Univalent Directed Type Theory

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Tribes

Let $\mathbf{E}$ be a category together with a subcategory of fibrations $\mathbf{F} \subseteq \mathbf{E}$.

- A morphism of called *anodyne* if it has the LLP with respect to the class of fibrations.

The category $\mathbf{E}$ is called a *tribe* if the following conditions are satisfied:

- $\mathbf{E}$ has a terminal object 1 and all objects are fibrant
- Pullbacks along fibrations exist and fibrations are stable under pullback
- Every morphism admits a factorization into an anodyne morphism followed by a fibration
- Anodyne morphisms are stable under pullback along fibrations

Tribe Axioms

We call the fibrations in $\mathbf{E}$ isofibrations. We further assume that $\mathbf{E}$ admits a hierarchy of universes: an infinite well-ordered set I and for each $U \in I$ a subtribe $\mathbf{E}_U \subseteq E$ such that

- $U_1 < U_2 \Rightarrow \mathbf{E}_{U_1} \subseteq \mathbf{E}_{U_2}$
- $\bigcup_{U \in I} \mathbf{E}_U = \mathbf{E}$
- for any finite family of objects $A_1, \ldots, A_n$ of $\mathbf{E}$ with each A_i belonging to $\mathbf{E}_{U_i}$ for some $U_i \in I$, any object of $\mathbf{E}$ obtained from them by applying the deduction rules of dependent type theory (when they make sense in $\mathbf{E}$) belongs to $\max\{U_i \mid 1 \leq i \leq n\}$.

Terminology:

An isofibration $p: X \to Y$ is called $\mathcal{U}$-small if for any pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

w/ B $\mathcal{U}$-small, the obj. A is $\mathcal{U}$-small

will not introduce all operations of MLTT / HoTT

in particular, not all dep prod exist.

Tribe Axioms

Universality of sums

For any finite set J and any family $(A_j)_{j \in J}$ of objects of **E** the sum

$$A := \coprod_{j \in J} A_j$$

exists and the pullback functor

$$\mathbf{E}(A) \to \prod_{j \in J} \mathbf{E}(A_j)$$

is an equivalence of tribes.

Tribe Axioms

Internal hom

The category **E** admits an internal hom, i.e. for any object A of **E** the functor $A \times (-)$ admits a right adjoint $\mathrm{Fun}(A, -)$.

LET
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$
BE A COMMUTATIVE SQUARE.

THEN IT'S CALLED A HOMOTOPY PULLBACK IF FOR ANY FACTORIZATION

$$X \xrightarrow{\sim} X' \longrightarrow Y$$

THE MAP $A \to B \times_Y X'$ IS AN EQUIVALENCE

A MAP $A \to B$ IS A HOMOTOPY MONOMORPHISM IF

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$
IS A HOMOTOPY PULLBACK

MORE GENERALLY: $A \to B$ IS n-TRUNCATED IF FOR ANY FIBRANT REPLACEMENT $A' \to B$

THE MAP

$$A' \longrightarrow A' \times_B A'$$

IS $(n-1)$-TRUNCATED.

AN OBJ. A IS n-TRUNCATED IF $A \to 1$ IS n-TRUNCATED

Interval axioms

Existence of an interval

There is an object $\mathbb{I}$ equipped with two maps

$$\partial_i : 1 \to \mathbb{I}, \quad i = 0, 1$$

such that

1. the diagonal $\mathbb{I} \to \mathbb{I} \times \mathbb{I}$ is both an isofibration and a homotopy monomorphism
2. the map $i = (\partial_1, \partial_0) : 1 \sqcup 1 \to \mathbb{I}$ is an isofibration and a homotopy monormorphism
3. dependent products along i exist.
4. the interval $\mathbb{I}$ is 0-truncated.

A RIGHT DEFORMATION RETRACT is a map

$$s : A \longrightarrow B$$

TOGETHER w/ MAPS $p : B \longrightarrow A$ AND $H : I \times B \longrightarrow B$

S.T.

- $ps = 1_A$

- $$\begin{tikzcd} B \arrow[dr, "d_0"] \arrow[drr, bend left, "id"] & & \\ & I \times B \arrow[r] & B \\ B \arrow[ur, "d_1"'] \arrow[urr, bend right, "sp"'] & & \end{tikzcd}$$ COMMUTES

  i.e. $H : 1_B \Rightarrow sp$

- $$\begin{tikzcd} I \times A \arrow[r, "pr"] \arrow[d, "1_I \times s"'] & A \arrow[d, "s"] \\ I \times B \arrow[r, "H"'] & B \end{tikzcd}$$ COMMUTES

DUALLY DEFINE LEFT DEFORMATION RETRACT

DEF: $w : 1 \longrightarrow A$ IS FINAL (INITIAL) IF IT IS A RIGHT (LEFT) DEFORMATION RETRACT

IDEA: WANT ADJ. BUT CAN ONLY DEFINE FULLY FAITHFUL RIGHT/LEFT ADJOINTS.

Interval axioms

Connections

The map

$$\partial_0 : 1 \to \mathbb{I}$$

is a *right deformation retract* and the map

$$\partial_1 : 1 \to \mathbb{I}$$

is a *left deformation retract*.

THIS IMPLIES :

- $\mathbb{I}$ HAS FINAL AND INITIAL OBJECTS
- $\mathbb{I}$ ADMITS CONNECTIONS

$$\max : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{I}$$
$$\min : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{I}$$

THE JOIN

RECALL WE HAVE $i : 1+1 \longrightarrow \mathbb{I}$ AND WE REQUIRED $i^* : \mathcal{E}/_{\mathbb{I}} \to \mathcal{E}/_{1+1} \cong \mathcal{E} \times \mathcal{E}$

TO HAVE A RIGHT ADJ. $i_* : \mathcal{E} \times \mathcal{E} \to \mathcal{E}/_{\mathbb{I}}$ WHICH IS A MORPHISM OF TRIBES

THUS FOR A, B OBJ. OF $\mathcal{E}$ WE GET AN (CO)FIBRATION

$$\begin{array}{c} i_*(A,B) \\ \downarrow \\ \mathbb{I} \end{array}$$

<u>DEFINE</u> $A * B = i_*(A, B)$

CALLED THE JOIN OF A AND B

$*$ is characterized by

- $(A,B) \longmapsto A*B$ is a functor
- $1*1 \cong \mathbb{I}$
- $\begin{array}{c} A*B \\ \downarrow ! \\ \mathbb{I} \end{array}$ is an isofibration
- Have canonical pullbacks

$$\begin{tikzcd} A \arrow[r] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & A*B \arrow[d] & B \arrow[l] \arrow[d, "!"] \arrow[dl, phantom, "\llcorner", very near start] \\ 1 \arrow[r, "\delta_1"'] & \mathbb{I} & 1 \arrow[l, "\delta_0"] \end{tikzcd}$$

- A map $X \to A*B$ is determined by

$$\begin{array}{c} X \\ \downarrow ! \\ \mathbb{I} \end{array} \quad \text{w/} \quad \begin{array}{c} X_0 \xrightarrow{\varphi_0} A \\ \\ X_1 \xrightarrow{\varphi_1} A \end{array}$$

THIS CORRESPONDS TO

$$
\begin{tikzcd}
X_0 \sqcup X_1 \arrow[rr, "\varphi_0 \sqcup \varphi_1"] \arrow[d] \arrow[dr] & & A \sqcup B \arrow[dl] \arrow[d] \\
X \arrow[rr, dashed] \arrow[dr] & 1 \sqcup 1 \arrow[d] & A + B \arrow[dl] \\
& I &
\end{tikzcd}
$$

Interval axioms

Simplicial structure

There is an isomorphism

$$a : 1 * \mathbb{I} \xrightarrow{\cong} \mathbb{I} * 1$$

such that the square below commutes

$$\begin{tikzcd}
1 \sqcup (1 \sqcup 1) \arrow[r, "\cong"] \arrow[d] & (1 \sqcup 1) \sqcup 1 \arrow[d] \\
1 * (1 \sqcup 1) \arrow[d, "id * i"'] & (1 \sqcup 1) * 1 \arrow[d, "i * id"] \\
1 * \mathbb{I} \arrow[r, "a"] & \mathbb{I} * 1
\end{tikzcd}$$

Moreover, these isomorphisms satisfy the Pentagon axiom.

EXPLANATION :

GIVEN OBJ. A, B, C. HAVE A COMMUTATIVE DIAGRAM

$$
\begin{tikzcd}
A \sqcup (B \sqcup C) \arrow[r, "\cong"] \arrow[d] & (A \sqcup B) \sqcup C \arrow[d] \\
A \sqcup (B * C) \arrow[d] & (A * B) \sqcup C \arrow[d] \\
A * (B * C) \arrow[r, dashed, "a_{A,B,C}", "\cong"'] \arrow[d] & (A * B) * C \arrow[d] \\
A * I \arrow[r] & I * 1
\end{tikzcd}
$$

THE INDUCED MAP $a_{A,B,C}$ IS AN ISOMORPHISM.

WE OBTAIN THE FOLLOWING DIAGRAM

$$
\begin{tikzcd}
 & (1 * 1) * (1 * 1) = \mathbb{I} * \mathbb{I} \arrow[dl, "a_{\mathbb{I},1,1}"'] & \\
(\mathbb{I} * 1) * 1 & & 1 * (1 * \mathbb{I}) \arrow[ul, "a_{1,1,\mathbb{I}}"'] \arrow[d, "1 * \ell"] \\
(1 * \mathbb{I}) * 1 \arrow[u, "u * 1"] & & 1 * (\mathbb{I} * 1) \arrow[ll, "a_{1,\mathbb{I},1}"]
\end{tikzcd}
$$

AND THE SIMPL. STRUCTURE AXIOMS ASSERTS THAT THIS COMMUTES

NOTE: THIS IMPLIES THAT $a_{A,B,C}$ SATISFY MACLANES PENTAGON,

I.E. THE JOIN DEFINES A MONOIDAL PRODUCT ON $\mathcal{E}$.

MOREOVER: 1 IS A MONOID WRTO THE JOIN.

SIMPLICES IN $\mathcal{E}$: $\Delta^n_{\underline{I}}$

- $\Delta^{-1}_{\underline{I}} = \emptyset$
- $\Delta^0_{\underline{I}} = 1$
- $\Delta^{n+1}_{\underline{I}} = \Delta^0_{\underline{I}} * \Delta^n_{\underline{I}}$

HAVE. $\emptyset \longrightarrow \Delta^0_{\underline{I}}$

. $\Delta^1_{\underline{I}} = \Delta^0_{\underline{I}} * \Delta^0_{\underline{I}} \longrightarrow \Delta^0_{\underline{I}}$

TURNING $\Delta^0_{\underline{I}}$ INTO A MONOID

IN PARTICULAR $\underline{I} \cong \Delta^1_{\underline{I}}$

WE OBTAIN

$$\begin{aligned} \Delta &\longrightarrow \mathcal{E} \\ [n] &\longmapsto \Delta^n_I \end{aligned}$$

AND IN PARTICULAR SIMPLICIAL MAPS IN $\mathcal{E}$

Category axioms

Segal map

For each isofibration $p : X \to Y$ and $0 \leq i \leq n$ the maps induced by ∂_i^n

$$\mathrm{Fun}(\Delta^n, X) \to \mathrm{Fun}(\Delta^n, Y) \times_{\mathrm{Fun}(\Delta^{n-1}, Y)} \mathrm{Fun}(\Delta^{n-1}, X)$$

is an isofibration. Furthermore, the Segal map

$$\mathrm{Fun}(\Delta^n, X) \to \mathrm{Fun}(\Delta^1, X) \times_X \cdots \times_X \mathrm{Fun}(\Delta^1, X)$$

is a trivial fibration.

WE OBTAIN THE USUAL COMPOSITION OPERATION BY CHOOSING SECTIONS

$$\begin{tikzcd} \mathrm{Fun}(\Delta^2, X) \arrow[r, "\delta_1^{2*}"] \arrow[d, "\sim"'] & \mathrm{Fun}(\Delta^1, X) \\ \mathrm{Fun}(\Delta^1, X) \times_X \mathrm{Fun}(\Delta^1, X) \arrow[u, bend right, dashed] & \end{tikzcd}$$

THIS ALLOWS US TO DEFINE HOMOTOPY CATEGORIES:

LET

$$\begin{tikzcd} X(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta^1} \arrow[d] \\ \Delta^0 \arrow[r] & X \times X \end{tikzcd}$$

DEFINE $h_0 X$

- ob $\Delta^0 \to X$
- MOR HOMOTOPY CLASSES $\Delta^0 \to X(a,b)$

<u>DEF</u>: $[A, B] = h_0 \mathrm{Fun}(A, B)$

GET A 2-CAT FROM $\mathcal{E}$

DEF: X is a <u>GRPD</u> if $X \to \mathrm{Fun}(\Delta^1, X)$ induced by $\Delta^1 \to \Delta^0$ is an equivalence.

Denote $\mathrm{Gr}(\mathcal{E}) \subseteq \mathcal{E}$ the full subcat. on groupoids.

Category axioms

Maximal groupoid

The inclusion $gr(\mathbf{E}) \hookrightarrow \mathbf{E}$ has a right adjoint

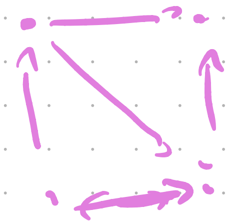
$$(-)^{\simeq} : \mathbf{E} \to gr(\mathbf{E})$$

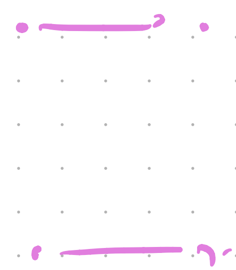
which is a morphism of tribes. Furthermore, we demand that we have an equivalence

$$\partial\Delta^1 \to (\Delta^1)^{\simeq}$$

NOW CONSIDER

$$\begin{tikzcd}
K \arrow[dr, dashed] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& \mathrm{Isinn}(X) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Fun}(\Delta^2, X) \times_{\mathrm{Fun}(\Delta^1, X)} \mathrm{Fun}(\Delta^2, X) \arrow[d] \\
& X \times X \arrow[rr] & & \mathrm{Fun}(\Delta^1, X) \times \mathrm{Fun}(\Delta^1, X)
\end{tikzcd}$$





Category axioms

Uniqueness of inverses

For any object C the canonical functor $C \to \mathsf{IsInv}(C)$ is an equivalence.

THIS IMPLIES THAT $\mathrm{IsInv}(C)$ IS (UP TO EQUIVALENCE) A PATH OBJECT.

SO WE HAVE

$$\begin{tikzcd}
 & C \arrow[dl, "\sim"'] \arrow[dr] & \\
\mathrm{IsInv}(C) \arrow[rr] & & C^{\Delta^1}
\end{tikzcd}$$

HENCE C IS A GROUPOID $\iff$ $\mathrm{IsInv}(C) \longrightarrow C^{\Delta^1}$ IS AN EQUIVALENCE, I.E $\iff$ MAPS HAVE INVERSES.

Category axioms

Full subtype former

Given an object C and a homotopy monomorphism of groupoids

$$j : X \hookrightarrow C^{\simeq}$$

then there exists a functor

$$C_X \hookrightarrow C$$

such that when restricted to maximal groupoids we obtain the original map

$$j : X = (C_X)^{\simeq} \to C^{\simeq}$$

and such that the map

$$\mathrm{Fun}(A, C_X) \to \mathrm{Fun}(A, C) \times_{\mathrm{Fun}(A^{\simeq}, C^{\simeq})} \mathrm{Fun}(A^{\simeq}, X)$$

is an equivalence.