UNIVALENT
DIRECTED
TYPE THEORY

LECTURE II

D.-C. CISINSKI, H.K. NGUYEN, T. WALDE
RECAP FROM LAST WEEK:

**Goal**: Define a dependent type theory with interpretation

**Approach**: Axiomatically introduce the syntactic category $E$ of this theory

**Setup**: $E$ is Joyal tribe

(isofibrations, path objs, htpy equivs, ...)

$$E \upu \triangleq$$

dependent type theory with

dependent sums & identity types

(but not all dependent products)

+ new constant $\Pi = \Lambda^+$ "interval"

and a (long) list of axioms
• interval $\Delta^0$ is equipped with

$\Delta^0 := 1 + 1 \xrightarrow{0,1} \Delta$

$\min / \max : \Delta^0 \times \Delta^0 \to \Delta^0$ def. retracts

• monoidal structure $- * - := i^* (- \otimes -)$

$\Delta^n := 1 + \ldots + 1$ + simpl. operators

• internal hom $\text{Fun}(A, B) = B^A$

$X$ is groupoid $\iff X \to X^{\Delta^1}$

$\text{gr}(E) := \{\text{groupoids}\}$ has all dependent products

$\text{gr}(E) \text{ model of HoTT}$

• groupoid core $(-)^\simeq \to \text{Map}(A, B) := (B^A)^\simeq$

$\forall X \text{ grpd} : \text{Map}(X, B) \simeq \text{Map}(X, B^\simeq)$

$\partial \Delta^1 \simeq (\Delta^1)^{\simeq}$

• Segal axiom $X^{\Delta^0} \simeq X^{\Delta^1 \times \ldots \times \Delta^1}$

• $C \to \text{islnuc} \subseteq \text{Fun}(\Delta^0, C)$

• $X \subseteq C \to \text{subgrpd} \to$ full subtype $\langle x \rangle \hookrightarrow C$

$\left\{ \begin{array}{c}
T \to C \\
T \simeq \ldots \to X \subseteq C
\end{array} \right\} =: \text{Map}(T, \langle x \rangle)$
The fundamental groupoid 

Define \( \text{Fun}^\sim(C, D) = \langle \text{Fun}(C, D^\sim) \rangle \hookrightarrow \text{Fun}(C, D) \)

full subtype of those \( \xymatrix{ C \ar[r] & D \ar@{-->}[r] & \text{D}^\sim } \) functors that factor

Axiom: For each type \( C \) there is
\[ C \rightarrow \Pi_{\omega} C \] with groupoid \( \Pi_{\omega} C \) s.t.
\[ \forall D : \text{Fun}(\Pi_{\omega} C, D) \hookrightarrow \text{Fun}^\sim(C, D) \]

\( \sim \rightarrow \Pi_{\omega} C : \text{Types} \rightarrow \text{Grpds} \)
right adjoint to inclusion up to homotopy.

Lemma: \( \Pi_{\omega} \Delta^1 \sim \mathbb{I} \)

Proof: \( \forall \) groupoid \( X \):

\[ \text{Fun}(\Pi_{\omega} \Delta^1, X) \sim \text{Fun}(\Delta^1, X) \sim \text{Fun}(\mathbb{I}, X) \]

\( \Rightarrow \) \( \Pi_{\omega} \Delta^1 \sim \mathbb{I} \) since both are groupoids
Localization

Construction: Let $W \leq \text{Map}(\Delta^0, C)$ subgroupoid

$\leadsto \text{Fun}_W(C, D) \leq \text{Fun}(C, D)$ full subtype of those $F : C \to D$ s.th.

$\text{Map}(\Delta^0, C) \to \text{Map}(\Delta^0, D)$

$\text{UI} \quad \text{UI}$

$W \to \text{is Inv}(D)$

Axiom: Given type $C$, subgroupoid $W \leq \text{Map}(\Delta^0, C)$ have $C \to \mathcal{L}_W C = C[\mathcal{W}^{-1}]$ s.th.

$\forall D : \text{Fun}(C[\mathcal{W}^{-1}], D) \simeq \text{Fun}_W(C, D)$

When $W = \text{Map}(\Delta^0, C)$, write $\mathcal{L} C := \mathcal{L}_W C$

Structure map $C \to \mathcal{T}_{\infty} C$ induces $\mathcal{L} C \to \mathcal{T}_{\infty} C$

because $\text{Map}(\Delta^0, C) \to \text{Map}(\Delta^0, \mathcal{T}_{\infty} C)$

$\simeq$

$\to \text{is Inv}(\mathcal{T}_{\infty} C)$

Axiom: $\mathcal{L} C \to \mathcal{T}_{\infty} C$
Notation: "objects" \(a : A\) are
\(a : X \to A\) with \(X = \{a\}\) a groupoid
denoted terms of \(A\) in context \(X\).

Construction: For \(a, b : A\) have hom space
\[
\begin{array}{c}
A(a, b) \xrightarrow{\cdot} \text{Fun}(\Delta^1, A) \\
\downarrow \quad \downarrow \text{isofibration}
\end{array}
\]
\[
\{(a, b)\} \to A \times A
\]

Problem: dependent types \(A \vdash B \text{ Type}\)
\(\triangleq \text{ isofibrations } B \to A\)
do not give rise to \(A \to \mathcal{U} = \text{Cat}\)
because \(f : A(a, a') \to B(a) \to B(a')\)

Idea: Introduce new "kinds of fibration"
corresponding to new kinds of dependent types
\(A \equiv B \text{ Type} \& A \equiv B \text{ Type}\)

Semantically: ccartesian/cartesian fibrations
Recall: **Right deformation retract** consists of:

\[ p: B \to A, \quad s: A \to B, \quad H: B \times \Delta^0 \to A \]

- \( H_0 = \text{id}_x, \quad H_1 = sp, \quad ps = \text{id}_A \)
- \( B \times \Delta^0 \xrightarrow{H} B \quad B \times \Delta^0 \xrightarrow{H} B \)
  \[
  \begin{array}{ccc}
  s \times \Delta^0 & \xleftarrow{\downarrow s} & p \times \Delta^0 \\
  \downarrow & & \downarrow \\
  A \times \Delta^0 & \xrightarrow{p_A} & A
  \end{array}
  \]

We say: \( s \) is **right adjoint section** of \( p \)

**Def:** object \( a: A \ (\triangleq X \to A) \) is **terminal** if it is right adjoint section of \( X \times A \to X \).

**Dually:** left adjoint sections \& initial objects

**Construction**

For \( f: B \to A \)

\[
\begin{array}{ccc}
\text{Fun}(\Delta^0, B) & \to & B \times_A A \\
\downarrow & & \downarrow \\
B \times B & \overset{p}{\to} & B \times A
\end{array}
\]

Informally:

\[
B \times A \Rightarrow \{(b: B, a: A, fb \to a)\}
\]

Special case: \( a: A \rightsquigarrow \text{slice} \ a/A := \{a\} \times A \)
**Def:** Let \( p : B \to A \) be isofibration.

If \( \text{Fun}(\Delta^1, B) \to B \overset{p}{\to} A \)
- has left adjoint section
  \( p \) is **cocartesian fibration**
- is equivalence
  \( \implies p \) is **left fibration**

**Example:** \( \text{Fun}(\Delta^1, A) \overset{\text{ev}_1}{\to} A \) is cocartesian
more generally: \( B \overset{f}{\to} A \overset{\text{ev}_0}{\to} A \) is cocartesian
\( a/A \to A \) is left fibration
**Lemma:** Cocartesian fibration \( p : B \to A \) is left fibration iff for all \( a : A \) fiber \( B_a \) is groupoid.

**Def:** Let \( p : B \to A \) a cocartesian fibration. Define \( \text{isCocart}_p \subseteq \text{Fun}(\Delta^1, B) \) essential image of section \( B \times A \xrightarrow{s} \text{Fun}(\Delta^1, B) \) "full subtype of \( p \)-cocartesian arrows."  

**Rem:** \( p : B \to A \) is left fibration iff every arrow of \( B \) is \( p \)-cocartesian.

**Lemma:** left/right deformation retracts, left/cocartesian fibrations are all stable under base-change.
<table>
<thead>
<tr>
<th><strong>semantics</strong></th>
<th><strong>syntax</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>( p: B \to A ) cocartesian</td>
<td>( A ^+ B ) Type</td>
</tr>
<tr>
<td>section ( s: A \to B ) of ( p )</td>
<td>( a: A \vdash s_a: B(a) )</td>
</tr>
</tbody>
</table>

In general, cannot expect

\[
\vdash a: A \vdash s_a: B(a) \Rightarrow a: A \vdash s_a: B(a)
\]

**Axiom** (Assembly for terminal sections)

Let \( p: B \to A \) be cartesian fibration

and assume for each \( a: A \)

- a terminal object \( s_a: B_a \)

Then require right adjoint section

\[
s: A \to B \text{ with } s(a) = s_a \quad (\forall a: A)
\]

**Note:** \( s: A \to B \) right adjoint section of \( p: B \to A \)

\[
\Rightarrow s(a): B_a \text{ is always terminal object } (\forall a: A)
\]
Theorem: Let $B' \xrightarrow{f} B$ be a morphism of cocartesian fibrations (i.e. $f$ preserves cocartesian arrows). Then $f$ is an equivalence iff $f_a : B'_a \to B_a$ is an equivalence for each $a : A$.

Corollary: A cocartesian fibration $p : B \to A$ is an equivalence iff for each $a : A$ the fiber $B_a$ is trivial, i.e. $B_a \sim \{a\}$.

Proof: Apply Thm to morphism $B \xrightarrow{p} A$. 

Apply
Proof of Thm:

\[ \{ (b', b, f(b') \to b) \mid p(b) \text{ invertible} \} \to \text{isInv}_A \]

\[ B' \times_f B \to \text{Fun}(\Delta^1, B) \to \text{Fun}(\Delta^1, A) \]

\( Q \) is cocartesian:

\[
\begin{array}{c}
\text{Q} \\
\downarrow \\
\text{B} \\
\downarrow \\
\text{A}
\end{array}
\quad
\begin{array}{c}
b'_0 \xrightarrow{v_1(b'_0)} \\
fb'_0 \xrightarrow{v_1(fb'_0)} \simeq f(v_1b'_0) \\
b_0 \xrightarrow{v} b_1 \\
\alpha_0 \xrightarrow{uv} \alpha_1
\end{array}
\]

For each \( b : B \), \( a = p(b) : A \) have

\[ Q_b \simeq B'_a / b := B'_a \times (B/b) \simeq B/b \]

which has terminal object.

Assembly right adjoint section \( Q \to B \)

Check: \( g : B \xrightarrow{s} Q \xrightarrow{ev_0} B' \) is homotopy inverse to \( f \)

\( \square \)
**Adjunctions**

**Def:** An adjunction consists of \( f : B \to A : g \) and an equivalence

\[
\{(b,a,fb\to a)\} = B \times_f A \xrightarrow{\sim} A \times_g B = \{(a,b,ga\leftarrow b)\}
\]

\[
\begin{array}{c}
B \times A \\
\downarrow
\end{array}
\xrightarrow{\sim}

\begin{array}{c}
A \times B
\end{array}
\]

of bifibrations.

\[\Delta \quad \text{functorial equivalence} \quad \]

\[A(fb,a) \sim B(b,ga)\]

\[\eta : B \to B \times_f A \xrightarrow{\sim} A \times_g B \to \text{Fun}(\Delta^!,B)\]

\[\eta : \text{id}_B \Rightarrow gf\]

**Lemma:** Adjunctions compose.

**Lemma:** Deformation retracts are adjunctions.
**Thm. (Pointwise formula for adjoints)**

Given \( f : B \to A \) and for each \( a : A \) terminal object \( s_a : B/a \)

Then have right adjoint \( g : A \to B \)

with \( s_a = (ga, \varepsilon_a : fga \to a) \).

**Proof:** Left deformation retract

\[
\min : \Delta^n \times \Delta^n \to \Delta^n \quad \text{(exhibiting } 0 : \Delta^n \text{ initial)}
\]

induces left def. retracts

\[
(b, fb, fb \Rightarrow fb) \quad \xrightarrow{\min} \quad B \times A \quad \xrightarrow{\eta} \quad \text{Fun}(\Delta^n, A)
\]

\[
\begin{array}{c}
\uparrow \\
\leftarrow \\
\downarrow \\
\end{array}
\begin{array}{c}
i \\
e_{\Delta^n} \\
e_{\Delta^n} \\
\end{array}
\]

\[
\begin{array}{c}
\text{co-cartesian} \\
\text{composite adjunction:} \\
\text{coassembly}
\end{array}
\]

\[
\begin{array}{c}
e_{\Delta^n} \\
i \\
\end{array}
\]

\[
B \times A \xrightarrow{\text{ev}_0} A
\]

\[
B \xrightarrow{f} A
\]

\[
\square
\]
Dualize everything starting from

**Def:** $p : B \to A$ cartesian/right fibration

\[ \iff \quad \text{Fun}(\Delta^1, B) \to B \times_A \]

has right adjoint section / is equivalence.

+ Assembly axiom for left adjoint sections

**Def:** An isofibration $p : B \to A_0 \times A_1$

is called a bifibration if

- $p_0 : B \to A_0$ is cartesian fibration
- $p_1 : B \to A_1$ is cocartesian fibration
- For $\beta : \Delta^1 \to B$ have
  - $\beta$ $p_0$-cartesian $\iff p_1(\beta)$ invertible
  - $\beta$ $p_1$-cocartesian $\iff p_0(\beta)$ invertible

**Rem:** bifibration corresponds to $A_0^\text{op} \times A_1 \to \text{Groupoids}$

**Example:** $\text{Fun}(\Delta^1, A) \to A \times A$ is bifibration
**Axiom**: (non-dependent directed path induction)

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & \exists! & \downarrow \\
\text{Fun}(\Delta^n, A) & \longrightarrow & A \times A
\end{array}
\]

(*we can probably prove this and other versions of directed path induction from other axioms*)

**Lemma**: \(B \rightarrow A_0 \times A_n\) bifibration

\((a_0, a_n): A_0 \times A_n \Rightarrow \text{fiber } B_{a_0, a_n}\) is groupoid.

**Corollary**: For every type \(A\), the mapping spaces \(A(a, a')\) are groupoids.

**Theorem**: Equivalences of bifibrations can be detected pointwise.
**Def:** A functor $f: C \rightarrow D$ is called **fully faithful** if square

$$
\begin{array}{ccc}
\text{Fun}(\Delta^n, C) & \rightarrow & \text{Fun}(\Delta^n, D) \\
\downarrow & & \downarrow \\
C \times C & \rightarrow & D \times D
\end{array}
$$

is (homotopy) pullback.

Equivalently: $C(c, c') \rightarrow D(fc, fc')$

equivalence of groupoids for all $c, c' : C$

**Theorem** A functor $f: C \rightarrow D$

is an equivalence if and only if it is

- fully faithful and
- induces a surjection

$\| C \|_{\infty} \rightarrow \| D \|_{\infty}$
**RELATIVE MAPPING TYPES**

Consider isofibrations $\xymatrix{ B \ar[r]^-{p} \ar[dr]_q & C \ar[d] \ar@{._>}[drr]^f \\ & A }$

If $p$ admits dependent product $p_* : E(B) \to E(A)$ (right adjoint to pullback), define $\boxed{ \text{Fun}_A(B,C) := p_* p^+(C \to A) }$

Note $\text{Map}(T, \text{Fun}_A(B,C))$

(not to be confused with)

$$\text{Fun}_A(B,D) = \left\{ \begin{array}{c} B \ar[r] \ar[dr] & D \\ \ar@{..>}[rr] & & A \end{array} \right\} = \text{global sections of } \text{Fun}_A(B,C)$$
**Axiom:** Dependent products exist along every cartesian and cocartesian fibration.

For cocartesian fibrations

\[
p : B \to A \quad \text{and} \quad q : C \to A
\]

\[
\vdash \mathsf{Fun}_A(B, C) \to A
\]

**Axiom**: Have (non-full) subtype

\[
\mathsf{Cocart}_A(B, C) \subset \mathsf{Fun}_A(B, C)
\]

classifying those

\[
\begin{array}{ccc}
T_x B & \to & T_x C \\
\downarrow \rho_T & & \downarrow q_T \\
T & & T
\end{array}
\]

which preserve cocartesian arrows.

\[
\text{\textit{*Actually, we introduce a more general}}
\]
\[
\text{\textit{(non-full) subtype former axiom}}
\]
\[
\text{\textit{of which this is a special case}}
\]
DIRECTED UNIVALENCE

Recall: We have a hierarchy of subtribes
\[ E_0 \leq E_1 \leq \ldots \leq E_\alpha \leq \ldots \leq E = E_\kappa \]

Axiom: For each \( \alpha < \kappa \) we have

- a type \( U_\alpha : E_{\alpha+1} \) and a \underline{univalent cocartesian fibration} \( p_\alpha : U^* \xrightarrow{\sim} U_\alpha \) satisfying:
  - for each \( A \rightarrow U_\alpha \) with \( A : E_\alpha \)
    have \( A \times U^* \xrightarrow{\sim} U_\alpha \) again in \( E_\alpha \)
  - for every cocartesian fibration \( p : B \rightarrow A \)
    with fibers in \( E_\alpha \) there is a unique (strict) pullback square
    \[
    \begin{array}{ccc}
    B & \xrightarrow{=} & U^*_\alpha \\
    \downarrow{p} & & \downarrow{p_\alpha} \\
    A & \xrightarrow{=} & U_\alpha \\
    \end{array}
    \]
Def: A cocartesian fibration \( p : B \rightarrow A \) is called univalent if the canonical map 
\[
(\ast) \quad \text{Fun}(\Delta^1, A) \xrightarrow{\simeq} \text{cocart}_{A \times A}(B \times A, A \times B)
\]
is an equivalence.

Explicit meaning for universe \( p : U^0 \rightarrow U \):

Given \( U \)-small types \( C, D \) have

\[
\begin{align*}
C \times D & \rightarrow U^0 \times U^0 \\
\downarrow & \downarrow \quad \gamma_{C^0 \times D^0} \\
1 \times 1 & \rightarrow U \times U
\end{align*}
\]

On fibers over \( \gamma_{C^0 \times D^0} \) \( (\ast) \) becomes:

\[
\begin{align*}
U(\gamma_{C^0 \times D^0}) & \rightarrow U^0 \times U \\
\downarrow & \downarrow \gamma_{C^0 \times D^0} \\
\text{Map}(C, D) & \rightarrow U \times U^0
\end{align*}
\]
Construction: Start with a cocartesian fibration
\[ p: A \to B \]
Have a tautological map
\[ A \to \text{cocart}_{A \times A} (B \times A, A \times B) \]
given by
\[ B \to B \times A \to A \xrightarrow{id} A \times A \]
\[ B \to A \times B \]

By path induction:
\[ A \to \text{cocart}_{A \times A} (B \times A, A \times B) \]
\[ \text{Fun}(\Delta^n, A) \to A \times A \]

Check: this is bifibration