Book Review:

Sketches of an Elephant: A Topos Theory Compendium

by

Peter T. Johnstone, Oxford Logic Guides,

vol.s, 43, 44.

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Book Review: *Sketches of an Elephant: A Topos Theory Compendium*, by Peter T. Johnstone, Oxford Logic Guides, vol.s 43, 44. Oxford University Press, 2002, xxii + 1160 pp.*

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The whimsically modest title "*Sketches of an Elephant*" (you know, the four blind men who think the elephant is like a snake, a tree trunk, etc.) fails to do justice to the ambitious scope of this project or its masterful execution. It could easily have been called "*Principia Toposophica*," perhaps reflecting its role in the logic of the new century.

Roughly speaking, the notion of a topos is to logic in general what a boolean algebra is to propositional logic, but this needs to be qualified in several ways. The original conception, due to A. Grothendieck, was mainly to be a vehicle for cohomology (see: Allyn Jackson, "*Comme Appelé du Néant*—As if Summond from the Void: The Life of Alexandre Grothendieck", *Notices Amer. Math. Soc.* **51**, 2004). Logically, such a "Grothendieck topos" is something like a universe of continuously variable sets. Before long, however, F.W. Lawvere and M. Tierney provided an elementary axiomatization more closely related to higher-order logic. These two aspects continue to be present, among several others referred to in the "elephant" image. Compared to traditional algebraic logic, however, the "algebraic" treatment of logic pursued in topos theory is closer in spirit to that found in "algebraic" topology, where the use of functorial methods is taken for granted. In topos theory, moreover, not only are such methods brought to bear on logic, but conversely, the topos approach permits the application of logic in a range of situations in topology, geometry and algebra — extending Grothendieck's original conception from cohomology.

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This scope of application, deriving from the generality of its methods, is what makes topos theory "foundational," in a sense that is neither the deductive-axiomatic one of Frege and Russell, nor entirely the descriptive-axiomatic one of Eilenberg & Steenrod. Be that as it may, it is clear that the book fits into the Oxford Logic Guides, and should be of great interest to readers of this BULLETIN. The purpose of this review, then, is to provide a field guide to this strange new territory, which may in places seem rather forbidding to the newcomer.

First, as to the nature of the beast, this pachyderm is foremost a compendium of the main results of research in the field since its Bourbakian beginnings in the 1960s, refined, related and organized into a body of theory that displays the maturity of the discipline as it now stands. No such corpus would have been possible even twenty years ago (before e.g. the work of A. Joyal and M. Tierney, "An Extension of the Galois Theory of Grothendieck", *Memoirs Amer. Math. Soc.* **309**, 1984), and as useful as was the author's early textbook *Topos Theory* (Academic Press, 1977), the difference between it and this book displays how much the field has ripened (but it also reflects the fact that the *Elephant* is not intended as a textbook). Although it includes some results not previously published (or hard to find), as well as many improved proofs of known results, the strength and value of the work really lies in its encyclopedic character and remarkable success in organizing and interrelating this amount of quite technical material into a coherent whole, with uniform notation, terminology, prerequisites and depth of rigor. Moreover, the high degree of deductive coherence, shared definitions and lemmas, etc., that are possible in such a book make it far more useful (and readable) than the sum of the individual research papers from which its results are mainly drawn.

The actual content is not restricted to topos theory proper, but also covers a range of results and methods from category theory that are related to toposes, and are employed in topos theory to one degree or another. This course of action not only serves to make the book largely self-contained, but contributes to the uniformity of notation and terminology and systematic organization as well. It also makes the book valuable as a resource for those related fields, such as general categorical logic; indexed, fibered, and internal category theory; allegories; locale theory; and several others.

The two volumes are divided into four main parts, two in each, corresponding roughly to different aspects of the topos concept, as follows:

A. Toposes as categories

B. 2-categorical aspects of topos theory

C. Toposes as spaces

D. Toposes as theories

The contents of the projected third volume are indicated to be:

E. Homotopy and cohomology

F. Toposes as mathematical universes.

Each of these parts falls into several chapters and sections, the general contents of which we will now survey individually.

A. Toposes as categories. A topos is by definition a cartesian closed category with a subobject classifier. To the logician, cartesian closure essentially means that the category is a model of the lambda-calculus, while the subobject classifier adds in a type of "propositions" or "truth-values". In the presence of equality relations for each type X (provided by the diagonal arrow $X \rightarrowtail X \times X$), this combination of resources permits not only the definition of the usual logical operations like negation, conjunction, and quantification, but also the formation of terms of all higher (finite) types, such as functionals and relational quantifiers. The notion of a topos thus corresponds logically to that of a system of higher-order predicate logic; but let us immediately ward off a possible misconception: topos theory is not chiefly concerned with the use of such systems of logic, in some form or other, to deduce fragments of mathematics, the way the logician might expect. Rather, it is occupied more with the study of the great variety of such toposes as they occur in nature, as it were, and their interrelations. Thus the perspective is closer to the way that group theory is about groups than the way analysis is about the real numbers.

For instance, the category $\mathcal{S}$ of all sets and functions is a topos, as is the category $\mathcal{S}_f$ of finite sets, but so also are the product category $\mathcal{S} \times \mathcal{S}_f$ and the functor category $\mathcal{S}^{\mathcal{S}_f}$. Indeed, the essentially algebraic character of the topos concept makes it closed under many important operations for constructing new objects out of old ones. An important class of examples is provided by the categories $\mathrm{Sh}(X)$ of all sheaves on a topological space X. From a logical point of view, such a topos can be regarded as consisting of sets varying continuously over the space X. Another kind of topos is the category $\mathcal{S}^G$ of sets A equipped with an action $A \times G \to A$ by a group G, and all equivariant functions; these can be seen as sets that in some sense "respect" the group structure. There are many more kinds of toposes, including ones constructed from various deductive logical systems, and ones determined by different notions of computability. Part A provides the basic definitions and examples, together with some of the most important general facts about the concepts involved. There is also a useful survey of P. Freyd's theory of "allegories", an abstraction of categories of relations which sheds light on the topos concept from a different angle.

The mappings of greatest interest in topos theory are the "geometric morphisms" between toposes, which arise quite naturally in a range of circumstances. If X and Y are topological spaces, for instance, each continuous function $X \to Y$ induces an associated geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$; similarly, a homomorphism of groups $G \to H$ gives rise to a geometric morphism of toposes $\mathcal{S}^G \to \mathcal{S}^H$. But the general notion of a topos then also makes possible geometric morphisms of the form $\mathrm{Sh}(X) \to \mathcal{S}^G$, relating a space X and a group G, along with many others relating logical systems, computation, topology, and algebra. In logic, set-theoretic forcing, boolean-valued and permutation models, and Kripke semantics can also all be described, related, and combined using toposes and geometric morphisms. The basic theory of geometric morphisms including some fundamental factorization theorems (akin to the Noether homomorphism theorem for groups) concludes Part A.

B. 2-categorical aspects of topos theory. This is the most technical part of the book, and will likely be the most difficult for the logical reader to appreciate. It consists of results on indexed, fibered, and internal categories that are motivated mainly by their use elsewhere in the theory. This not to say that the content is of no logical interest, but only that its interest will be more apparent in its applications. The important notion of an internal category in a category, for instance, is related to the idea that a logical system can be formalized as an axiomatic theory in a (meta-)language, i.e. another logical system. This makes it possible to describe the interpretations of a "theory" represented by an internal category $\mathbb{C}$ in the "meta-language" category (perhaps a topos) $\mathcal{S}$ as functors of the form $\mathbb{C} \to \mathcal{S}$, so that the category of all models of the "theory" is a certain subcategory of the functor category $\mathcal{S}^{\mathbb{C}}$ (then also a topos). Such "functorial semantics" (developed in Part D) requires some of the notions developed here. Accordingly, the general reader is advised to refer to results in this part of the book as required for the comprehension of other parts.

The specialist, on the other hand, will welcome the systematic and authoritative presentation of these tools, the development of which was scattered across the journals. In this presentation, for instance, the theory of the 2-category $\mathfrak{BTop}/\mathcal{S}$ of toposes bounded over an arbitrary base topos $\mathcal{S}$ now seems to be quite mature and satisfactory.

C. Toposes as spaces. After the spade-work of the previous two Parts, this volume begins to develop the real fruits of the theory. According to the original conception of Grothendieck and his followers, a topos is a generalized topological space, and the category of toposes and geometric morphisms can replace spaces and continuous mappings for many purposes, thereby extending topological and geometric ideas to algebraic and other situations that give rise to toposes. Accordingly, we first consider the relation between classical topological concepts and toposes.

In passing from a space X to the topos $\mathrm{Sh}(X)$ of all sheaves of sets on X, one needs only the poset $\mathcal{O}(X)$ of open sets of X, not the point set. Specifically, a sheaf F on X is a contravariant functor on $\mathcal{O}(X)$, thus taking each pair of open subsets $U \subseteq V$ to a function between sets $FV \to FU$, satisfying the expected condition on compositions for $V \subseteq W$. To be a sheaf, such a functor must satisfy a "patching condition" for open covers $U = \bigcup_i U_i$ that essentially says it is "continuous," in the sense that FU is determined by the values FU_i. A classical example is the sheaf of continuous, real-valued functions $FU = \mathcal{C}(U, \mathbb{R})$, for which $U \subseteq V$ acts on any $g : V \to \mathbb{R}$ just by restriction to U. Here it is clear that given an open cover $U = \bigcup_i U_i$, there is a unique continuous function $f : U \to \mathbb{R}$ for every family of such $f_i : U_i \to \mathbb{R}$, provided only that all f_i and f_j agree on every $U_i \cap U_j$. What is required here is just the lattice structure of $\mathcal{O}(X)$; thus one also has toposes built from complete Heyting algebras or "frames", which have (many of) the same lattice-theoretic properties as such posets $\mathcal{O}(X)$. A frame can be regarded as a "pointless space," and they share many properties with classical spaces, while encompassing also many other kinds of posets like boolean algebras, Scott domains and Lindenbaum algebras of logical theories (suitably completed). There is an intimate connection between toposes and such "pointless spaces"; logically, boolean-valued models of set theory arise as toposes of this kind.

In this scheme of "pointless topology" a space is determined by a system of open subspace inclusions instead of by a set of points. One can further generalize to "spaces" determined by "covering systems" of maps in a category. There is an associated natural generalization of the notion of a sheaf for such generalized covering families, and the topos of all such sheaves then plays the role of the "space" determined by the system of covering families. An early example of this idea was in algebraic geometry, where covering families of étale maps of schemes were considered, but Cohen forcing provides another example, as do Beth models of intuitionistic logic.

Regarded thus as generalized spaces, toposes and their mappings enjoy many familiar properties of classical spaces. Open maps, connectedness, compactness, and similar concepts apply, and behave in ways analogous to their classical counterparts. In some cases, classical theorems for spaces follow from, and are subsumed under, much more general ones about toposes, which reveal aspects that were not evident in the topological setting. This is the case for the theory of open and proper maps. The appearance of classifying toposes in the treatment of exponentiability reveals connections between logic and topology that are also hinted at in the relation between exponentiability, compactness and definability.

The descent theorems and groupoid representations are a bold extension of Galois theory to toposes, with connections to logical definability and invariance. Beth's definability theorem from elementary model theory is actually a miniature version of some of the results developed here (in this connection see M. Makkai, "Duality and Definability in First-Order Logic", *Memoirs Amer. Math. Soc.* **503**, 1993).

D. Toposes as theories. This final part (to date) is of course the most overtly "logical" of the four, but also connects the logical aspect to the geometrical and algebraic ones.

The basic propositional operations of conjunction, disjunction, negation, etc. of course have an order-theoretic, algebraic meaning, which has long been known and studied, but it was not until the pioneering work of F.W. Lawvere in the 1960s (see e.g. "Adjointness in Foundations", *Dialectica* **23**, 1969, pp. 281–296) that it was known how to extend this treatment to full first-order logic. Lawvere's discovery that *all* of the logical operations are instances of adjointness made it possible for the first time to give a truly algebraic treatment of logic, since adjoint functors are "algebraic" in a precise sense. The following two-way rule of inference for the existential quantifier, for instance,

$$\frac{\varphi(x,y) \vdash \psi(y)}{\exists x.\varphi(x,y) \vdash \psi(y)}$$

says that the existential quantification of x is left adjoint to adding a dummy-variable x, in a calculus of entailments $\vartheta(x,\dots,y) \vdash \vartheta'(x,\dots,y)$ between formulas with a common "context of variables" (a list $(x,\dots,y)$ of which variables may occur freely). Semantically, this is just the fact that taking direct images $f(U)$ of subsets $U \subseteq X$ along a function $f : X \to Y$ is left adjoint to taking inverse images $f^{-1}(V)$ of subsets $V \subseteq Y$,

$$U \subseteq f^{-1}(V) \quad \text{iff} \quad f(U) \subseteq V\,.$$

Given the resulting description of a first-order logical theory $\mathcal{T}$ as an algebra of a certain kind (a suitably structured category), a model M of $\mathcal{T}$ in another such algebra $\mathcal{S}$ then becomes simply a homomorphism $M : \mathcal{T} \to \mathcal{S}$. If $\mathcal{S}$ is e.g. the category of all sets, then the category $\mathrm{Hom}_{\mathrm{Alg}}(\mathcal{T},\mathcal{S})$ of all such algebra homomorphisms is essentially the same as the category of all classical $\mathcal{T}$-models (the notion of natural transformation of functors usually gives the right definition of a homomomorphism *between* $\mathcal{T}$-models). In this framework of "functorial semantics," the sorts of theories usually studied in logic occur as the "free algebras", and more generally, the algebras presented by generators and relations corresponding to the basic non-logical constants of the language and the axioms of the theory.

As in the theory of classifying spaces in topology, every logical theory has a "classifying topos", the geometric morphisms into which correspond uniquely (up to isomorphism) to models of the theory. This remarkable fact establishes a fundamental connection between functorial semantics, classifying spaces in topology, and the theory of algebraic extensions, that cuts three ways, as it were. It makes it possible to import logical methods into algebra and topology, and conversely, in ways that are only beginning to be exploited (see e.g. I. Moerdijk, "Classifying Spaces and Classifying Topoi", *Lecture Notes in Math.* **1616**, Springer-Verlag, 1995).

One issue that deserves mentioned here is the appearance of intuitionistic logic in connection with categorical semantics and toposes. It appears, not for any philosophical reasons, but for practical, mathematical ones, related to the effect of continuous variation and the "internalization" of logic. That intuitionism should thus reappear is a surprising and fascinating mathematical development which should be of especially great interest to logicians, who can thereby relate their methods and results to naturally occurring phenomena in other branches of mathematics.

The higher-order and type-theoretic structure of toposes results, as already mentioned, from cartesian closure, which is the algebraic basis underlying the so-called "Curry-Howard Isomorphism" familiar to logicians. Toposes provide convenient models for systems of higher-order logic, as well as various type theories like the typed lambda-calculus and the dependent type-theory of Martin-Löf. Moreover, these theories are actually *deductively complete* with respect to topos models (Gödel-incompleteness not withstanding!).

But perhaps even more importantly, the usual systems of type theory and higher-order logic are, of course, also *sound* with respect to topos semantics, permitting the use of such theories to reason about objects and structures in any toposes. It can thus be of real practical benefit to, say, a geometer using toposes for some geometric purpose to have at his disposal the logical calculus of dependent type theory or higher-order logic. This sort of approach has been pursued in some parts of algebraic topology and geometry, and more systematically in what is called "synthetic" differential geometry. Some of these topics will presumably be addressed in Volume Three, which should be of particular interest to readers of this BULLETIN.

Throughout, the exposition is rigorous without being formal; it is also respectful and pleasantly literary. The style overall is more Halmos than Lang, and it meets the standard of clarity set by the author's lucid *Stone Spaces* (Cambridge University Press, 1982). Definitions and proofs are transparent, and diagrams are used effectively, if not profusely, in the contemporary style (as opposed to the more diagrammatic fashion evident in some older category theory). The mathematical prerequisites are moderately high (this is no textbook), but appropriate to the intended audience of research logicians and mathematicians who are not specialists in category theory; Mac Lane's *Categories for the Working Mathematician* (Springer-Verlag, 1971) surely suffices. While not intended to document the historical developments or priority in results, the suggestions for further reading provided in each section form a useful guide to the literature, and the bibliography is the most comprehensive one currently available in the field.

With the completion of this mammoth undertaking, Professor Johnstone will have performed an invaluable service to the topos theory community by bringing in the harvest of their collective labors over the last four decades. May Logic reap some of the fruits.