\mathcal{A} \boxtimes \mathrm{Set}, \mathcal{F}) \cong \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{Topoi}(\mathrm{Set}, \mathcal{F}))$$

$$\mathrm{Topoi}(\mathrm{S}(\mathcal{A}), \mathcal{F}) \cong \mathrm{Acc}_\omega(\mathcal{A}, \mathrm{pt}(\mathcal{F})).$$

Now we finally move to logic. We are interested in **Syntax-Semantics dualities**. Some might call them reconstruction theorems, or completeness-like theorems, depending on the background and inclination.

$$\begin{tikzcd} \mathrm{Syn}^{\circ} \arrow[r, bend left, dashed, "\mathrm{Mod}"] & \mathrm{Sem} \arrow[l, bend left, dashed, "\mathrm{Th}"] \end{tikzcd}$$

Now we finally move to logic. We are interested in **Syntax-Semantics dualities**.

$$\begin{tikzcd}
 & \mathsf{LBIon} \arrow[dl, dashed, bend right, "\mathbb{O}"'] \\
\mathsf{Topoi} \arrow[ur, "\mathsf{pt}"] \arrow[dr, "\mathsf{pt}"'] & \\
 & \mathsf{Acc}_\omega \arrow[ul, dashed, bend left, "\mathsf{S}"] \arrow[uu, dashed, "\mathsf{ST}"']
\end{tikzcd}$$

Classifying topoi

By a theory, here we intend a geometric theory. We identify them with lex-geometric Sketches.

$$
\begin{tikzcd}
 & \mathrm{LGSketches} \arrow[d, "\mathrm{Mod}"] \arrow[ddl, bend right, "\mathbb{M}od"'] \arrow[ddr, bend left, "\beth"] & \\
 & \mathrm{Acc}_\omega \arrow[dl, dashed, bend right, "\mathrm{ST}"'] \arrow[dr, dashed, bend left, "\mathrm{S}"] & \\
\mathrm{BIon} \arrow[rr, dashed, bend right, "\mathbb{O}"'] & & \mathrm{Topoi} \arrow[ll, "pt"'] \arrow[ul, "\mathrm{pt}"']
\end{tikzcd}
$$

Classifying topoi

$$
\begin{tikzcd}
 & \mathsf{LGSketches} \arrow[d, "\mathsf{Mod}"] \arrow[ddl, bend right, "\mathbb{M}od"'] \arrow[ddr, bend left, "\gimel"] & \\
 & \mathsf{Acc}_\omega \arrow[dl, dashed, bend right, "\mathsf{ST}"'] \arrow[dr, dashed, bend left, "\mathsf{S}"] & \\
\mathsf{BIon} \arrow[rr, dashed, bend right, "\mathbb{O}"'] & & \mathsf{Topoi} \arrow[ll, "pt"'] \arrow[ul, "\mathsf{pt}"]
\end{tikzcd}
$$

We wonder whether S and $\mathbb{O}$ can reconstruct the classifying topos of a theory $\gimel(\mathbb{T})$ when applied to its category (or ionad) of models $\mathsf{Mod}(\mathbb{T})$.

In this new setting we can reformulate our previous discussion in the following mathematical question:

$$\daleth(-) \overset{?}{\cong} \mathsf{SMod}(-).$$

$$\daleth(-) \overset{?}{\cong} \mathbb{O}\mathsf{Mod}(-).$$

Thm. (DL)

The following are equivalent:

- $\daleth(\mathcal{S})$ has enough points;
- $\daleth(\mathcal{S})$ coincides with $\mathbb{O}\mathbb{M}od(\mathcal{S})$.

Thm. (DL)

The following are equivalent:

- $\daleth(\mathcal{S})$ has enough points;
- $\daleth(\mathcal{S})$ coincides with $\mathbb{OM}od(\mathcal{S})$.

- This result strongly resonates with Makkai's **Stone duality for first-order logic**, and in a sense, it is a generalization of his result, in that every **ultracategory** that he considers can be seen as a generalized bounded ionad.
- One of the best achievements of this observation is to acknowledge a logical status to ionads, which were previously confided to topology.

$$
\begin{tikzcd}
 & \mathsf{LGSketches} \arrow[d, "\mathsf{Mod}"] \arrow[ddl, bend right, "\mathbb{M}od"'] \arrow[ddr, bend left, "\gimel"] & \\
 & \mathsf{Acc}_{\omega} \arrow[dl, dashed, bend right, "\mathsf{ST}"'] \arrow[dr, dashed, bend left, "\mathsf{S}"] & \\
\mathsf{BIon} \arrow[rr, dashed, bend right, "\mathbb{O}"'] & & \mathsf{Topoi} \arrow[ll, "pt"] \arrow[ul, "\mathsf{pt}"]
\end{tikzcd}
$$

Categorical model theory is a subfield of categorical logic aiming to describe the relevant **categorical properties of the categories of models of some theory**. It was extensively developed by Makkai and Paré in their well known book [80s].

Motto: Categorical model theory $\leftrightarrow$ accessible categories

Since then, some **hypotheses** have very often been **added** in order to smooth the theory and obtain the same results of the classical model theory:

1. amalgamation property;
2. directed colimits;
3. a nice enough fogetful functor $U : \mathcal{A} \to \text{Set}$;
4. every map is a monomorphism;
5. ...

Meanwhile, in a galaxy far far away...

Model theorists (Shelah '70s) introduced the notion of Abstract elementary class (AEC), which is how a classical logician approaches to axiomatic model theory.

Thm. (Rosicky, Beke, Lieberman)

A category $\mathcal{A}$ is equivalent to an abstract elementary class iff:

1. it is an accessible category with directed colimits;
2. every map is a monomorphism;
3. it has a *structural* functor $U : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is finitely accessible and U is iso-full, nearly full and preserves directed colimits and monomorphisms.

Quite not what we were looking for, uh?!

This looks a bit artificial, unnatural and not elegant.

Our aim

1. Have a **conceptual understanding** of those accessible categories in which model theory blooms naturally.
2. When an accessible category with directed colimits admits such a nice forgetful functor?

Thm. (DL)

The Scott adjunction restricts to locally decidable topoi and AECs.

$$\mathsf{S} : \mathsf{AECs} \leftrightarrows \mathsf{LDTopoi} : \mathsf{pt}$$

Thm. (Henry, DL)

The unit $\eta : A \to \mathsf{pt}\mathsf{S}A$ is faithful precisely when A has a faithful functor into Set preserving directed colimits.

Thm. (Henry)

There is an accessible category with directed colimits which cannot be axiomatized by a geometric theory.

This problem was originally proposed by Rosicky in his talk "Towards categorical model theory" at the CT2014 in Cambridge: *Show that the category of uncountable sets and monomorphisms between cannot be obtained as the category of point of a topos. Or give an example of an abstract elementary class that does not arise as the category points of a topos.*