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UNIVALENT DIRECTED TYPE THEORY

LECTURE II

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RECAP FROM LAST WEEK:

Goal: Define a dependent type theory with interpretation $\boxed{\text{Type} = \infty\text{-category}}$

Approach: Axiomatically introduce the syntactic category $\mathcal{E}$ of this theory

Setup: $\mathcal{E}$ is Joyal tribe (isofibrations, path objs, htpy equivs, ...)

$\triangleq$ $\boxed{\text{dependent type theory with dependent sums \& identity types}}$

(but not all dependent products)

+ $\boxed{\text{new constant } \mathbb{I} = \Delta^1}$ "interval"

and a (long) list of axioms

- interval Δ^1 is equipped with

$$\boxed{\begin{array}{ll} i : & \partial\Delta^1 := \mathbb{1} + \mathbb{1} \xrightarrow{0,1} \Delta^1 \\ \text{min/max} : & \Delta^1 \times \Delta^1 \longrightarrow \Delta^1 \end{array}} \quad \text{def. retracts}$$

- monoidal structure $- \star - := i_*(- \amalg -)$

$$\rightsquigarrow \boxed{\Delta^n := \mathbb{1} \star \dots \star \mathbb{1}} + \text{simpl. operators}$$

- $\boxed{\text{internal hom } \mathrm{Fun}(A, B) = B^A}$

$$\boxed{X \text{ is groupoid} :\Longleftrightarrow X \xrightarrow{\sim} X^{\Delta^1}}$$

$\mathrm{gr}(E) := \{\text{groupoids}\}$ has <u>all</u> dependent products

$$\rightsquigarrow \boxed{\mathrm{gr}(E) \text{ model of HoTT}}$$

- $\boxed{\text{groupoid core } (-)^{\simeq}} \rightsquigarrow \mathrm{Map}(A, B) := (B^A)^{\simeq}$

$$\forall X \text{ grpd}: \ \mathrm{Map}(X, B) \simeq \mathrm{Map}(X, B^{\simeq})$$

- $\boxed{\partial\Delta^1 \xrightarrow{\sim} (\Delta^1)^{\simeq}}$

- Segal axiom $\boxed{X^{\Delta^n} \xrightarrow{\sim} X^{\Delta^1} \underset{X}{\times} \cdots \underset{X}{\times} X^{\Delta^1}}$

- $\boxed{C \xrightarrow{\sim} \mathrm{isInv}_C} \subseteq \mathrm{Fun}(\Delta^1, C)$

- $X \subseteq C^{\simeq}$ subgrpd $\rightsquigarrow$ $\boxed{\text{full subtype } \langle X \rangle \hookrightarrow C}$

$$\left\{ \begin{tikzcd} T \arrow[r] & C \\ T^{\simeq} \arrow[u] \arrow[r, dashed] & X \subseteq C^{\simeq} \arrow[u] \end{tikzcd} \right\} =: \mathrm{Map}(T, \langle X \rangle)$$

The fundamental groupoid

Define $\mathrm{Fun}^{\simeq}(C,D) = \left\langle \mathrm{Fun}(C, D^{\simeq}) \right\rangle \hookrightarrow \mathrm{Fun}(C,D)$

full subtype of those functors that factor
$$\begin{tikzcd} C \arrow[rr] \arrow[dr, dotted] & & D \\ & D^{\simeq} \arrow[ur] & \end{tikzcd}$$

Axiom: For each type C there is

$C \longrightarrow \Pi_\infty C$ with groupoid $\Pi_\infty C$ s.th.

$\forall D$: $\mathrm{Fun}(\Pi_\infty C, D) \xrightarrow{\sim} \mathrm{Fun}^{\simeq}(C,D)$

$\rightsquigarrow$ $\Pi_\infty C$: Types $\longrightarrow$ Grpds

right adjoint to inclusion up to homotopy.

Lemma: $\Pi_\infty \Delta^1 \simeq \mathbb{1}$

Proof: $\forall$ groupoid X :

$$\mathrm{Fun}(\Pi_\infty \Delta^1, X) \simeq \mathrm{Fun}(\Delta^1, X) \simeq \mathrm{Fun}(\mathbb{1}, X)$$

$\Rightarrow$ $\Pi_\infty \Delta^1 \simeq \mathbb{1}$ since both are groupoids

Localization

<u>Construction</u>: Let $W \subseteq \mathrm{Map}(\Delta^1, C)$ subgroupoid

$\rightsquigarrow \mathrm{Fun}_W(C,D) \subseteq \mathrm{Fun}(C,D)$ full subtype

of those $F: C \longrightarrow D$ s.th.

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, C) \arrow[r] & \mathrm{Map}(\Delta^1, D) \\ W \arrow[u, phantom, "\cup"] \arrow[r, dashed] & \mathrm{isInv}(D) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

<u>Axiom</u>: Given type C, subgroupoid $W \subseteq \mathrm{Map}(\Delta^1, C)$

have $C \longrightarrow \mathcal{L}_W C = C[W^{-1}]$ s.th.

$$\forall D : \quad \mathrm{Fun}(C[W^{-1}], D) \xrightarrow{\simeq} \mathrm{Fun}_W(C,D)$$

When $W = \mathrm{Map}(\Delta^1, C)$, write $\mathcal{L} C := \mathcal{L}_W C$

Structure map $C \to \Pi_\infty C$ induces $\mathcal{L} C \longrightarrow \Pi_\infty C$

because

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, C) \arrow[r] \arrow[dr, dashed] & \mathrm{Map}(\Delta^1, \Pi_\infty C) \\ & \mathrm{isInv}(\Pi_\infty C) \arrow[u, "\simeq"] \end{tikzcd}$$

<u>Axiom</u>: $\mathcal{L} C \xrightarrow{\simeq} \Pi_\infty C$

Notation: "objects" $a : A$ are

$$a : X \longrightarrow A \quad \text{with } X =: \{a\} \text{ a groupoid}$$

$\triangleq$ terms of A in context X.

Construction: For $a, b : A$ have hom space

$$\begin{tikzcd} A(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, A) \arrow[d, "\text{isofibration}"] \\ \{(a,b)\} \arrow[r] & A \times A \end{tikzcd}$$

Problem: dependent types $A \vdash B$ Type

$\triangleq$ isofibrations $\mathcal{B} \longrightarrow A$

do not give rise to $A \longrightarrow \mathcal{U} = \mathcal{C}at$

because $f : A(a,a') \not\rightsquigarrow B(a) \longrightarrow B(a')$

Idea: Introduce new "kinds of fibration"

corresponding to new kinds of dependent types

$$A \vDash^{+} B \text{ Type} \quad \& \quad A \vDash B \text{ Type}$$

Semantically: cocartesian/cartesian fibrations

Recall: <u>Right deformation retract</u> consists of:

$p: B \to A$, $s: A \to B$, $H: B \times \Delta^1 \to A$

- $H_0 = id_X$, $H_1 = sp$, $ps = id_A$
-

$$\begin{tikzcd} B \times \Delta^1 \arrow[r, "H"] & B \\ A \times \Delta^1 \arrow[u, "s \times \Delta^1"] \arrow[r, "pr_A"'] \arrow[ur, phantom, "\cong"] & A \arrow[u, "s"'] \end{tikzcd} \qquad \begin{tikzcd} B \times \Delta^1 \arrow[r, "H"] \arrow[d, "p \times \Delta^1"'] \arrow[dr, phantom, "\cong"] & B \arrow[d, "p"] \\ A \times \Delta^1 \arrow[r, "pr_A"'] & A \end{tikzcd}$$

We say: s is <u>right adjoint section</u> of p

> **<u>Def</u>:** object $a : A$ $(\triangleq X \to A)$ is <u>terminal</u> if it is right adjoint section of $X \times A \to X$.

<u>Dually</u>: Left adjoint sections & initial objects

<u>Construction</u> For $f: B \to A$

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, B) \arrow[r] \arrow[d] & B \underset{f}{\vec{\times}} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, A) \arrow[d] \\ B \times B \arrow[r, "B \times p"] & B \times A \arrow[r, "p \times B"] & A \times A \end{tikzcd}$$

Informally: $\boxed{B \underset{f}{\vec{\times}} A = \{(b : B, a : A, fb \to a)\}}$

Special case: $a : A \rightsquigarrow$ <u>slice</u> $a/A := \{a\} \vec{\times} A$

Def: Let $p : B \to A$ be isofibration.

If $\mathrm{Fun}(\Delta^1, B) \longrightarrow B \overrightarrow{\times}_p A$

- has left adjoint section

  $\rightsquigarrow$ p is <u>cocartesian fibration</u>

- is equivalence

  $\rightsquigarrow$ p is <u>left fibration</u>

Cocartesian fibration:

$$
\begin{tikzcd}
B \arrow[d] & b \arrow[r, dashed, "s(\alpha)"] & f_! b \\
A & pb \arrow[r, "\alpha"] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "\beta"] \arrow[d, equal] & b' & \beta \\
b \arrow[r] & (p\beta)_! b \arrow[u, dashed] & H(\beta) \arrow[u] \\
 & & sp(\beta) \\
pb \arrow[r, "p\beta"'] & pb' &
\end{tikzcd}
$$

Example: $\mathrm{Fun}(\Delta^1, A) \xrightarrow{ev_1} A$ is cocartesian

more generally: $B \overrightarrow{\times}_f A \xrightarrow{ev_1} A$ is cocartesian

$a/A \longrightarrow A$ is left fibration

Lemma. Cocartesian fibration $p : B \to A$
is left fibration
iff for all $a : A$ fiber B_a is groupoid.

Def: Let $p : B \to A$ cocartesian fibration.
Define $\mathrm{isCocart}_p \subseteq \mathrm{Fun}(\Delta^1, B)$ essential
image of section $B \vec{\times} A \xrightarrow{s} \mathrm{Fun}(\Delta^1, B)$
"full subtype of p-cocartesian arrows"

Rem: $p : B \to A$ is left fibration
iff. <u>every</u> arrow of B is p-cocartesian.

Lemma: Left/right deformation retracts,
left/cocartesian fibrations
are all stable under base-change

semantics	syntax
$p : B \longrightarrow A$ cocartesian	$A \vDash^{\dagger} B$ Type
section $s : A \longrightarrow B$ of p	$a : A \vDash^{\dagger} s_a : B(a)$
section $\begin{array}{ccc} & & B \\ & \overset{s}{\nearrow} & \downarrow \\ A^{\simeq} & \longrightarrow & A \end{array}$	$a : A \vdash s_a : B(a)$

In general, cannot expect

$$a : A \vdash s_a : B(a) \rightsquigarrow a : A \vDash^{\dagger} s_a : B(a)$$

<u>Axiom</u> (Assembly for terminal sections)

Let $p : B \longrightarrow A$ be cocartesian fibration and assume for each $a : A$

a terminal object $s_a : B_a$

Then require right adjoint section

$$s : A \longrightarrow B \quad \text{with} \quad s(a) = s_a \quad (\forall a : A)$$

<u>Note</u>: $s : A \to B$ right adjoint section of $p : B \to A$

$\Rightarrow$ $s(a) : B_a$ is always terminal object $(\forall a : A)$

Theorem: Let

$$\begin{tikzcd} B' \arrow[rr, "f"] \arrow[dr, "p"'] & & B \arrow[dl, "p'"] \\ & A & \end{tikzcd}$$

be morphism of cocartesian fibrations (i.e. f preserves cocartesian arrows)

Then f is an equivalence

iff $f_a : B'_a \to B_a$

is an equivalence for each $a : A$

Corollary: A cocartesian fibration $p : B \to A$ is an equivalence iff for each $a : A$ the fiber B_a is trivial,

i.e. $B_a \xrightarrow{\sim} \{a\}$

Proof Apply Thm to morphism

$$\begin{tikzcd} B \arrow[rr, "p"] \arrow[dr, "p"'] & & A \arrow[dl, "="] \\ & A & \end{tikzcd}$$

<u>Proof of Thm</u>:

$$\begin{tikzcd}
Q := \{(b', b, fb' \xrightarrow{\beta} b) \mid p(\beta) \text{ invertible}\} \arrow[d] \arrow[rr] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{isInv}_A \arrow[d] \\
B' \vec{\times}_f B \arrow[r] & \mathrm{Fun}(\Delta^1, B) \arrow[r] & \mathrm{Fun}(\Delta^1, A)
\end{tikzcd}$$

Q is cocartesian :

$$\begin{tikzcd}
Q \arrow[dd] & b'_0 \arrow[rr] & & v_!(b'_0) \\
& fb'_0 \arrow[d] \arrow[r] & v_!(fb'_0) \simeq f(v_! b'_0) \arrow[d, dashed] & \\
& b_0 \arrow[r, "v"] & b_1 & \\
B \arrow[d, "p"] & b_0 \arrow[r, "v"] & b_1 & \\
A & a_0 \arrow[r, "u = p v"] & a_1 &
\end{tikzcd}$$

For each $b : B$, $a = p(b) : A$ have

$$Q_b \simeq B'_a / b := B'_a \times_{B_a} (B/b) \simeq B/_b$$

which has terminal object.

Assembly $\rightsquigarrow$ right adjoint section $Q \xrightarrow{\quad} B$, $s : B \dashrightarrow Q$

Check: $g : B \xrightarrow{s} Q \xrightarrow{ev_0} B'$ is homotopy inverse to f $\square$

Adjunctions:

Def: An adjunction consists of

$$f : B \longrightarrow A : g \quad \text{and an equivalence}$$

$$\begin{tikzcd} \{(b,a,fb\to a)\} = B \overset{\to}{\underset{f}{\times}} A \arrow[r, "\simeq"] \arrow[d] & A \overset{\leftarrow}{\underset{g}{\times}} B = \{(a,b,ga\leftarrow b)\} \arrow[d] \\ B \times A \arrow[r, "="] & A \times B \end{tikzcd}$$

of bifibrations.

$\triangleq$ functorial equivalence

$$A(fb,a) \simeq B(b,ga)$$

$\rightsquigarrow$ unit $\quad \eta : B \longrightarrow B \overset{\to}{\underset{f}{\times}} A \simeq A \overset{\leftarrow}{\underset{g}{\times}} B \longrightarrow \mathrm{Fun}(\Delta^1, B)$

$$\eta : \mathrm{id}_B \Rightarrow gf$$

counit $\quad \varepsilon : A \longrightarrow \mathrm{Fun}(\Delta^1, A)$

$$\varepsilon : fg \Rightarrow \mathrm{id}_A$$

Lemma: Adjunctions compose.

Lemma: Deformation retracts are adjunctions.

Thm (Pointwise formula for adjoints)

Given $f : B \to A$ and

for each $a : A$

terminal object $s_a : B/a$

Then have right adjoint $g : A \to B$

with $s_a = (ga, \varepsilon_a : fga \to a)$.

Proof: Left deformation retract

$\min : \Delta^1 \times \Delta^1 \to \Delta^1$ (exhibiting $0 : \Delta^1$ initial)

induces left def. retracts

$$
\begin{tikzcd}
(b, fb, fb \Rightarrow fb) & B \overset{\rightarrow}{\underset{f}{\times}} A \arrow[rr] \arrow[d, "ev_0"] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(\Delta^1, A) \arrow[d, "ev_0"] \\
b \arrow[u, mapsto] & B \arrow[u, dashed, bend left, "i"] \arrow[rr] & & A \arrow[u, dashed, bend left]
\end{tikzcd}
$$

$\rightsquigarrow$ Composite adjunction:

$$
\begin{tikzcd}
& B \overset{\rightarrow}{\underset{F}{\times}} A \arrow[dl, bend right, "ev_0"'] \arrow[dr, "ev_0"] & \\
B \arrow[ur, bend right, "i"'] \arrow[u, phantom, "\perp"] \arrow[rr, "f"'] & & A \arrow[ul, dashed, bend right, "\text{Assembly}"'] \arrow[u, phantom, "\perp"]
\end{tikzcd}
$$

cocartesian

$\square$

Dualize everything starting from

Def: $p : B \to A$ cartesian/right fibration

$:\Longleftrightarrow$ $\mathrm{Fun}(\Delta^1, B) \to B \overleftarrow{\times}_p A$

has right adjoint section / is equivalence.

+ Assembly axiom for left adjoint sections

Def: An isofibration $p : B \to A_0 \times A_1$ is called a bifibration if

- $p_0 : B \to A_0$ is cartesian fibration
- $p_1 : B \to A_1$ is cocartesian fibration
- For $\beta : \Delta^1 \to B$ have
  - β p_0-cartesian $\Longleftrightarrow$ $p_1(\beta)$ invertible
  - β p_1-cocartesian $\Longleftrightarrow$ $p_0(\beta)$ invertible

Rem: bifibration corresponds to

$A_0^{op} \times A_1 \longrightarrow \mathrm{Groupoids}$

Example: $\mathrm{Fun}(\Delta^1, A) \longrightarrow A \times A$ is bifibration

Axiom* (non-dependent directed path induction)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "\text{bifibration}"] \\ \mathrm{Fun}(\Delta^1, A) \arrow[r] \arrow[ur, dotted, "\exists!"] & A \times A \end{tikzcd}$$

(*we can probably prove this and other versions of directed path induction from other axioms)

Lemma: $B \longrightarrow A_0 \times A_1$ bifibration
$(a_0, a_1) : A_0 \times A_1 \implies$ fiber B_{a_0, a_1} is groupoid.

Corollary: For every type A, the mapping spaces $A(a, a')$ are groupoids.

Theorem: Equivalences of bifibrations can be detected pointwise.

Def: A functor $f : C \to D$ is called fully faithful if square

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, C) & \longrightarrow & \mathrm{Fun}(\Delta^1, D) \\ \downarrow & & \downarrow \\ C \times C & \longrightarrow & D \times D \end{array}$$

is (homotopy) pullback

Equivalently: $C(c,c') \to D(fc, fc')$ equivalence of groupoids for all $c, c' : C$

Theorem A functor $f : C \to D$ is an equivalence if and only if it is

- fully faithful and
- induces a surjection
  $\| C^{\simeq} \|_0 \longrightarrow \| D^{\simeq} \|_0$

RELATIVE MAPPING TYPES

Consider isofibrations
$$\begin{tikzcd} B \arrow[dr, "p"] & & C \arrow[dl, "q"'] \\ & A & \end{tikzcd}$$

If p admits dependent product

$p_*: E(B) \to E(A)$ (right adjoint to pullback)

define $\boxed{\underline{\mathrm{Fun}}_A(B,C) := p_* p^*(C \to A)}$

Note $\mathrm{Map}(T, \underline{\mathrm{Fun}}_A(B,C))$

$$\simeq \left\{ \begin{tikzcd} T \times_A B \arrow[rr] \arrow[dr] \arrow[dd, dashed] \arrow[drr, phantom, "\lrcorner"] & & B \arrow[dr] & \\ & T \arrow[rr] & & A \\ T \times_A C \arrow[ur] \arrow[rr] \arrow[urr, phantom, "\urcorner"] & & C \arrow[ur] & \end{tikzcd} \right\}$$

(not to be confused with
$$\mathrm{Fun}_A(B,D) = \left\{ \begin{tikzcd} B \arrow[rr, dashed] \arrow[dr] & & D \arrow[dl] \\ & A & \end{tikzcd} \right\} = \text{global sections of } \underline{\mathrm{Fun}}_A(B,C))$$

Axiom: Dependent products exist along every cartesian and cocartesian fibration.

For cocartesian fibrations

$p : B \to A$ and $q : C \to A$

$\rightsquigarrow \quad \underline{\mathrm{Fun}}_A(B, C) \longrightarrow A$

Axiom*: Have (non-full) subtype

$\underline{\mathrm{Cocart}}_A(B, C) \subset \underline{\mathrm{Fun}}_A(B, C)$ classifying

those

$$\begin{tikzcd} T \times_A B \arrow[rr] \arrow[dr, "p_T"'] & & T \times_A C \arrow[dl, "q_T"] \\ & T & \end{tikzcd}$$

which preserve cocartesian arrows.

(* Actually, we introduce a more general (non-full) subtype former axiom of which this is a special case)

DIRECTED UNIVALENCE

Recall: We have a hierarchy of subtribes

$$E_0 \subseteq E_1 \subseteq \dots \subseteq E_\alpha \subseteq \dots \subseteq E = E_\kappa$$

Axiom: For each $\alpha < \kappa$ we have

a type $U_\alpha : E_{\alpha+1}$ and a

univalent cocartesian fibration

$p_\alpha : U_\alpha^\bullet \longrightarrow U_\alpha$ satisfying:

- for each $A \longrightarrow U_\alpha$ with $A : E_\alpha$

  have $A \times_{U_\alpha} U_\alpha^\bullet$ again in E_α

- for every cocartesian fibration $p : B \to A$

  with fibers in E_α there is a unique

  (strict) pullback square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & U_\alpha^\bullet \arrow[d] \\ A \arrow[r, "\ulcorner p \urcorner"] & U_\alpha \end{tikzcd}$$

Def: A cocartesian fibration $p: B \to A$ is called <u>univalent</u> if the canonical map

$$(*) \quad \mathrm{Fun}(\Delta^1, A) \xrightarrow{\simeq} \underline{\mathrm{cocart}}_{A\times A}(B\times A, A\times B)$$

is an equivalence

Explicit meaning for universe $p: \mathcal{U}^\circ \to \mathcal{U}$:

Given $\mathcal{U}$-small types C, D have

$$\begin{tikzcd}
C \times D \arrow[r] \arrow[d] & \mathcal{U}^\circ \times \mathcal{U}^\circ \arrow[d] \\
\mathbb{1} \times \mathbb{1} \arrow[r, "\ulcorner C \urcorner \times \ulcorner D \urcorner"] & \mathcal{U} \times \mathcal{U}
\end{tikzcd}$$

On fibers over $\ulcorner C \urcorner \times \ulcorner D \urcorner$ $(*)$ becomes:

$$\boxed{\begin{tikzcd}
\mathcal{U}(\ulcorner C \urcorner, \ulcorner D \urcorner) \arrow[d, "\simeq"] \\
\mathrm{Map}(C, D)
\end{tikzcd}} = \left\{ \begin{tikzcd}
C \arrow[r] \arrow[dr] \arrow[dd, dotted] & \mathcal{U}^\circ \times \mathcal{U} \arrow[dr] & \\
& \mathbb{1} \arrow[r, "\ulcorner C \urcorner \times \ulcorner D \urcorner"] & \mathcal{U} \times \mathcal{U} \\
D \arrow[ur] \arrow[r] & \mathcal{U} \times \mathcal{U}^\circ \arrow[ur] &
\end{tikzcd} \right\}$$

Construction: Start with a cocartesian fibration

$$p : A \to B$$

Have a tautological map

$$A \longrightarrow \underline{\mathrm{cocart}}_{A\times A}(B\times A, A\times B)$$

given by

$$\begin{tikzcd}
B \arrow[rr] \arrow[dr] \arrow[dd, dotted, "id"'] \arrow[drr, phantom, "\lrcorner", very near start] & & B\times A \arrow[dr] & \\
& A \arrow[rr] & & A\times A \\
B \arrow[ur] \arrow[rr] \arrow[urr, phantom, "\urcorner", very near start] & & A\times B \arrow[ur] &
\end{tikzcd}$$

$\rightsquigarrow$ by path induction:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \underline{\mathrm{cocart}}_{A\times A}(B\times A, A\times B) \arrow[d] \\
\mathrm{Fun}(\Delta^1, A) \arrow[r] \arrow[ur, dashed, "\exists !"] & A\times A
\end{tikzcd}$$

check: this is bifibration