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ABSTRACT. Moerdijk has introduced a topos of sheaves on a category of filters. Following his suggestion, we prove that its double negation subtopos is the topos of sheaves on the subcategory of ultrafilters - the *ultrasheaves*. We then use this result to establish a double negation translation of results between the topos of ultrasheaves and the topos on filters.

1. INTRODUCTION

In 1993 I. Moerdijk [6] introduced a model of constructive nonstandard arithmetic in the topos Sh($\mathbb{F}$), of sheaves on a category of filters for a certain Grothendieck topology J. Further contributions to this model were made by I. Moerdijk and E. Palmgren [7] and Palmgren [9, 10, 11, 12]. A previous work by the second author [2] studies the sheaves on the full subcategory of ultrafilters, $\mathbb{U}$, hence called *ultrasheaves.* The resulting topos is Boolean, so its internal logic is no longer intuitionistic, but it is a model of nonstandard set theory. In fact it is a model of Nelsons internal set theory, see Nelson [8], an axiomatization of nonstandard set theory.

The question arises what the exact relationship is between the topos of ultrasheaves, Sh($\mathbb{U}$), and Sh($\mathbb{F}$)? The subcategory $\mathbb{U}$ is "large" in $\mathbb{F}$, in the sense that it is a generating family for $\mathbb{F}$. We also know that "many" sheaves (namely the representable ones) on $\mathbb{F}$ are still sheaves when restricted to $\mathbb{U}$. Moerdijk conjectured that Sh($\mathbb{U}$) is the double negation subtopos of Sh($\mathbb{F}$) and in this paper we show that this is true.

Given a (intuitionistic) logic one can force it to become classical by adding the law of excluded middle to the assumptions. For a topos of sheaves there is a corresponding transformation, namely by adding the double negation topology to the underlying site. Not all of the original sheaves will be sheaves with respect to the new topology, but the internal logic in the resulting topos of sheaves will be classical.

In the second section of this paper we collect some definitions and results we will need subsequently. Then, in the third section, we prove that the topos Sh($\mathbb{U}$) is equivalent to a topos of sheaves on $\mathbb{F}$ for a finer

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topology than J, thereby showing that $\mathrm{Sh}(\mathbb{U})$ is in fact a subtopos of $\mathrm{Sh}(\mathbb{F})$. This is, of course, also useful in a setting (e.g. constructive mathematics) where you want to avoid using ultrafilters.

In the following foutrth section we prove that this smaller topos on $\mathbb{F}$ is in fact equivalent to the double negation subtopos of $\mathrm{Sh}(\mathbb{F})$. Finally, in the fifth section we establish a double negation translation of results between $\mathrm{Sh}(\mathbb{U})$ and $\mathrm{Sh}(\mathbb{F})$.

2. Preliminary definitions and results

Definition 2.1. The category $\mathbb{F}$ has as *objects* pairs $(A, \mathcal{F})$, where A is a set and $\mathcal{F}$ a filter on A. The *morphisms* $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ are equivalence classes of partial functions $\alpha : A \to B$ such that

(i) α is defined on some $F \in \mathcal{F}$,
(ii) $\alpha^{-1}(G) \in \mathcal{F}$, for all $G \in \mathcal{G}$.

Two such partial functions $\alpha : F \to B$ and $\alpha' : F' \to B$ are equivalent if there is $E \subseteq F \cap F'$ in $\mathcal{F}$ such that $\alpha|_E = \alpha'|_E$.

This category of filters $\mathbb{F}$ was introduced by V. Koubek and J. Reiterman [4] and studied further by A. Blass [1].

Note that for almost all equivalence class $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ there is a total continuous function $f : A \to B$ representing α. The only exception is if B is the empty set. Then there is a morphism $\alpha : (A, \mathcal{F}) \to (\emptyset, \{\emptyset\})$ only if the filter $\mathcal{F}$ contains $\emptyset$ (we say that $\mathcal{F}$ is *improper*). In this case, α is the unique such morphism and an isomorphism $\mathcal{F} \cong (\emptyset, \{\emptyset\})$. The filter on the empty set, $(\emptyset, \{\emptyset\})$, is the initial object 0 in $\mathbb{F}$. Terminal object 1 is $(\{0\}, \{\{0\}\})$.

From Koubek and Reiterman [4] we have the following useful characterizations:

Proposition 2.2. *For morphisms $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ we have:*

(i) *α is mono if and only if there is a $F \in \mathcal{F}$ such that α is injective on F,*
(ii) *α is epi if and only if $\alpha(\mathcal{F}) = \mathcal{G}$.*

These characterizations hold true also in $\mathbb{U}$, but the situation is further simplified by the fact that all morphisms in $\mathbb{U}$ are epi.

Moerdijk (in [6]) defined a subcanonical Grothendieck topology J on $\mathbb{F}$ as follows:

Definition 2.3. A finite family $\{\alpha_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ is a *J-covering* if the induced map

$$\mathcal{G}_1 + \ldots + \mathcal{G}_n \to \mathcal{F}$$

is an epimorphism.

Over the resulting site he studied, in particular, the representable sheaves of the form ${}^*S = \mathrm{Hom}_{\mathbb{F}}(-, (S, \{S\}))$. At any filter $\mathcal{F}$, ${}^*S(\mathcal{F})$ is the reduced power of S over $\mathcal{F}$. Thus restricting the underlying category to the full subcategory $\mathbb{U}$ one can study ultrapowers as sheaves.

For the ultrafilters in $\mathbb{F}$ we have the following result from Palmgren [10]:

Theorem 2.4.

(i) *Any morphism from a proper filter to an ultrafilter is a covering map.*

(ii) *Any cover of an ultrafilter contains a single map covering the ultrafilter.*

The topology induced on $\mathbb{U}$ by $(\mathbb{F}, J)$ is the atomic topology. In Eliasson [2] it is proved that all representable sheaves on $\mathbb{F}$ are still sheaves when restricted to $\mathbb{U}$. Thus the atomic topology is subcanonical.

We now turn our interest to the internal logics of the topoi $\mathrm{Sh}(\mathbb{F})$ and $\mathrm{Sh}(\mathbb{U})$. For more details see Palmgren [10] and Eliasson [2].

Let L be a first order language and $I = \langle S, R_1, R_2, \ldots, f_1, \ldots, c_1, \ldots \rangle$ an L-structure. Let ${}^*I$, the *-transform of I, be the L-structure in $\mathrm{Sh}(\mathbb{U})$ defined as follows:

- Set S: ${}^*S$ the representable sheaf previously defined.
- Constant $s \in S$: ${}^*s$ constant function
$$\lambda x.s \in {}^*S(\mathcal{U}).$$
- Relation $R \subseteq S$: ${}^*R$ subsheaf of ${}^*S$ given at $\mathcal{U}$ by
$$\alpha \in {}^*R(\mathcal{U}) \iff (\exists U \in \mathcal{U})(\forall x \in U)\alpha(x) \in R.$$
- Function $f : T \to S$: ${}^*f$ representable natural transformation from ${}^*T$ to ${}^*S$ given at $\mathcal{U}$ by
$${}^*f_{\mathcal{U}}(\alpha) = \lambda x.f(\alpha(x)).$$

We also define what it means to be *standard* for a $\gamma \in {}^*S(\mathcal{U})$:

- $\mathrm{St}(\gamma)$ if and only if γ is constant on some $U \in \mathcal{U}$.

Thus every L-structure I (in Sets) gives rise to an $L \cup \{\mathrm{St}\}$-structure ${}^*I$ in $\mathrm{Sh}(\mathbb{U})$.

We have the usual interpretation of the the logical symbols in the two Grothendieck topoi. Below we give the sheaf semantics for $\mathrm{Sh}(\mathbb{U})$ in full detail. For the more complicated case $\mathrm{Sh}(\mathbb{F})$ we refer the reader to Palmgren [10].

Theorem 2.5. *Let $\mathcal{U}$ be an ultrafilter, Φ and Ψ arbitrary formulas and $\alpha \in {}^*T(\mathcal{U})$. Then*

(i) $\mathcal{U} \Vdash \Phi(\alpha) \wedge \Psi(\alpha)$ *if and only if* $\mathcal{U} \Vdash \Phi(\alpha)$ *and* $\mathcal{U} \Vdash \Psi(\alpha)$,

(ii) $\mathcal{U} \Vdash \Phi(\alpha) \vee \Psi(\alpha)$ *if and only if* $\mathcal{U} \Vdash \Phi(\alpha)$ *or* $\mathcal{U} \Vdash \Psi(\alpha)$,

(iii) $\mathcal{U} \Vdash \Phi(\alpha) \to \Psi(\alpha)$ *if and only if* $\mathcal{U} \Vdash \Phi(\alpha)$ *implies* $\mathcal{U} \Vdash \Psi(\alpha)$,

(iv) $\mathcal{U} \Vdash \neg\Phi(\alpha)$ *if and only if* $\mathcal{U} \nVdash \Phi(\alpha)$,

(v) $\mathcal{U} \Vdash (\exists x \in {}^*S)\Phi(\alpha, x)$ *if and only if for some* $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$

$$\mathcal{V} \Vdash \Phi(\alpha \circ \beta, \delta),$$

(vi) $\mathcal{U} \Vdash (\forall x \in {}^*S)\Phi(\alpha, y)$ *if and only if for all* $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$

$$\mathcal{V} \Vdash \Phi(\alpha \circ \beta, \delta).$$

As is evident in the theorem above, the internal logic in $\mathrm{Sh}(\mathbb{U})$ is classical, i.e. the topos is Boolean. Now we state the fundamental theorem for topoi on filters:

Theorem 2.6 (Moerdijk). *Let* $\mathcal{F}$ *be a filter,* Θ *an L-formula and* $\alpha \in {}^*S(\mathcal{U})$*. Then*

$$\mathcal{F} \Vdash {}^*\Theta(\alpha) \textit{ if and only if } (\exists F \in \mathcal{F})(\forall x \in F)\Theta(\alpha(x)).$$

This result is proved by Moerdijk in [6] for $\mathrm{Sh}(\mathbb{F})$ and by the second author in [2] for $\mathrm{Sh}(\mathbb{U})$. The theorem plays a role analogous to Łoś's theorem, which follows from it, see Eliasson [3].

3. $\mathrm{Sh}(\mathbb{U})$ is equivalent to a topos of sheaves on $\mathbb{F}$

We will study the topos $\mathrm{Sh}(\mathbb{U})$ of ultrasheaves and its relation to sheaves on the category $\mathbb{F}$ of filters. For clarity let A be the atomic topology on $\mathbb{U}$. We first define a new topology J_∞ on $\mathbb{F}$.

Definition 3.1. A basis for the J_∞*-topology* are small families $\{\alpha_i : \mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ (for any set I) such that the induced morphism

$$\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$$

is epic.

Note that from Blass [1] we know that the category $\mathbb{F}$ has all coproducts. Now the following theorem holds:

Theorem 3.2. $\mathrm{Sh}(\mathbb{U}, A) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$.

To prove the theorem we will need three lemmas.

Lemma 3.3. $(\mathbb{F}, J_\infty)$ *is a subcanonical site.*

Proof. Any epi in $\mathbb{F}$ is regular [6, Lemma 1.2]. Hence the covering map $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is regular, and the topology subcanonical. □

Lemma 3.4. *The collection of ultrafilters in* $\mathbb{F}$ *generates* $\mathbb{F}$*.*

See Eliasson [2, Prop. 2.2] for a proof, the details of which also imply the following.

Lemma 3.5. *Every object in* $\mathbb{F}$ *is covered (in the sense of* J_∞*) by objects in* $\mathbb{U}$*.*

Now the theorem follows by the *Comparison Lemma* (see, for instance, Mac Lane and Moerdijk [5]). It gives that the restriction $\mathbf{Sets}^{\mathbb{F}^{op}} \to \mathbf{Sets}^{\mathbb{U}^{op}}$ induces an equivalence of categories $\mathrm{Sh}(\mathbb{U}, A) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$.

4. $\mathrm{Sh}(\mathbb{U})$ is the double negation subtopos of $\mathrm{Sh}(\mathbb{F}, J)$

In this section we prove that $\mathrm{Sh}(\mathbb{U})$ is the double negation subtopos of $\mathrm{Sh}(\mathbb{F}, J)$. Instead of working with sheaves relative the Grothendieck topology J we will work with the (equivalent) Lawvere-Tierney topology j on $\mathbf{Sets}^{\mathbb{F}^{op}}$.

A presheaf F in $\mathbf{Sets}^{\mathbb{F}^{op}}$ is a *j-sheaf*, with respect to a topology j, if for every dense monomorphism $m : A \to E$ in $\mathbf{Sets}^{\mathbb{F}^{op}}$, every map $A \to F$ extends uniquely to a map $E \to F$.

We will prove that the $j_{\neg\neg}$-sheaves are the same as the j_∞-sheaves in two steps. First we prove that a subpresheaf of a representable sheaf is dense with respect to the topology $j_{\neg\neg}$ if and only if the $\neg\neg$-closure of it is j-dense. Then we prove that the latter are exactly the dense subobjects with respect to j_∞. Note that it is enough to prove this for subobjects of representable sheaves.

We will prove both lemmas working with sieves on a filter, rather than in the Heyting algebra of subobjects. So, we will list some sieve formulations of topological and algebraical concepts.

- A sieve on $\mathcal{F}$ is a subpresheaf $A \to \mathbf{y}(\mathcal{F})$.
- The j-closure of A, which is the sheafification of A, is the set:

$$\begin{aligned}\overline{A}^j =& \{h : \mathcal{G} \to \mathcal{F} \mid h^*A \in J(\mathcal{G})\} \\ =& \{h : \mathcal{G} \to \mathcal{F} \mid \exists \{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n \in J(\mathcal{G}) \\ & \text{ such that } h \circ g_i \in A,\ i = 1, \dots n\}.\end{aligned}$$

- A is j-dense if and only if A is a J-covering sieve of $\mathcal{F}$.
- If B is also a sieve on $\mathcal{F}$ then

$$\begin{aligned}A \Rightarrow B = \{h : \mathcal{G} \to \mathcal{F} \mid \forall g : \mathcal{H} \to \mathcal{G} \\ h \circ g \in A \Rightarrow h \circ g \in B\},\end{aligned}$$

  which is a sieve on $\mathcal{F}$.

We know that the double negation closure of a subpresheaf A, $\neg\neg A$, is $(A \Rightarrow 0) \Rightarrow 0$ and this can be calculated as

$$\neg\neg A = \{h : \mathcal{G} \to \mathcal{F} \mid \forall g : \mathcal{H} \to \mathcal{G}\, \exists f : \mathcal{H}' \to \mathcal{H} \text{ such that } h \circ g \circ f \in A\}.$$

Moreover, from Mac Lane and Moerdijk [5, VI Lemma 1.2], we have that the double negation (in $\mathrm{Sh}(\mathbb{F}, J)$) of a j-sheaf E is $(E \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j$.

Here $\overline{0}^j$ is the sheafification of the empty presheaf which is isomorphic to $\mathbf{y}(0)$, the initial object in $\mathrm{Sh}(\mathbb{F}, J)$. To be precise, as a subobject of

$\mathbf{y}(\mathcal{F})$:

$$\overline{0}^j(\mathcal{G}) = \begin{cases} \{!_{\mathcal{F}} \circ f\} & \text{if } \mathcal{G} \text{ is improper (i.e. isomorphic to 0),} \\ \emptyset & \text{if } \mathcal{G} \text{ is proper.} \end{cases}$$

Here $f : \mathcal{G} \to 0$ is an isomorphism.

We will prove that, for a subpresheaf $A \to \mathbf{y}(\mathcal{F})$ of a j-sheaf $\mathbf{y}(\mathcal{F})$ we have:

$$(\overline{A}^j \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j = 1_{\mathbf{y}(\mathcal{F})} \text{ if and only if } \overline{(A \Rightarrow 0) \Rightarrow 0}^j = 1_{\mathbf{y}(\mathcal{F})}.$$

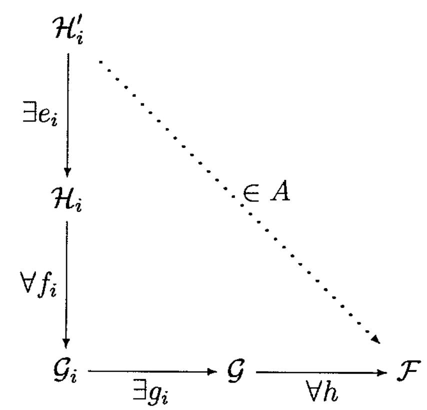
First we assume that $\mathcal{F}$ is proper.

The righthand side says that for all $h : \mathcal{G} \to \mathcal{F}$ we have $h \in \overline{(A \Rightarrow 0) \Rightarrow 0}^j$. Hence $\forall h : \mathcal{G} \to \mathcal{F}\ \exists\{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n \in J(\mathcal{G})$ such that $h \circ g_i \in \neg\neg A$ for all $i = 1, \ldots, n$.

Hence we get the following condition:

$$\begin{aligned} \forall h : \mathcal{G} \to \mathcal{F}\, \exists\{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n \in J(\mathcal{G}) \text{ such that, for any } i \in \{1, \ldots, n\}, \\ \forall f_i : \mathcal{H}_i \to \mathcal{G}_i\, \exists e_i : \mathcal{H}'_i \to \mathcal{H}_i \text{ such that } h \circ g_i \circ f_i \circ e_i \in A. \quad (1) \end{aligned}$$

We illustrate this in a commutative diagram:



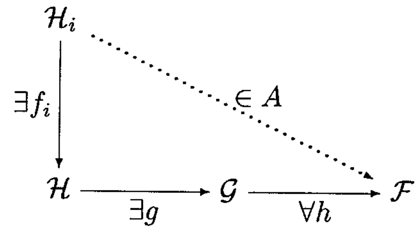
Remember that $\{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n$ is a J-cover of $\mathcal{G}$.

The lefthand side is equivalent to $\overline{A}^j \Rightarrow \overline{0}^j \leq \overline{0}^j$. We will study it pointwise, at a filter $\mathcal{G}$, i.e. $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G}) \leq \overline{0}^j(\mathcal{G})$. We see that $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G}) = \{h : \mathcal{G} \to \mathcal{F} \mid \forall g : \mathcal{H} \to \mathcal{G}\ h \circ g \in \overline{A}^j(\mathcal{H}) \Rightarrow h \circ g \in \overline{0}^j(\mathcal{H})\}$. If the filter $\mathcal{G}$ is proper, then $\overline{0}^j(\mathcal{G})$ is empty, and hence $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G})$ is also empty.

We then have that:

$$\begin{aligned} &\forall h : \mathcal{G} \to \mathcal{F}\, \exists g : \mathcal{H} \to \mathcal{G} \text{ such that } h \circ g \notin \overline{0}^j(\mathcal{H}) \\ &\qquad \text{and } \exists\{f_i : \mathcal{H}_i \to \mathcal{H}\}_{i=1}^n \in J(\mathcal{H}) \text{ such that,} \\ &\qquad\qquad \text{for any } i \in \{1, \ldots, n\}, h \circ g \circ f_i \in A. \quad (2) \end{aligned}$$

We illustrate this case too, with a commutative diagram:



Here we assume that $\mathcal{G}$ is proper.

Lemma 4.1. *For a subpresheaf $A \to \mathbf{y}(\mathcal{F})$ of a j-sheaf $\mathbf{y}(\mathcal{F})$ we have:*

$$(\overline{A}^j \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j = 1_{\mathbf{y}(\mathcal{F})} \text{ if and only if } \overline{(A \Rightarrow 0) \Rightarrow 0}^j = 1_{\mathbf{y}(\mathcal{F})}.$$

Proof. If the filter $\mathcal{F}$ is improper then $\mathbf{y}(\mathcal{F})$ is isomorphic to its subsheaf $\overline{0}^j$. But both $(\overline{A}^j \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j$ and $\overline{(A \Rightarrow 0) \Rightarrow 0}^j$ are j-sheaves and, thus, greater than or equal to $\overline{0}^j$. Hence both sides of the equation are true, and therefore equivalent.

Now assume $\mathcal{F}$ is proper. Then we have the descriptions ((1) and (2) above) of the left- and righthand sides of the relation, and the scene is set for proving the equivalence:

"$\Longrightarrow$": Note that it is enough to find a cover on $\mathcal{F}$ (because of the stability of the topology J). Let $\mathcal{F}$ be covered by the identity $i : \mathcal{F} \to \mathcal{F}$. Take any $f : \mathcal{G} \to \mathcal{F}$ and prove that there is a $e : \mathcal{H} \to \mathcal{G}$ such that $i \circ f \circ e \in A$.

If $\mathcal{G}$ is improper then $f : \mathcal{G} \to \mathcal{F}$ is already in A, and you can take e as the identity. If $\mathcal{G}$ is proper then by assumption, given $f : \mathcal{G} \to \mathcal{F}$, there is $g : \mathcal{H} \to \mathcal{G}$ and $f_1 : \mathcal{H}_1 \to \mathcal{H}$ such that $f \circ g \circ f_1 \in A$. Hence, let $e = g \circ f_1$.

"$\Longleftarrow$": Take any $h : \mathcal{G} \to \mathcal{F}$. If $\mathcal{G}$ is improper prove that $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G}) \leq \overline{0}^j(\mathcal{G})$. Note that $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G})$ contains at most one map, since there is only one map $h : \mathcal{G} \to \mathcal{F}$. But this map $!_{\mathcal{F}} \circ f : \mathcal{G} \to \mathcal{F}$ (where $f : \mathcal{G} \to 0$ isomorphism) is in $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G})$ since, for any $g : \mathcal{H} \to \mathcal{G}$, $(!_{\mathcal{F}} \circ f) \circ g = !_{\mathcal{F}} \circ (f \circ g) \in \overline{0}^j(\mathcal{H})$.

If $\mathcal{G}$ is proper then find a $g : \mathcal{H} \to \mathcal{G}$ and a cover $\{f_i\}$ of $\mathcal{H}$ such that $h \circ g \circ f_i \in A$. By assumption there are $g_1 : \mathcal{G}_1 \to \mathcal{G}$ and $e_1 : \mathcal{H}_1 \to \mathcal{G}_1$ such that $h \circ g_1 \circ id \circ e_1 \in A$. Let $g = g_1 \circ e_1$ and the identity $i : \mathcal{H}_1 \to \mathcal{H}_1$ be a covering. Then we have $h \circ g \circ i \in A$. □

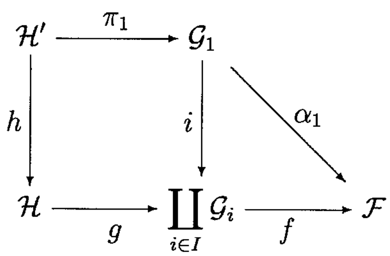
Our second lemma proves that the righthand side in the lemma above is equivalent to being j_∞-dense.

Lemma 4.2. *A subpresheaf A of a j-sheaf $\mathbf{y}(\mathcal{F})$ is j_∞-dense if and only if $\neg\neg A$ is j-dense.*

Proof. If the filter $\mathcal{F}$ is improper then $\mathbf{y}(\mathcal{F})$ is isomorphic to its subsheaf $\overline{0}^j$. But both $\overline{A}^{j_\infty}$ and $\overline{(A \Rightarrow 0) \Rightarrow 0}^j$ are j-sheaves and, thus, greater than or equal to $\overline{0}^j$. Hence both sides of the equation are true, and therefore equivalent. Now assume $\mathcal{F}$ is proper.

"$\Longrightarrow$": Take $\{\alpha_i : \mathcal{G}_i \to \mathcal{F}\}_{i\in I}$ a J_∞-covering in A. Prove that the induced map $f : \coprod_{i\in I} \mathcal{G}_i \to \mathcal{F}$ is in $\neg\neg A$. Take any $g : \mathcal{H} \to \coprod_{i\in I} \mathcal{G}_i$. Consider $i : \mathcal{G}_1 \to \coprod_{i\in I} \mathcal{G}_i$ (observe that we have $f \circ i = \alpha_1$).

Next take the pullback of $g : \mathcal{H} \to \coprod_{i\in I} \mathcal{G}_i$ and $i : \mathcal{G}_1 \to \coprod_{i\in I} \mathcal{G}_i$. Call the pullback $\mathcal{H}'$ and the projection on $\mathcal{H}$, $h : \mathcal{H}' \to \mathcal{H}$ as indicated in the diagram.

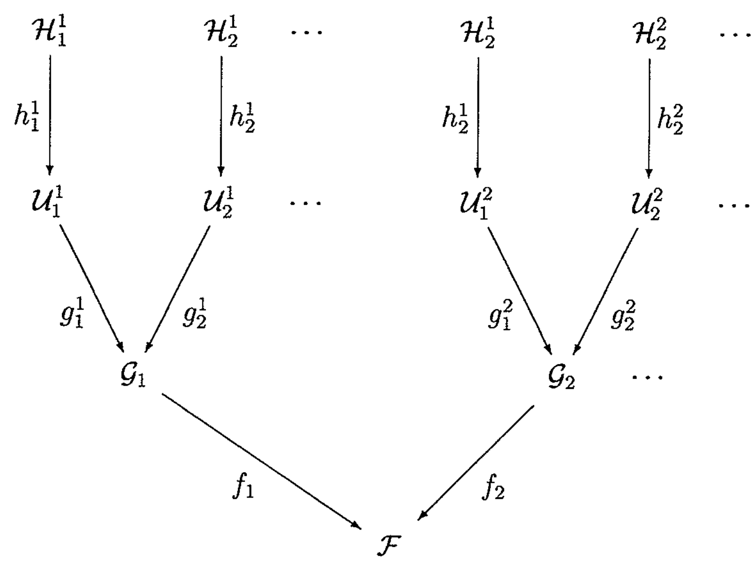


Then we have $f \circ (g \circ h) = (f \circ i) \circ \pi_1 = \alpha_1 \circ \pi_1 \in A$, since A is a sieve and $\alpha_1 \in A$. But $f : \coprod_{i\in I} \mathcal{G}_i \to \mathcal{F}$ is an epimorphism, and hence a J-covering of $\mathcal{F}$.

"$\Longleftarrow$": Take $\{f_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ a J-covering in $\neg\neg A$. We know that for every $\mathcal{G}_i$ there is a J_∞-covering of ultrafilters $\{g^i_j : \mathcal{U}^i_j \to \mathcal{G}_i\}_{j\in I_i}$ (Lemma 3.5).

Now since f_i is in $\neg\neg A$ there are $h^i_j : \mathcal{H}^i_j \to \mathcal{U}^i_j$, for $i = 1, \ldots, n$, $j \in I_i$, such that $f_i \circ g^i_j \circ h^i_j \in A$. But the families $\{f_i\}$ and $\{g^i_j\}$ are jointly epimorphic and the h^i_j:s are epimorphisms (since $\mathcal{U}^i_j$ ultrafilter) and, hence, the family $\{f_i \circ g^i_j \circ h^i_j : \mathcal{H}^i_j \to \mathcal{F}\}$ is jointly epimorphic and a J_∞-covering of $\mathcal{F}$.

The proof is illustrated in this commutative diagram.



□

By Lemma 4.2 we have that a subpresheaf A of a representable sheaf is j_∞-dense if and only if its $\neg\neg$-closure is j-dense. By Lemma 4.1 we have that the $\neg\neg$-closure is j-dense if and only if the j-closure of A is $\neg\neg$-dense (in $\mathrm{Sh}(\mathbb{F}, J)$). Hence, maps from j_∞-dense subobjects of a sheaf extends to F if and only if maps from $j_{\neg\neg}$-dense subobjects extends to F. This gives that $\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$. Together with the result from section 3 we get the desired result:

Theorem 4.3. *A presheaf F is in $\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J)$ if and only if it is in $\mathrm{Sh}(\mathbb{F}, J_\infty)$, and $\mathrm{Sh}(\mathbb{F}, J_\infty)$ is equivalent to $\mathrm{Sh}(\mathbb{U})$, thus*

$$\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J) \cong \mathrm{Sh}(\mathbb{U}).$$

5. The double negation translation

In this section we show how the previous result can be used to translate true formulas between the topoi $\mathrm{Sh}(\mathbb{U})$ and $\mathrm{Sh}(\mathbb{F})$. Since this is a translation between classical and intuitionistic logic it takes the form of a double negation translation. The translation is fairly routine, but we have not found it in the literature on $\neg\neg$-sheaves.

Between $\mathrm{Sh}_{\neg\neg}(\mathbb{F})$ and $\mathrm{Sh}(\mathbb{F})$ there is a geometric morphism

$$\mathrm{Sh}_{\neg\neg}(\mathbb{F}) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Sh}(\mathbb{F})$$

consisting of the factors sheafification (with respect to the topology $\neg\neg$) $a : \mathrm{Sh}(\mathbb{F}) \to \mathrm{Sh}_{\neg\neg}(\mathbb{F})$ and inclusion $i : \mathrm{Sh}_{\neg\neg}(\mathbb{F}) \to \mathrm{Sh}(\mathbb{F})$.

For any sheaf F in $\mathrm{Sh}(\mathbb{F})$ we have the corresponding maps

$$\mathrm{Sub}_{\neg\neg}(F) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Sub}(F).$$

The sheafification now corresponds to closure with respect to the double negation topology, previously written $\neg\neg(\cdot) : \mathrm{Sub}(F) \to \mathrm{Sub}_{\neg\neg}(F)$. The inclusion map of course acts as the identity on the closed subobjects of F. But given a first order formula $\Theta(\alpha)$, with a free variable of the sort F, the interpretations of $\Theta(\alpha)$ in $\mathrm{Sub}_{\neg\neg}(F)$ and $\mathrm{Sub}(F)$ will be different, since the interpretations of the logical symbols are different in the two topoi.

Therefore we will write $[\Theta(\alpha)]$ for the formula $\Theta(\alpha)$ interpreted in $\mathrm{Sub}(F)$ (intuitionistic logic) and $<\Theta(\alpha)>$ for the same formula interpreted in $\mathrm{Sub}_{\neg\neg}(F)$ (classical logic).

We will now prove translation results in both directions: if a formula is true in $\mathrm{Sh}_{\neg\neg}(\mathbb{F})$ then its double negation translation is true in $\mathrm{Sh}(\mathbb{F})$ and, if a formula without any universal quantifier is true in $\mathrm{Sh}(\mathbb{F})$ then it is also true in $\mathrm{Sh}_{\neg\neg}(\mathbb{F})$. We will also give an example of why we can not have a full translation result in the second direction.

Theorem 5.1. *Let $\Theta(\alpha)$ be a first order formula with a free variable of the sort F. Then, if $\Theta(\alpha)$ is true in $\mathrm{Sh}_{\neg\neg}(\mathbb{F})$, its double negation translation $\Theta'(\alpha)$ is true in $\mathrm{Sh}(\mathbb{F})$.*

Here $\Theta'(\alpha)$ is the well-known double negation translation, defined below.

Definition 5.2. Given a formula $\Theta(\alpha)$ we define $\Theta'(\alpha)$ by the structural induction:

$$\begin{aligned}
&\Theta(\alpha) \equiv \top \text{ then } \Theta'(\alpha) = \top,\\
&\Theta(\alpha) \equiv \bot \text{ then } \Theta'(\alpha) = \bot,\\
&\Theta(\alpha) \equiv \Phi(\alpha) \wedge \Psi(\alpha) \text{ then } \Theta'(\alpha) = \Phi'(\alpha) \wedge \Psi'(\alpha),\\
&\Theta(\alpha) \equiv \Phi(\alpha) \vee \Psi(\alpha) \text{ then } \Theta'(\alpha) = \neg(\neg\Phi'(\alpha) \wedge \neg\Psi'(\alpha)),\\
&\Theta(\alpha) \equiv \Phi(\alpha) \to \Psi(\alpha) \text{ then } \Theta'(\alpha) = \Phi'(\alpha) \to \Psi'(\alpha),\\
&\Theta(\alpha) \equiv \neg\Phi(\alpha) \text{ then } \Theta'(\alpha) = \neg\Phi'(\alpha),\\
&\Theta(\alpha) \equiv \exists x\, \Phi(\alpha, x) \text{ then } \Theta'(\alpha) = \neg\forall x\, \neg\Phi'(\alpha, x),\\
&\Theta(\alpha) \equiv \forall x\, \Phi(\alpha, x) \text{ then } \Theta'(\alpha) = \forall x\, \Phi'(\alpha, x).
\end{aligned}$$

Lemma 5.3. *Let $\Theta(\alpha)$ be a first order formula. Then $i(<\Theta'(\alpha)>) = [\Theta'(\alpha)]$.*

Proof. We know that $i : \mathrm{Sub}_{\neg\neg}(F) \to \mathrm{Sub}(F)$ preserves 1 (i.e. $\top$), $\wedge$, $\to$ and $\forall x$ since it preserves Π-functors. Note that it also preserves 0 (i.e. $\bot$). The inclusion i also preserves equations and, hence, all atomic formulas. Using the bracket notation introduced above we therefore have: $i(<\bot>) = [\bot]$, $i(<\Phi \wedge \Psi>) = i(<\Phi>) \wedge i(<\Phi>)$ etc.

Now we will prove the lemma by induction on the definition of Θ'. We will only look at the case of the existential quantifier:

$$\begin{aligned}
i(<\neg\forall x\, \neg\Phi'(\alpha, x)>) &= \neg i(<\forall x\, \neg\Phi'(\alpha, x)>)\\
&= \neg\forall x\, i(<\neg\Phi'(\alpha, x)>)\\
&= \neg\forall x\, \neg i(<\Phi'(\alpha, x)>)\\
&= \neg\forall x\, \neg[\Phi'(\alpha, x)] \text{ (from the IH)}\\
&= [\neg\forall x\, \neg\Phi'(\alpha, x)].
\end{aligned}$$

□

Now we will prove the theorem.

Proof. We want to prove that if $<\Theta(\alpha)>= 1$ then $[\Theta'(\alpha)] = 1$.

Assume $<\Theta(\alpha)> = 1$. Since classical logic holds in $\mathrm{Sub}_{\neg\neg}(F)$ we know that $<\Theta(\alpha)> = <\Theta'(\alpha)>$. Hence $<\Theta'(\alpha)> = 1$. By the lemma above $[\Theta'(\alpha)] = i(<\Theta'(\alpha)>) = i(<\top>) = [\top] = 1$. □

For the translation in the other direction we will need a lemma.

Lemma 5.4. *Let $\Theta(\alpha)$ be a formula without universal quantifiers that is built up from double negation stable predicates. Then $a([\Theta(\alpha)]) = <\Theta(\alpha)>$.*

Proof. We know that $a : \mathrm{Sub}(F) \to \mathrm{Sub}_{\neg\neg}(F)$ preserves $\top$, $\wedge$, $\vee$ and $\exists x$. The functor a also preserves equations and, hence, all atomic formulas.

But a corresponds to double negation closure $(\neg\neg(\cdot))$ and also preserves $\perp$, since $a([\perp]) = \neg\neg 0 = 0 =<\perp>$, $\neg$, since negation in $\mathrm{Sub}_{\neg\neg}(F)$ is $(\cdot) \to \neg\neg 0 = (\cdot) \to 0 = \neg(\cdot)$, and it preserves implication, since $\neg\neg(\Phi \to \Psi) = \neg\neg\Phi \to \neg\neg\Psi$.

Using the assumption that all predicates are $\neg\neg$-stable, we can conclude, using induction, that $a([\Theta(\alpha)]) =<\Theta(\alpha)>$. □

Theorem 5.5. *Let $\Theta(\alpha)$ be a first order formula with a free variable of the sort F. Assume $\Theta(\alpha)$ is without universal quantifiers and has double negation stable predicates. Then, if $\Theta(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{F})$, *$\Theta(\alpha)$ is also true in* $\mathrm{Sh}_{\neg\neg}(\mathbb{F})$.

Proof. We assume that $[\Theta(\alpha)] = 1$. Then $<\Theta(\alpha)>= a([\Theta(\alpha)]) = 1$. □

Of course, classically any formula is equivalent to a formula without universal quantifiers, so we have as an easy corollary:

Corollary 5.6. *For every first order $\Theta(\alpha)$ formula with a free variable of the sort F and with double negation stable predicates, there is a classically equivalent formula $\Theta^+(\alpha)$ such that if $\Theta^+(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{F})$ *then $\Theta(\alpha)$ is true in* $\mathrm{Sh}_{\neg\neg}(\mathbb{F})$.

Theorem 5.5 cannot be extended to include universal quantifiers, as can be seen by considering the following fact. In Moerdijk [6] it is shown that

$$\mathrm{Sh}(\mathbb{F}) \models \neg(\forall x \in {}^*\mathbb{N})[\mathrm{St}(x) \vee \neg\mathrm{St}(x)].$$

Note that the ${}^*\mathbb{N}$ and standard predicates are double negation stable.

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