VENTURING FURTHER INTO EPISTEMIC TOPOLOGY

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joint work with Alexandru Baltag

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Formal Epistemology Seminar, CMU
Background: Information in logic
Knowledge and Information in Logic

Information as range
Epistemic logic

Jaakko Hintikka

Information as correlation
Situation theory

Jon Barwise
Further Varieties of Information in Logic

also **procedural information** (about process of inquiry)

information as code (deduction, computing)

and more …
II

Enter Dependence
Dependence and Information

physical dependence of $y$ on $X$ in the world
determined, or correlated behavior
covers a family of notions/intuitions

epistemic dependence

if you know the values of $X$ you know the value of $y$

related epistemic view: implication between questions
# A Dependence Table

Toy example: assignment of objects to variables

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
</table>
| 0  | 0  | 0  | **red**
| 0  | 0  | 1  |
| 1  | 1  | 1  | **red**
| 2  | 1  | 0  |

**global dependence**

fixing value of $x$ fixes that of $y$

$x \rightarrow y$: $y$ depends on $x$

dependence graph

$x \rightarrow y$, not $y \rightarrow x$

not $z \rightarrow x$, $\{y, z\} \rightarrow x$

**local dependence**

x depends on $y$ at $(0, 0, 0)$

but not at $(1, 1, 1)$

more basic!
Dependence Models

\[ \mathcal{M} = (V, O, S, P) \]

- **V** variables
- **O** objects (possible values of variables)
- **S** states: family of functions from \( V \) to \( O \)
  - need not be full \( O^V \): **gaps** = dependence
- **P** predicates of objects

\[ D^s_{xy} \quad \forall t \in S: \text{if } s =_x t, \text{ then } s =_y t \quad D^s_{xy} \quad \forall s \in S: D^s_{xy} \]

with

\[ s =_x t : s(x) = t(x), \quad s =_x t : \forall x \in X: s =_x t \]

also

\[ D^s_{XY} : \forall y \in Y: D^s_{xy} \]
functional definability

useful shift in perspective: \( x(s) := s(x) \)
variables as maps from states to objects
states might now be more than assignments

a simple factorization result

\( D_{xy} \) holds globally in a dependence model iff there exists a (partial) function \( F \) from tuples of \( X \)-values to \( y \)-values s.t. for all \( s \):

\[
y(s) = F(X(s)) \quad \text{‘} y = F \circ X \text{’}
\]
Modal Dependence Logic LFD
Language and Semantics
A Simple Logic of Functional Dependence

Alexandru Baltag & Johan van Benthem


a modal logic of functional dependence
Basic Modal-Style Dependence Logic

**assignment of values to variables:** state of the world

dependence because of gaps in full function space

**dependence model:** family of assignments

propositional language plus $D_x \phi \mid D_{xy}$

$D_{xy}$ local dependence at state $s$: if $s =_x t$, then $s =_y t$

$D_x \phi$ local modality/quantifier: for all $t$ s.t. $s =_x t$, $t \models \phi$

universal modality, global dependence definable
Expressive Power

• Changing $x$ implies changing $y$ $D_{y|x}$

more complex versions with ‘(pre-causal) influence’

• $D_{\emptyset \varphi}$ is the universal modality $U\varphi$

• Global dependence defined from local: $UD_{x|y}$

• Dependence as value restriction:
if $x$ lies within some range, so does $y$: $U (Q_1 x \rightarrow Q_2 y)$

• universal quantifier $\forall X \varphi$: $D_{\text{VAR}-X} \varphi$
Other dependence logics: CRS and DL

Extends **CRS**:
Generalized FOL semantics drops independence assumptions:
   - decidable sublogic remains
   - now add explicit dependence atoms

Different take from **DL**:
   - not second-order logic on teams
   - classical propositional base
   - (switching teams would require dynamic modalities)
CRS First-Order Logic

Dependence models: ‘generalized assignment models’

\[ M = (D, \mathcal{V}, I) \text{ with } \mathcal{V} \text{ set of ‘available’ assignments} \]

\[ M, s \models \exists x. \phi \iff \text{there exists } t \text{ in } \mathcal{V} \text{ with } s =^x t \text{ and } M, t \models \phi \]

\[ s =^x t : s(y) = t(y) \text{ for all variables } y \text{ distinct from } x \]

**Theorem** The first-order validities on generalized assignment models are recursively axiomatizable and decidable.

Drops **independence** principles like \( \exists x. \exists y. \phi \rightarrow \exists y. \exists x. \phi \): these impose existential confluence properties on the set \( \mathcal{V} \).
Translation Into FOL on Standard Models

**Thm** There is a translation $tr$ from the language of LFD into first-order logic making the following equivalent for modal formulas $\phi$:

(a) $\phi$ is satisfiable in a dependence model,

(b) $tr(\phi)$ is satisfiable in a *standard* first-order model.

**Trick:** finitely many variables $x$, code that a tuple of values for $x$ is an available assignment with new *dedicated predicate* $Ux$.

**Corollary** All logics that we will consider are RE (axiomatizable).

**Caveat** Not reduction of object-language, but meta-language.
Axiom System for LFD

The proof system **LFD** consist of

(a) The principles of modal S5 for each separate $D_X\varphi$

(b) Monotonicity $D_X\varphi \rightarrow D_{X\cup Y}\varphi$

(c) Reflexivity, Transitivity, Monotonicity for atoms $D_Xy$

(d) **Transfer axiom** $(D_XY \land D_Y\varphi) \rightarrow D_X\varphi$

(e) Invariance $(\neg)Qx \rightarrow D_X(\neg)Qx$, $(\neg)D_Xy \rightarrow D_X(\neg)D_Xy$
IV

Independence and Correlation
Independence and Correlation

Independence is not the negation of dependence $\neg D_{X,y}$.

Natural sense of independence of $y$ from $X$: fixing the values of $X$ leaves $y$ free to take on any value it can take in the model (‘knowing $X$ implies no knowledge about $y$’).

Can be formalized as an independence atom $I_{X,y}$.

$\neg I_{X,y}$ some local correlation, $D_{X,y}$ total correlation

spectrum of (quantitative) correlations in between
V

Completeness and Other System Results
Some Results About the System

Thm  LFD is sound, complete, decidable
Thm  LFD + \text{I} axiomatizable, undecidable
Open  PL + just I decidable?

some follow up results

LFD + terms + = undecidable  Graedel & Puetzstueck
includes general analysis of localization and decidability
bisimulation analysis LFD  Puetzstueck, Koudijs
LFD has Finite Model Property  Koudijs
Ongoing Related Projects

mutual interpretability LFD and Guarded Fragment (solves computational complexity LFD)  
Koudijs, ten Cate

dependence in dynamical systems  
link up with temporal logic  
Ba & vB with Dazhu Li

dependence & independence in Linear Algebra/Matroid Theory  
modal logics of vector spaces  
vB & Nick Bezhanishvili

Why mention all these technical LFD topics?
Interesting to see later which ones make sense  
(and if so, how) in a topological setting
VI

Empirical Inquiry, Measurement, Topology
From Sharp Values to Empirical Measurement

LFD semantics assumes sharp values and suggests that we can know these this works for many epistemic puzzles and scenarios but it is highly idealized in many settings empirical inquiry yields only approximate measurements
Epistemic Topology

open sets
outcomes of possible observations
approximation of values now essential
many versions of this idea:
Vickers, Parikh & Moss
also Intuitionistic Logic since 1930s

Topology of information states, or:
stages in temporal history of inquiry
Background: Logics of Space

[Image of the cover of the book "Handbook of Spatial Logics"]
VII

Topological Dependence Logic
Topo-Dependence Models

\[ M = (S, O, \{O_x\}_{x \in \text{VAR}, V}) \]

set of admissible states \( S \)
variables \( x \) map states to objects \( x(s) \) in their ranges \( O_x \)
these ranges carry topologies \( O_x \)
measurements of \( x \) yield opens in \( O_x \)
* separation axioms make epistemic sense: \( T_0 \)
* lift to finite sets of variables using product topology

topologies on values can be retracted to \( O_x \) on states:
the smallest topology that makes the map \( x \) continuous
often makes for more intuitive formulations
Knowledge and Knowability

local modal semantic format \( M, s \models \varphi \)

\( U \varphi \) what is true in all states is hard knowledge
[can be made more sophisticated in various standard ways]

\( K_x \varphi \) local knowability under new information
\( \varphi \) true in some open \( X \)-neighborhood of \( s \)

the standard topological interpretation of modal logic
\( S4 \) instead of \( S5 \) lots of theory in epistemic logic
(also on modal logics for product topologies)
Important Special Topologies

discrete topology contains our earlier LFD approach

Alexandrov topology each point has smallest open NBD
~ standard relational models: specialization pre-orders

Proposition. In an Alexandroff topo-dependence model $M$, the above semantical clauses are equivalent to the following relational versions:

$$s \models_M K_X \varphi \iff \forall t \in S(s \leq_X t \Rightarrow t \models \varphi),$$

$$s \models_M K_X Y \iff \forall t, w \in S(s \leq_X t \leq_X w \Rightarrow t \leq_Y w),$$

where $\leq_X$ is the specialization preorder for the $X$-topology $\tau_X^S$ on states.

technical uses our completeness proofs all pass via representation theorems for such relational models
Alternative/Richer Frameworks

topo-logic (Parikh & Moss)

dynamic-epistemic update logics like PAL and DEL for explicit changes of topological models

temporal logics with topological structure (dynamic topological logic DTL)

here we will stick with the simplest base setting
Continuity and Learnability
we may know that a dependence exists but it may not be useful for approximation.

$k_x y$ approximable dependence of $y$ on $X$:

get arbitrarily close to $y(s)$ by measuring $X(s)$ locally, continuous function at state $s$

may still not yield knowledge:

we do not know which state $s$ we are at

local continuity can be fragile
maybe the more natural topological sense is locality$_2$:
truth throughout some open neighborhood of $s$

knowable dependence $K_{xy}$ $y$ depends on $X$
continuously over some open neighborhood $U$ of $s$

i.e. dependence function $y(X)$ on states factors over
continuous map from $X$-values to $y$-values on the set $U$

similar factoring views for our other notions to come
Connections to Formal Learning Theory

if the spaces are **first-countable** (countable basis for open NBHs)
continuous \( y \) is **gradually learnable** from \( X \) under some \( X \)-data stream

**Proposition**

Let \( X : S \to (\mathbb{D}_X, \tau_X) \) and \( Y : S \to (\mathbb{D}_Y, \tau_Y) \) be observational variables. The following are equivalent:

1. \( Y \) is gradually learnable from \( X \)-observations;
2. there exists a continuous function \( F : (\mathbb{D}_X, \tau_X) \to (\mathbb{D}_Y, \tau_Y) \) s.t. \( F \circ X = Y \);
3. there is a known epistemic dependence between the variables \( X : S \to (\mathbb{D}_X, \tau_X) \) and \( Y : S \to (\mathbb{D}_Y, \tau_Y) \), i.e. we have \( S \models K(X; Y) \).

Here \( K(X ; Y) = U k_{x,y} \)
IX

The System LCD
The Logic of Continuous Dependence

Alexandru Baltag & Johan van Benthem

Knowability as Continuous Dependence

Alexandru Baltag, ILLC, Amsterdam

Based on joint work with Johan van Benthem.

conceptual/mathematical analysis of epistemic notions
design of simple modal base logics
sequence of completeness and decidability results
uniform approach via representation theorems
\[ M, s \models K_X \varphi \iff \exists U \in O_X : X(s) \in U \land \forall t : t(X) \in U : M, t \models \varphi \]

Equivalent in our earlier terms: some open NBH in the retracted \( O_X \)-topology at \( s \) is contained in \([\varphi]\)

the usual topological interpretation of modal logic

\[ M, s \models k_X y \iff \forall U \in O_y \text{ s.t. } s(y) \in U \exists V : \forall t : t(X) \in V : t(y) \in U \]

Or: every \( y \)-open NBD of \( s \) contains an \( X \)-open NBD of \( s \)

Again too fragile? Replace by local\(_2\) version

\[ M, s \models K_X y \iff M, s \models K_X K_X y \]
Digression: First-Order Perspective

what can we expect a priori?

LFD and its related systems were all translatable effectively into first-order logic explains the axiomatizability of these logics definition of topological space is second-order, but: if we think of the outcomes of measurements as a base attractive three-sorted FOL-translatable versions exist
**LCD Proof System**

<table>
<thead>
<tr>
<th>(I)</th>
<th>Axioms and rules of LFD</th>
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<tbody>
<tr>
<td>(II)</td>
<td>S4 Axioms for knowability:</td>
</tr>
<tr>
<td>(K-Necessitation)</td>
<td>From $\varphi$, infer $K_X \varphi$</td>
</tr>
<tr>
<td>(K-Distribution)</td>
<td>$K_X (\varphi \rightarrow \psi) \rightarrow (K_X \varphi \rightarrow K_X \psi)$</td>
</tr>
<tr>
<td>(Veracity)</td>
<td>$K_X \varphi \rightarrow \varphi$</td>
</tr>
<tr>
<td>(Positive Introspection)</td>
<td>$K_X \varphi \rightarrow K_X K_X \varphi$</td>
</tr>
<tr>
<td>(Knowable Determination)</td>
<td>$K_X \varphi \Rightarrow D_X \varphi$</td>
</tr>
<tr>
<td>(Knowledge of Necessity)</td>
<td>$A \varphi \Rightarrow K \varphi$</td>
</tr>
</tbody>
</table>

(Axioms for knowable dependence:

- $K_X Y$, provided that $Y \subseteq X$
- $(K_X Y \land K_X Z) \rightarrow K_X (Y \cup Z)$
- $(K_X Y \land K_Y Z) \rightarrow K_X Z$
- $K_X Y \Rightarrow (D_X Y \land K_X K_X Y)$
- $C(Y) \Rightarrow K Y$
- $K_X Y \Rightarrow (K_Y \varphi \rightarrow K_X \varphi)$

Table 1: The proof system **LCD**, where we used the notations $A \varphi := D_{\emptyset} \varphi$, $C(Y) := D_{\emptyset} Y$, and $K := K_{\emptyset}$ (for both formulas $\varphi$ and sets $Y \subseteq V$).

**explanation:** look like LFD axioms but different content

{t | s $=_{X} t$} contained in every X-open around s, but need not itself be open.

KT expresses continuity of dependence, even modal correspondence-style.
LCD Results

**LCD** modal logic of knowability plus knowable continuity

**Thm** LCD is complete for validity

**Open** Axiomatize full language with local continuity

**Thm** LCD is decidable

**Key proof steps** Standard modal completeness for ‘general relational models’. Representation theorem general as standard relational models: yields Alexandrov topologies. Decidability via finite general relational model property.

**mathematical logic** Generalize Tarski-McKinsey Thm for S4 now not just produce the reals, but also continuous maps over them
Computability and Domain Theory
Toward Computable Dependence

Specialization Computability in **Domain Theory**

complete lattices of ‘information pieces’

**Scott topology** with base of finite information pieces (not the upset topology for inclusion)

Scott continuity ~ abstract **computability**

**Thm** Complete and decidable logic is **LCD +** one extra axiom for bottom elements in domain structure

\[ K(K_X \varphi \vee K_X \psi) \rightarrow (K \varphi \vee K \psi). \]

Representation proof more complex so as to produce domains!

K is universal modality, look at bottom element for X-value domain
XI

Independence in Topological Models
Independence in Epistemic Topology

topological independence

\[ s \models Ig_X Y \iff Y(U) = D_Y \text{ for all } U \in \tau_X^S(s). \]

subtly different weaker than LFD-style independence which wants to see all \( y \)-values on the \( X \)-equal states

also symmetric(!) global versions

\[ I(X; Y) \leftrightarrow I(Y; X), \quad Ig(X; Y) \leftrightarrow Ig(Y; X). \]

scope for modal logic: some connections

\[ I_X Y \Rightarrow Ig_X Y, \quad I(X; Y) \Rightarrow Ig(X; Y) \]
\[ I(X; Y) \Rightarrow I_X Y, \quad Ig(X; Y) \Rightarrow Ig_X Y \]

open problem Topological I-logic axiomatizable, decidable?
Everywhere Surjective Functions

Y as function of X is **everywhere surjective**  Lebesque 1904

sort of opposite to continuous functions

**Thm** The following are equivalent:

1. $D(X;Y) \land Ig(X;Y)$ holds in the model $\mathcal{M}$ (at any/all states);

2. there exists some everywhere-surjective map $F : D_X \rightarrow D_Y$ s.t. $F \circ X = Y$.

interesting recent mathematical theory
Bernardi and Rainaldi 2018

Anti-Learning Theory?
How to logicize **more general accuracy correlations** in between independence and functional dependence?

And also: **inverse accuracy correlations** like that between position and momentum in Heisenberg’s Uncertainty Principle?
Epistemic Know-How: Uniform Continuity
From Continuity to Uniform Continuity

epistemic know-how in empirical inquiry

knowing an approximation that works: uniform continuity

\[ \forall \varepsilon > 0 \, \forall s \in S \, \exists \delta > 0 \, \forall t \in S \ (d_X(X(s), X(t)) \leq \delta \Rightarrow d_Y(Y(s), Y(t)) \leq \varepsilon). \]

\[ \forall \varepsilon > 0 \, \exists \delta > 0 \, \forall s \in S \, \forall t \in S \ (d_X(X(s), X(t)) \leq \delta \Rightarrow d_Y(Y(s), Y(t)) \leq \varepsilon). \]

LCD-style modal logic on metric dependence models

\[ U(X; Y) \quad \text{uniformly continuous dependence map } Y(X) \]

\[ U_X Y \quad \text{locally uniform continuous dependence} \]

**Proposition**

Given empirical variables \( X : S \rightarrow (D_X, d_X) \) and \( Y : S \rightarrow (D_Y, d_Y) \), the following are equivalent:

1. \( U(X; Y) \) holds;

2. there exists a uniformly continuous map \( F_{X,Y} : (D_X, d_X) \rightarrow (D_Y, d_Y) \) s.t. \( F \circ X = Y \) holds on \( S \).
The Complete Logic LUD

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<td>(U-Inclusion)</td>
<td>( U(X; Y) ), provided that ( Y \subseteq X )</td>
</tr>
<tr>
<td>(U-Additivity)</td>
<td>((U(X; Y) \land U(X; Z)) \rightarrow U(X; Y \cup Z))</td>
</tr>
<tr>
<td>(U-Transitivity)</td>
<td>((U(X; Y) \land U(Y; Z)) \rightarrow U(X; Z))</td>
</tr>
<tr>
<td>(Uniform Dependence is Known)</td>
<td>( U(X; Y) \rightarrow KU(X; Y) )</td>
</tr>
<tr>
<td>(Uniformity implies Continuity)</td>
<td>( U(X; Y) \rightarrow K(X; Y) )</td>
</tr>
</tbody>
</table>

some explanations

**Thm**  LUAD is complete and decidable.

proof runs via ‘pseudo metric models’:

requires yet more complex representation argument

**open problem**  how to deal with \( U_X Y \)
Uniform Spaces

uniform spaces with entourages  Weil 1937

Family $\mathcal{U}$ of ref-sym relations $\sim$ plus refinement closure:
for every $R \in \mathcal{U}$ there is $S \in \mathcal{U}$ with $S \circ S \subseteq R$

Sets $\{y \mid y \sim x\}$ generate a topology

qualitative uniform continuity

$$\forall U \in \mathcal{U} \exists V \in \mathcal{V} \forall x \in D \forall y \in D \ ((x, y) \in V \Rightarrow (F(x), F(y)) \in U)$$

Analogy with margin-of-error relations in epistemology
How to best bring into our kind of logic?
One Option: Modal Logic of Accuracy Dynamics

u assigns binary relations $\sim_x$ to variables $x$

$r \leq_x r'$ more refined relations on all $X$-variables

$r \models S_x \varphi \quad \varphi$ true in all $U(X)$-close points

$U_x y \quad y$ uniformly continuous in $X$

dynamic modalities $[\downarrow_x] \varphi$

$\varphi$ true for all $X$-refinements of current $r$

logic now includes reasoning about accuracy refinements

$\sim$ modal logics of relation change, e.g.:

The Modal Logic of Stepwise Removal
Johan van Benthem, Krzysztof Mierzewski & Francesca Zaffora Blando
Review of Symbolic Logic.
Discussion and Conclusions
Limitations 1

Our topological models consider global objects ‘known’
a global dependence function is known in this sense

But what if we want to learn a dependence function?
need to lift our semantics to families of dependence models
LFD paper: ‘dependence universes’
optimal logic still to be developed

analogies with learning the content of a set:
suggests lifting of public announcement logic
plus richer static base language
Limitations 2

our topological models assume sharply defined functions

but is not this at odds with the imprecise measurement setting?

possible alternative: point-free topology

primitive approximation maps run backwards

‘as if’ representation results

create points out of primitive opens,

produce continuous function out of approximations
General Enterprise: Epistemize Analysis

develop parallel mathematical notions ~ epistemic notions

various directions:

• common sense analogues of math notions

continuity/uniform continuity: move to knowing how?

• common sense epistemic principles as high-level expressions of mathematical notions

e.g., $K_\gamma y \rightarrow U_\gamma y$ (epistemic de dicto to de re)/local compactness?
Digression: Where is the Dynamics?

epistemic actions
model change, LFD + PAL
dependence universes

in the topology/metrics:
DEL accuracy dynamics

causality: dependence
plus interventions that
change causal models
Summary: Richer Epistemic Topology

Exploit more topology explicitly in epistemic logic

Unify topological and computational aspects

Venture beyond topology into Analysis

not discussed today:

Add further structure beyond dependence: causality

Step up abstraction: category theoretic-framework for LCD style logics (current work by Ye Lingyuan)
Questions and Answers: Just a Few Points

$K_{xy}$ Why is ‘knowable’ $k_{xy}$ only based on more $X$-info?

Yes, could be based on measuring other variables $Z$. Simple case: when $X$ depends on $Z$. In general, we would need $K_Z k_{xy}$ which is not yet in our language…

What are we learning?

Just the actual state $s$ of the system. Like the actual world in epistemic scenarios. Learning the state space itself, or a dependence function, requires richer models. We can see this as LCD over more structured states, but we might want a richer language describing the structural content of states.
Questions and Answers: Just a Few Points

Could not we also learn about $y$ by other means than measurement of other variables? [Hope I got this right]

Even in LCD, measuring $X$ gives information about $y$ in combination with another source of information: the structure of the state space and its assignment gaps. And yes of course, other informational events could take place, PAL- or DEL-style, not based on measurement.

How to capture more general intuitions about approximate knowledge in terms of ‘getting enough information about $X$ produces enough information about $y$’?
Questions and Answers: Continued

Spelling out the ‘enough’ is interesting as a typical qualitative device in stating the gist of mathematical notions. In which direction does it run here: from $y$ to $X$, or vice versa? Could entourages offer a good way of studying this?

Is ‘knowability’ a felicitous term? Better ‘will be known’ or some other term from current learning-theoretic topology?

Our framework is more about information than knowledge, and yes, there may be better names for what we study.

Connections with knowability in intuitionistic logic?

May be helpful to bring out intuitionistic structures inside LCD.
Questions and Answers: Continued

Note: intuitionistic mathematics make heavy use of (uniform) continuity in its logical analysis of the informational surplus in constructive statements and proofs.

Why is LFD independence complex qua logic?
What about topological independence?

In LFD: independence atoms can enforce a full Cartesian product for three variables, and FO(3) is undecidable. Topo-independence is weaker, and no such reduction may exist, leaving room for a decidable D + I logic.

Aside: independence makes computational practice easier, but this regular structure is precisely what can make for a more complex logical theory.
Questions and Answers: Aftermath

for posing further questions or getting better answers:

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