Logics of Knowability

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Overview

Some Notable Papers on Logic and Knowability

- "A Logical Analysis of Some Value Concepts" (Fitch 1963)
- "What One May Come to Know" (van Benthem 2004)
- "Can truth out?" (Burgess 2008)
- "*Knowable* as *known after an announcement*" (Balbiana, Baltag, van Ditmarsch, Herzig, Hoshi, de Lima 2008)
- "Everything is knowable — how to get know *whether* a proposition is true" (van Ditmarsch, van der Hoek, and Iliev 2012)
- "Logic and Topology for Knowledge, Knowability, and Belief" (Bjorndahl and Özgün 2017)
- "Logics for knowability" (Liu, Fan, van Ditmarsch, and Kuijer 2022)

Approaches So Far

EAL ETL	$\Diamond K\varphi$ $FK\varphi$
DEL	$\exists A\langle A!\rangle\varphi$
Primitive	$\Box\varphi$

$\Box\varphi$ "one could come to know what φ used to express (before you came to know it)" (Bjorndahl and Özgün 2017)

"[W]e...leave further investigation of this subtlety to future work."

Motivation

$\Box\varphi$ "one could come to know what φ used to express (before you came to know it)"

- Once you accept positive introspection, $\Box\varphi \to \Box K\varphi$ sounds pretty good.
- $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ sounds pretty bad.
  - Limited resources
  - Measurements at the quantum scale

The Epistemic Temporal Language

We define $\mathcal{L}_{KG}$ by the following formal grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid G\varphi$$

- A model will be birelational frame equipped with a valuation.
- All truth clauses are standard.

ET Frames

Definition 1

An *ET diagram* is a triple $\mathcal{D} = (\mathcal{I}, Q_{-}, P_{-,-})$ such that the following hold:

1. $\mathcal{I} = (I, \preccurlyeq)$ is a linear order;
2. Q_{-} assigns to each $\tau \in I$ a quasi-order Q_{τ};
3. $P_{-,-}$ assigns to each pair (τ, σ) with $\tau \preccurlyeq \sigma$ an order-preserving map $P_{\tau,\sigma} : Q_{\sigma} \to Q_{\tau}$ where $P_{\tau,\tau} = \mathrm{id}_{Q_{\tau}}$ for all $\tau \in I$ and $P_{\tau,\sigma} \circ P_{\sigma,\rho} = P_{\tau,\rho}$.

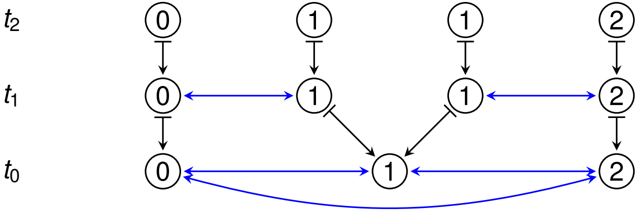
- The order-preserving requirement corresponds to a perfect-recall-like condition.
- We can obtain a relational frame from an ET diagram.

$$\mathcal{D} \rightsquigarrow F(\mathcal{D})$$

Such frames will be called *ET frames*.

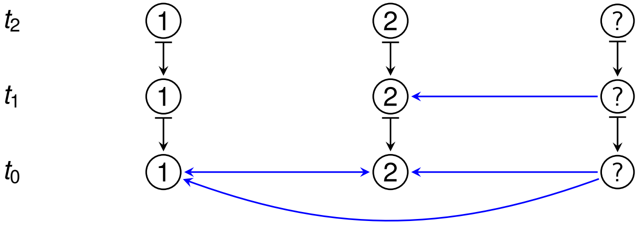
A Simple Example

- Consider an urn containing two marbles, and assume that Betty knows that each marble is exactly black or white.
- She then retrieves the marbles from the urn, one at a time.
- The relevant times are $t_0 \prec t_1 \prec t_2$.



A non-S5ish Example

- In a room, there is a locked box on a table with two keys (1 and 2) on it. The actual key to the to the lock is a third key hidden under the carpet.
- Otto is reliably informed that a key in the room will open the box.
- Otto enters the room and tries key 1 and then key 2.

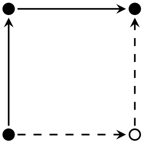


Weak ET Frames

Definition 2

A *weak ET frame* is a triple $(W, R, \leqslant)$ where W is a set, R and $\leqslant$ are quasi-orders, and the following condition holds:

- (Progressivity) If $w \leqslant x \; R \; y$, then there exists $z \in W$ such that $x \; R \; z \leqslant y$.



The Logic of Knowledge and the Future

The Logic LKF	
C	any substitution instance of a theorem of CPC
$\mathbf{K}_K$	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
$\mathbf{T}_K$	$K\varphi \to \varphi$
$\mathbf{4}_K$	$K\varphi \to KK\varphi$
$\mathbf{K}_G$	$G(\varphi \to \psi) \to (G\varphi \to G\psi)$
$\mathbf{T}_G$	$G\varphi \to \varphi$
$\mathbf{4}_G$	$G\varphi \to GG\varphi$
P	$KG\varphi \to GK\varphi$
MP	from $\varphi \to \psi$ and φ infer ψ
$\mathtt{Nec}_K$	from φ infer $K\varphi$
$\mathtt{Nec}_G$	from φ infer $G\varphi$

Soundness and Completeness for LKF

Theorem 3

LKF *is sound with respect to the class of weak ET diagrams.*

Theorem 4

LKF *is strongly complete with respect to the class of ET diagrams.*

- We incur minimal logical costs by working with ET frames.
- The proof is by model construction.

Histories

Definition 5

Let C and D be sets of $\mathcal{L}_{KG}$-formulas. We say that the pair (C, D) is *coherent* just in case C and D are both LKF-consistent and for all formulas φ, if $C \vdash_{\text{LKF}} G\varphi$, then $D \vdash_{\text{LKF}} \varphi$.

Definition 6

A *history* $h = \langle \Gamma_1, \ldots, \Gamma_n \rangle$ is a finite sequence of maximally consistent theories such that for any $1 \leq i < n$ the pair (Γ_i, Γ_{i+1}) is coherent.

- We write $h_\bullet$ for $h(\ell(h))$.
- For two history h and i with $\ell(h) = \ell(h') = n$, we write $h \; R_n \; h'$ if for all $1 \leq i \leq n$, whenever $K\varphi \in h(i)$, $\varphi \in h'(i)$.

The Canonical ET Model

Definition 7

The *canonical diagram for* LKF is $\mathcal{D}^{\mathsf{LKF}} = (\omega, Q_{-}^{\mathsf{LKF}}, P_{-,-}^{\mathsf{LKF}})$ where

1. Q_n^{LKF} is the set of all histories of length n ordered by R_n, and
2. $P_{n,m}^{\mathsf{LKF}} : H_m \to H_n$ is given by $h \mapsto h|_n$.

Definition 8

The *canonical ET model for* LKF is $\mathcal{M}^{\mathsf{LKF}} = (W^{\mathsf{LKF}}, R^{\mathsf{LKF}}, \leqslant^{\mathsf{LKF}}, V^{\mathsf{LKF}})$ where

1. $(W^{\mathsf{LKF}}, R^{\mathsf{LKF}}, \leqslant^{\mathsf{LKF}}) = F(\mathcal{D}^{\mathsf{LKF}})$, and
2. $V^{\mathsf{LKF}}(p) = \{h \in W^{\mathsf{LKF}} : p \in h_\bullet\}$.

Theorem 9

For any $\varphi \in$ LKF *and* $h \in W^{\mathsf{LKF}}$, $\mathcal{M}^{\mathsf{LKF}}, h \vDash \varphi$ *if and only if* $\varphi \in h_\bullet$.

The Language

- We define $\mathcal{L}_{K\Box}$ by the following formal grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K\varphi \mid \Box\varphi$$

- We write $\mathcal{L}_{\Box}$ for the K-free fragment of $\mathcal{L}_{K\Box}$.
- We fix a translation $t : \mathcal{L}_{K\Box} \to \mathcal{L}_{KG}$, which is defined as follows:
    - $t(p) = p$;
    - $t(\neg\varphi) = \neg t(\varphi)$;
    - $t(\varphi \wedge \psi) = t(\varphi) \wedge t(\psi)$;
    - $t(K\varphi) = Kt(\varphi)$;
    - $t(\Box\varphi) = FKt(\varphi)$.
- We write $t_{\Box}$ for $t|_{\mathcal{L}_{\Box}}$.

The Game Plan

- Our goal is axiomatize the logics of the class of ET frames.
- By our results on LKF and some basic facts about the translations, it is enough to prove completeness with respect to weak ET frames.

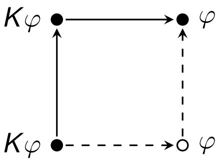
Toward Recovering the Temporal Dimension

- We want to give a syntactic characterization of the persistent formulas.

Definition 10

We define the set of *sharp formulas* inductively as follows:

- $\top$ and $\bot$ are sharp;
- For any $\varphi \in \mathcal{L}_{K\Box}$, $\Diamond\varphi$ is sharp;
- If φ and ψ are sharp, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are sharp;
- If φ is sharp, then $K\varphi$ is sharp.



- We can dualize this definition to obtain a definition of *flat formula*.

The Logic of Knowledge and Knowability

The Logic LKKa		
C	any substitution instance of a theorem of CPC	
$\mathbf{K}_K$	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	
$\mathbf{T}_K$	$K\varphi \to \varphi$	
$\mathbf{4}_K$	$K\varphi \to KK\varphi$	
A	$K\varphi \to \Box\varphi$	
MP	from $\varphi \to \psi$ and φ infer ψ	
$\mathtt{Nec}_K$	from φ infer $K\varphi$	
Sh	from $\chi \to \hat{K}\psi$ infer $\chi \to \Diamond\psi$	χ is sharp

Theorem 11

LKKa *is sound with respect to the class of weak ET frames.*

Completeness of LKKa

Theorem 12

LKKa *is strongly complete with respect to the class of weak ET frames.*

Definition 13

The canonical model for LKKa is defined as $(W^{\text{LKKa}}, R^{\text{LKKa}}, \leqslant^{\text{LKKa}}, V^{\text{LKKa}})$ where

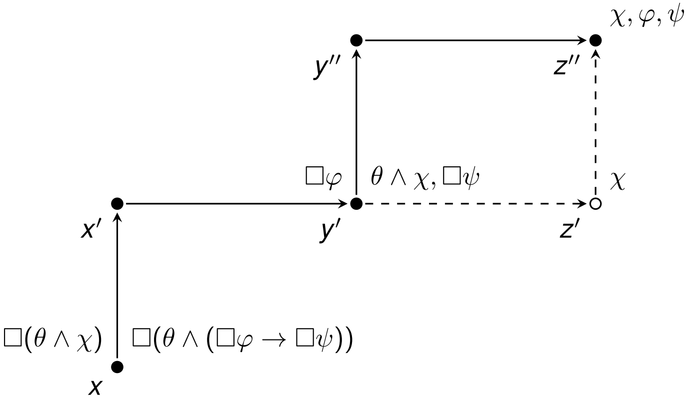
1. W^{LKKa} is the set of all LKKa-maximally consistent theories;
2. $\Gamma\ R^{\text{LKKa}}\ \Delta$ just in case for all φ, if $K\varphi \in \Gamma$ then $\varphi \in \Delta$;
3. $\Gamma \leqslant^{\text{LKKa}} \Delta$ just in case for all sharp formulas φ, if $\varphi \in \Gamma$, then $\varphi \in \Delta$;
4. $V^{\text{LKKa}}(p) = \{\Gamma \in W^{\text{LKKa}} : p \in \Gamma\}$.

The Logic of Pure Knowability

The Logic of Knowability		
C	any substitution instance of a theorem of CPC	
$\mathbf{Ka}_{\Box}$	$\Box\top$	
$\mathbf{T}^{\flat}_{\Box}$	$\Box\varepsilon \to \varepsilon$	ε is flat
$\mathbf{4}^{*}_{\Box}$	$\Box\varphi \to \Box(\varphi \wedge \Box\varphi)$	
MP	from $\varphi \to \psi$ and φ infer ψ	
Reg	from $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$	
PR	from $\chi \to (\varphi \to \psi)$ infer $\Box(\theta \wedge \chi) \to \Box(\theta \wedge (\Box\varphi \to \Box\psi))$	χ is sharp

The Rule PR

Where χ is sharp, from $\chi \to (\varphi \to \psi)$ infer $\Box(\theta \wedge \chi) \to \Box(\theta \wedge (\Box\varphi \to \Box\psi))$.



Soundness and Completeness for LPKa

Theorem 14

LPKa *is sound with respect to the class of weak ET frames.*

Theorem 15

LPKa *is strongly complete with respect to the class of weak ET frames.*

- The proof is again by model construction.
- The techniques needed will be a bit more delicate because our language is fairly poor.

Epistemic Theories

Definition 16

A theory $\mathcal{E} \subseteq \mathcal{L}_\Box$ is called *epistemic* if it satisfies the following conditions

1. (Consistency) $\bot \notin \mathcal{E}$;
2. (Box closure) If $\alpha \in \mathcal{E}$, then $\Box\alpha \in \mathcal{E}$;
3. (Sharp regularity) If $\chi \in \mathcal{E}^\sharp$ and $\vdash \chi \to (\beta \to \gamma)$, then $\Box\beta \to \Box\gamma \in \mathcal{E}$.

Definition 17

A pair of theories $(\Gamma, \mathcal{E})$ is called *concordant* if Γ is a maximally consistent theory, $\mathcal{E}$ is an epistemic theory, and $\mathcal{E} \subseteq \Gamma$.

Model Construction

Definition 18

We define the model $\mathcal{M}^{\mathsf{LPKa}} = (W^{\mathsf{LPKa}}, R^{\mathsf{LPKa}}, \leqslant^{\mathsf{LPKa}}, V^{\mathsf{LPKa}})$ as follows:

1. W^{LPKa} is the set of concordant pairs.
2. $(\Gamma, \mathcal{E})\ R^{\mathsf{LPKa}}\ (\Delta, \mathcal{F})$ just in case $\mathcal{E} \subseteq \mathcal{F}$.
3. $(\Gamma, \mathcal{E}) \leqslant^{\mathsf{LPKa}} (\Delta, \mathcal{F})$ just in case $\Gamma^\sharp \subseteq \Delta^\sharp$ and $\mathcal{E}^\sharp \subseteq \mathcal{F}^\sharp$.
4. $V^{\mathsf{LPKa}}(p) = \{(\Gamma, \mathcal{E}) \in W : p \in \Gamma\}$.

Theorem 19

For any concordant pair $(\Gamma, \mathcal{E})$, $\varphi \in \Gamma$ if and only if $\mathcal{M}, (\Gamma, \mathcal{E}) \vDash \varphi$.

Epistemic Theory Construction

- We want to show that if $(\Gamma, \mathcal{E})$ is concordant and $\Box\varphi \in \Gamma$, then there is a concordant $(\Delta, \mathcal{F})$ such that $\varphi \in \mathcal{F}$.
- Fix an enumeration $(\psi_i)_{i<\omega}$ of all formulas and an enumeration $(\theta_i)_{i<\omega}$ of all formulas whose outermost connective is a conditional. For convenience, write $\theta_i = \beta_i \to \gamma_i$.
- Set $\alpha_0 = \varphi$.
- If $n > 0$ is odd, let k_n be the least natural such that $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \vdash \psi_{k_n}$ but $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \nvdash \Box\psi_{k_n}$. Set $\alpha_n = \alpha_{n-1} \wedge \psi_{k_n}$.
- If $n > 0$ is even, let k_n be the least natural such that $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \vdash \chi$ and $\vdash \chi \to \theta_{k_n}$ but $\mathcal{E}^\sharp \cup \{\alpha_{n-1} \wedge \Box\alpha_{n-1}\} \nvdash \Box\beta_i \to \Box\gamma_i$. Set $\alpha_n = \alpha_{n-1} \wedge (\Box\beta_i \to \Box\gamma_i)$.
- Set $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$ where $\mathcal{F}_n = \mathrm{cl}(\mathcal{E}^\sharp \cup \{\alpha_n \wedge \Box\alpha_n\})$.

Toward Consistency

There are three lemmas that we use to verify that the construction produces a consistent theory.

Lemma 20

Suppose that $(\Gamma, \mathcal{E})$ *is concordant and that* $\Box\alpha \in \Gamma$*. Then* $\mathcal{E}^\sharp \cup \{\alpha \wedge \Box\alpha\}$ *is consistent.*

Lemma 21

Suppose that $(\Gamma, \mathcal{E})$ *is concordant and that* $\Box\alpha \in \Gamma$*. If* $\mathcal{E}^\sharp \vdash (\alpha \wedge \Box\alpha) \to \beta$*, then* $\Box(\alpha \wedge \beta) \in \Gamma$*.*

Lemma 22

Suppose that $(\Gamma, \mathcal{E})$ *is concordant and that* $\Box\alpha \in \Gamma$*. If* $\mathcal{E}^\sharp \vdash (\alpha \wedge \Box\alpha) \to \chi$ *and* $\vdash \chi \to (\beta \to \gamma)$ *for some sharp formula* χ*, then* $\Box(\alpha \wedge (\Box\beta \to \Box\gamma)) \in \Gamma$*.*

Some Possible Further Directions

- Different views on knowledge: change the assumptions on the objects in ET diagrams.
  - Drop transitivity.
  - Add symmetry.
- Bounded agents: drop the order-preserving assumption on ET diagrams.
- Limits: can we say anything interesting about the limit space?
- Scientific knowability: add eternal propositional variables (e_i) and adopt the axiom $\Box\lambda \wedge \Box\mu \to \Box(\lambda \wedge \mu)$ where λ and μ are eternal literals."
  - Limits again: we probably can say something about the limit space!
- Connections to DEL

Thank you!