Tribes

Let $E$ be a category together with a subcategory of fibrations $F \subseteq E$.

- A morphism of called *anodyne* if it has the LLP with respect to the class of fibrations.

The category $E$ is called a *tribe* if the following conditions are satisfied:

- $E$ has a terminal object $1$ and all objects are fibrant
- Pullbacks along fibrations exist and fibrations are stable under pullback
- Every morphism admits a factorization into an anodyne morphism followed by a fibration
- Anodyne morphisms are stable under pullback along fibrations
We call the fibrations in $E$ isofibrations. We further assume that $E$ admits a hierarchy of universes: an infinite well-ordered set $I$ and for each $U \in I$ a subtribe $E_U \subseteq E$ such that

- $U_1 < U_2 \Rightarrow E_{U_1} \subseteq E_{U_2}$
- $\bigcup_{U \in I} E_U = E$
- for any finite family of objects $A_1, \ldots, A_n$ of $E$ with each $A_i$ belonging to $E_{U_i}$ for some $U_i \in I$, any object of $E$ obtained from them by applying the deduction rules of dependent type theory (when they make sense in $E$) belongs to $\max\{U_i \mid 1 \leq i \leq n\}$. 
**Terminology:**

An isofibration $p: X \to Y$ is called $U$-small if for any pullback diagram

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \downarrow p \\
B & \rightarrow & Y
\end{array}
\]

with $B$ $U$-small, the obj. $A$ is $U$-small.

We will not introduce all operations of $U$-TT/TT in particular, not all deep prod exist.
Tribe Axioms

Universality of sums
For any finite set $J$ and any family $(A_j)_{j \in J}$ of objects of $\mathbf{E}$ the sum

$$A := \coprod_{j \in J} A_j$$

exists and the pullback functor

$$\mathbf{E}(A) \to \coprod_{j \in J} \mathbf{E}(A_j)$$

is an equivalence of tribes.
Internal hom
The category $\mathbf{E}$ admits an internal hom, i.e. for any object $A$ of $\mathbf{E}$ the functor $A \times (\_)$ admits a right adjoint $\text{Fun}(A, \_)$.
Let $\begin{bmatrix} A & \to & X \end{bmatrix}$ be a commutative square.

Then it's called a homotopy pullback if for any factorization $\begin{bmatrix} X & \to & Y \end{bmatrix}$, the map $A \to B \times_X Y$ is an equivalence.

A map $A \to B$ is a homotopy monomorphism if $A = A$ is a homotopy pullback.
More generally: $A \to B$ is $n$-truncated if for any fibrant replacement $A' \to B$ the map

$$A' \to A' \times_B A$$

is $(n-1)$-truncated.

An obj. $A$ is $n$-truncated if $A \to 1$ is $n$-truncated.
Interval axioms

Existence of an interval
There is an object \( \mathbb{I} \) equipped with two maps

\[
\partial_i : 1 \rightarrow \mathbb{I}, \quad i = 0, 1
\]

such that

1. the diagonal \( \mathbb{I} \rightarrow \mathbb{I} \times \mathbb{I} \) is both an isofibration and a monomorphism
2. the map \( i = (\partial_1, \partial_0) : 1 \sqcup 1 \rightarrow \mathbb{I} \) is an isofibration and a monomorphism
3. dependent products along \( i \) exist.
4. the interval \( \mathbb{I} \) is 0-truncated.
A right deformation retract is a map

\[ \delta : A \rightarrow \mathbb{S} \]

together with maps \( p : \mathbb{S} \rightarrow A \) and \( H : \mathbb{I} \times \mathbb{S} \rightarrow \mathbb{S} \)

subject to

- \( p \circ \delta = 1_A \)
- \( \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I} \)
- \( H : \mathbb{I} \times \mathbb{S} \rightarrow \mathbb{S} \)

i.e., \( H : 1_{\mathbb{S}} \rightarrow sp \)

\[ \mathbb{I} \times \mathbb{S} \rightarrow \mathbb{S} \]

commutes

\[ \mathbb{I} \times \mathbb{A} \rightarrow \mathbb{A} \]

commutes

\[ \mathbb{I} \times \mathbb{B} \rightarrow \mathbb{B} \]
Define left deformation retract.

**Def:** $w: A 	o A$ is final (initial) if it is a right (left) deformation retract.

**Idea:** Want adj. but can only define fully faithful right/ left adjoints.
Interval axioms

Connections

The map

\[ \partial_0 : 1 \rightarrow \mathbb{I} \]

is a right deformation retract and the map

\[ \partial_1 : 1 \rightarrow \mathbb{I} \]

is a left deformation retract.
This implies:

1. It has final and initial objects
2. It admits connections

\[ \text{max: } I \times I \to I \]
\[ \text{min: } I \to I \]

The join

Recall we have \( i : 1 \to I \) and we required \( i^* : E/I \to E \)

To have a right adj. \( i_* : E \to E/I \) which is a morphism of fibrations

Thus for \( A,B \) obj. of \( E \) we get an fibration

\[ i_*(A,B) \]

Define \( A*B = i_*(A,B) \)

Called the join of \( A \) and \( B \)
It is characterized by

- $(A, B) \mapsto A \times B$ is a functor
- $1 \times A = A$
- $A \times B$ is an internalization
- \[
\begin{array}{c}
A \\
\downarrow
\end{array}
\Rightarrow
\begin{array}{c}
X_0 \\
\downarrow
\end{array}
\xrightarrow{\psi_0}
\begin{array}{c}
A
\end{array}
\]

- A map $X \mapsto A \times B$ is determined by

- have canonical pullbacks

\[
\begin{array}{ccc}
A & \rightarrow & A \times B \\
\downarrow & & \downarrow \\
X_0 & \rightarrow & A
\end{array}
\]

\[
\begin{array}{ccc}
\delta_0 & & \delta_0 \\
\downarrow & & \downarrow \\
X_1 & \rightarrow & A
\end{array}
\]
This corresponds to

\[ X \cup Y \cup Z \]

\[ \subseteq \]

\[ A \cup B \]

\[ \subseteq \]

\[ A \cup A \]

\[ \cup \]

\[ X \]

\[ \subseteq \]

\[ A \cup B \]

\[ \subseteq \]

\[ A \cup B \]
Interval axioms

Simplicial structure
There is an isomorphism

$$a : 1 \ast \mathbb{I} \xrightarrow{\cong} \mathbb{I} \ast 1$$

such that the square below commutes

\[
\begin{array}{ccc}
1 \sqcup (1 \sqcup 1) & \xrightarrow{\cong} & (1 \sqcup 1) \sqcup 1 \\
\downarrow & & \downarrow \\
1 \ast (1 \sqcup 1) & & (1 \sqcup 1) \ast 1 \\
\downarrow id \ast i & & \downarrow i \ast id \\
1 \ast \mathbb{I} & \xrightarrow{a} & \mathbb{I} \ast 1
\end{array}
\]

Moreover, these isomorphisms satisfy the Pentagon axiom.
EXPLANATION:

Given obj. A, B, C, have a commutative diagram

\[
\begin{array}{c}
\text{Au}(B\cup C) \quad \xrightarrow{=} \quad (\text{AuB})\cup C \\
\downarrow \\
\text{Au}(B\times C) \\
\downarrow \\
\text{Au}(B \times C) \\
\downarrow \\
A \times I \\
\end{array}
\]

The induced map \( q_{A,B,C} \) is an isomorphism.
We obtain the following diagram.

\[
(1 \times 1) \times (1 + 1)
\]

\[
\begin{array}{ccc}
\alpha_1 & \times & \beta_1 \\
\alpha_1 & \downarrow & \beta_1 \\
\alpha_1 \times (1 + 1) & \times & (1 + 1) \\
\end{array}
\]

And the simpl. structure axioms asserts that this commutes.
**Note:** This implies that \( a.b.c \) satisfy MacLane's Pentagon,

i.e. the \( \mathcal{F} \) defines a monoidal product on \( \mathcal{E} \).

Moreover: \( \mathcal{I} \) is a monoid under the \( \mathcal{F} \).

**Simplices in \( \mathcal{E} \):** \( \Delta^n_j \)

- \( \Delta^{-1}_0 = 0 \)
- \( \Delta^0_0 = 1 \)
- \( \Delta^{n+1}_j = \Delta^n_0 \times \Delta^j_1 \)

These have \( \phi \rightarrow \Delta^0_j \)

- \( \Delta'_{ij} = \Delta^0_i \times \Delta^j_1 \)

Turning \( \Delta^0_i \) into a monoid

In particular \( \mathcal{I} \) is \( \Delta^0_1 \)
we obtain

\[ \Delta \to \ominus \]

\[ [n] \to \Delta^n_p \]

and in particular simplicial maps in \( \ominus \)
Category axioms

Segal map
For each isofibration $p : X \rightarrow Y$ and $0 \leq i \leq n$ the maps induced by $\partial_i^n$

\[ \text{Fun}(\Delta^n, X) \rightarrow \text{Fun}(\Delta^n, Y) \times_{\text{Fun}(\Delta^{n-1}, Y)} \text{Fun}(\Delta^{n-1}, X) \]

is an isofibration. Furthermore, the Segal map

\[ \text{Fun}(\Delta^n, X) \rightarrow \text{Fun}(\Delta^1, X) \times_X \cdots \times_X \text{Fun}(\Delta^1, X) \]

is a trivial fibration.
We obtain the usual composition operations by choosing recollements:

\[ \text{Fun}(A^2, X) \xrightarrow{\delta^*_A} \text{Fun}(A^3, X) \]

This allows us to define homotopy categories:

let \( A \in \text{K} \) be a homology class \( A \in \text{K} \)

**Def:** \( [A, B] \in \text{Fun}(A, B) \)

Get \( \mathcal{Q} \)-Cat from \( \mathcal{E} \)
DEF: $x$ is a GDPR if $x \in \text{Fun}(\Delta^1 \times \Delta^0)$ induced by $\Delta^1 \rightarrow \Delta^0$ is an equivalence.

Denote $Gr(\Sigma)$ as the full subcat. on Groupdos.
Maximal groupoid

The inclusion $\text{gr}(E) \hookrightarrow E$ has a right adjoint

$(-)\cong : E \to \text{gr}(E)$

which is a morphism of tribes. Furthermore, we demand that we have an equivalence

$\partial \Delta^1 \to (\Delta^1)\cong$
Consider a map $\text{IsInw}(X) \rightarrow \text{Fun}(\Delta^2, X) \times \text{Fun}(\Delta^3, X)$.

Define $\text{Fun}(1,1) = \text{Fun}(\Delta^0, X)$.
Uniqueness of inverses

For any object $C$ the canonical functor $C \to \text{IsInv}(C)$ is an equivalence.
This implies that \( \text{IsLaw}(c) \) is (up to equivalence) a path object.

Hence \( c \) is a groupoid \( \Rightarrow (\text{IsLaw}(c) \rightarrow C^{op1}) \) is an equivalence, i.e. \( C \to \) maps have inverses.
Category axioms

Full subtype former

Given an object $C$ and a homotopy monomorphism of groupoids

$$j : X \hookrightarrow C^\sim$$

then there exists a functor

$$C_X \hookrightarrow C$$

such that when restricted to maximal groupoids we obtain the original map

$$j : X = (C_X)^\sim \rightarrow C^\sim$$

and such that the map

$$\text{Fun}(A, C_X) \rightarrow \text{Fun}(A, C) \times_{\text{Fun}(A^\sim, C^\sim)} \text{Fun}(A^\sim, X)$$

is an equivalence.