Directed Type Theory

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Axioms for a synthetic theory of categories

with univalence, in order to work up to equivalence

$\rightsquigarrow$ synthetic theory of $(\infty,1)$-categories

- through a fragment of dependent type theory
- with the goal of implementing in proof assistants

Desiderata: • a type theory where „type" means „category" (as opposed to „set" or „homotopy type")

• should be expressive enough to do ordinary $(\infty,1)$-category theory — all of it!

S = free completion of the point by small colimits
$S_\bullet$ = pointed version of S = $S_{*/}$

X groupoid $\Leftrightarrow$
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & S_\bullet \arrow[d] \\ * \arrow[r] & S \end{tikzcd}$$
functorially in X
(directed univalence of $S_\bullet \to S$)

A small $\quad F: A^{op} \to S \quad a : A$, $h_a = \mathrm{Hom}_A(-,a) : A^{op} \to S$
$\hat{A} = \mathrm{Fun}(A^{op}, S)$

$\mathrm{Hom}_{\hat{A}}(h_a, F) \xrightarrow{\sim} F(a)$ in S functorially in both a and F

$\rightsquigarrow$ theory of (pointwise) Kan extensions in the context of functors with values in (co)complete types.

• straightening / unstraightening. Universal cocartesian fibration with small fibers

$$\mathrm{Cat}_\bullet \to \mathrm{Cat}$$

$X \xrightarrow{p} A$ cocartesian fibration with small fibers $\Leftrightarrow$
$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Cat}_\bullet \arrow[d] \\ A \arrow[r] & \mathrm{Cat} \end{tikzcd}$$
functorially
(directed univalence)

- characterization of equivalences as fully faithful and essentially surjective functors
- the fragment of the theory corresponding to groupoids (or homotopy types) should be a semantic interpretation of univalent HoTT.
- ordinary theory of $(\infty,1)$-category theory should be a semantic interpretation (under the form of quasi-categories or of complete Segal spaces)

  In fact, the theory of $(\infty,1)$-categories internally to any ∞-topos should be a semantic interpretation. More generally, a whole class of cosmoi in the sense of Riehl-Verity.

A directed analogue of the HoTT Book should include:

HTT $\left\{ \begin{array}{l} \text{• the theory of presentable categories / types} \\ \text{• the theory of topoi} \end{array} \right.$

HA $\left\{ \begin{array}{l} \text{• the theory of stable categories, t-structures, weight structures.} \\ \text{• the theory of operads} \\ \text{• Beck's theorem} \end{array} \right.$

• the theory of model categories and Quillen functors

  $\hookrightarrow$ derived functors as (absolute) Kan extensions

  $\hookrightarrow$ derived adjunctions

• K-theory of „Waldhausen types" — additivity theorem, localizing invariants,

  functoriality with respect to polynomial functors ($\rightsquigarrow$ λ-operations)...

  Theorem of the heart: D stable with bounded t-structure $\Rightarrow$ $K(D^\heartsuit) \xrightarrow{\sim} K(D)$

• (∞, n)-category theory ...

The theory will swallow itself: we can use it to produce semantic interpretations of any type theory.

There is a syntactic category $\mathcal{E}$.

Given any type C there is the full subcategory $RFib(C)$ of $\mathcal{E}/_C$ with objects those conservative cartesian fibrations

$$\begin{array}{c} X \\ \downarrow \\ C \end{array}$$

$RFib(C)$ is the syntactic category corresponding to the logic of the type of functors

$$C^{op} \longrightarrow S$$

Let $Rep(C)$ be the full subcategory of $RFib(C)$ spanned by those $\begin{array}{c} X \\ \downarrow \\ C \end{array}$ such that X has a terminal object.

Then $Rep(C)$ is the syntactic category of the internal logic of C (corresponding to representable functors $C^{op} \longrightarrow S$)

Remark: subcategories of the syntactic category $\mathcal{E}$ correspond to fragments of our directed type theory — we do not really need to use ordinary category theory to express/use them.

In particular, we can speak of a (locally) cartesian closed type C and express in $Rep(C)$ the corresponding logic, consider Awodey's natural models of type theory (study structures of a representable fibration in $RFib(C)$).

This means we can study (dependent) type theory intrinsically, without using ordinary category theory.

Remark: the assignment $C \longmapsto Rep(C)$ will rectify pullbacks and will lead to strict universes (as opposed to Tarski ones) but dependent products will only be well defined up to homotopy.

However, will present the axioms using category theory.

We will work with a tribe in the sense of Joyal $\mathcal{E}$ — the syntactic category associated to our logic.

Maps in $\mathcal{E}$ are called *functors*.

$\mathcal{E}$ is equipped with a class of fibrations $\twoheadrightarrow$ often called *isofibrations.*

Isofibrations are closed under composition and any isomorphism is an isofibration.

For any isofibration $p: X \twoheadrightarrow Y$ and any map $g: Y' \longrightarrow Y$ there is a pullback square

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

with p' an isofibration.

Any functor $f: X \longrightarrow Y$ can be factored into an anodyne map $X \overset{\sim}{\rightarrowtail} Z$

followed by an isofibration $Z \twoheadrightarrow Y$, where anodyne maps are those with the left lifting property with respect to all isofibrations.

Pulling back anodyne maps along isofibrations gives anodyne maps

$$\begin{tikzcd} X' \arrow[r, tail, "\sim"] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, two heads] \\ Y' \arrow[r, tail, "\sim"'] & Y \end{tikzcd}$$

We will call equivalences $\xrightarrow{\sim}$ homotopy equivalences.

(those maps that become invertible in the localization of $\mathcal{E}$ by anodyne maps)

Example: $\mathcal{E}$ = full subcategory of sSet spanned by quasi-categories.

isofibrations = fibrations of the Joyal model structure.

The list of axioms is long (~ 20).

They are sound because my book on higher categories together with my joint work with Kim Nguyen contain a proof that the theory of quasi-categories is a semantic interpretation (the same way Kan complexes form a semantic interpretation of univalent HoTT, after Voevodsky).

The axioms are designed so that we can write the directed analogue of the HoTT book.