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Directed Type Theory III

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Reminder on directed univalence

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & A & \end{tikzcd}$$

cocartesian fibrations

$$\begin{tikzcd} A' \times_A X \arrow[rr, dashed, "\varphi"] \arrow[dr] & & A' \times_A Y \arrow[dl] \\ & A & \end{tikzcd}$$

$$\begin{tikzcd} & \underline{\mathrm{Hom}}_A(X,Y) \arrow[d, "\text{isofibration}"] \\ A' \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r] & A \end{tikzcd}$$

$\tilde{\varphi}$ factors through $\underline{\mathrm{cocart}}_A(X,Y) \xrightarrow[\text{monomorphism}]{\text{homotopy}} \underline{\mathrm{Hom}}_A(X,Y)$

iff φ preserves cocartesian edges.

Construction of the canonical functor associated to a cocartesian fibration

We have

$$p: X \longrightarrow A:$$

We want:

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, A) \arrow[rr, dashed] \arrow[dr] & & \underset{A\times A}{\mathrm{cocart}}(X\times A, A\times X) \arrow[dl] \\
& A\times A &
\end{tikzcd}$$

Construct:

$$
\begin{tikzcd}
W \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, "p"] \\
\Delta^1 \times \mathrm{Fun}(\Delta^1, A) \arrow[r] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
W_\varepsilon \arrow[r] \arrow[d, "\pi_\varepsilon"'] \arrow[dr, phantom, very near start, "\lrcorner"] & W \arrow[d, "\pi"] \\
\mathrm{Fun}(\Delta^1, A) \arrow[r, hook, "\varepsilon"] & \Delta^1 \times \mathrm{Fun}(\Delta^1, A)
\end{tikzcd}
$$

$\varepsilon = 0, 1$

Observe:

$$
\begin{tikzcd}
W_0 \arrow[r] \arrow[d, "\pi_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \times A \arrow[d, "p \times 1"] \\
\mathrm{Fun}(\Delta^1, A) \arrow[r] & A \times A
\end{tikzcd}
\qquad
\begin{tikzcd}
W_1 \arrow[r] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \times X \arrow[d, "1 \times p"] \\
\mathrm{Fun}(\Delta^1, A) \arrow[r] & A \times A
\end{tikzcd}
$$

Suffices to produce

$$\begin{tikzcd} W_0 \arrow[rr, dashed] \arrow[dr, "\pi_0"'] & & W_1 \arrow[dl, "\pi_1"] \\ & \mathrm{Fun}(\Delta^1, A) & \end{tikzcd}$$

preserving cocartesian edges.

$$\begin{tikzcd}
\{0\} \times W_0 \arrow[r] \arrow[d, hook] & W \arrow[d, "\pi"] \\
\Delta^1 \times W_0 \arrow[r, hook, "1 \times \pi_0"'] \arrow[ur, dashed] & \Delta^1 \times \mathrm{Fun}(\Delta^1, A) =: C
\end{tikzcd}$$

$\mathrm{Fun}(\Delta^1 \times W_0, W) \xrightarrow{ev_0} {}^{\pi}/C$ has a fully faithful right adjoint

Take the fiber over

$$\{1\} \times \mathrm{Fun}(\Delta^1, A) \hookrightarrow \Delta^1 \times \mathrm{Fun}(\Delta^1, A)$$

and we get $W_0 \longrightarrow W_1$.

Definition. The cocartesian fibration $p : X \longrightarrow A$ is (directed) univalent if the canonical functor

$$\mathrm{Fun}(\Delta^1, A) \longrightarrow \underline{\mathrm{cocart}}_{A\times A}(X\times A, A\times X)$$ is an equivalence.

Axiom. There is a hierarchy of univalent cocartesian fibrations. In particular, any cocartesian fibration is a pullback of a directed univalent cocartesian fibration.

Generic notation for a directed univalent universe:
$$\begin{tikzcd} \mathrm{Cat}_\bullet \arrow[d] \\ \mathrm{Cat} \end{tikzcd}$$

$$\boxed{\mathrm{Fun}(\Delta^1, \mathrm{Cat}) \xrightarrow{\sim} \underline{\mathrm{coCart}}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat}\times\mathrm{Cat}_\bullet)}$$

Remark. The full subtype of $\underline{\mathrm{coCart}}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat}\times\mathrm{Cat}_\bullet)$ classifying equivalences between cocartesian fibrations

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr] & & Y \arrow[dl] \\ & A & \end{tikzcd}$$

is then equivalent to Cat. In other words, directed univalence implies/subsumes „ordinary univalence".

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathrm{Fun}(A, \mathrm{Cat})) \arrow[rr, "\sim"] \arrow[dr] & & \mathrm{Fun}(A, \underline{\mathrm{coCart}}_{\mathrm{Cat}\times\mathrm{Cat}}(\mathrm{Cat}_\bullet \times \mathrm{Cat}, \mathrm{Cat}\times\mathrm{Cat}_\bullet)) \arrow[dl] \\
& \mathrm{Fun}(A, \mathrm{Cat}) \times \mathrm{Fun}(A, \mathrm{Cat}) &
\end{tikzcd}$$

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d, "p_i"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d] \\
A \arrow[r, "F_i"'] & \mathrm{Cat}
\end{tikzcd}$$

$i = 1, 2$

$$\mathrm{Fun}(\Delta^1, \mathrm{Cat})(F_1, F_2) \xrightarrow{\sim} \mathrm{Fun}_A(X_1, X_2)^{\simeq}$$

Directed univalence is a way to express straightening / unstraightening

$$\begin{array}{c} X \\ \Big\downarrow \text{cocart. fib.} \\ A \end{array} \quad \longleftrightarrow \quad A \xrightarrow{\text{functor}} \mathrm{Cat}$$

For A a groupoid we have:

- any isofibration $X \longrightarrow A$ is cocartesian
- $\mathrm{Fun}(A, \mathrm{Cat}) \simeq \mathrm{Cat}_{/A}$

This is why type dependency is expressed through slices when all types are groupoids (ordinary type theory).

$$\begin{tikzcd}
\mathcal{S}_\bullet^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{S}_\bullet \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d] \\
\mathcal{S}^{\simeq} \arrow[r] & \mathcal{S} \arrow[r, hook] & \mathrm{Cat}
\end{tikzcd}$$

universal left & right fibration (usual univalent universe of groupoids)

universal left fibration

universal cocart. fib.

In particular, groupoids form a semantic interpretation of HoTT with univalence.

Arithmetic axiom.

There is a natural number object $\mathbb{N}$

Consequence: $\Delta^n \rightsquigarrow \Delta^0 * \Delta^n = \Delta^{n+1}$ induces a map of groupoids

$$\begin{array}{ccc} \mathbb{N} \longrightarrow \mathrm{Cat}^{\simeq} \\ n \longmapsto [n] \end{array} \quad \text{with} \quad \begin{tikzcd} \Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Cat}_{\bullet} \arrow[d] \\ \Delta^0 \arrow[r, "{[n]}"'] & \mathrm{Cat} \end{tikzcd}$$

(it is fully faithful by univalence).

Define Δ as the full subtype of Cat with objects $[n]$ for $n : \mathbb{N}$.

$$\mathbb{N} \simeq \Delta^{\simeq}$$

$\mathbb{N}$ is the collection of finite cardinals

Δ is the theory of finite ordinals.

Fat join axiom. For any types A and B there is a canonical homotopy pushout square

$$\begin{tikzcd} A \times \partial\Delta^1 \times B \arrow[r] \arrow[d] & A \times \Delta^1 \times B \arrow[d] \\ A \amalg B \arrow[r] & A * B \end{tikzcd}$$

induced by

$$\begin{tikzcd} (A \times B) \amalg (A \times B) \cong A \times \partial\Delta^1 \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A \times \Delta^1 \times B \arrow[d, "\text{projection}"] \\ \partial\Delta^1 \arrow[r] & \Delta^1 \end{tikzcd}$$

$$\begin{tikzcd} (A \times B) \amalg (A \times B) \arrow[d, "pr_1 \amalg pr_2"'] \\ A \amalg B \arrow[r] & \partial\Delta^1 \end{tikzcd}$$

Recall:

A commutative square
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
is defined to be homotopy cocartesian if

$$\begin{tikzcd} \mathrm{Fun}(D, E) \arrow[r] \arrow[d] & \mathrm{Fun}(C, E) \arrow[d] \\ \mathrm{Fun}(B, E) \arrow[r] & \mathrm{Fun}(A, E) \end{tikzcd}$$

is homotopy catesian for all E.

Consequence of the fat join axiom:

There are canonical equivalences:

$$\lrcorner \xrightarrow{\sim} \partial\Delta^1 * \Delta^0 \quad \text{and} \quad \ulcorner \xrightarrow{\sim} \Delta^0 * \partial\Delta^1$$

$$\ulcorner * \Delta^0 \xleftarrow{\sim} \square \quad \text{and} \quad \Delta^0 * \lrcorner \xleftarrow{\sim} \square$$

where $\square = \Delta^1 \times \Delta^1$, $\ulcorner$ is the full subtype of $\square$ with objects the pairs $(0,0), (0,1), (1,0)$, and dually for $\lrcorner$.

Remark: the comparison maps are induced by

$$\Delta^1 \xleftarrow{\min} \square \quad \text{and} \quad \Delta^1 \xleftarrow{\max} \square$$

This implies that there is a homotopy pushout

$$\begin{tikzcd} \Delta^1 \arrow[r, hook, "\delta_1^2"] \arrow[d, hook, "\delta_1^2"'] & \Delta^2 \arrow[d, hook] \\ \Delta^2 \arrow[r, hook] & \square \end{tikzcd}$$

expressing the fact that a commutative square $c : \square \longrightarrow A$ is a pair of commutative triangles having a common composition:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr] & X_2 \arrow[d] \\ X_3 \arrow[r] & X_4 \end{tikzcd}$$

All these are the computation needed to reason about pullbacks in the ways we are used to:

- universal property of pull backs

$$\begin{tikzcd}
S \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dashed] & & \\
& T \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& T \arrow[r] & Y
\end{tikzcd}$$

- $T \times_Y X$ as the limit of the $\lrcorner$-indexed diagram $T \longrightarrow Y \longleftarrow X$

- $T \times_Y X \longrightarrow T$ seen as the image of $X \longrightarrow Y$ through the right adjoint of the functor $(W \longrightarrow T) \longmapsto (W \longrightarrow T \longrightarrow Y)$.

$\Rightarrow$ <u>all usual properties of pullbacks hold true.</u>

Optional axiomata

Axiom. There is an opposite operator — a functor $X \longmapsto X^{op}$ with

$$X = (X^{op})^{op} \text{ (functorially)}$$

and an isomorphism $(\Delta^1)^{op} \cong \Delta^1$ compatible with the permutation

$$\begin{aligned} \partial\Delta^1 &\longrightarrow \partial\Delta^1 \\ 0 &\longmapsto 1 \\ 1 &\longmapsto 0 \end{aligned}$$

induces

$$\begin{array}{ccc} (X * Y)^{op} & \cong & Y^{op} * X^{op} \\ \downarrow & & \downarrow \\ (\Delta^1)^{op} & \cong & \Delta^1 \end{array}$$

$$\mathrm{Fun}(X,Y)^{op} \cong \mathrm{Fun}(X^{op}, Y^{op})$$

Axiom (Yoneda)

There, for each type A a right fibration

$$\begin{array}{c} Tw(A) \\ \downarrow{\scriptstyle \pi_A} \\ A \times A^{op} \end{array}$$

(given functorially in A and preserving size)

such that all fibers of each map

$$Tw(A) \longrightarrow A \times A^{op} \xrightarrow{pr_1} A \quad \text{and} \quad Tw(A) \longrightarrow A \times A^{op} \xrightarrow{pr_2} A^{op}$$

have <u>initial</u> objects.

$$\begin{array}{c} (Tw(A))^{op} \\ \downarrow \\ A^{op} \times A = (A \times A^{op})^{op} \end{array} \iff Hom_A : A^{op} \times A \longrightarrow S \subseteq Cat$$

By transposition: $h_A : A \longrightarrow \hat{A} := Fun(A^{op}, S)$ the Yoneda embedding.

We could introduce the operators $(-)^{op}$ and Tw directly.

Define A^{op} as the full subtype of $\mathrm{Fun}(A, S)$ spanned by functors $F : A \to \mathrm{Cat}$ such that $\int F$ has an initial object.

$$\begin{tikzcd} \int F \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\ulcorner", very near start] & S_\bullet \arrow[d] \\ A \arrow[r, "F"'] & S \end{tikzcd}$$

$A^{op} \subseteq \mathrm{Fun}(A, S) \iff \mathrm{Hom}_A : A^{op} \times A \to S$

$$\iff \begin{tikzcd} S(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_\bullet \arrow[d] \\ A^{op} \times A \arrow[r] & S \end{tikzcd}$$

Define $\mathrm{Tw}(A) := S(A)^{op}$.

We can prove that $\mathrm{Tw}(A)$ is $\mathcal{U}$-small whenever A is $\mathcal{U}$-small, using

Lemma.

A type A is $\mathcal{U}$-small if and only if $A^{\simeq}$ is $\mathcal{U}$-small and $A(x,y)$ is $\mathcal{U}$-small for any $(x,y) : T \longrightarrow A \times A$ with T a $\mathcal{U}$-small groupoid.
(e.g. $T = A^{\simeq} \times A^{\simeq}$).

Warning: here $\mathcal{U}$-small means there is a homotopy pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Cat}_{\mathcal{U},\bullet} \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Cat}_{\mathcal{U}} \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

q isofibration

$$\begin{tikzcd} \mathrm{Fun}_C(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & \mathrm{Fun}(X,Y) \arrow[d, "q_*"] \\ \Delta^0 \arrow[r, "\ulcorner p \urcorner"'] & \mathrm{Fun}(X,C) \end{tikzcd}$$

Def. Weak equivalence: functor $A \to B$ inducing $\pi_\infty(A) \xrightarrow{\sim} \pi_\infty(B)$
weakly contractible: type A such that $A \to \Delta^0$ is a weak equivalence.

Theorem. Let $u : A \to B$ be a functor.
The following conditions are equivalent.
(i) for any $b : B$ the slice A/b is weakly contractible.
(ii) for any functor $B \to C$ and any left fibration $X \to C$

$$\mathrm{Fun}_C(B, X) \xrightarrow{\sim} \mathrm{Fun}_C(A, X)$$

(iii) the functor u is a retract of the composition of a fully faithful right adjoint followed by a localization.

Definition: such maps are called <u>left anodyne</u>
(or limit-cofinal,
or initial ...)

Observation.

$\mathcal{S} \hookrightarrow \mathrm{Cat}$ has a left adjoint $\pi_\infty : \mathrm{Cat} \to \mathcal{S}$.

$\mathrm{Fun}(A, \mathcal{S}) \hookrightarrow \mathrm{Fun}(A, \mathrm{Cat})$ has a left adjoint $\pi_\infty(-/A)$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\text{cocart. fib.}"'] & \pi_\infty(X/A) \arrow[dl, "\text{left fib.}"] \\ \mathrm{Cat} & \end{tikzcd}$$

$\pi_\infty(X/A)$ is fiberwise the localization by all maps.

Lemma. $\pi_\infty(X/A) \simeq \mathcal{L}_W(X)$ with W the collection of all edges of X that are sent to identities in A (or isomorphisms...)

$\Rightarrow$ $X \longrightarrow \Pi_\infty(X/A)$ is a localization and this remains true after any pullback along any functor $A' \longrightarrow A$.

For a general functor $X \xrightarrow{f} A$ we get:

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[dr, "f"'] & \tilde{X} \arrow[r, "\substack{\text{fully faithful}\\ \text{left adj.}}"] \arrow[d, "\tilde{f}"', "\text{iso. fib.}"] & Y \arrow[r, "\text{localization}"] \arrow[dl, "\varphi"', "\text{cocart. fib.}"] & \Pi_\infty(Y/A) \arrow[dll, "\text{left fib.}"] \\
& A & &
\end{tikzcd}$$

Conclusion:

$$\mathrm{Fun}(A, S) \simeq \mathrm{LFib}(A) \underset{\text{full}}{\subseteq} \mathrm{Cat}_{/A}$$

has a left adjoint.

Equivalently: there is a weak factorization system

left anodyne maps / left fibrations.

Theorem.
Both $\mathcal{S}$ and Cat are bicomplete.

$\rightsquigarrow$ theory of (pointwise) Kan extensions, distributors ... .

Consequence

$$\begin{aligned} N : \mathrm{Cat} &\longrightarrow \mathrm{Fun}(\Delta^{op}, \mathcal{S}) \\ A &\longmapsto \mathrm{Hom}_{\mathrm{Cat}}([-], A) \end{aligned}$$

has a left adjoint $c : \mathrm{Fun}(\Delta^{op}, \mathcal{S}) \longrightarrow \mathrm{Cat}$.

Theorem. The nerve functor is fully faithful with essential image spanned by complete Segal objects in $\mathrm{Fun}(\Delta^{op}, S)$.

In terms of syntax this means that we get an inductive way to determine functors $u : A \to B$ using Reedy-like inductions:

inductive constraints explaining how to turn a rule

$$a : \Delta^n \to A \quad \longmapsto \quad u(a) : \Delta^n \to B$$

into a functor $u : A \to B$.

Such rules could be introduced earlier in the syntax in order to be able to construct maps explicitly to begin with.

The Yoneda Lemma.

$S(A) := \mathrm{Tw}(A)^{op}$ twisted diagonal.

A „small"

$\hat{A} = \mathrm{Fun}(A^{op}, S)$

$h : A \longrightarrow \hat{A}$ Yoneda embedding

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d] & S_\bullet \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near start] \\
\Delta^\circ \arrow[r] & \mathrm{Cat} & S \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
S(A) \arrow[r, "w"] \arrow[d] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S(\hat{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_\bullet \arrow[d] \\
A^{op} \times A \arrow[r, "1_{A^{op}} \times h"'] & A^{op} \times \hat{A} \arrow[r, "h^{op} \times 1_{\hat{A}}"'] & \hat{A}^{op} \times \hat{A} \arrow[r, "\mathrm{Hom}_{\hat{A}}"'] & S
\end{tikzcd}$$

$$\begin{tikzcd}
S(A) \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_\bullet \arrow[d] \\
A^{op} \times A \arrow[r, "1_{A^{op}} \times h"'] \arrow[rr, bend right, "\mathrm{Hom}_A"'] & A^{op} \times \hat{A} \arrow[r, "\text{evaluation}"'] & S
\end{tikzcd}$$

Theorem. Both functors defined above $S(A) \longrightarrow V$ and $S(A) \longrightarrow V$ are left anodyne

In other words both functors

$$(a, F) \longmapsto F(a) \quad \text{and} \quad (a, F) \longmapsto \mathrm{Hom}(h(a), F)$$

are left Kan extensions of Hom_A along $1_{A^{op}} \times h : A^{op} \times A \to A^{op} \times \hat{A}$

In particular, there is a unique isomorphism of functors

$F(a) \cong \mathrm{Hom}(h(a), F)$ extending the isomorphism

$$\mathrm{Hom}_A(a, b) \cong \mathrm{Hom}_{\hat{A}}(h(a), h(b))$$

expressing the fully faithfulness of the Yoneda embedding.

We could develop the theory further.

But instead, we can develop the theory formally within itself.

A directed type theory is given by a cartesian closed type $\mathbb{C}$ with universal finite sums as well as an interval object I so that all the axioms of directed type theory hold in $\mathbb{C}$ (declaring all maps of $\mathbb{C}$ to be isofibrations).

Proving theorems in $\mathbb{C}$ will prove theorems for our original theory $\mathcal{E}$ (using directed univalence for $\mathrm{Cat} = \mathbb{C}$).

But this will also prove theorems in $\mathbb{C} = \mathrm{Fun}(A, \mathrm{Cat})$.

Of course, this means that we must prove that our axioms hold true in the type $\mathrm{Fun}(A, \mathrm{Cat})$ for any $A : \mathrm{Cat}$.

The idea is that we can develop directed type theory within such $\mathbb{C}$ for all $\mathbb{C}$:

- presentable categories
- topoi
- operads
- (algebraic) geometry
- K-theory
- motivic sheaves
- representation theory
- spectra of stable categories...

Theorem.

The theory of quasi-categories is a semantic interpretation of directed type theory.

Proof. The first part of the axioms is literally what Joyal has been working on to begin with (the fact that groupoids are the same thing as Kan complexes, that invertibility of natural transformations can be detected objectwise, all this through a careful study of the join operation and of slicing, leading to fully developped theory of left/right fibrations).
The second part: construction of a directed univalent universe of cocartesian fibration is done in a joint work with Kim Nguyen (the case of a universe of left fibrations is worked out in my book on higher categories).

Note: Lurie's Kerodon is a presentation of the theory of $(\infty,1)$-categories that is very much compatible with the axiomatic point of view.

Nima Rasekh's work includes constructions of universes of cocartesian fibrations. Using the explicit Quillen equivalences relating quasi-categories and Rezk's complete Segal spaces, it is easy to deduce that complete Segal spaces define a semantic interpretation of directed type theory.

Remark on the proof of the theorem.

When we say that the proof is in the literature, we cheat slightly.

In the axioms, elements $a : A$ really mean functors $a : X \to A$ with X any groupoid.

That means essentially that, for any Kan complex X, we have to prove that the theory of quasi-categories C equipped with an isofibration $C \to X$ is a semantic interpretation of directed type theory, with:

$$\begin{array}{ccc} \Delta^1 \times X & \text{interval} & \\ \downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} X \times Cat_\bullet & & \\ \downarrow & \text{universe} & \\ X \times Cat & & \end{array}$$

But that really is a straightforward exercise!

This is developed more formally in the work of Louis Martini on $(\infty,1)$-category theory internally in a Grothendieck ∞-topos.

Example of computation:

A Basic tool: Quillen's theorem B

There is a subtype former:

Given a class of maps closed under composition

$$W \xrightarrow[\text{mono.}]{\text{homotopy}} \mathrm{Fun}(\Delta^1, C)^{\simeq}$$

there is a universal homotopy monomorphism

$$\langle W \rangle \longrightarrow C \qquad \text{with} \qquad W = \mathrm{Fun}(\Delta^1, \langle W \rangle)^{\simeq}.$$

Consider $W \longrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\Delta^1, S))^{\simeq}$

defined as those maps in $\mathrm{Fun}(\Delta^1, S)$ corresponding

of pullback squares in S

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & y \arrow[d] \\ z \arrow[r] & t \end{tikzcd}$$

We get

$$\mathrm{Fun}_{\mathrm{cart}}(\Delta', S) \hookrightarrow \mathrm{Fun}(\Delta', S).$$

Theorem. $\mathrm{Fun}_{\mathrm{cart}}(\Delta', S)$ has small colimits and the embedding functor $\mathrm{Fun}_{\mathrm{cart}}(\Delta', S) \hookrightarrow \mathrm{Fun}(\Delta', S)$ commutes with them

(in other words, descent theory holds in S).

Proof: Let $\mathcal{U} : S^{op} \longrightarrow S$, $x \longmapsto (S/_x)^{\simeq}$. Then

(up to size issues)

$$\begin{tikzcd} \mathrm{Fun}_{\mathrm{cart}}(\Delta', S) \arrow[r, "\simeq"] \arrow[d, "\mathrm{dom}"'] & S/_{\mathcal{U}} \arrow[dl] \\ S & \end{tikzcd}$$

$\Rightarrow$ existence of colimits + agreement with colimits in S through the domain functor.

Use universality of colimits in S (because S is locally cartesian closed). ∎

Quillen's Theorem B

Let $u : A \longrightarrow B$ be a functor. Assume that, for any map $f : b_0 \Rightarrow b_1$ in B the induced functor $A/b_0 \longrightarrow A/b_1$ is a weak equivalence. Then, for any $b : B$ the commutative square

$$\begin{tikzcd}
\pi_\infty(A/b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \pi_\infty(A) \arrow[d] \\
\Delta^0 \simeq \pi_\infty(B/b) \arrow[r] & \pi_\infty(B)
\end{tikzcd}$$

is a homotopy pullback square

(i.e. a pullback in the universe S).

Proof. $\operatorname*{colim}_{b \text{ in } B} A/b \simeq A$.

Indeed the colimit of $B \xrightarrow{\text{Yoneda}} \mathrm{Fun}(B^{op}, S)$ is the terminal object.

Since $\mathrm{Fun}(B^{op}, S) \simeq \mathrm{RFib}(B) \underset{\text{full}}{\subseteq} \mathrm{Cat}/_B$ commutes with colimits as well as with limits, we get

$$\operatorname*{colim}_{b} B/b \simeq B \text{ over } B \text{ in } \mathrm{Cat}.$$

But $u^*: \mathrm{Fun}(B^{op}, S) \longrightarrow \mathrm{Fun}(A^{op}, S)$ corresponds to $u^*: \mathrm{Cat}/_B \longrightarrow \mathrm{Cat}/_A$

defined as
$$\begin{array}{c} Y \\ \downarrow \\ B \end{array} \longmapsto \begin{array}{c} A \times_B Y \\ \downarrow \\ A \end{array}$$

hence $\operatorname{colim}_b u^* \mathrm{Hom}_B(-,b) = \operatorname{colim}_b \mathrm{Hom}_B(u(-),b) \cong * $ in $\mathrm{Fun}(A^{op}, S)$

corresponds to

$$\operatorname{colim}_b A/b \cong A \quad \text{in} \quad \mathrm{Cat}/A .$$

Hence $\boxed{\operatorname{colim}_b \pi_\infty(A/b) \cong \pi_\infty(A)}$ in S (π_∞ commutes with colimits since it is a left adjoint).

(this subsumes Quillen's theorem A)

$$b \longmapsto \begin{array}{c} \pi_\infty(A/b) \\ \downarrow \\ \pi_\infty(B/b) \end{array} \quad \text{is a functor} \quad B \longrightarrow \mathrm{Fun}_{\mathrm{Cart}}(\Delta^1, S).$$

its colimit in $\mathrm{Fun}_{\mathrm{Cart}}(\Delta', S)$ that agrees with its colimit in $\mathrm{Fun}(\Delta', S)$.

In particular, for each $b : B$

$$\begin{tikzcd}
\pi_\infty(A/b) \arrow[r] \arrow[d] & \pi_\infty(A) \cong \operatorname*{colim}_b \pi_\infty(A/b) \arrow[d] \\
\pi_\infty(B/b) \arrow[r] & \pi_\infty(B) \cong \operatorname*{colim}_b \pi_\infty(B/b)
\end{tikzcd}$$

is a map in $\mathrm{Fun}_{\mathrm{Cart}}(\Delta', S)$ i.e. a pullback square.

Comments on syntax.

There are several level of deduction rule.

The way they appear distinguish the various ways dependent type may be introduced.

$\Gamma \vdash A : \mathrm{Type}$ means a type A is declared in context Γ.

$\Gamma \models A : \mathrm{Cat}$ means a type A is declared <u>functorially</u> in context Γ

A basic rule must be

$$\frac{\Gamma \models A : \mathrm{Cat}}{\Gamma \vdash A : \mathrm{Type}}$$

The word „functorially" means that $\Gamma \vdash A : \mathrm{Type}$ holds and that the corresponding fibration

$$\begin{array}{c} A \\ {\scriptstyle p}\big\downarrow \\ \Gamma \end{array}$$

is cocartesian.

To define „cocartesian" we only need an interval I, that is a type I equipped with two elements $0 : I$ and $1 : I$.

We can the define „left/right I-deformation retracts" and being cocartesian means that

$$\mathrm{Fun}(I, A) \xrightarrow{ev_0} \mathrm{Fun}(I, \Gamma) \times_\Gamma A$$ is a left I-deformation retract.

Therefore

$\Gamma \vDash A : \mathrm{Cat}$ could be writen $\Gamma \vdash_{I} A : \mathrm{Type}$

for $I = *$ the terminal object, $\vdash_{*}$ is the usual deduction rule.

In a type A „elements" or „objects" are written $a : A$.

There are maps $f : a \Rightarrow b$ in A corresponding to $\varepsilon : I \models f(\varepsilon) : A$

$$\varepsilon : I \vdash f(\varepsilon) : A \quad \text{introduce a maps}$$

$$\partial I = I^{\simeq} \longrightarrow A$$

i.e. a pair $(f(0), f(1)) = (a, b)$.

Whereas $\varepsilon : I \models f(\varepsilon) : A$ introduces $f : I \longrightarrow A$.

Given $f : A \longrightarrow B$ and $g : A \longrightarrow B$

$$x : X \models \alpha(x) : f(x) \Rightarrow g(x)$$

thus means

$$x : X \models \tilde{\alpha}(x) : I \longrightarrow \mathrm{Fun}(A, B), \; \tilde{\alpha}(x)_0 = f, \; \tilde{\alpha}(x)_1 = g$$

hence produces

$$I \times X \longrightarrow \mathrm{Fun}(A, B)$$

For $a : A$ and $b : A$ the type $a \Rightarrow b$ is thus the type of maps from a to b in A.

We introduce

$$A : \mathrm{Type} \vdash A^{op} : \mathrm{Type}$$

with rules: $(A^{op})^{op} = A$

$I^{op} = I$ with 0 and 1 intertwined

We can also define I^{op} as the interval defined to be I but with end points $1 : I$ and $0 : I$ <u>in this order</u>.

$\Gamma \models^{op} A : \mathrm{Cat}$ is then defined to be $\Gamma \vdash_{I^{op}} A : \mathrm{Type}$.

If „Γ is groupoid" holds then

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma \vDash A : \mathrm{Cat}}$$

(since groupoids do not see the difference between I and $*$ they do not see the difference between the inference rules associated to them)

We can introduce usual notations

$$\frac{a : A \vdash u(a) : B}{a : A^{\simeq} \vdash u(a) : B}$$

(hence $a : A \vdash u(a) : B$ will only produce $u : A^{\simeq} \longrightarrow B$)

$$\frac{a : A \vDash u(a) : B}{u : A \longrightarrow B}$$

We may also introduce directed univalence

$$\frac{\Gamma,\ a : A \vDash u(a) : B}{I \times \Gamma \vDash W_u : \mathrm{Type}}$$

(together with rules expressing that
the pullback of W_u at 0 is A
" " " " at 1 is B)

and then the rules expressing that the preceding one gives a homotopy inverse of the one obtained from the cocartesian fibration on $\Delta^1 \times \mathrm{Fun}(\Delta^1, \mathrm{Cat})$ obtained by pulling back along the evaluation functor.

Impose axiom:

$u : A \longrightarrow B$ isofibration

$$\sum_{\substack{\partial\Delta^1 \xrightarrow{(a_0,a_1)} A \\ \downarrow \qquad \downarrow u \\ \Delta^1 \xrightarrow[f]{} B}} \mathrm{Fiber}\Big(a_0 \Rightarrow a_1 \longrightarrow u(a_0) \Rightarrow u(a_1) \Big) \stackrel{\text{isomorphism}}{\cong} \mathrm{Fun}(\Delta^1, A)^{\simeq}$$

Axiom

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, A)^{\simeq} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, A) \arrow[d] \\ \mathrm{Fun}(\partial\Delta^1, A)^{\simeq} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, A) \end{tikzcd}$$ is a homotopy pullback.

Replacing pullbacks by homotopy pullbacks (in the Segal condition and in in the definition of cocartesian fibrations) this should be ok – suffices to rectify sections up to homotopy into actual sections when needed.

All these precautions could be avoided if there where a nice way to produce dependent type theories in which we can easily force a given family of maps to be a fibration... Maybe is it the case?