Technical Report CMU-Phil-72

Using D-separation to Calculate Zero Partial Correlations in Linear Models with Correlated Errors

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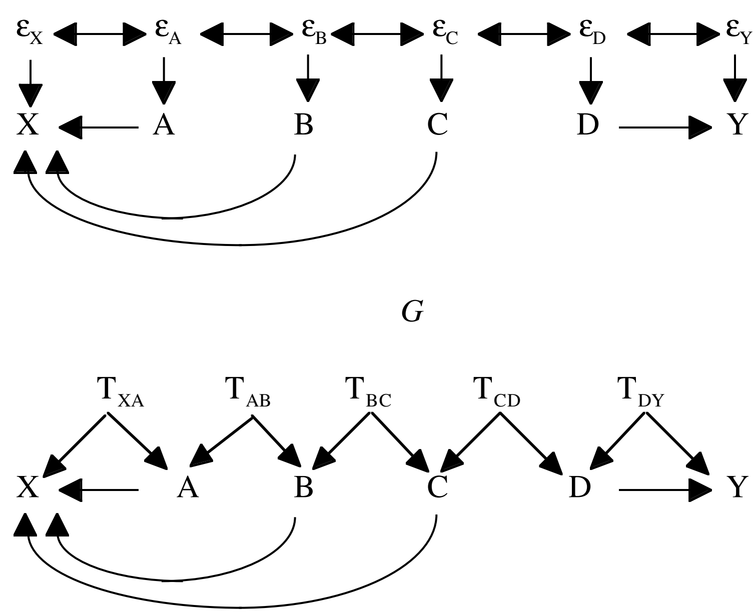
Abstract

It has been shown in Spirtes(1995) that X and Y are d-separated given **Z** in a directed graph associated with a recursive or non-recursive linear model without correlated errors if and only if the model entails that $\rho_{XY.Z} = 0$. This result cannot be directly applied to a linear model with correlated errors, however, because the standard graphical representation of a linear model with correlated errors is not a directed graph. The main result of this paper is to show how to associate a directed graph with a linear model L with correlated errors, and then use d-separation in the associated directed graph to determine whether L entails that a particular partial correlation is zero.

In a linear structural equation model (SEM) some partial correlations may be equal to zero for *all* values of the model's free parameters (for which the partial correlation is defined). (When we refer to "all values" of the free parameters, we assume that there are no constraints upon the models parameters except for the coefficients and the correlations among the error variables that are fixed at zero.) In this case we will say that the SEM **linearly entails** that the partial correlation is zero. It has been shown in Spirtes(1995) that X and Y are d-separated given **Z** in a directed graph associated with a recursive or non-recursive linear model with uncorrelated errors if and only if the model linearly entails that $\rho_{XY.Z} = 0$. This result cannot be directly applied to a linear model with correlated errors, however, because the standard graphical representation of a linear model with correlated errors is not a directed graph. The main result of this paper is to show how to associate a directed graph with a linear model L with correlated errors, and then use d-separation in the associated directed graph to determine whether L linearly entails that a particular partial correlation is zero. The standard graph terminology in this paper, the standard terminology for linear structural equation models, and the relationship between the two terminologies are described in the Appendix.

If G is the graph of SEM L with correlated errors, let **Transform(*G*)** be the graph resulting from replacing a double headed arrow between correlated errors ε_i and ε_j with a new latent variable T_{ij} ($i < j$) and edges from T_{ij} to X_i and X_j, and then removing the error terms from the graph. See Figure 1. A **trek** between X_i and X_j is an undirected path between X_i and X_j that contains no colliders. If there is a trek $X_i \leftarrow T_{ij} \rightarrow X_j$ in Transform(*G*), we will say that X_i and X_j are **d-adjacent** in Transform(*G*). A trek $X_j \leftarrow T_{ij} \rightarrow X_i$ is called a **correlated error trek** in Transform(*G*). In Transform(*G*), a **correlated error trek sequence** is a sequence of vertices $\langle X_i, .., X_k \rangle$ such that no pair of vertices adjacent in the sequence are identical, and for each pair of vertices X_r and X_s adjacent in the sequence, there is a correlated error trek between X_r and X_s. For example in Figure 1, the sequence of vertices <X,A,B,C,D,Y> is a correlated error trek sequence between X and Y.



Transform(*G*)

Figure 1

It might at first glance appear that for every parameterization of G, there is a parameterization of Transform(G) with the same covariance matrix. The following theorem shows that this is not the case.

Theorem 1: There exists a SEM L with measured variables $\mathbf{X}$, correlated errors, graph G, and correlation matrix $\Sigma(\mathbf{X})$ such that no linear parameterization of Transform(G) has marginal correlation matrix $\Sigma(\mathbf{X})$.

Proof. Assume that L has no structural equations, but every pair of errors is correlated in L. G and Transform(G) are shown in Figure 2.

Figure 2

Suppose that the marginal correlation matrix $\Sigma(\mathbf{X})$ is the following:

$$\Sigma(X) = \begin{pmatrix} 1.0 & 0.99 & 0.99 \\ 0.99 & 1.0 & 0.99 \\ 0.99 & 0.99 & 1.0 \end{pmatrix}$$

Every parameterization of Transform(G) is of the following form:

$$\begin{aligned} X_1 &= a_{11}T_1 + a_{13}T_3 + b_{11}\varepsilon'_1 \\ X_2 &= a_{21}T_1 + a_{22}T_2 + b_{22}\varepsilon'_2 \\ X_3 &= a_{32}T_2 + a_{33}T_3 + b_{33}\varepsilon'_3 \end{aligned} \qquad (1)$$

Suppose first that the variance of each of the variables is equal to 1. It follows then that

$$\begin{aligned}
\mathrm{var}(X_1) &= 1 = a_{11}{}^2 + a_{13}{}^2 + b_{11}{}^2 \\
\mathrm{var}(X_2) &= 1 = a_{21}{}^2 + a_{22}{}^2 + b_{22}{}^2 \\
\mathrm{var}(X_3) &= 1 = a_{32}{}^2 + a_{33}{}^2 + b_{33}{}^2
\end{aligned} \tag{2}$$

$$\begin{aligned}
\mathrm{corr}(X_1, X_2) &= 0.99 = a_{11}a_{21} \\
\mathrm{corr}(X_2, X_3) &= 0.99 = a_{22}a_{32} \\
\mathrm{corr}(X1, X_3) &= 0.99 = a_{13}a_{33}
\end{aligned} \tag{3}$$

From (1), the absolute values of each of the coefficients is less than one. From (2), it follows that a_{11}, a_{21}, a_{22}, a_{32}, a_{13}, and a_{33} all have absolute values greater than 0.99. Hence var(T_1) is greater than 1, which is a contradiction. It follows that there are no solutions to (2) and (3).

Suppose now that we do not fix the variances of the exogenous variables at one. We will show that if the corresponding set of equations has a solution, then so do (2) and (3), which is a contradiction.

$$\begin{aligned}
\mathrm{var}(X_1) &= 1 = a'_{11}{}^2 \,\mathrm{var}(T_1) + a'_{13}{}^2 \,\mathrm{var}(T_3) + b'_{11}{}^2 \,\mathrm{var}(\varepsilon'_1) \\
\mathrm{var}(X_2) &= 1 = a'_{21}{}^2 \,\mathrm{var}(T_1) + a'_{22}{}^2 \,\mathrm{var}(T_2) + b'_{22}{}^2 \,\mathrm{var}(\varepsilon'_2) \\
\mathrm{var}(X_3) &= 1 = a'_{32}{}^2 \,\mathrm{var}(T_2) + a'_{33}{}^2 \,\mathrm{var}(T_3) + b'_{33}{}^2 \,\mathrm{var}(\varepsilon'_3)
\end{aligned} \tag{2'}$$

$$\begin{aligned}
\mathrm{corr}(X_1, X_2) &= 0.99 = a'_{11}\, a'_{21} \,\mathrm{var}(T_1) \\
\mathrm{corr}(X_2, X_3) &= 0.99 = a'_{22}\, a'_{32} \,\mathrm{var}(T_2) \\
\mathrm{corr}(X_1, X_3) &= 0.99 = a'_{13}\, a'_{33} \,\mathrm{var}(T_3)
\end{aligned} \tag{3'}$$

Suppose now that 2' and 3' have a solution. Then set

$$\begin{array}{lll}
a_{11} = a'_{11}\sqrt{\mathrm{var}(T_1)} & a_{21} = a'_{21}\sqrt{\mathrm{var}(T_1)} & a_{22} = a'_{22}\sqrt{\mathrm{var}(T_2)} \\
a_{32} = a'_{32}\sqrt{\mathrm{var}(T_2)} & a_{13} = a'_{13}\sqrt{\mathrm{var}(T_3)} & a_{33} = a'_{33}\sqrt{\mathrm{var}(T_3)} \\
b_{11} = b'_{11}\sqrt{\mathrm{var}(\varepsilon'_1)} & b_{22} = b'_{22}\sqrt{\mathrm{var}(\varepsilon'_2)} & b_{33} = b'_{33}\sqrt{\mathrm{var}(\varepsilon'_3)}
\end{array}$$

These now form a solution to (2) and (3), which is a contradiction. ∴

Lemma 1: If Σ is a positive definite matrix, then there exists a positive definite matrix Σ' = Σ - δI, where δ is a real positive number.

Proof. Suppose that Σ is a positive definite matrix. It follows then that for all solutions of det(Σ - λI) = 0, λ is positive. Let the smallest solution of det(Σ - λI) = 0 be λ_1. Let δ be less than λ_1 and greater than 0. Let $\Sigma' = \Sigma - \delta I$. We will now show that all of the solutions of $\det(\Sigma' - \lambda' I) = 0$ are positive. $\Sigma' - \lambda' I = \Sigma - \delta I - \lambda' I = \Sigma - (\lambda' + \delta)I$. If we set $\lambda' = \lambda - \delta$, then for each solution of $\det(\Sigma - \lambda I) = 0$, there is a solution of $\det(\Sigma - (\lambda' + \delta)I) = 0$. Since $\lambda' = \lambda - \delta$, and δ is less than λ_1, the smallest solution of $\det(\Sigma' - \lambda' I) = 0$ is greater than 0. $\therefore$

A linear transformation of a set of random variables is **lower triangular** if and only if there is an ordering of the variables such that the matrix representing the transformation is zero for all entries a_{ij}, when $j > i$.

Lemma 2: If $X_1, \ldots, X_n$ have a joint normal distribution $N(0,\Sigma)$, where Σ is positive definite, then there is a set of n mututally independent standard normal variables $T_1, \ldots, T_n$, such that $X_1, \ldots, X_n$ are a lower triangular linear transformation of $T_1, \ldots, T_n$ and for each i, the coefficient of T_i in the equation for X_i is not equal to zero.

Proof. For every positive definite correlation matrix Σ, a complete directed graph can be given a linear parameterization that represents Σ (Spirtes et al. 1993). The reduced form of a complete directed graph is a lower triangular transformation of independent error variables that is non-zero on the diagonal, because Σ is positive definite. $\therefore$

Theorem 2: If G is the graph of SEM L with measured variables $\mathbf{X}$, normally distributed correlated errors, and marginal correlation matrix $\Sigma(\mathbf{X})$, $\{X, Y\} \cup \mathbf{Z} \subseteq \mathbf{X}$, and X is d-separated from Y given $\mathbf{Z}$ in Transform(G), then $\rho_{XY.\mathbf{Z}} = 0$ in $\Sigma(\mathbf{X})$.

Proof. First we will construct a latent variable model of $\varepsilon_1,\ldots,\varepsilon_n$. Then we will use this model to form the latent variable model L' with graph G' that has marginal correlation matrix $\Sigma(\mathbf{X})$ but no correlated errors, and in which X is d-separated from Y given $\mathbf{Z}$ in G'. It follows that $\rho_{XY.\mathbf{Z}} = 0$ in $\Sigma(\mathbf{X})$.

Order the variables so that X is first, Y is second, followed by each variable with a descendant in $\mathbf{Z}$, followed by any remaining variables that have X or Y as descendants, followed by the rest of the variables. Given this ordering, we will now refer to the variables as $X_1,\ldots,X_n$, where for all i, X_i is the i^{th} variable in the ordering. Suppose for the graph in Figure 1 we are interested in whether $\rho_{XY} = 0$ (i.e. $\mathbf{Z} = \emptyset$). One renaming of the variables for the graph in Figure 1 that is compatible with the ordering rules given above is shown in Figure 3.

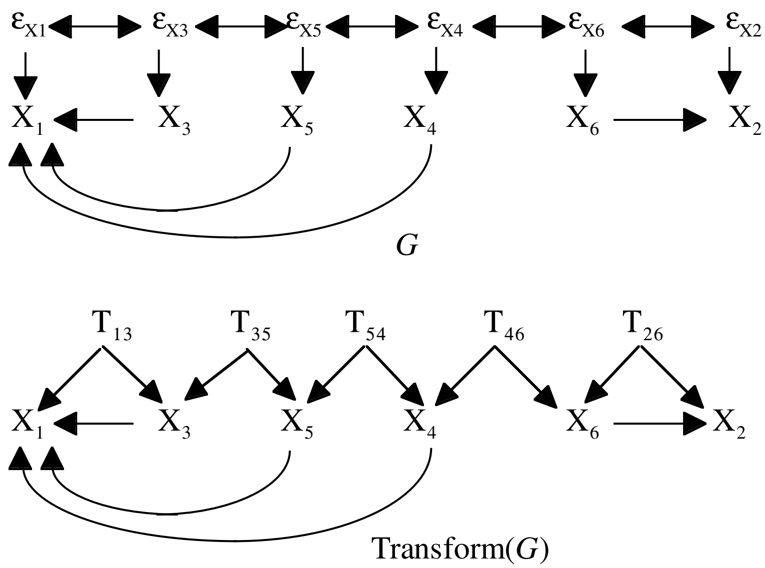


Figure 3

Suppose that the correlation matrix among the error terms of L is Σ. We will show that there is a latent variable model of Σ of the form

$$\varepsilon_i = \sum_{j \le i} a_{ij} T_j + \varepsilon''_i$$

where each of the T_i and ε''_i are uncorrelated.

By hypothesis, Σ is a positive definite matrix. By Lemma 1, there is a set of variables $\varepsilon'_1,\dots,\varepsilon'_n$ with positive definite matrix $\Sigma' = \Sigma - \delta I$. As a first step to constructing a latent variable model of ε, we will construct a latent variable model of ε', represented by a directed graph *H*. Note that *H* does not contain any of the X variables or ε variables.

By Lemma 2, there is a set of variables $T_1, \dots, T_n$ such that $\varepsilon'_1, \dots, \varepsilon'_n$ with correlation matrix $\Sigma'(\varepsilon')$ are a lower triangular linear transformation of $T_1,\dots, T_n$ and for each i, the coefficient of T_i in the equation for ε'_i is not equal to zero. That is

$$\varepsilon'_i = \sum_{j<i} a_{ij} T_j$$

where $a_{ii} \ne 0$. The transformation can be represented by a directed graph *H* in which for each i, there are edges from T_i to ε'_j, $j \ge i$.

From the construction of H, there are no edges from T_j to ε'_1 unless j = 1. Hence, for every $j \neq 1$, in H every every trek between ε'_1 and ε'_j contains T_1. It follows that there is at most one trek between ε'_1 and ε'_j. The edge from T_1 to ε'_1 is not zero. Hence if ε_1 and ε_j are not correlated in L (i.e. X_1 and X_j are not d-adjacent in Transform(G)) then the edge from T_1 to ε_j is zero. In the example from Figure 3, $a_{12} = a_{14} = a_{15} = a_{16} = 0$.

Applying this strategy to each of the T_i variables in turn, we can now show that for each i and r > i, if there is no trek between ε'_r and ε'_i containing a variable T_j, where j < i, and X_r is not d-adjacent to X_i in Transform(G) (i.e. ε_i and ε_j are uncorrelated in L), then the $T_i \rightarrow \varepsilon'_r$ edge can be removed from the graph (i.e. a_{ir} can be set to zero.) Suppose on the contrary that there is no trek between ε'_r and ε'_i containing a variable T_j, where j < i, and X_r is not d-adjacent to X_i in Transform(G) (i.e. ε_i and ε_j are uncorrelated in L), but the $T_i \rightarrow \varepsilon'_r$ edge is not removed from the graph by this procedure (i.e. a_{ir} is not set to zero.) By the construction of H, if k > i, then there is no edge from T_k to ε'_i. It follows that if in H there is no trek between ε'_r and ε'_i containing a variable T_j, where j < i, then every trek between ε'_i and any other variable contains the edge from T_i to ε'_i, which is not equal to zero.The $T_i \rightarrow \varepsilon'_r$ edge exists by hypothesis, so there is exactly one trek between ε'_i and ε'_r in H. Hence ε'_i and ε'_r are correlated in every parameterization of H. (Note that this could not be claimed if there were more than one trek between ε'_i and ε'_r since in that case the treks might cancel each other.) Since the covariances between distinct ε' variables are equal to the correlations between the corresponding ε variables, it follows that ε_i and ε_r are correlated in L', and hence d-adjacent in Transform(G). This is a contradiction. The end result of this process of edge removal for the graph in Figure 3 is shown in Figure 4.

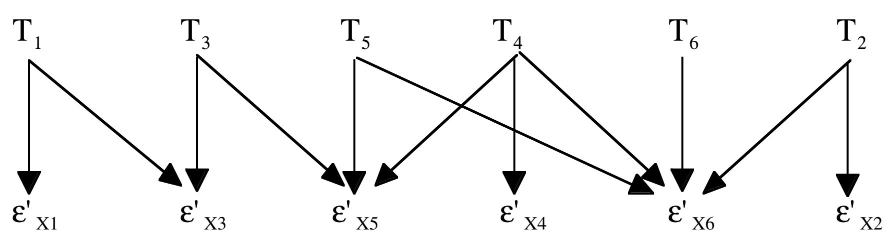


Figure 4: *H* after extra edges are removed

From the latent variable model without correlated errors of the ε' variables, we can now form a latent variable model without correlated errors of the ε variables. For each i, let ε''_i be a normally distributed variable with variance δ that is independent of all of the T_i, and all of the other ε'' variables. It follows then that

$$\varepsilon_i = \sum_{j \le i} a_{ij} T_j + \varepsilon''_i$$

because the addition of the ε''_i term does not change any of the correlations, and adds δ to the variance of ε'_i.

From the latent variable model without correlated errors of the ε variables, we can now form a latent variable model without correlated errors of $\Sigma(\mathbf{X})$. If we use the above equation to replace each ε_i in the SEM L, we form a SEM L' which has no correlated errors, but has the same marginal covariance matrix as L. If the equations in L are:

$$X_i = \sum_{j \ne i} b_{ij} X_j + \varepsilon_i$$

then the equations in L' are:

$$X_i = \sum_{j \ne i} b_{ij} X_j + \sum_{j < i} a_{ij} T_j + \varepsilon''_i$$

L' has a graph *G*' obtained from *G* and *H* in the following way: Remove each error variable from *G* (because all of the error terms are uncorrelated in L'), add each of the T_i variables to *G*, and add an edge from T_i to X_j if in *H* there is an edge from T_i to ε'_j. Note that the ancestor relations among the **X** variables in *G*' is the same as the ancestor relations among the **X** variables in Transform(*G*). Given the graph *G* from Figure 3 and the graph *H* from Figure 4, the end result is shown in Figure 5. As in Transform(*G*), we will call a trek $X_j \leftarrow T_m \rightarrow X_i$ that contains a T variable a **correlated error trek** in *G*'.

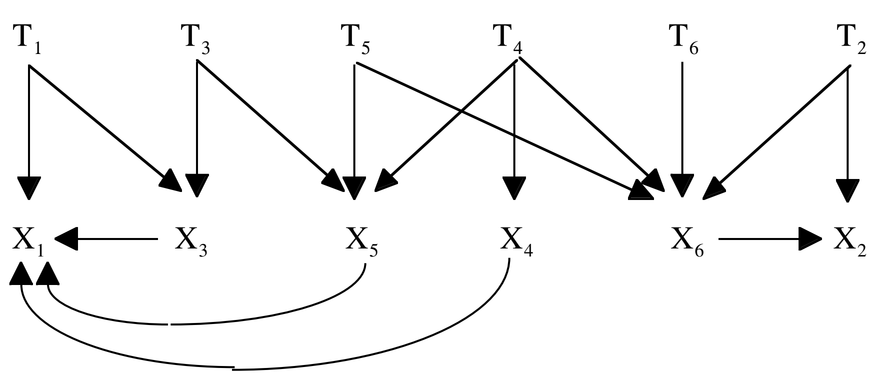


Figure 5: *G'*

We will now show that if there is a correlated error trek between X_i and X_j in *G*' that contains a variable T_r, then in Transform(*G*) there is a correlated error trek sequence between X_i and X_j, such that every variable in the correlated error trek sequence, with the possible exception of the endpoints, has index (i.e. subscript) less than or equal to r (henceforth referred to as the correlated error trek sequence in Transform(*G*) corresponding to the correlated error trek between X_i and X_j in *G*'.) The proof is by induction on r. Suppose first that r = 1. If there is a correlated error trek between X_i and X_j in *G*' that contains T_1 then there are correlated error treks between X_i and X_1 in Transform(*G*), and between X_j and X_1. The concatenation of these two correlated error treks forms a correlated error trek sequence in which (trivially) every variable in the sequence except for the endpoints has an index less than or equal to 1. The induction hypothesis is that for all $r \leq n$, if there is a correlated error trek between X_i and X_j in *G*' that contains T_r, then in Transform(*G*) there is a correlated error trek sequence between X_i and X_j, such that every variable in the sequence, with the possible exception of the endpoints has an index less than r. Suppose now that in *G*' there is a correlated error trek between X_i and X_j such that the trek contains T_{n+1}, where $i, j \geq n+1$. Since the edge between T_{n+1} and X_i exists in G', it follows from the method of construction of G' that either there is a correlated error trek between X_i and X_{n+1} in *G*' that contains some T_r, $r < n+1$, or X_{n+1} and X_i are d-adjacent in Transform(*G*). In the former case, by the induction hypothesis there is a correlated error trek sequence between X_i and X_{n+1} that, except for the endpoints, contains only vertices whose indices are less than or equal to n+1. In the latter case, $<X_i,X_{n+1}>$ is a correlated error trek sequence between X_i and X_{n+1}. Similarly, there is a correlated error trek sequence between X_{n+1} and X_j that, except for the endpoints, contains only vertices whose indices are less than or equal to n+1. These two correlated error trek sequences can be concatenated to form a correlated error trek sequence between X_i and X_j that, except for the endpoints, contains only vertices whose indices are less than or equal to n+1. For the *G*' shown in Figure 5, there is a correlated error trek between X_5 and X_6, and a corresponding correlated error trek sequence $<X_5,X_4,X_6>$ in the graph Transform(*G*) in Figure 3.

We will now show that if X_1 and X_2 are d-connected given **Z** in *G*', then X_1 and X_2 are d-connected given **Z** in Transform(*G*) using Lemma 3.3.1+ (Richardson 1994, which is an extension to the cyclic case of Lemma 3.3.1 in Spirtes et al. 1993). Lemma 3.3.1+ states that there is a path in a directed graph G that d-connects X and Y given **Z** if and only if there is a sequence of vertices **Q** and a set **P** of paths in *G* between pairs of adjacent vertices in **Q** that have the following properties: (i) For each occurrence of a pair of adjacent variables X_i and X_j in **Q,** $i \neq j$, and there is a unique path in **P** that d-connects X_i and X_j

given $\mathbf{Z}\backslash\{X_i,X_j\}$; (ii) if $<X_i,X_j,X_k>$ is a subsequence of **Q**, the corresponding path between X_i and X_j in **P** is into X_j, and the corresponding path between X_j and X_k in **P** is into X_j (in which case we say that the occurrence of X_j is a **collider** in **Q**) then X_j has a descendant in **Z**; and (iii) if there is an occurrence of X_j that is a non-collider in **Q**, then X_j is not in **Z**. Note that we do not require that a vertex occur only once in **Q**. Hence one occurrence of a vertex in **Q** may be a collider, and another occurrence of the same vertex in **Q** may be a non-collider.

Suppose now that there is an undirected path U that d-connects X_1 and X_2 given **Z** in *G*'. Intuitively, in *G*' we would like to form **Q** and **P** by breaking U into pieces, such that each correlated error trek occurs as a separate piece. More formally, form a sequence **Q** of vertices and an associated sequence **P** of paths in *G*' with the following properties: (i) every vertex in **Q** is in **X** and occurs on U; (ii) no vertex occurs in **Q** more than once; (iii) if A occurs before B in **Q**, then A occurs before B on U; (iv) if the subpath of U between A and B is a correlated error trek, then A and B both occur in that order in **Q**. The path in **P** associated with a pair A and B of adjacent vertices in **Q** is the subpath of U between A and B. In the example in Figure 5, in *G*' the d-connecting path between X_1 and X_2 given $\mathbf{Z} = \emptyset$ is $X_1 \leftarrow X_5 \leftarrow T_4 \rightarrow X_6 \rightarrow X_2$, $\mathbf{Q} = <X_1,X_5,X_6,X_2>$, and $\mathbf{P} = <X_1 \leftarrow X_5,\ X_5 \leftarrow T_4 \rightarrow X_6,\ X_6 \rightarrow X_2>$. In this example, there are no colliders in **Q**.

Because U is a path that d-connects X_1 and X_2 given **Z** in G', it is easy to see that the paths in **P** have the following properties in *G*': (i) Each path in **Q** d-connects its endpoints X_i and X_j given $\mathbf{Z}\backslash\{X_i,X_j\}$; (ii) if there is an occurrence of X_i in **Q** that is a collider then X_i has a descendant in **Z**; and (iii) if there is an occurrence of X_i in **Q** that is a non-collider, then X_i is not in **Z**.

We will now show how to construct a sequence of vertices **Q'** and a set **P'** of paths in Transform(*G*) between pairs of adjacent vertices in **Q'** that have the following properties: (i) For each occurrence of a pair of adjacent variables X_i and X_j in **Q'** there is a unique path in **P'** that d-connects X_i and X_j given $\mathbf{Z}\backslash\{X_i,X_j\}$; (ii) if there is an occurrence of X_i in **Q'** that is a collider, then X_i has a descendant in **Z**; and (iii) if there is an occurrence of X_i in **Q'** that is a non-collider, then X_i is not in **Z**. It will follow from Lemma 3.3.1+ that X and Y are d-connected given **Z** in Transform(*G*).

We will create **Q'** by several modifications of **Q**. Step (1) in creating **Q'** is to replace each subsequence $<X_r,X_s>$ of **Q** such that X_r and X_s are on a correlated error trek in **Q**, with the corresponding correlated error trek sequence $<X_r, \ldots, X_s>$ in Transform(*G*). Note that each occurrence of X_k between <X_r, …,X_s> is a collider in **Q'**. In the example, after the first step **Q'** = <X_1,X_5,X_4,X_6,X_2> and **P'** = <$X_1 \leftarrow X_5$, $X_5 \leftarrow T_{45} \rightarrow X_4$, $X_4 \leftarrow T_{46} \rightarrow X_6$, $X_6 \rightarrow X_2$>, i.e. we replaced the subsequence <X_4,X_6> in **Q** by <X_5,X_4,X_6>.

Recall that the ancestor relations among the **X** variables (which includes the variables in **Z**) in *G*' is the same as the ancestor relations among the **X** variables in Transform(*G*). After stage (1) in creating **Q'**, if X_k is not an ancestor of **Z** in Transform(*G*) (or in *G*'), but has an occurrence in **Q'** that is a collider, it follows that X_k was added to **Q'** by replacing a subsequence <X_r,X_s> of **Q** by a corresponding correlated error trek sequence <X_r, …,X_s> in Transform(*G*). Hence any such X_k lies between some pair of vertices X_r and X_s that are adjacent in **Q**. Because every vertex in <X_r, …, X_s> in **Q'** (except for X_r and X_s) has an index less than r and s, and X_k is not an ancestor of **Z** in *G*', it follows from the ordering of the variables that we chose, that X_r and X_s are not ancestors of **Z** in *G*'. Because X_r and X_s are on U but not ancestors of **Z** in *G*', there is a subpath of U that is a directed path from X_r to X_1 and a subpath of U that is a directed path X_s to X_2, or vice versa. In either case, in *G*', X_r is an ancestor of X_1 and X_s is an ancestor of X_2, or X_r is an ancestor of X_2 and X_s is an ancestor of X_1. Because in *G*', X_r is an ancestor of X_1 and X_s an ancestor of X_2 or vice-versa, and k < r and s, it follows from the ordering of the variables that X_k is also an ancestor of X_1 or X_2 in *G*'. Hence X_k is an ancestor of X_1 or X_2 in Transform(*G*). In the example, in Transform(*G*) X_4 is not an ancestor of the empty set but is an ancestor of X_1, and it is between two vertices X_5 and X_6 which also are not ancestors of the empty set but are ancestors of X_1 or X_2.

If there is some vertex X_k in **Q'** that is not an ancestor of **Z**, but occurs in **Q'** as a collider, suppose without loss of generality that there is a vertex that is an ancestor of X_1 but not of **Z**, that occurs as a collider in **Q'**. Let X_a be the last occurrence of a collider in **Q'** that is an ancestor of X_1 but not of **Z**, if there is one, otherwise let $X_a = X_1$. Step (2) in forming **Q'** and **P'** is to replace the subsequence <X_1,…,X_a> by <X_1,X_a> if $X_a \neq X_1$, and replacing the corresponding paths in **P'** by a directed path from X_a to X_1 if $X_a \neq X_1$. (Such a directed path exists if $X_a \neq X_1$ because X_a is an ancestor of X_1.) This removes all occurences of vertices between X_1 and X_a that are not ancestors of **Z**, but are colliders in **Q'**. In the example, $X_a = X_4$, and after step 2, **Q'** = <X_1,X_4,X_6,X_2> and **P'** = <$X_1 \leftarrow X_4$, $X_4 \leftarrow T_{46} \rightarrow X_6$, $X_6 \rightarrow X_2$>.

By definition, every vertex that occurs as a collider between X_a and X_2 in **Q'** is an ancestor of **Z** or of X_2. Let X_b be the first vertex after X_a in **Q'** that is an ancestor of X_2 but not of **Z**, if there is one, otherwise let $X_b = X_2$. Step (3) in forming **Q'** and **P'** is to replace the subsequence <X_b, …,X_2> by <X_b,X_2> if $X_b \neq X_2$, and replacing the corresponding paths in **P'** by a directed path from X_b to X_2 if $X_b \neq X_2$. This removes all occurrences of colliders between X_b and X_2 that are not ancestors of **Z**. Note that all occurrences of colliders that are left are between X_a and X_b, and every occurrence of a collider between X_a and X_b is an ancestor of **Z** by construction. In the example, $X_b = X_2$, and after step (3), **Q'** and **P'** are unchanged.

We will now show that every path between a pair of variables X_u and X_v in **P'** d-connects X_u and X_v given **Z**\{X_u,X_v}. If the path between X_u and X_v is also in **P**, then it d-connects X_u and X_v given **Z**\{X_u,X_v} because every path in **P** has this property. If the path between X_u and X_v is not in **P**, but was added in step (1) of the formation of **P'**, then the path between X_u and X_v is a correlated error trek, which d-connects X_u and X_v given **Z**\{X_u,X_v} because no T variable is in **Z**. If the path between X_u and X_v is not in **P**, but was added in step (2) of the formation of **P'**, then $X_u = X_1$, $X_v = X_a$, and the path between X_u and X_v is a directed path from X_a to X_1 that does not contain any member of **Z**. Hence the path d-connects X_u and X_u given **Z**. Similarly, if the path between path between X_u and X_v is not in **P**, but was added in step (3) of the formation of **P'**, then $X_u = X_b$, $X_v = X_2$, and the path between X_u and X_v is a directed path from X_b to X_2 that does not contain any member of **Z**. Hence the path d-connects X_u and X_v given **Z**.

We will now show that every vertex that occurs as a collider in **Q'** has a descendant in **Z**, and every vertex that occurs as a non-collider in **Q'** is not in **Z**. Every vertex that occurs as a collider in **Q'** is an ancestor of **Z**, because steps (2) and (3) in the formation of **Q'** removed all occurrences of colliders that were not ancestors of **Z**. Every vertex that occurs as a non-collider in **Q** and as a non-collider in **Q'** is not in **Z**, because every vertex that occurs as a non-collider in **Q** is not in **Z**. The only vertices that may occur as non-colliders in **Q'** but not in **Q** are X_a and X_b. X_a is not in **Z**, because either it is equal to X_1 or X_2, neither of which is in **Z**, or it is not an ancestor of **Z** by construction. Similarly, X_b is not in **Z**.

Hence **Q'** is a sequence of paths that satisfy properties (i), (ii), and (iii). It follows from Lemma 3.3.1+ that X_1 and X_2 are d-connected given **Z** in Transform(*G*).

By contraposition, since X_1 and X_2 are d-separated in Transform(*G*), they are d-separated given **Z** in *G*'. Because *G*' is the directed graph of a latent variable model L' with correlation matrix that has marginal $\Sigma(\mathbf{X})$, no correlated errors, and X_1 and X_2 are d-separated given **Z** in *G*', it follows from Theorem 3 (Spirtes 1995) that $\rho_{XY.Z} = 0$ in $\Sigma(\mathbf{X})$. $\therefore$

Theorem 3: If *G* is the graph of SEM L with normally distributed correlated errors and marginal correlation matrix $\Sigma(X)$, and X is d-connected to Y given **Z** in Transform(*G*) then L does not linearly entail that $\rho_{XY.\mathbf{Z}} = 0$.

Proof. If X is not d-separated from Y given **Z** in Transform(*G*), then by Theorem 3 (Spirtes 1995) there is a parameterization of Transform(*G*) with correlation matrix $\Sigma(X)$ such that $\rho_{XY.\mathbf{Z}} \neq 0$. By the convention adopted for new latent variables names in Transform(*G*), no new latent variable was called T_{ji} where j >i. For the sake of notational convenience, we will also use the name T_{ji} to refer to T_{ij}. In that parameterization,

$$X_i = \sum_{j<i} b_{ij} X_j + \sum_{j \neq i} a_{ij} T_{ij} + \varepsilon'_i$$

Now define

$$\varepsilon_i = \sum_{j \neq i} a_{ij} T_{ij} + \varepsilon'_i$$

It follows then that

$$X_i = \sum_{j<i} b_{ij} X_j + \varepsilon_i$$

which is a parameterization of L in which $\rho_{XY.\mathbf{Z}} \neq 0$. $\therefore$

Because the covariance matrix of the non-error variables in a linear SEM does not depend upon whether the error terms are normally distributed, but depends only upon the linear coefficients and the covariance matrix among the errors, Theorem 2 and Theorem 3 can obviously be extended to the case where the error terms are not normally distributed.

Appendix

Sets of variables and defined terms are in boldface. A **directed graph** is an ordered pair of a finite set of vertices **V**, and a set of directed edges **E**. A directed edge from A to B is an ordered pair of distinct vertices $\langle A,B \rangle$ in **V** in which A is the **tail** of the edge and B is the **head**; the edge is **out of** A and **into** B, and A is **parent** of B and B is a **child** of A. A sequence of edges $\langle E_1,\ldots,E_n \rangle$ in G is an **undirected path** if and only if there exists a sequence of vertices $\langle V_1,\ldots,V_{n+1} \rangle$ such that for $1 \leq i < n$ either $\langle V_i,V_{i+1} \rangle = E_i$ or $\langle V_{i+1},V_i \rangle = E_i$. A path U is **acyclic** if no vertex occurring on an edge in the path occurs more than once. A sequence of edges $\langle E_1,\ldots,E_n \rangle$ in G is a **directed path** if and only if there exists a sequence of vertices $\langle V_1,\ldots,V_{n+1} \rangle$ such that for $1 \leq i < n$ $\langle V_i,V_{i+1} \rangle = E_i$. If there is an acyclic directed path from A to B or B = A then A is an **ancestor** of B, and B is a **descendant** of A. A directed graph is **acyclic** if and only if it contains no directed cyclic paths.[1]

Vertex X is a **collider** on an acyclic undirected path U in directed graph G if and only if there are two edges on U that are directed into X. Three disjoint sets **X**, **Y**, and **Z**, **X** and **Y** are **d-separated** given **Z** in G if and only if there is no acyclic undirected path U from a member of **X** to a member of **Y** such that every non-collider on U is not in **Z**, and every collider on U has a descendant in **Z**. For three disjoint sets **X**, **Y**, and **Z**, **X** and **Y** are **d-connected** given **Z** in G if and only if **X** and **Y** are not **d-separated** given **Z**.

The variables in a linear structural equation model (SEM) can be divided into two sets, the "error variables" or "error terms," and the substantive variables. Corresponding to each substantive variable X_i is a linear equation with X_i on the left hand side of the equation, and the direct causes of X_i plus the error term ε_i on the right hand side of the equation. Since we have no interest in first moments, without loss of generality each variable can be expressed as a deviation from its mean.

[1] An undirected path is often defined as a sequence of vertices rather than a sequence of edges. The two definitions are essentially equivalent for acyclic directed graphs, because a pair of vertices can be identified with a unique edge in the graph. However, a cyclic graph may contain more than one edge between a pair of vertices. In that case it is no longer possible to identify a pair of vertices with a unique edge.

Consider, for example, two SEMs S_1 and S_2 over $\mathbf{X} = \{X_1, X_2, X_3\}$, where in both SEMs X_1 is a direct cause of X_2 and X_2 is a direct cause of X_3. The structural equations[2] in Figure 6 are common to both S_1 and S_2.

$$X_1 = \varepsilon_1$$
$$X_2 = \beta_1 X_1 + \varepsilon_2$$
$$X_3 = \beta_2 X_2 + \varepsilon_3$$

Figure 6: Structural Equations for SEMs S_1 and S_2

where β_1 and β_2 are free parameters ranging over real values, and ε_1, ε_2 and ε_3 are error terms. In addition suppose that $\varepsilon_1, \varepsilon_2$ and ε_3 are distributed as multivariate normal. In S_1 we will assume that the correlation between each pair of distinct error terms is fixed at zero. The free parameters of S_1 are $\theta = \langle\beta, \mathbf{P}\rangle$, where β is the set of linear coefficients $\{\beta_1, \beta_2\}$ and $\mathbf{P}$ is the set of variances of the error terms. We will use $\Sigma_{\mathbf{S1}}(\theta_1)$ to denote the covariance matrix parameterized by the vector θ_1 for model S_1, and occasionally leave out the model subscript if the context makes it clear which model is being referred to. If all the pairs of error terms in a SEM S are uncorrelated, we say S is a SEM with **uncorrelated errors**.

S_2 contains the same structural equations as S_1, but in S_2 we will allow the errors between X_2 and X_3 to be correlated, i.e., we make the correlation between the errors of X_2 and X_3 a free parameter, instead of fixing it at zero, as in S_1. In S_2 the free parameters are $\theta = \langle\beta, \mathbf{P'}\rangle$, where β is the set of linear coefficients $\{\beta_1,\beta_2\}$ and **P'** is the set of variances of the error terms and the correlation between ε_2 and ε_3. If the correlations between any of the error terms in a SEM are not fixed at zero, we will call it a SEM with **correlated errors**.[3]

It is possible to associate with each SEM with uncorrelated errors a directed graph that represents the causal structure of the model and the form of the linear equations. For example, the directed graph associated with the substantive variables in S_1 is $X_1 \rightarrow X_2 \rightarrow X_3$, because X_1 is the only substantive variable that occurs on the right hand side of the equation for X_2, and X_2 is the only substantive variable that appears on the right hand side of the equation for X_3. We generally do not include error terms in the directed graph

[2] We realize that it is slightly unconventional to write the trivial equation for the exogenous variable X_1 in terms of its error, but this serves to give the error terms a unified and special status as providing all the external sources of variation for the system

[3] We do not consider SEMs with other sorts of constraints on the parameters, e.g., equality constraints.

associated with a SEM unless the errors are correlated. We enclose measured variables in boxes, latent variables in circles, and leave error variables unenclosed.

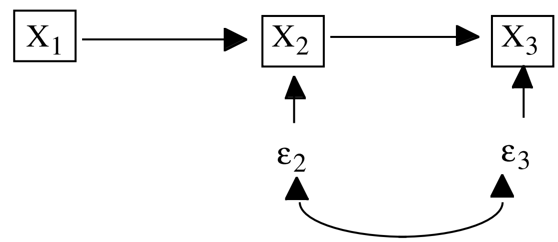


Figure 7. SEM S_2 with correlated errors

The typical path diagram that would be given for S_2 is shown in Figure 7. This is not strictly a directed graph because of the double-headed arrow between error terms ε_2 and ε_3, which indicates that ε_2 and ε_3 are correlated. It is generally accepted that correlation is to be explained by some form of causal connection. Accordingly if ε_2 and ε_3 are correlated we will assume that either ε_2 causes ε_3, ε_3 causes ε_2, some latent variable causes both ε_2 and ε_3, or some combination of these. In other words, double-headed arrows are an ambiguous representation of a causal connection.

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