When No Price Is Right

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Abstract

We address three problems in decision theory, providing more general solutions than previously available. First, we show how to extend a partial preference over a set of gambles to a full preference (weak order) without assuming an Archimedean property. Second, we show how to represent a partial preference as the consensus of a collection of full preferences. Third, we show how to represent a full non-Archimedean preference by a nonstandard utility function. The types of gambles that we consider are of two well-known varieties: (i) real-valued random variables such as those for which de Finetti developed a theory of coherent prevision and (ii) lotteries over general qualitative prizes such as those for which von Neumann, Morgenstern, Anscombe and Aumann produced expected-utility representations. Non-Archimedean preferences arise when some gambles have no fair price. Two common situations give rise to non-Archimedean preferences: gambles whose values are required to be greater than every real number, and strict preferences between gambles whose values differ by more than zero but less than every positive real number. In this paper, we present a common framework in which to deal with both of the above types of gambles. Within that framework, we show (a) how to extend a partial preference over a subset of all gambles of interest, (b) how to represent such a partial preference via a set of weak orders over the full set of gambles, and (c) how to represent non-Archimedean preferences over sets of gambles via nonstandard models of the reals.

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1. Introduction

1.1. Motivation

The primary goal of this paper is to extend two well-known theories of decision making to allow for non-Archimedean (discontinuous) preferences. One of the Archimedean theories that we extend is the theory of previsions of [1]. In that theory, a bookie provides a real-valued prevision (like an expected value) to each random variable in some set. Here is a simple example in which no such prevision is available.

Example 1. Let $\Omega = \{H, T\}$ with each point representing the side of a nearly-fair coin that is being tossed. Let $\mathcal{G}$ consist of all real-valued functions defined on $\Omega$. Each such function $X$ is equivalent to an ordered pair $(x, y)$, where $x = X(H)$ and $y = X(T)$. Let $(x_1, y_1) \prec (x_2, y_2)$ express that an agent is willing to pay $(x_1, y_1)$ in order to receive $(x_2, y_2)$ but not the other way around. The following example of a strict preference relation (denoted by $\prec$) satisfies our assumptions. Say that $(x_1, y_1) \prec (x_2, y_2)$ if either of the following hold:

- $x_1 + y_1 < x_2 + y_2$,
- $x_1 + y_1 = x_2 + y_2$ and $x_2 > x_1$.

Note that, for every two distinct pairs, either $(x_1, y_1) \prec (x_2, y_2)$ or $(x_2, y_2) \prec (x_1, y_1)$, but never both. For example, $(-1, 1) \prec (0, 0) \prec (1, -1)$. Even though the coin is nearly fair, it appears to be slightly biased in favor of $H$ according to the agent being modeled here. To see the discontinuity of these preferences, look at the set of pairs of the form $(-1 + x, 1)$ for real $x$. We have $(0, 0) \prec (-1 + x, 1)$ for all $x > 0$, but $(-1 + x, 1) \prec (0, 0)$ for $x \leq 0$. Preference jumps from strict in one direction to strict in the other direction without passing through indifference. There is no real-valued prevision that ranks the elements of $\mathcal{G}$ in the order of the stated preferences. (A proof appears in Section 1.2.) On the other hand, a nonstandard-valued prevision can provide such a ranking.
The other theory that we extend is that of [2, 3] for decisions about lotteries. This theory starts with a preference relation that satisfies an Archimedean axiom and produces a real-valued utility that ranks the lotteries the same as the preference relation. Here is an example in which no such utility is available.

Example 2. Let \( \mathcal{P} = \{a, b, c\} \) be a set of prizes, of which there are but three. Let \( \mathcal{G} \) be the set of all lotteries amongst these prizes. Each lottery corresponds to a probability vector \((p_a, p_b, p_c)\), were \( p_j \) is the probability that the agent receives price \( j \) for \( j \in \{a, b, c\} \). We can denote the lotteries as \( p_a a + p_b b + p_c c \). In particular, each prize can be identified with the lottery that assigns that prize with probability 1. Suppose that the agent expresses the strict preferences \( a \prec b \prec c \). The independence axiom of von Neumann and Morgenstern (see Definition 3 in Section 1.5) implies that

\[
\alpha c + (1 - \alpha) a \prec \beta c + (1 - \beta) a,
\]

whenever \( \beta \geq \alpha \). If preferences were Archimedean, there would exist \( \alpha \) such that \( \alpha c + (1 - \alpha) a \sim b \). Suppose, to the contrary, that the agent expresses the preferences that \( \alpha c + (1 - \alpha) a \prec b \) for \( \alpha < 0.5 \) and \( b \prec \alpha c + (1 - \alpha) a \) for \( \alpha \geq 0.5 \). In this case, there is no \( \alpha \) such that \( \alpha c + (1 - \alpha) a \sim b \).

Both of the examples above fail to satisfy the Archimedean axiom below. A definition is needed first.

**Definition 1.** Let \( \mathcal{M} \) be a nonempty set. We call \( \mathcal{M} \) a mixture set if, for every \( m_1, m_2 \in \mathcal{M} \) and every standard \( \alpha \in [0, 1] \), \( \alpha m_1 + (1 - \alpha) m_2 \) is well defined and is an element of \( \mathcal{M} \).

**Axiom 1** (Archimedean). Let \( \mathcal{M} \) be a mixture set with a preorder \( \preceq \). Let \( m_1 \sim m_2 \) mean \((m_1 \preceq m_2) \land (m_2 \preceq m_1)\). For all \( m_1, m_2, m_3 \in \mathcal{M} \) such that \( m_1 \preceq m_2 \preceq m_3 \), there exists standard \( \alpha \in [0, 1] \) such that \( m_2 \sim \alpha m_1 + (1 - \alpha) m_3 \).

In this paper, we do not assume that preferences satisfy the Archimedean axiom, whose justification in other theories is primarily mathematical rather than based on principles. One popular way to represent non-Archimedean preferences is via lexicographic probability.

**Example 3** (Continuation of Example 1). Let \( P_1 \) be the probability on \( \Omega \) such that \( P_1(\{H\}) = P_1(\{T\}) = 1/2 \), and let \( P_2 \) be such that \( P_2(\{H\}) = 1 \). The preference in Example 1 can be represented as follows. First declare a
strict preference between all pairs of random variables that have different expectations under \( P_1 \). For those pairs whose expectations are the same under \( P_1 \), declare a strict preference according to their expectations under \( P_2 \).

Example 3 is a simple lexicographic probability. More general lexicographic probabilities involve longer sequences of comparisons to break ties that persist after expectations have been calculated for two or more probabilities. Arbitrary well-ordered sets of probabilities can be used to define a lexicographic probability. If \( \Omega \) is infinite, there may be non-Archimedean preferences that cannot be represented by lexicographic probabilities.

Example 4. Let \( \Omega \) be an infinite set of arbitrary cardinality. Suppose that we want to express a strict preference for the indicator \( I_\omega \) of every singleton \( \{ \omega \} \) over the constant 0 as well as indifference between each pair \( I_\omega_1 \) and \( I_\omega_2 \). Every set \( \{ P_\alpha \}_{\alpha \in \mathbb{N}} \) of probabilities on \( \Omega \) that preserves the indifferences between all of the singleton indicators \( I_\omega \) will have \( P_\alpha(I_\omega) = 0 \) for all \( \alpha \) and all \( \omega \). Hence, the strict preference for \( I_\omega \) over 0 will never appear in the lexicography. Another way to look at the issue is that every probability \( P \) that allows for the strict preference for \( I_{\omega_0} \) over 0 for even a single \( \omega_0 \) will violate the indifference between all of the \( I_\omega \)'s. Hence, no such \( P \) can appear in the lexicography.

In this paper, we take the approach of using a non-Archimedean number system, i.e. a nonstandard model of the reals, to represent non-Archimedean preferences. In [4], Narens develops a theory of measurement without Archimedean assumptions. The theory leads to measurements whose values lie in nonstandard models of the reals. Narens' measurement systems have a number of features in common with probability and preference, so it is not surprising that nonstandard numbers are useful for representing non-Archimedean preference structures. Furthermore, we show later (see Lemma 14) that all lexicographic preferences can be represented by a nonstandard utility. In this way, we can represent all non-Archimedean preferences using the same framework. See [5] for a more thorough comparison of the uses of lexicographic probabilities and nonstandard numbers in representing preferences.

Although not apparent from the statement, the Archimedean axiom has consequences for preferences that have an unbounded nature.

Example 5. Let \( \Omega = \mathbb{Z}^+ \), the positive integers. Let \( I_n(\omega) \) be the indicator of the event \( \{ \omega = n \} \) for each integer \( n \). Let \( X(\omega) = 2^\omega \), and let \( Z \) be a bounded function defined on \( \Omega \). Suppose that \( P(I_n) = 2^{-n} \) for each \( n \). Then no finite
number can be $P(X)$. Let $Y = X + Z$. Although $P(X) = P(Y) = \infty$, there could be a strict preference between $Y$ and $X$ depending on $P(Z)$, and this preference is not reflected in the values of $P(X)$ and $P(Y)$.

For cases like Example 5, instead of requiring the agent to assign a constant value $P(X)$ to each random variable $X$, we assume (as in the lottery case) that an agent is willing to engage in some trades in which the agent gives up one random variable in order to receive another.

One thing that our theory has in common with de Finetti’s is that, if an agent is willing to trade $X_1$ for $Y_1$ and is willing to trade $X_2$ for $Y_2$, then the agent is willing to trade $X_1 + X_2$ for $Y_1 + Y_2$.

**Example 6.** Let $\Omega = \mathbb{Z}^+$, and let $X_n(\omega) = I_{\{n\}}(\omega)$, the indicator of the event that $\omega$ equals $n$, for $n \geq 1$. An agent wishes to express the following opinions:

- If $X \geq Y$, then $X$ is at least as valuable as $Y$.
- All $X_n$ ($n \geq 1$) are equally valuable, and each is strictly more valuable than the constant 0.

As in Example 1, this preference is not representable by a standard-valued function, but it is representable by a nonstandard-valued function. The proof will appear in Section 1.2.

1.2. When no Price is Available

That there are no standard-valued representations for the preferences in Examples 1 and 6 follows from Theorem 3.1 of [6]. That theorem relies on the following definition.

**Definition 2.** Let $\rho$ be a binary relation on a set $\mathcal{Y}$. Then $\mathcal{Z} \subseteq \mathcal{Y}$ is $\rho$-order dense in $\mathcal{Y}$ if and only if, whenever $x \rho y$ and $x, y \in \mathcal{Y} \setminus \mathcal{Z}$, there is $z \in \mathcal{Z}$ such that $x \rho z$ and $z \rho y$.

**Theorem 1** (Fishburn 1970, Theorem 3.1). Let $\preceq$ be a weak order on a set $\mathcal{Y}$, and let $\sim$ be the strict partial order on $\mathcal{Y}/\sim$ given by Lemma 25 in Appendix B.2 of the supplemental article [7]. There is a standard-valued function $u$ on $\mathcal{Y}$ such that

\[
x \preceq y \text{ if and only if } u(x) \leq u(y), \text{ for all } x, y \in \mathcal{Y},
\]

if and only if $\preceq$ is a weak order and there is a countable subset of $\mathcal{Y}/\sim$ which is $\prec'$-order dense in $\mathcal{Y}/\sim$. 

Example 7 (Continuation of Example 1). We have $[X] = \{X\}$ for each $X \in \mathcal{G}$. If a set $Z$ is $\prec'$-order dense in $\mathcal{G}$, it must contain infinitely many elements from every subset of $\mathcal{G}$ of the form $L_c = \{(x, y) : x+y = c\}$ for every standard $c$. Since there are uncountably many disjoint sets of the form $L_c$, $Z$ can’t be countable.

Example 8 (Continuation of Example 6). The preference relation includes $0 \prec X_1$, hence $c \prec c + X_1$ for all standard $c$. For each standard $c$, let $Z_c = \{Z \in \mathcal{G} : c \leq Z \leq c + X_1\}$. We show next that $X_1 \prec d$ for all standard $d > 0$. Suppose, to the contrary, that there is standard $d > 0$ such that $d \leq X_1$. Let $n > 1/d$ be a standard integer so that $1 \prec nd$ as constant functions. Since all $X_n$ are indifferent,

$$nd \preceq X_1 + \cdots + X_n \prec 1,$$

which contradicts $1 \prec nd$. So, we have $c \prec c + X_1 \prec d$ for all $c < d$. It follows that $c \neq d$ implies $Z_c \cap Z_d = \emptyset$. Let $Z$ be $\prec'$-order dense in $\mathcal{G}/\sim$. Then, for every real $c$, $Z$ must contain at least one element of $Z_c$. Since there are uncountably many disjoint $Z_c$, $Z$ is uncountable.

1.3. Notation

Throughout this paper, $\Omega$ will denote a state space, $\mathcal{G}$ will denote a set of gambles which are functions from $\Omega$ to some other space $\mathcal{E}$. We consider two types of gambles. One type is that of real-valued random variables. In this case, $\mathcal{E} = \mathbb{R}$ and $\mathcal{G}$ is some linear subspace $\mathcal{O}$ of $\mathbb{R}^\Omega$. The other type of gamble is that of horse lotteries based on the theory of [2, 3]. In this case, $\mathcal{E} = \mathcal{R}$, the set of (finitely-additive) probabilities over a set $\mathcal{P}$ of prizes, and $\mathcal{G}$ is a subset $\mathcal{H}$ of $\mathcal{R}^\Omega$. The elements of $\mathcal{H}$ are called horse lotteries. Subsets of $\Omega$ will be called events.

1.4. What is New

With regard to the theory of [2, 3], we drop the assumption that preferences are Archimedean. That is, we do not assume Axiom 1. Non-Archimedean preference structures are known to not be representable by real-valued functions. We use nonstandard models of the reals in order to represent such preferences. For the remainder of the paper, we will refer to the familiar real numbers in the set $\mathbb{R}$ as standard numbers to distinguish them from the nonstandard numbers that we introduce in Appendix
C of the supplemental article [7]. Readers needing a more thorough understanding of nonstandards could read one of the many treatments such as [8]. Other treatments of probability and/or decision theory that make use of nonstandard numbers include [9, 10, 11, 12]. Section 3.2 of [9] has extensive references along with some details of some of them attempt to make use of nonstandard-valued probabilities. Unlike the present paper, [10] starts with the familiar decision theory setup (loss functions, bayes rules, minimax rules, admissible rules, etc.) and obtains new results by allowing probabilities to take nonstandard values. The approach of [11, 12] is primarily to define probabilities that take infinitesimal values. For a probability on a set \( \Omega \) to be a “non-Archimedean Probability,” in their terminology, they impose a condition that requires all singletons \( \{\omega\} \in \Omega \) to have probabilities that are standard multiples of a common infinitesimal \( \epsilon \). That is, there is an infinitesimal \( \epsilon \) such that for every \( \omega \in \Omega \), there is a standard \( a_\omega > 0 \) such that \( P(\{\omega\}) = a_\omega \epsilon \). This assumption places severe restrictions on the forms of non-Archimedean preferences that can be expressed. For those familiar with nonstandard models of the reals, all of our analysis will be external. The main reason for this is that the nonstandards are non-Archimedean from an external perspective, but are Archimedean from an internal perspective.

In our approach, preference amongst options is taken as fundamental, and we derive constructs such as probability and utility in manners more akin to those of [2, 3, 1]. We model partial preferences that an agent has between the elements of \( G \). In the case of lotteries, [2, 3] used weak orders to express the preferences of agents. Weak orders express a complete preference in the sense that, for every pair \( (X, Y) \) of elements of \( G \), exactly one of the following holds:

- The agent is indifferent between \( X \) and \( Y \).
- The agent strictly prefers \( X \) to \( Y \).
- The agent strictly prefers \( Y \) to \( X \).

We assume a weaker form of stated preference that we outline in Section 1.5. In addition to the three types of stated preference above, we assume that for some pairs \( (X, Y) \) of gambles, the agent expresses willingness to trade away \( X \) in order to receive \( Y \) without ruling out the possibility of being willing to trade \( Y \) to receive \( X \). For other pairs, the agent is allowed to forgo stating a willingness to trade the gambles in the pair.
De Finetti [1] started with an arbitrary set of random variables and assumed that an agent could choose a standard value $P(X)$ such that he/she would be willing to trade away either $X$ or $P(X)$ in order to receive the other one. Specifically, the change in fortune $\alpha[X - P(X)]$ is considered fair for all real $\alpha$. Although de Finetti’s theory deals only in fair trades (indifference,) there is an implicit assumption that “more is better,” which is built into the notion of coherence: Previsions are coherent if there is no finite sum of fair trades that is uniformly below zero, i.e., less than some negative constant. In de Finetti’s theory the agent is willing to accept

$$\alpha[X - P(X)] - \alpha[Y - P(Y)] = \alpha(X - Y) + \alpha[P(Y) - P(X)],$$

(1)

for all real $\alpha$. If $P(X) = P(Y)$, the right-hand side of (1) is $\alpha(X - Y)$, and the agent is implicitly willing to trade $X$ for $Y$ in either direction. If $P(X) \neq P(Y)$, there is an implicit strict preference in one direction, e.g., if $P(Y) > P(X)$, the agent would require a “kickback” of $P(Y) - P(X)$ in order to trade $Y$ to get $X$. Although not explicitly part of de Finetti’s theory, if the agent were offered $Y - X$, the agent would be willing to accept it. Indeed the agent already stated a willingness to accept (1) with $\alpha = -1$, which is uniformly smaller than $Y - X$. But the agent would not be willing to accept $X - Y$.

1.5. Expressed Preference

In this paper, we assume that the first step that an agent takes in expressing trading preferences is to specify a subset $\mathcal{J}$ of $\mathcal{G}^2$ such that $(X,Y) \in \mathcal{J}$ means that the agent is willing to trade (give) $X$ in order to receive $Y$. For example, in de Finetti’s framework, if an agent assesses a fair price $p$ for a random variable $X$, then both $(X,p)$ and $(p,X)$ will be elements of $\mathcal{J}$. As de Finetti does, we also close the set $\mathcal{J}$ under a number of operations specified by certain assumptions. Appendix D of the supplemental article [7] contains the results on how a set $\mathcal{J}$ can be closed under the assumptions.

Assumption 1. For all $X \in \mathcal{G}$, the agent is willing to trade $X$ for $X$.

Assumption 1 says that $(X,X) \in \mathcal{J}$ for all $X \in \mathcal{G}$, and it has the affect of avoiding vacuous trading preferences.

Assumption 2. Suppose that the agent is willing to trade $X_j$ to get $Y_j$ for $j = 1, \ldots, n$ and $(\alpha_1, \ldots, \alpha_n)$ is a probability vector. Then the agent is willing to trade $\alpha_1 X_1 + \cdots + \alpha_n X_n$ to get $\alpha_1 Y_1 + \cdots + \alpha_n Y_n$. 

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Assumption 2 says that if \((X_j, Y_j) \in J\) for \(j = 1, \ldots, n\), then
\[
\left( \sum_{j=1}^{n} \alpha_j X_j, \sum_{j=1}^{n} \alpha_j Y_j \right) \in J,
\]
(2)
for each probability vector \((\alpha_1, \ldots, \alpha_n)\).

Assumption 2 makes mathematical sense in both the random-variable and lottery cases despite the fact that the arithmetic involved in the two cases must be interpreted differently. The same can be said of Assumption 3 below, whose statement makes use a definition.

**Definition 3.** Let \(\rho\) be a binary relation on the set of gambles \(\mathcal{G}\). Suppose that, for all \(X, Y, Z \in \mathcal{G}\) and all standard \(\alpha \in (0, 1]\), \(X \rho Y\) if and only if \(\alpha X + (1-\alpha)Z \rho \alpha Y + (1-\alpha)Z\). Then, we say that \(\rho\) satisfies the independence axiom.

It is straightforward to see that the “only if” direction of the independence axiom is implied by Assumptions 1 and 2. (Let \(n = 2\) and \(X_2 = Y_2 = Z\) in Assumption 2.) The “if” direction needs to be assumed explicitly.

**Assumption 3.** For all \(X, Y, Z \in \mathcal{G}\) and \(\alpha \in (0, 1]\), if the agent is willing to trade \(\alpha X + (1-\alpha)Z\) to receive \(\alpha Y + (1-\alpha)Z\), then the agent is willing to trade \(X\) to receive \(Y\).

Assumption 3 says that for all \(X, Y, Z \in \mathcal{G}\) and all standard \(\alpha \in (0, 1]\), if \((\alpha X + [1 - \alpha]Z, \alpha Y + [1 - \alpha]Z) \in J\), then \((X, Y) \in J\).

There is one further assumption that will be stated later because it requires more definitions.

Throughout this paper, the more lengthy proofs will appear in Appendix A of the supplemental article [7].

2. Trading Systems

In this section, we introduce the general structure that we use to represent an agent’s preferences over trades of random variables and horse lotteries.

2.1. Structure Common to Both Cases (Part 1)

In both the random-variable and the lottery cases, we assume that willingness to trade is expressed using the binary relation \(\preceq\) and equivalently using the subset \(J\) of \(\mathcal{G}^2\) consisting of pairs \((X, Y)\) for which \(X \preceq Y\), i.e.,
the agent is willing to trade $X$ to get $Y$. When an agent is willing to trade $X$ and $Y$ in both directions, we say that the agent is indifferent between $X$ and $Y$. We express indifference between $X$ and $Y$ by $X \sim Y$. If $X \sim Y$, both $(X,Y)$ and $(Y,X)$ are in $\mathcal{J}$, and $Y \sim X$.

When $X \preceq Y$ but $\neg(X \sim Y)$, the agent is willing to trade $X$ to get $Y$, but is unwilling to trade $Y$ to get $X$. In this case, it is useful to distinguish two different types of willingness to trade in only one direction. We say that an agent has a strict preference for $Y$ over $X$ if the agent has given sufficient thought to the relative values of $X$ and $Y$ and categorically refuses to consider trading $Y$ to get $X$. In this case, we express strict preference for $Y$ over $X$ by $X \prec Y$. We say an agent has a semi-preference for $Y$ over $X$ if the agent has not given sufficient thought to the relative values of $X$ and $Y$ and might, upon further reflection, be willing to trade $Y$ to get $X$. In this case, we express semi-preference for $Y$ over $X$ by $X \triangleleft Y$. For both strict preference or semi-preference for $Y$ over $X$, we have $(X,Y) \in \mathcal{J}$ and $(Y,X) \notin \mathcal{J}$. The difference between strict preference and semi-preference is not reflected in the trades that the agent is willing to make, but rather in the degree of openness to allowing a trade in the opposite direction. However, to make the distinction more mathematically precise, we require the following operational distinction between the two types of preference:

**Assumption 4.** Strict preference $\prec$ is transitive. Also,

$$[(X \prec Y) \land (Y \triangleleft Z)] \lor [(X \triangleleft Y) \land (Y \prec Z)]$$

implies $X \prec Z$.

Assumption 4 says that $\prec$ has a type of precedence over $\triangleleft$. In particular, we do not assume that semi-preference $\triangleleft$ is transitive. It is straightforward, however, to show that the disjunction of the two is transitive. That is, if we define $\leq$ by

$$X \leq Y \text{ means } (X \prec Y) \lor (X \triangleleft Y),$$

then $\leq$ is transitive. Example 10 below is a case in which $\prec$ and $\leq$ are transitive, but $\triangleleft$ is not. In particular, “further reflection” that causes the agent to replace one instance of $X \triangleleft Y$ by $X \sim Y$ leads to other instances of $\triangleleft$ becoming $\prec$. We don’t concern ourselves with closing a set $\mathcal{J}$ under Assumption 4 for the following reason. If a set $\mathcal{J}$ satisfies the first three assumptions, then Assumption 4 deals only with the labels ($\prec$ or $\triangleleft$) that attach to pairs that are already in $\mathcal{J}$ rather than requiring that additional
pairs must belong to \( J \). For example, it is acceptable for every pair \((X, Y) \in J\) with \((Y, X) \notin J\) to be labeled \( \prec \). Similarly, it is acceptable for every such pair to be labeled \( \triangleleft \). In the latter case, if an agent wants to switch a \( \triangleleft \) label to \( \prec \) in order to express strict preference, Lemma 9 in Section 3.2 shows how to identify which other labels must also be switched in order to satisfy Assumption 4. Lemma 8 in Section 3.1 shows what must change if an agent wishes to change an instance of \( \triangleleft \) to \( \sim \).

Note that the symbol \( \preceq \) is equivalent to the disjunction of \( \leq \) and \( \sim \). That is,

\[ X \preceq Y \text{ is equivalent to } (X \leq Y) \lor (X \sim Y). \quad (3) \]

Finally, if an agent has not expressed willingness to trade \( X \) and \( Y \) in either direction, then neither \((X, Y)\) nor \((Y, X)\) is in \( J \), and we write \( X \diamond Y \) and \( Y \diamond X \).

We summarize the above description in the following definition.

**Definition 4.** A **partial preference relation** on \( G \) consists of four binary relations as follows:

1. An indifference between \( X \) and \( Y \), denoted as \( X \sim Y \). This indicates that the agent is willing to give \( X \) to receive \( Y \) and is willing to give \( Y \) to receive \( X \).
2. A strict preference for \( Y \) over \( X \), denoted as \( X \prec Y \). This indicates that the agent is willing to give \( X \) to receive \( Y \), but is resolved not to give \( Y \) to receive \( X \).
3. A semi-preference for \( Y \) over \( X \), denoted as \( X \triangleleft Y \). This indicates that the agent is willing to give \( X \) to receive \( Y \) but is unresolved whether or not to be willing to give \( Y \) to receive \( X \).
4. Remaining silent on whether or not to trade \( X \) and \( Y \) in either direction, denoted as \( X \diamond Y \).

A partial preference relation that satisfies Assumptions 1–4, is called a **partial trading system**. A partial trading system for which \( \triangleleft \) and \( \diamond \) are empty is called a **full trading system**. We denote a partial trading system by \((\sim, \prec, \triangleleft)\) since \( \diamond \) is defined implicitly by the other three relations. A full trading system will be denoted \((\sim, \prec)\).

**Example 9.** Let \( X \) be a linear space, and define \( \sim \) on \( X \) by \( X \sim Y \) if and only if \( X = Y \). Let \( \prec \) and \( \triangleleft \) be empty. Then \((\sim, \prec, \triangleleft)\) is a partial trading system on \( X \). It is clear that this is the smallest partial trading system on \( X \).
Example 10. An agent is considering a coin flip that could result in either head (H) or tail (T). Let Ω = \{H, T\}, and let \(G\) be the set of all standard-valued functions defined on Ω. Denote such functions as ordered pairs, where the first coordinate corresponds to H. Assume that the agent prefers larger functions to smaller functions. The agent believes that the coin could have a probability of H anywhere in the closed interval [0.5, 0.75]. This translates to the following preferences, where \((x_1, x_2)\) is to be understood as distinct from \((y_1, y_2)\):

- \((x_1, x_2) \sim (x_1, x_2)\) for all \(x_1, x_2\),
- \((x_1, x_2) \prec (y_1, y_2)\) if \(y_2 - x_2 > \max\{x_1 - y_1, 3(x_1 - y_1)\}\),
- \((x_1, x_2) \preceq (y_1, y_2)\) if \(y_2 - x_2 = \max\{x_1 - y_1, 3(x_1 - y_1)\}\),
- \((x_1, x_2) \diamond (y_1, y_2)\) for all other distinct pairs.

For example, \((-1, 1) \prec (0, 0) \prec (-0.5, 1.5)\), but \((-1, 1) \prec (-0.5, 1.5)\), because paying \((-1, 1)\) to receive \((-0.5, 1.5)\) yields the positive constant \((0.5, 0.5)\). From a different perspective, the agent could later decide either that \((-1, 1) \sim (0, 0)\) or that \((0, 0) \sim (-0.5, 1.5)\), but not both. If the agent later decided that \((-1, 1) \sim (0, 0)\) it would have the effect of deciding that the probability of H is actually 0.5, thereby expanding both \(\sim\) and \(\prec\) to form a weak order defined by

\[(x_1, x_2) \preceq (y_1, y_2)\text{ if and only if } x_1 + x_2 \leq y_1 + y_2.\]

We give some evidence of this last claim at the end of the example. If the agent chose to let \((0, 0) \sim (-0.5, 1.5)\), it would have the effect of deciding that the probability of H is 0.75. If the agent chose to make both instances of \(\prec\) into \(\preceq\), that would amount to ruling out the two endpoint probabilities.

To conclude, suppose that the agent decides to convert the partial preference \((-1, 1) \prec (0, 0)\) into indifference. Then the agent is willing to trade either \((1, -1)\) or \((-1, 1)\) in order to receive \((0, 0)\). Because the agent also is willing to engage in finitely many trades, the agent is also willing to trade \((1, -1)\) for \((-1, 1)\) in either direction by combining individual swaps with the random variable \((0, 0)\). Hence, every numerical function that represents this agent’s preference will assign \((-1, 1)\) and \((1, -1)\) the same value even though the agent did not explicitly express \((-1, 1) \sim (1, -1)\).
Note that a full trading system \((\sim, \prec)\) is equivalent to its corresponding weak order \(\preceq\) because

- \(X \sim Y\) if and only if \((X \preceq Y) \land (Y \preceq X)\), and
- \(X \prec Y\) if and only if \((X \preceq Y) \land \neg(Y \preceq X)\).

The following result relates the above types of trading to the four assumptions that we make.

**Lemma 1.** Let \(J \subseteq G^2\) satisfy Assumptions 1–4. Then

- \(\sim\) is an equivalence relation, hence it partitions \(G\) into equivalence classes,
- \(\prec\) is a strict partial order,
- \(\triangleleft\) is asymmetric,
- \(\prec\) and \(\triangleleft\) are codirectional (see Definition 23 in Appendix B.1 of the supplemental article [7]) and defined on the equivalence classes of \(\sim\), and
- \(\preceq\) expressed in (3) is a preorder defined on the equivalence classes of \(\sim\).

**Proof.** Start with \(\sim\). Assumption 1 says that \(\sim\) is reflexive. If the agent is willing to trade \(X\) for \(Y\) in both directions, \(\sim\) must be symmetric. If \(X \sim Y\) and \(Y \sim Z\), then Assumption 2 says that \(0.5X + 0.5Y \sim 0.5Y + 0.5Z\), which is the same as trading \(X\) to get \(Z\) in both directions according to Assumption 3. Hence, \(X \sim Z\), and \(\sim\) is transitive.

For \(\prec\), if \(X \prec Y\) then \(\neg(Y \prec X)\) because the agent is not willing to trade in both directions, so \(\prec\) is asymmetric. That \(\prec\) is transitive is part of Assumption 4. To see that \(\prec\) is defined on the equivalence classes of \(\sim\), suppose that \(X \prec Y\), \(X' \sim X\), and \(Y' \sim Y\). Then the agent is willing to trade \((1/3)X' + (1/3)X + (1/3)Y\) to get \((1/3)X + (1/3)Y + (1/3)Y'\) (Assumption 2), hence, the agent is willing to trade \(X'\) to get \(Y'\) (Assumption 3).

For \(\triangleleft\), if \(X \triangleleft Y\) then \(\neg(Y \triangleleft X)\) because the agent is not willing to trade in both directions, so \(\triangleleft\) is asymmetric. To see that \(\triangleleft\) is defined on the equivalence classes of \(\sim\), suppose that \(X \triangleleft Y\), \(X' \sim X\), and \(Y' \sim Y\). Then the agent is willing to trade \((1/3)X' + (1/3)X + (1/3)Y\) to get
\[(1/3)X + (1/3)Y + (1/3)Y',\] hence, the agent is willing to trade \(X'\) to get \(Y'\). That \(\prec\) and \(\triangleleft\) are codirectional, follows directly from Assumption 4.

That \(\leq\) expressed in (3) is a preorder defined on the equivalence classes of \(\sim\) follows from Lemma 27 in Appendix B.2 of the supplemental article [7].

There is additional structure that is common to both the lottery and random-variable cases, but we must first show how to map a set of horse lotteries into a linear space \(X\) that will behave, in many ways, like a set of random variables. The mapping appears in Section 2.2.

2.2. Structure Peculiar to the Lottery Case

Two features are the main differences between the structures of the lottery and random-variable cases. One is the fact that random variables are elements of a linear space whereas horse lotteries are elements of a convex set. This means that the arithmetic that is available in one case differs from the arithmetic that is available in the other case. The other feature that is different is the fact that random variables are standard-valued functions with some natural partial orders based on the ordering of standard numbers. The prizes involved in horse lotteries can be quite general objects with no natural quantitative properties. The main feature that we need in order to represent partial preferences is the arithmetic that is available in a linear space. To that end, this section provides a mapping of horse lotteries into a linear space that preserves Assumptions 1–4 and the partial preference structure described in Section 1.5. The mapping that we provide is a generalization of what appears in [13] for the same purpose in a simpler setting. The same mapping was used by [14] in a similar setting.

Let \(Q\) be the set of all bounded finitely-additive signed measures over the set of prizes \(P\), and let \(R\) be the set of all finitely-additive probabilities over \(P\). Then \(Q\) is a linear space that contains \(R\) as a convex subset. Let \(H\) be a mixture set of horse lotteries. Then

\[H \subseteq R^\Omega \subseteq Q^\Omega,\]

the last of which is a linear space. Let \(K_0\) be the linear span of those elements of \(Q^\Omega\) of the form \(h_2 - h_1\) where \(h_1, h_2 \in H\). The following is straightforward to prove.

**Proposition 1.** Each element of \(K_0\) has the form \(\alpha(h_2 - h_1)\) with \(\alpha > 0\) standard and \(h_1, h_2 \in H\).
The representation of each $k \in K_0$ as $\alpha(h_2 - h_1)$ is far from unique. For example, if $h_1, h_2, h_3 \in H$ for $j = 1, 2, 3$ and $\alpha \in (0, 1)$, then

\[
\alpha(h_2 - h_1) = [\alpha h_2 + (1 - \alpha) h_3] - [\alpha h_1 + (1 - \alpha) h_3]. \tag{4}
\]

Also, the 0 element of $K_0$ has many representations as $\alpha(h - h)$ for every $\alpha > 0$ and every $h \in H$. Due to such lack of uniqueness, some proofs about $K_0$ will require us to prove that concepts are well defined when they are based on representations of elements of $K_0$ as $\alpha(h_2 - h_1)$.

The set $K_0$ plays an important role in our treatment of trading horse lotteries. It allows us to do much of the same sort of arithmetic that we can do in the random-variable case. The only parts of the random variable analysis that we do not duplicate for horse lotteries are those derived from inequalities between random variables. (See Definition 7 in Section 2.4.)

The first use of $K_0$ is to help us close a set $J \subseteq H^2$ under Assumption 3 if $J$ satisfies Assumptions 1 and 2. The second use of $K_0$ is to unify the statements and proofs of our results that apply to both the lottery and the random-variable cases. These results will be stated in terms of a linear space $\mathcal{X}$. In the random-variable case, $\mathcal{X} = \mathcal{O}$, the linear space of random variables under consideration. In the lottery case, $\mathcal{X} = K_0$. In order for $K_0$ to be useful as the space $\mathcal{X}$, it must have properties similar to those that $\mathcal{O}$ has. In particular, we need binary relations on $K_0$ that correspond to the partial preference on $H$.

For each binary relation $\rho$ on a mixture set $H$ of horse lotteries, define the binary relation $\rho'$ on $K_0$ as follows. For each $k_1, k_2 \in K_0$, express $k_2 - k_1 = \alpha(h_2 - h_1)$ with $\alpha > 0$ and $h_1, h_2 \in H$. Then say that $k_1 \rho k_2$ if and only if $h_1 \rho h_2$. Since the representation $k_2 - k_1 = \alpha(h_2 - h_1)$ is not unique, we need to show that $\rho'$ is well defined.

**Lemma 2.** Assume that $\rho$ satisfies Assumptions 1–3. Then $\rho'$ described above is well defined.

**Proof.** To see that $\rho'$ is well defined, suppose that

\[
k_2 - k_1 = \alpha(h_1 - h_2) = \alpha'(h_1' - h_2'), \tag{5}
\]

where $\alpha, \alpha' > 0$ and $h_1, h_2, h_1', h_2' \in H$. We need to show that $h_1 \rho h_2$ implies $h_1' \rho' h_2'$. It follows from (5) that, as elements of $H$,

\[
\frac{\alpha}{\alpha + \alpha'} h_1 + \frac{\alpha'}{\alpha + \alpha'} h_2 = \frac{\alpha}{\alpha + \alpha'} h_2 + \frac{\alpha'}{\alpha + \alpha'} h_1'. \tag{6}
\]
Start with $h_1 \rho h_2$ and use Assumption 2 to mix $h'_2$ with weight $\alpha'/(\alpha + \alpha')$ into both sides. The result is

$$\frac{\alpha}{\alpha + \alpha'} h_1 + \frac{\alpha'}{\alpha + \alpha'} h'_2 \rho \frac{\alpha}{\alpha + \alpha'} h_2 + \frac{\alpha'}{\alpha + \alpha'} h'_2.$$ 

Combine this with (6) to get

$$\frac{\alpha}{\alpha + \alpha'} h_2 + \frac{\alpha'}{\alpha + \alpha'} h'_1 \rho \frac{\alpha}{\alpha + \alpha'} h_2 + \frac{\alpha'}{\alpha + \alpha'} h'_2.$$ 

Assumption 3 then implies that $h'_1 \rho h'_2$. 

Imagine that we apply Lemma 2 to each $\rho$ in the set $\{\sim, \prec, \prec\}$, to produce $\sim'$, $\prec'$, and $\prec'$ on $K_0$. In Appendix E of the supplemental article [7], we show that, in each application of Lemma 2, the relations $\rho$ and $\rho'$ inherit each other’s properties of satisfying the three assumptions, as well as being reflexive, symmetric, asymmetric, and/or transitive. Also, $\prec$ and $\prec$ are codirectional if and only if $\prec'$ and $\prec'$ are codirectional.

After mapping $H$ to $K_0$, we will extend partial trading systems on $K_0$ using the same theorems that extend partial trading systems on $O$. In the lottery case, after extending a partial trading system, we can translate results back to $H$. For a binary relation $\rho'$ on $K_0$, we can recover the corresponding extension on $H$ as follows. For $h_1, h_2 \in H$ and $\alpha > 0$, let $k = \alpha(h_2 - h_1) \in K_0$. Then $h_1 \rho h_2$ if and only if $k \rho' 0$.

2.3. Structure Common to Both Cases (Part 2)

Now that we have mapped a mixture set $H$ of horse lotteries to a linear space, we are prepared to complete the presentation of the structure that is common to both the lottery and random-variable cases. In the random-variable case, $X$ will stand for $O$, the linear space of random variables. In the lottery case, $X$ will stand for $K_0$, the linear space introduced in Section 2.2.

When dealing with a linear space, there is a condition that is equivalent to the independence axiom (see Definition 3) and is somewhat more convenient mathematically.

**Definition 5.** Let $\rho$ be a binary relation on $X$. We call $\rho$ affine if for $X, Y, Z \in X$ and all standard $\alpha > 0$, $X \rho Y$ if and only if $\alpha X + Z \rho \alpha Y + Z$. Let $A_\rho$ be the subset of $X^2$ that represents $\rho$ as in Definition 21 in Appendix B.1 of the supplemental article [7]. We will call $A_\rho$ affine if $\rho$ is affine.
Here are some simple properties of an affine relation.

**Proposition 2.** If $\rho$ is affine and $X \rho Y$, then $-Y \rho -X$. If $\sim$ is an affine equivalence relation and $X \sim Y$, then $\alpha X \sim \alpha Y$ for all standard $\alpha$.

The following result is useful in proving the equivalence of the independence axiom and being affine.

**Lemma 3.** Let $X$ be a linear space and assume that $\rho$ satisfies the independence axiom. For all $X, Y \in X$ and standard $c > 0$, $X \rho Y$ if and only if $cX \rho cY$.

**Proof.** Let $X, Y \in X$, $c > 0$, and $Z = 0$. If $c \leq 1$, apply the independence axiom with $\alpha = c$. If $c > 1$, apply the independence axiom with $\alpha = 1/c$ so that $cX \rho cY$ if and only if $\alpha(cX) + (1 - \alpha)Z \rho \alpha(cY) + (1 - \alpha)Z$ if and only if $X \rho Y$. \hfill $\Box$

**Lemma 4.** A binary relation $\rho$ on a linear space $X$ satisfies the independence axiom if and only if $\rho$ is affine.

**Proof.** For the “if” direction, assume that $\rho$ is affine, that is, for all $X, Y, Z \in X$ and standard $\beta > 0$,

$$X \rho Y \text{ if and only if } \beta X + Z \rho \beta Y + Z.$$  \hspace{1cm} (7)

We need to show that $\rho$ satisfies the independence axiom. That is, we must show that for all $X, Y, Z \in X$ and standard $\alpha \in (0, 1]$,

$$X \rho Y \text{ if and only if } \alpha X + (1 - \alpha)Z \rho \alpha Y + (1 - \alpha)Z.$$ \hspace{1cm} (8)

Let $X, Y, Z \in X$ and $\alpha \in (0, 1]$. Let $\beta = \alpha$ and $Z' = (1 - \alpha)Z$. Since $\rho$ is affine, $X \rho Y$ if and only if $\beta X + Z' \rho \beta Y + Z'$ if and only if $\alpha X + (1 - \alpha)Z \rho \alpha Y + (1 - \alpha)Z$.

For the “only if” direction, assume that $\rho$ satisfies the independence axiom. We need to show that $\rho$ is affine. Let $X, Y, Z \in X$ and $\beta > 0$. We need to show that (7) holds. If $\beta < 1$, let $Z' = Z/(1 - \beta)$, so that $X \rho Y$ if and only if $\beta X + (1 - \beta)Z' \rho \beta Y + (1 - \beta)Z'$ if and only if $\beta X + Z \rho \beta Y + Z$. If $\beta \geq 1$, apply Lemma 3 with $c = 2\beta^2$ so that $X \rho Y$ if and only if $2\beta^2 X \rho 2\beta^2 Y$. Let $\alpha = 1/(2\beta)$ and $Z' = 2\beta Z/(2\beta - 1)$. Since $\rho$ satisfies the independence axiom, $\beta^2 X \rho \beta^2 Y$ if and only if $\alpha \beta^2 X + (1 - \alpha)Z' \rho \alpha \beta^2 Y + (1 - \alpha)Z'$ if and only if $\beta X + Z \rho \beta Y + Z$. \hfill $\Box$
Definition 6. Let \(\mathcal{T} = (\sim, \prec, \lhd)\) be a partial trading system. We call a trade fair if it is a sum of finitely many elements of \(\mathcal{X}\) of the form \(\alpha(Y - X)\), where \(X \sim Y\) and \(\alpha\) is standard. Denote the set of fair trades \(\mathcal{F}_T\). We call a trade acceptable if it is the sum of finitely many elements of \(\mathcal{X}\) of the form \(\alpha(Y - X)\) with \(X \preceq Y\) and \(\alpha > 0\) is standard. Denote the set of acceptable trades as \(\mathcal{V}_T\).

The following is immediate from the definitions.

Proposition 3. Let \(\mathcal{T}\) be a partial trading system. Then \(\mathcal{F}_T\) is a linear space, \(0\) is a fair trade, and \(\mathcal{V}_T\) is a convex cone.

The following result summarizes the properties of trades.

Lemma 5. Let \(\mathcal{T} = (\sim, \prec, \lhd)\) be a partial trading system. Then

\(\begin{align*}
\bullet & \quad V \in \mathcal{F}_T \text{ if and only if } 0 \sim V, \\
\bullet & \quad V \in \mathcal{V}_T \text{ if and only if } 0 \preceq V.
\end{align*}\)

Proof. We will prove the claim about \(\mathcal{V}_T\). For the “if” direction, assume that \(0 \preceq V\), then \(V = 1(V - 0) \in \mathcal{V}_T\). For the “only if” direction, assume that \(V \in \mathcal{V}_T\). Write \(V = \sum_{j=1}^{n} \alpha_j(Y_j - X_j)\) where \(X_j \preceq Y_j\) for \(j = 1, \ldots, n\), \(\alpha_j > 0\) for each \(j\) such that \((X_j \prec Y_j) \lor (X_j \lhd Y_j)\). First, note that, for all \(j\), \(0 \preceq \alpha_j(Y_j - X_j)\) because \(\preceq\) is affine. For \(j = 1\), we have \(0 \preceq \alpha_1(Y_1 - X_1)\). Because \(\preceq\) is affine, we can add \(\alpha_2(Y_2 - X_2)\) to both sides to get

\[0 \preceq \alpha_2(Y_2 - X_2) \preceq \alpha_1(Y_1 - X_1) + \alpha_2(Y_2 - X_2).\]

Continuing in this manner \(n\) times, we get \(0 \preceq V\). The proof of the claims about \(\mathcal{F}_T\) is similar. \(\square\)

There is an alternative way to denote partial trading systems. An equivalence relation on a set \(\mathcal{X}\) corresponds to a partition \(\mathcal{C}\) of \(\mathcal{X}\), where each element \(C \in \mathcal{C}\) is called an equivalence class. For each equivalence class \(C\), we have \(X, Y \in C\), if and only if \(X \sim Y\). (See Definition 23 in Appendix B.1 of the supplemental article [7].) Let \(\rho\) be defined on the equivalence classes of \(\sim\), and for each \(X \in \mathcal{X}\), we let \([X]_{\sim}\) denote the equivalence class that contains \(X\). We will allow ourselves to use the notation \([X]_{\sim} \rho [Y]_{\sim}\) to mean that \(X' \sim Y'\) for every \(X' \sim X\) and \(Y' \sim Y\).

For each pair of equivalence classes \([X]_{\sim}\) and \([Y]_{\sim}\), exactly one of the following is true:

\(\begin{align*}
[X]_{\sim} & = [Y]_{\sim}, \quad [X]_{\sim} \prec [Y]_{\sim}, \quad [Y]_{\sim} \prec [X]_{\sim}, \\
[X]_{\sim} & \lhd [Y]_{\sim}, \quad [Y]_{\sim} \lhd [X]_{\sim}, \quad [X]_{\sim} \lhd [Y]_{\sim}, \quad [Y]_{\sim} \lhd [X]_{\sim}, \quad [X]_{\sim} \lhd [Y]_{\sim}.
\end{align*}\) (9)
2.4. Structure Peculiar to the Random-Variable Case

In this section $G$ is a linear space $O$ of random variables of interest. The idea that “more is better” in the random-variable case can be formalized in several ways.

Definition 7. Let $X, Y \in O$. We say that $Y$ weakly dominates $X$ or $X$ is weakly dominated by $Y$ if $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$ and there is $\omega \in \Omega$ such that $X(\omega) < Y(\omega)$. We say that $Y$ strictly dominates $X$ or $X$ is strictly dominated by $Y$ if $X(\omega) < Y(\omega)$ for all $\omega \in \Omega$. We say that $Y$ uniformly dominates $X$ or $X$ is uniformly dominated by $Y$ if there exists a standard $\epsilon > 0$ such that $X(\omega) \leq Y(\omega) - \epsilon$ for all $\omega \in \Omega$.

It is trivial to see that weak dominance is an extension of strict dominance which, in turn, is an extension of uniform dominance. Many of our results do not depend on which version of dominance we choose. For those results that depend on which form of dominance we choose (in Section 5), we will be explicit about which form is needed. We use $X \prec_{\text{Dom}} Y$ to denote “$Y$ dominates $X$” in whichever sense one chooses.

Definition 8. Let $\prec$ be a strict partial order on $O$. We say that $\prec$ extends dominance if $X \prec_{\text{Dom}} Y$ implies $X \prec Y$.

There is an order relationship even weaker than weak dominance to which we also need to pay attention. All three versions of $X \prec_{\text{Dom}} Y$ imply $X \leq Y$. Since (following de Finetti) we will include a respect for dominance in our definition of coherence, we must also be sure that coherence attends to cases of $X \leq Y$, even if they don’t satisfy $X \prec_{\text{Dom}} Y$. In particular, coherence will need to place restrictions on what trades are allowed when

$$(X \leq Y) \land [\neg(X \prec_{\text{Dom}} Y)]. \quad (10)$$

For instance, every case in which $Y$ weakly dominates $X$ but $Y$ does not strictly dominate $X$ satisfies (10) under both strict and uniform dominance. Under weak dominance, (10) never occurs for $X$ and $Y$ distinct.

Since an agent who wishes to be coherent does not want to accept a trade that is dominated by zero, we define coherence to entail that no acceptable trade is dominated by zero. However, because the agent, upon further reflection, might change a $\prec$ relation to $\sim$, we need coherence to entail even more.

Definition 9. We call a partial trading system $\mathcal{T}$ almost coherent if no acceptable trade is dominated by zero. We call an almost coherent partial trading system coherent if
• \( \preceq \) extends \( \leq \), and
• \( \prec \) extends dominance.

If \( T = (\sim, \prec) \) is a coherent full trading system, we call the associated weak order \( \preceq \) a coherent order.

**Example 11.** Let \( \mathcal{O} \) be a linear space of standard-valued functions defined on a space \( \Omega \). Define \( \sim \) on \( \mathcal{O} \) by \( X \sim Y \) if and only if \( X = Y \). Choose one of the definitions of dominance, and define \( \prec \) on \( \mathcal{O} \) by \( X \prec Y \) if and only if \( X \prec_{\text{dom}} Y \). Define \( \lhd \) on \( \mathcal{O} \) by \( X \lhd Y \) if and only if \( (X \leq Y) \land [\neg((X \sim Y) \lor (X \prec Y))] \). Since every acceptable trade is non-negative, \( T = (\sim, \prec, \lhd) \) is an almost coherent partial trading system. By construction, \( \preceq \) extends \( \leq \), and \( \prec \) is dominance. So \( T \) is coherent.

The following example illustrates how the coherence of a trading system can depend on the chosen definition of dominance.

**Example 12.** Let \( \Omega = \mathbb{Z}^+ \), the positive integers, and let \( \mathcal{O} \) be the set of all bounded standard-valued functions. Let \( P \) be a (possibly finitely-additive) probability on \( 2^\Omega \), and rank all elements of \( \mathcal{O} \) by their expected values under \( P \). That is,

- \( X \sim Y \) if and only if \( P(X) = P(Y) \), and
- \( X \prec Y \) if and only \( P(X) < P(Y) \).

These bullets define a full trading system \( T = (\sim, \prec) \). If \( P(\{\omega\}) > 0 \) for all \( \omega \in \Omega \), then \( T \) is coherent under all three definitions of dominance. If there is at least one \( \omega \in \Omega \) with \( P(\{\omega\}) > 0 \) and at least one with \( P(\{\omega\}) = 0 \), then \( T \) is coherent under strict and uniform dominance, but not under weak dominance. If \( P(\{\omega\}) = 0 \) for all \( \omega \), then \( T \) is coherent only under uniform dominance.

If an agent has an almost coherent partial trading system \( T \), there is a way to close \( T \) under the two bulleted conditions in Definition 9 and make the result coherent. Lemma 39 in Appendix D of the supplemental article [7] shows how this is done.

Although \( \prec_{\text{dom}} \) is generally not defined on the equivalence classes of an equivalence relation \( \sim \), if \( \sim \) is part of a coherent partial trading system, then there is a natural analog to \( \prec_{\text{dom}} \) that is defined on the equivalence classes.
Lemma 6. Let $\mathcal{T} = (\sim, \prec, \ll)$ be a coherent partial trading system. Let $\mathcal{C} = \mathcal{X}/\sim$ be the partition that corresponds to $\sim$. For each $X \in \mathcal{X}$, let $[X]$ be the element of $\mathcal{C}$ that contains $X$. Define $[X] \prec_{\text{Dom}}' [Y]$ to mean that there exist $X' \in [X]$ and $Y' \in [Y]$ such that $X' \prec_{\text{Dom}} Y'$. Then $\prec_{\text{Dom}}'$ is a strict partial order that is defined on the equivalence classes of $\sim$.

Proof. Note that $\prec_{\text{Dom}}'$ is defined directly on the equivalence classes of $\sim$. To see that $\prec_{\text{Dom}}'$ is asymmetric, assume to the contrary that there are $X, Y \in \mathcal{C}$ with $X \preccurlyeq_{\text{Dom}}' Y$ and $Y \preccurlyeq_{\text{Dom}}' X$. Then, there exist $X', X* \in [X]$ and $Y', Y* \in [Y]$ such that $X' \prec_{\text{Dom}} Y'$ and $Y* \prec_{\text{Dom}} X*$. Since $\sim$ and $\prec_{\text{Dom}}'$ are affine and transitive, we have

$$Y* \prec_{\text{Dom}} X* = X' + X* - X' \prec_{\text{Dom}} Y' + X* - X' \in [Y],$$

which contradicts $\mathcal{T}$ being coherent. To see that $\prec_{\text{Dom}}'$ is transitive, assume that $[X] \prec_{\text{Dom}}' [Y]$ and $[Y] \prec_{\text{Dom}}' [Z]$. Let $X' \in [X]$, $Y', Y* \in [Y]$, and $Z* \in [Z]$ be such that $X' \prec_{\text{Dom}} Y'$ and $Y* \prec_{\text{Dom}} Z*$. Then $X' + Y* - Y' \in [X]$, and

$$X' + Y* - Y' \prec_{\text{Dom}} Y* \prec_{\text{Dom}} Z*.$$

So $[X] \prec_{\text{Dom}}' [Z]$. \qed

A result similar to Lemma 6 holds if we replace $\prec_{\text{Dom}}'$ with $\preccurlyeq_{\text{Dom}}$. 

Lemma 7. Assume the conditions of Lemma 6. Define $[X] \preceq_{\text{Dom}} [Y]$ to mean that there exist $X' \in [X]$ and $Y' \in [Y]$ such that $X' \preceq Y'$. Then $\preceq_{\text{Dom}}$ is a preorder that is defined on the equivalence classes of $\sim$.

Proof. Note that $\preceq_{\text{Dom}}$ is defined directly on the equivalence classes of $\sim$. Clearly, $\preceq_{\text{Dom}}$ is reflexive. To see that $\preceq_{\text{Dom}}$ is transitive, assume that $[X] \preceq_{\text{Dom}} [Y]$ and $[Y] \preceq_{\text{Dom}} [Z]$. Let $X' \in [X]$, $Y', Y* \in [Y]$, and $Z* \in [Z]$ be such that $X' \preceq Y'$ and $Y* \preceq Z*$. Then $X' + Y* - Y' \in [X]$, and

$$X' + Y* - Y' \preceq Y* \preceq Z*.$$

So $[X] \preceq_{\text{Dom}} [Z]$. \qed

If $\mathcal{C} = \mathcal{X}/\sim$ is the set of equivalence classes of $\sim$, and $(\mathcal{C}, \prec, \ll)$ is a partial trading system, we will take the liberty of using the notation $[X] \ll_{\sim} [Y]$ to mean that $[X] \prec_{\text{Dom}} [Y]$ and $[X] \ll_{\sim} [Y]$. And we will use $[X] \ll_{\sim} [Y]$ to mean that $[X] \ll_{\sim} [Y]$. 

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There is no obvious analog to coherence in the lottery case, and our results that apply in both cases will need extra clauses such as “If coherence is desired, . . . ” Alternatively, some results can be stated for both cases by referring to the partial trading systems as “(coherent) partial trading systems.”

3. Extending a Partial Trading System

In this section we assume that $X$ is a linear space of functions defined on $\Omega$. In the random-variable case, $X = \mathcal{O}$. In the lottery case, $X = \mathcal{K}_0$. We show that a partial trading system on $X$ can be extended to a full trading system on $X$.

Definition 10. Let $T_1 = (C_1, \prec_1, \triangleleft_1)$ and $T_2 = (C_2, \prec_2, \triangleleft_2)$ be partial trading systems. We say that $T_2$ is an extension of $T_1$ or $T_2$ extends $T_1$ if

1. $C_2$ is a coarsening of $C_1$,
2. for all $X, Y \in X$, $[X]_{C_1} \prec_1 [Y]_{C_1}$ implies $[X]_{C_2} \prec_2 [Y]_{C_2}$, and
3. for all $X, Y \in X$, $[X]_{C_1} \preceq_1 [Y]_{C_1}$ implies $[X]_{C_2} \preceq_2 [Y]_{C_2}$, where (for $j = 1, 2$) $[X]_{C_j} \preceq_j [Y]_{C_j}$ means $([X]_{C_j} = [Y]_{C_j}) \lor ([X]_{C_j} \prec_j [Y]_{C_j}) \lor (\lnot [X]_{C_j} \prec_j [Y]_{C_j})$.

We will take the liberty of saying that $\prec_2$ extends $\prec_1$ if 2 above holds, and $\preceq_2$ extends $\preceq_1$ if 3 above holds.

As an example, every coherent full trading system is an extension of the $T$ given in Example 11. Note that it is possible for $T_2$ to extend $T_1$ even if $\triangleleft_2$ does not extend $\triangleleft_1$.

The following result is straightforward from the definition of coherent.

**Proposition 4.** A partial trading system that extends a coherent partial trading system is also coherent.

There are two things that we can do to extend a partial trading system. One is to add more fair trades by making equivalence classes larger, which is the same as coarsening the corresponding partition. Another is to add more acceptable trades that are not fair by extending the strict partial order. There are restrictions on which new trades can be declared acceptable. In order to extend to a full trading system, the $\triangleleft$ relation needs to be pared down to empty by converting each of its instances either to an equivalence or to a strict preference in the same direction as $\prec$. 

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Theorem 2 starts with a (coherent) partial trading system. It then introduces a new comparison between $X$ and $Y$ if and only if it is allowed by $\lhd$, and the result will be a (coherent) partial trading system. (An instance of $X \lhd Y$ precludes extending $\prec$ to make $Y \prec X$.) Theorem 2 is an analog to the fundamental theorem of prevision. (See de Finetti, 1974, Section 3.10.) The fundamental theorem shows how to extend a coherent prevision from a set of random variables to a larger set of random variables while maintaining coherence. The key to the extension is to answer the question “which numbers can serve as a coherent prevision for a random variable that doesn’t yet have a prevision?” In our setting the analogous question is “which equivalence classes can we merge or to which pairs can we assign strict preference while maintaining coherence?” The equivalence classes play two roles in our above analysis. They stand in for previsions and for random variables. In analogy to the fundamental theorem, the equivalence classes (as previsions) that can be coherently merged with an equivalence class (as random variable) are those that don’t stand in a strict preference relation to each other. The ones that can be assigned an order are those that are not already indifferent or in the opposite order.

3.1. Extending an Equivalence Relation

In this section, $\mathcal{X}$ is a general linear space. Extending an equivalence relation is equivalent to coarsening the corresponding partition by merging distinct (i.e., disjoint) equivalence classes. The proof of Lemma 8 appears in Appendix A.1 of the supplemental article [7].

Lemma 8. Let $\mathcal{T} = (\mathcal{C}, \prec, \lhd)$ be a (coherent) partial trading system. Let $[Q_1]_C$ and $[Q_2]_C$ be disjoint elements of $\mathcal{C}$ such that neither of the following is true:
\[ [Q_1]_C \prec [Q_2]_C, \quad [Q_2]_C \prec [Q_1]_C. \]  
(11)
There is a (coherent) partial trading system $\mathcal{T}_* = (\mathcal{C}_*, \prec_*, \lhd_*)$ that extends $\mathcal{T}$ and includes the comparison $Q_1 \sim_* Q_2$.

3.2. Extending a Strict Partial Order

The most straightforward way to extend a strict partial order is to add a strict relation that neither exists already nor is already contradicted. For example, $X \prec Y$ is already contradicted if $X \sim Y$, $Y \prec X$, $Y \lhd X$, or $Y \prec_{\text{Dom}} X$. The proof of Lemma 9 appears in Appendix A.2 of the supplemental article [7].
Lemma 9. Let \( \mathcal{T} = (\mathcal{C}, <, \triangleleft) \) be a (coherent) partial trading system. Let \([X]_c\) and \([Y]_c\) be elements of \(\mathcal{C}\) such that all of the following fail:
\[
[X]_c = [Y]_c, \ [X]_c < [Y]_c, \ [Y]_c < [X]_c, \ [Y]_c \triangleleft [X]_c. \tag{12}
\]
There is a (coherent) partial trading system \( \mathcal{T}^* = (\mathcal{C}, <^*, \triangleleft^*) \) that extends \( \mathcal{T} \) and that includes the comparison \([X]_c <^* [Y]_c\).

3.3. Extending to a Full Trading System

Theorem 2 shows how to extend a (coherent) partial trading system to a (coherent) full trading system. Because of the appeal to transfinite induction, we don’t save any steps in the proof by using the \((\mathcal{C}, <, \triangleleft)\) notation over the \((\sim, <, \triangleleft)\) notation. The proof of Theorem 2 appears in Appendix A.3 of the supplemental article [7].

Theorem 2. Let \( \mathcal{T} = (\sim, <, \triangleleft) \) be a (coherent) partial trading system. There exists a (coherent) full trading system \( \mathcal{T}^* = (\sim^*, <^*) \) that extends \( \mathcal{T} \).

4. Representing a Partial Trading System

In this section, we present two representations of partial trading systems. The first representation is by a (possibly nonstandard) numerical function.

Definition 11. A numerical function \( U \) on \( \mathcal{G} \) represents a partial trading system \( \mathcal{T} = (\sim, <, \triangleleft) \) if
- \( X \sim Y \) implies \( U(X) = U(Y) \),
- \( X < Y \) implies \( U(X) < U(Y) \), and
- \( X \triangleleft Y \) implies \( U(X) \leq U(Y) \).

The second representation is by a consensus of a set of weak orders. (See Appendix B.3 of the supplemental article [7] for details on consensus.) Briefly, there is a collection \( \mathcal{A} \) of full trading systems such that
- \( X \sim Y \) if and only if \( X \sim_* Y \) for all \((\sim_*, <_*) \in \mathcal{A}\),
- \( X < Y \) if and only if \( X <_* Y \) for all \((\sim_*, <_*) \in \mathcal{A}\),
- \( X \triangleleft Y \) if and only if both (i) \( X \preceq_* Y \) for all \((\sim_*, <_*) \in \mathcal{A}\), and (ii) \( \neg[(X \sim Y) \lor (X < Y)] \).
4.1. Representation by a Nonstandard Function

From Definition 11, a function \( U \) represents a partial trading system if and only if \( aU + b \) represents that same system for all \( a > 0 \) and all \( b \). For full trading systems there is a simpler characterization of a representing function.

**Lemma 10.** For a full trading system \( T = (\sim, \prec) \), \( U \) represents \( T \) if and only if

\[
U(X) < U(Y) \text{ if and only if } X \prec Y. \tag{13}
\]

**Proof.** For the “only if” direction, assume that \( U \) represents \( T \). The second bullet in Definition 11 implies the “if” part of (13). For the “only if” direction of (13), assume that \( U(X) < U(Y) \). We need to show that \( X \prec Y \). Since \( T \) is a full trading system, precisely one of \( X \prec Y, Y \prec X, \) or \( X \sim Y \) holds. The first bullet of Definition 11 rules out \( X \sim Y \), and the second bullet rules out \( Y \prec X \). Hence \( X \prec Y \).

For the “if” direction, assume that (13) is true. We need to prove that the three bullets of Definition 11 hold. For the first bullet, assume that \( X \sim Y \). Then (13) says that neither \( U(X) < U(Y) \) nor \( U(Y) < U(X) \). Hence \( U(X) = U(Y) \). The second bullet is the “if” direction of (13). The third bullet is vacuous because \( \prec \) empty for a full trading system. \( \square \)

**Example 13.** Let \( \Omega = \mathbb{Z}^+ \). Let \( \mathcal{X} \) be the linear span of all bounded functions and one non-negative unbounded function \( X_U \), for example, \( X_U(\omega) = \omega \) for all \( \omega \). Elements of \( \mathcal{X} \) are all uniquely represented as \( X = q(X)X_U + X' \), where \( q(X) \) is a standard real, and \( X' \) is bounded. Define \( \preceq \) as follows:

\( X_1 \preceq X_2 \) if either (i) \( q(X_1) < q(X_2) \) or (ii) both \( q(X_1) = q(X_2) \) and \( \sum_{\omega \in \Omega} [X_1(\omega) - X_2(\omega)]2^{-\omega} \leq 0 \). Also, define \( U(q(X)X_U + X') = \Delta q(X) + \sum_{\omega \in \Omega} X'(\omega)2^{-\omega} \), where \( \Delta \) is some externally infinite nonstandard. That \( U \) represents \( \preceq \) follows from the fact that \( q(X_1) > q(X_2) \) implies that \( \Delta q(X_1) + c_1 > \Delta q(X_2) + c_2 \) for standards \( c_1, c_2 \).

The structure of trading systems allows us to represent them using a relatively simple type of function.

**Definition 12.** A space \( \mathcal{W} \) of functions is a *standard-linear space* if \( \alpha Y + \beta Z \in \mathcal{W} \) for all standard \( \alpha, \beta \) and all \( Y, Z \in \mathcal{W} \). A nonstandard-valued function \( U \) on a standard-linear space \( \mathcal{W} \) is called a *standard-linear function* if \( U(\alpha Y + \beta Z) = \alpha U(Y) + \beta U(Z) \) for all \( Y, Z \in \mathcal{W} \) and all standard \( \alpha, \beta \).

Notice that Definition 12 restricts the coefficients in linear combinations to be standard even though the values of \( U \) might be nonstandard. Also,
notice that every linear space (such as $X$) is a standard-linear space. Finally, notice that $U(0) = 0$ for every standard-linear function $U$.

**Lemma 11.** If $U$ is a standard-linear function defined on a linear space $X$, then $U$ represents a full trading system on $X$.

**Proof.** Assume that $U$ is a standard-linear function on a linear space $X$. Define the weak order $\preceq$ on $X$ by $X \preceq Y$ if and only if $U(X) \leq U(Y)$. Let $\sim$ and $\prec$ correspond to $\preceq$ via Lemma 28 in Appendix B.1 of the supplemental article [7]. Then $\prec$ is a strict partial order defined on the equivalence classes of $\sim$. Let $\prec$ be a strict partial order defined on the equivalence classes of $\sim$. Let $T = (\sim, \prec)$. We need only prove that $T$ satisfies Assumptions 1–4. Since $U(X) = U(Y)$ if $X \sim Y$, Assumption 1 holds. Since $U$ is standard linear, $U(\alpha_1X_1 + \cdots + \alpha_nX_n) = \alpha_1U(X_1) + \cdots + \alpha_nU(X_n)$, for all standard $\alpha_1, \ldots, \alpha_n$ and $X_1, \ldots, X_n \in X$. This implies that Assumption 2 holds. Let $X, Y, Z \in X$ and $\alpha \in (0, 1]$. If $U(\alpha X + [1 - \alpha]Z) \leq U(\alpha Y + [1 - \alpha]Z)$, then $U(X) \leq U(Y)$, and Assumption 3 holds. Since $\prec$ is empty for a full trading system, Assumption 4 holds.

**Lemma 12.** A standard-linear function $U$ represents a partial trading system $T = (\sim, \prec, \triangleleft)$ if and only if $U$ represents a full trading system $T'$ that extends $T$.

**Proof.** For the “if” direction, assume that $U$ represents $T' = (\sim', \prec', \triangleleft)$ where $T'$ extends $T = (\sim, \prec, \triangleleft)$. We need to show that the three bullets in Definition 11 hold in $T$. If $X \sim Y$, then $X \sim' Y$ and $U(X) = U(Y)$. If $X \prec Y$, then $X \prec' Y$ and $U(X) < U(Y)$. If $X \triangleleft Y$, then either $X \sim' Y$ or $X \prec' Y$. In either case $U(X) \leq U(Y)$.

For the “only if” direction, assume that $U$ represents $T = (\sim, \prec, \triangleleft)$. Define $T' = (\sim', \prec', \triangleleft)$ to be the full trading system represented by $U$ according to Lemma 11. To see that $T'$ extends $T$, note that $X \sim Y$ implies $U(X) = U(Y)$ which implies $X \sim' Y$, $X \prec Y$ implies $U(X) < U(Y)$ which implies $X \prec' Y$, and $X \triangleleft Y$ implies $U(X) \leq U(Y)$ which implies either $X \sim' Y$ or $X \prec' Y$.

In light of Lemma 12 and Theorem 2, the collection of all standard-linear functions that represent a partial trading system can be identified (and proven to be non-empty) by showing that every full trading system has a standard-linear function that represents it. Theorem 3 will show that every
full trading system can be represented by a standard-linear function. The proof of Theorem 3 relies heavily on Lemma 13 below.

In Lemma 13 and Theorem 3, we deal with “extension” in a different direction than we did in Section 3. In Section 3, all of the results deal with a single linear space $\mathcal{X}$ which supports a number of different partial trading systems. The extensions we do in that section extend partial trading systems until we reach a full trading system. In this section, we start with a subspace $\mathcal{Y}$ of $\mathcal{X}$ and a standard-linear function $U$ that represents a full trading system on $\mathcal{Y}$. We then expand $\mathcal{Y}$ through a series of linear spaces until we expand all the way to $\mathcal{X}$. For each expansion, we extend the numerical function $U$ to a larger domain so that it represents each successive full trading system.

Lemma 13 is the part of the proof of Theorem 3 that most closely resembles de Finetti’s fundamental theorem of prevision. The main step in both proofs is constructing bounds for the possible values of the representing function (prevision in de Finetti’s case, $U$ in Theorem 3) at a new random variable $Z$ given previously chosen values of the representing function. In de Finetti’s theorem, one uses existing previsions of random variables $X$ for which either $X \leq Z$ or $Z \leq X$. In Lemma 13, we replace prevision by a representing function $U$, and we replace $X \leq Z$ and $Z \leq X$ by $X \prec Z$ and $Z \prec X$ respectively. Additional steps are needed to deal with strict preferences of a non-Archimedean nature and with sets of nonstandards that don’t have suprema and/or infima. The proofs of Lemma 13 and Theorem 3 appear in Appendix A.4 and Appendix A.5 respectively of the supplemental article [7].

**Lemma 13.** Assume the following structure:

- $\mathcal{X}$ is a linear space of functions defined on a set $\Omega$.
- $\mathcal{T}$ is a full trading system on $\mathcal{X}$.
- $\mathcal{Y}$ is a subspace of $\mathcal{X}$.
- $\mathcal{T}_0 = (\approx, \prec)$ is the restriction of $\mathcal{T}$ to $\mathcal{Y}$.
- $\mathbb{IR}$ is a standard or nonstandard model of the reals.
- $U : \mathcal{Y} \to \mathbb{IR}$ is a standard-linear function that represents $\mathcal{T}_0$.
- $Z \in \mathcal{X} \setminus \mathcal{Y}$, and $Z$ is the linear span of $\mathcal{Y} \cup \{Z\}$.
• $T_1$ is the restriction of $T$ to $Z$.

Then $U$ can be extended to a standard-linear function $U'$ from $Z$ to a non-standard model $*\mathbb{R}'$ that contains $*\mathbb{R}$ and that represents $T_1$.

**Theorem 3.** Let $\mathcal{T} = (\sim, \prec)$ be a full trading system on a linear space $\mathcal{X}$. There exists a standard-linear function $U$ that represents $\mathcal{T}$.

**Example 14 (Continuation of Example 1).** It is relatively straightforward to apply the transfinite induction argument in the proof of Theorem 3. If $X_1 = 1$, then we can let $U(1) = 1$ so that $U(c) = c$ for every standard constant $c$. Next, we find $U(I\{H\})$. Since each $(x, y)$ can be written as $(x - y)I\{H\} + y$, this will be the final step in the induction. We have $0.5 < U(I\{H\}) < c$ for all standard $c > 0.5$. It follows that $U(I\{H\}) = 0.5 + \epsilon$ for some positive infinitesimal $\epsilon$. So, $U((x, y)) = 0.5(x + y) + \epsilon(x - y)$ for all $(x, y) \in \mathbb{R}^2$.

**Example 15 (Continuation of Example 6).** It is easy to see that $U(X_n)$ must be the same positive infinitesimal $\epsilon$ for all $n$. Other random variables need to be handled on a case-by-case basis. For example, let $Z(\omega) = 1/\omega$. It is clear that $Z \geq \sum_{j=1}^{n} X_j/j$ for all finite $n$. Hence $U(Z) \geq \sum_{j=1}^{n} \epsilon/j$ for all $n$. Note that $\epsilon_n = \sum_{j=1}^{n} \epsilon/j$ is infinitesimal for all $n$. Let $E_n = \{1, \ldots, n\}$ so that $Z = ZE_n + ZE_n^C$. Since $ZE_n^C < 1/n$ for all $n$, we have

$$U(Z) = U(ZE_n) + U(ZE_n^C) < \epsilon_n + \frac{1}{n},$$

for all $n$. Hence $U(Z)$ is also infinitesimal. However, we can also see that $U(Z)/\epsilon > \sum_{j=1}^{n} 1/j$ for all $n$, hence $U(Z)/\epsilon$ is externally infinite. Some infinitesimals are much bigger than others.

Here is a useful corollary to Theorem 3 in the random-variable case.

**Corollary 1.** Let $\mathcal{T}$ be a coherent full trading system on a linear space of standard-valued random variables. Let $U$ be a standard-linear function that represents $\preceq$. If $U(c) = c$ for every standard constant $c$, then $X \sim U(X)$ for every $X \in \mathcal{X}$ for which $U(X)$ is standard.

Corollary 1 begs the question “If $U(X)$ is nonstandard, to what constant is $X$ indifferent?” The answer is “There is no such constant.” Recall that only standard constants are random variables in $\mathcal{X}$. If $U(X)$ is nonstandard but externally finite, then the set $\mathcal{X}_0$ of constant random variables splits into two disjoint sets: those that are preferred to $X$ and those to which $X$ is preferred.
The standard part of $U(X)$ is in precisely one of those two sets depending on whether $U(X)$ is greater than or less than its standard part. If $U(X)$ is externally infinite ($+\infty$)\(^1\) then $x \prec X$ for all standard finite $x$. If $U(X)$ is externally negatively infinite, then $X \prec x$ for all standard finite $x$. What the externally infinite nonstandard values of $U(X)$ tell us is which random variables are preferred to which (obvious from the definition of represents) and by how much each is preferred to another. In particular, if $U(X)$ and $U(Y)$ are both externally infinite (regardless of sign) but $U(aX+bY)$ is finite and standard (both $a$ and $b$ non-zero and standard,) then there is a finite constant $c$ such that $aX+bY \sim c$, even though there are no finite prices for which the agent would be willing buy or sell $aX$ or $bY$. If $U(aX+bY)$ is externally finite but nonstandard, then there is a constant $c$, the standard part of $U(aX+bY)$, such that the agent would be willing to pay anything strictly less than $c$ to get $aX+bY$ and would be willing to sell $aX+bY$ for anything strictly greater than $c$. If the price were precisely $c$, the agent would be willing to enter into only one of the two trades depending on whether $U(aX+bY)$ is larger or smaller than its standard part.

4.2. Lexicographic Preference

Earlier, we promised to show that every lexicographic preference is represented by a nonstandard utility.

**Lemma 14.** If preference is expressed through a lexicography of expected utilities, there is a standard-linear function that represents the preference.

**Proof.** A lexicographic preference on a linear space $\mathcal{X}$ is defined in terms of a well-ordered set $\{U_\gamma\}_{\gamma \in \Gamma}$ of utility functions defined on $\mathcal{X}$. Each $U_\gamma$ provides a preference relation on $\mathcal{X}$ that satisfies Assumptions 1–4 as well as the Archimedean axiom. The lexicographic preference is defined as follows. For each $X, Y \in \mathcal{X}$, we say that $X \prec Y$ if the first $\gamma$ for which $U_\gamma(X) \neq U_\gamma(Y)$, satisfies $U(X) < U(Y)$. If $U_\gamma(X) = U_\gamma(Y)$ for all $\gamma$, we say $X \sim Y$. With this structure it is straightforward to see that the lexicographic preference ($\sim, \prec$) satisfies the four assumptions that make it a full trading system. Theorem 3 says that there is a standard-linear function that represents the full trading system. \hfill \Box

\(^1\)All values of $U$ are internally finite by construction.
4.3. Representation by Consensus

The following result gives a simple representation of a partial trading system. In the statement and proof, there are parenthetical references to coherence and/or dominance. Ignore those references to obtain the theorem and proof for the lottery case. Include the references to coherence and/or dominance to obtain the theorem and proof for the random-variable case. The proof of Theorem 4 appears in Appendix A.6 of the supplemental article [7].

**Theorem 4.** \( T = (\sim, \prec, \lhd) \) is a (coherent) partial trading system (where \( \prec \) extends dominance and \( \preceq \) extends \( \leq \)) if and only if there is a collection \( A \) of (coherent) full trading systems on \( X \) such that the type-2 consensus relation of the \( \preceq \) relations that correspond to the elements of \( A \) is \( T \).

5. Probability and Expected Utility

In this section, we show how to construct a probability over \( \Omega \) from a full trading system \( T = (\sim, \prec) \) and how to interpret a function \( U \) that represents \( T \) as an expected utility. Due to the very different natures of prizes (in the lottery case) and numerical random variables (in the random-variable case) the means by which we construct a probability and interpret an expected utility will differ in the two cases. We deal with random variables in Section 5.1 and with lotteries in Section 5.2. There are a few concepts that are common to both cases.

**Definition 13.** Let \( \Sigma \) be a field of subsets of \( \Omega \). Suppose that, for each \( B \in \Sigma \) and \( Z, Y \in X \), \( S_{Z,B} \in X \), where

\[
S_{Z,B}(\omega) = \begin{cases} 
Z(\omega) & \text{if } \omega \in B, \\
0(\omega) & \text{if } \omega \in B^C.
\end{cases}
\]

For \( X, Y \in X \) we say that \( Y \) is conditionally preferred to \( X \) conditional on \( B \) (denoted \( X \prec Y|B \)) if \( 0 \prec S_{Y-X,B} \). If \( 0 \preceq S_{Y-X,B} \) and \( 0 \preceq S_{X-Y,B} \), we write \( X \sim Y|B \) and say that \( X \) and \( Y \) are conditionally indifferent conditional on \( B \).

**Definition 14.** An event \( B \) (subset of \( \Omega \)) is null if \( X \sim Y \) for all \( X, Y \in X \) such that \( X(\omega) = Y(\omega) \) for all \( \omega \in B^C \). An event is non-null if it is not null.

The concept of expected value in a standard-valued finitely-additive setting is described in detail in [15, Appendix A] and Section 2 of the online resource for [15]. We extend this concept to nonstandard-valued functions.
Definition 15. Let \( W \) be a standard-linear space of standard or nonstandard valued functions that contains all standard constants. Let \( W \) be a standard-linear function on \( W \). We say that \( W \) acts as an expected value on \( W \) if it satisfies

- \( W(1) = 1 \), and
- if \( X \preceq Y \), then \( W(X) \leq W(Y) \).

The following is straightforward.

Proposition 5. Let \( W \) act as an expected value on a standard-linear space \( W \) of functions defined on a set \( Z \). If \( W \) contains the indicators of the sets in a field \( \Sigma \) of subsets of \( Z \), then \( P(B) = W(I_B) \) for \( B \in \Sigma \) defines a finitely-additive probability on \( \Sigma \).

Definition 16. Under the conditions and notation of Proposition 5, we say that \( W \) acts as an expected value with respect to \( P \).

Unlike the countably-additive setting, there can be multiple functions \( W_1, W_2, \ldots \) that act as expected values with respect to a common finitely-additive probability \( P \).

5.1. The Random-Variable Case

In this section \( T = (\sim, \prec) \) will be a full trading system on a linear space \( O \) of standard-valued random variables. Let \( \preceq \) be the corresponding weak order on \( O \). Let \( U \) be a standard-linear function that represents \( T \) and which has been normalized so that \( U(1) = 1 \). The results apply to all three definitions of coherence unless explicitly stated otherwise.

5.1.1. Probability

We take probability to be finitely additive in Lemma 15 and elsewhere due to the difficulty in defining countable sums of nonstandard numbers. See the results and examples in Appendix C.5 of the supplemental article [7].

Lemma 15. Let \( \Sigma \) be a field of subsets of \( \Omega \). Suppose that \( O \) contains the indicator functions of all elements of \( \Sigma \). Define \( P(B) = U(I_B) \) for each \( B \in \Sigma \). Then \( P \) is a finitely-additive (possibly nonstandard-valued) probability on \( \Sigma \).

Proof. Let \( B \in \Sigma \) be an event. Since \( 0 \preceq I_B \), we have \( P(B) \geq 0 \). Since \( I_\Omega \equiv 1 \), \( P(\Omega) = 1 \). Suppose that \( B_1 \) and \( B_2 \) are disjoint events. Then \( I_{B_1} + I_{B_2} = I_{B_1 \cup B_2} \). Since \( U \) is standard-linear, we have \( P(B_1) + P(B_2) = P(B_1 \cup B_2) \). \( \square \)
5.1.2. Expected Utility

In the random-variable case, the justification for the coherence assumption is that the numerical values of the random variables carry a utility interpretation. Definition 15 makes the following straightforward.

**Lemma 16.** Let \( U \) be a standard-linear function that represents a coherent full trading system \( T = (\sim, \prec) \) on \( O \) that contains all standard constants. (See Theorem 3.) Suppose that \( U \) has been normalized so that \( U(1) = 1 \). Then \( U \) acts as an expected value.

**Proof.** The only part of the definition of expected value that is not explicitly assumed of \( U \) in the statement of the lemma is the second bullet of the definition. Since \( T \) is assumed to be coherent, \( X \preceq Y \) implies \( X \leq Y \). Since \( U \) represents \( T \), \( X \preceq Y \) implies \( U(X) \leq U(Y) \). \( \square \)

5.2. The Lottery Case

In this section, we assume that \( T = (\sim, \prec) \) is a full trading system on a linear space \( K_0 \), as constructed in Section 2.2. It is straightforward to convert a representing function defined on \( K_0 \) to a function defined on \( H \) that represents a full trading system on the set \( H \) of horse lotteries. According to Proposition 1, each element of \( K_0 \) has the form \( \alpha (h_1 - h_2) \) where \( \alpha > 0 \) and \( h_1, h_2 \in H \). The correspondence between elements of \( K_0 \) and \((\alpha, h_1, h_2)\) triples is not unique. If \( U \) is a standard-linear function that represents a full trading system \((\sim', \prec')\) on \( K_0 \), define \( V \) on \( H \) as follows. First choose an arbitrary element \( h_0 \) of \( H \) to satisfy \( V(h_0) = 0 \). Then, for each \( h \in H \), define \( V(h) = U(h - h_0) \). It is straightforward to show that \( V \) satisfies: for all \( h_1, h_2 \in H \),

- \( h_1 \sim h_2 \) if and only if \( V(h_1) = V(h_2) \), and
- \( h_1 \prec h_2 \) if and only if \( V(h_1) < V(h_2) \),

where \( \sim \) and \( \prec \) correspond to \( \sim' \) and \( \prec' \) as in Section 2.2.

In the random-variable case, the numerical value of each random variable at a state, \( X(\omega) \), acts as the utility to the agent of the random variable \( X \) when state \( \omega \) occurs. In the lottery case, there are no intrinsic numerical utilities associated with lotteries or prizes. For example, prize \( p_1 \) may be worth more than prize \( p_2 \) in one state while \( p_2 \) is worth more than \( p_1 \) in some other state. Under conditions to be developed below, we introduce notation...
\( V(\omega, p) \) to stand for the value of prize \( p \) in state \( \omega \). This will allow us to construct an expected utility interpretation of a representing function \( V \) on \( \mathcal{H} \) (or equivalently \( U \) on \( \mathcal{K}_0 \).) Throughout this section, we take the liberty of referring to a full trading system and its corresponding binary relations as if they applied to elements of \( \mathcal{K}_0 \) or elements of \( \mathcal{H} \), whichever is most convenient for deriving each result.

5.2.1. Probability

**Definition 17.** Let \( \Sigma \) be a field of subsets of \( \Omega \). For each \( B \in \Sigma \) and \( X \in \mathcal{X} \), define \( S_{X,B} \) as in (14). Suppose that there exists \( X \in \mathcal{X} \) such that \( S_{X,B} \in \mathcal{X} \) for all \( B \in \Sigma \). We say that \( X \) is a numeraire on \( \Sigma \) if \( 0 \prec X \mid B \) for all non-null \( B \in \Sigma \).

If a numeraire on \( \Sigma \) exists, it can be used to identify a probability on \( \Sigma \). (In the random variable case, we treated the constant 1 as if it were a numeraire.)

**Lemma 17.** Let \( \Sigma \) be a field of subsets of \( \Omega \). Let \( X \in \mathcal{X} \) be such that for all \( B \in \Sigma \), \( S_{X,B} \in \mathcal{X} \) and \( 0 \leq S_{X,B} \). Let \( U : \mathcal{X} \to \ast \mathbb{R} \) be a standard-linear function that represents \( \mathcal{T} \), normalized so that \( U(X) = 1 \). Define \( P(B) = U(S_{X,B}) \) for each \( B \in \Sigma \). Then \( P \) is a finitely-additive (possibly nonstandard-valued) probability on \( \Sigma \). If \( X \) is a numeraire on \( \Sigma \), then \( P(B) > 0 \) if and only if \( B \) is non-null.

**Proof.** Let \( B \in \Sigma \) be an event. Since \( 0 \leq S_{X,B} \), we have \( P(B) \geq 0 \). By construction \( P(\Omega) = 1 \). Suppose that \( B_1 \) and \( B_2 \) are disjoint events. Then \( 0.5S_{X,B_1} + 0.5S_{X,B_2} = 0.5S_{X,B_1 \cup B_2} + 0.5 \times 0 \). Since \( U \) is standard-linear, we have \( P(B_1) + P(B_2) = P(B_1 \cup B_2) \). If \( B \) is null, then \( 0 \sim S_{X,B} \), so \( P(B) = 0 \). If \( X \) is a numeraire and \( B \) is non-null, then \( 0 < P(B) \). \( \Box \)

5.2.2. Expected Utility

Let \( X \) be a numeraire with representation \( X = h_1 - h_0 \) for two horse-lotteries \( h_0 \) and \( h_1 \). We will assume that \( V \) is defined so that \( V(h_0) = 0 \). For each state \( \omega \) and each horse-lottery \( h \), define \( S_{h - h_0, \{\omega\}} \) as in (14). Note that \( S_{h - h_0, \{\omega\}} \) depends on \( h \) only through \( h(\omega) \).

**Definition 18.** The state-dependent utility of \( h \) in a non-null state \( \omega \) is

\[
V(\omega, h) = \frac{U(S_{h - h_0, \{\omega\}})}{U(S_{h_1 - h_0, \{\omega\}})}.
\] (15)
If \( h(\omega) \) is the lottery \( r \), we will take the liberty of using the notation \( V(\omega, r) \) to stand for \( V(\omega, h) \). Similarly, if \( h(\omega) \) is the lottery that assigns prize \( p \) with probability 1, we will take the liberty of using the notation \( V(\omega, p) \) to stand for \( V(\omega, h) \). Note that the denominator in (15) is also \( P(\{\omega\}) \), where \( P \) is constructed as in Lemma 17.

The following example, a modification of [16, Example 2.2], illustrates a problem that can arise if non-simple lotteries are allowed, even in the case of a finite state space.

**Example 16.** Let \( \Omega = \{1, \ldots, \ell\} \) and let \( P \) be the discrete uniform distribution over \( \Omega \). Let \( \mathcal{P} \) be the closed interval \([0, 1]\). For each lottery \( r \) over \( \mathcal{P} \) (element of \( \mathcal{R} \)), define

\[
a(r) = \lim_{t \to \infty} t \cdot r([0, 1/t])/2,
\]

if the limit exists. For example, if \( r \) is a continuous distribution with continuous density function \( f \) over \( \mathcal{P} \), then \( a(r) = \lim_{x \to 0} f(x)/2 \). If \( r \) is a simple distribution, then \( a(r) = 0 \).

Let \( \mathcal{H} \) be the set of horse lotteries \( h \) such that, for every \( \omega \in \Omega \):

- \( h(\omega) \) is a countably-additive probability over \( \mathcal{P} \), and
- \( a(h(\omega)) \) exists and is finite.

It is straightforward that

\[
a[\alpha h_1(\omega) + (1 - \alpha) h_2(\omega)] = \alpha a[h_1(\omega)] + (1 - \alpha) a[h_2(\omega)],
\]

for all \( h_1, h_2 \in \mathcal{H} \) and all \( \omega \). It follows that \( \mathcal{H} \) is a mixture set. For each \( h \in \mathcal{H} \), define

\[
V(h) = P \left[ \int_{\mathcal{P}} p h(\cdot)(dp) + a[h(\cdot)] \right].
\]

That is, for each \( h \in \mathcal{H} \), compute \( V(h) \) as follows:

1. For each \( \omega \) find the mean of \( p \) with respect to \( h(\omega) \), and call the result \( u_h(\omega) \).
2. Find the mean \( v_h \) of \( u_h(\cdot) \) with respect to \( P \).
3. Find the mean \( w_h \) of \( a[h(\cdot)] \) with respect to \( P \).
4. Set \( V(h) = v_h + w_h \).
Define a full trading system on $\mathcal{K}_0$ as follows: Define $U(\alpha[h_2 - h_1]) = \alpha[V(h_2) - V(h_1)]$. That $U$ represents a full trading system follows from Lemma 11.

Next, consider the following two elements of $\mathcal{H}$:

- For each $\omega$, $h_1(\omega)$ assigns probability 1 to $\{1\}$.
- For each $\omega$, $h_0(\omega)$ assigns probability 1 to $\{0\}$.

Then $u_{h_1}(\omega) = 1$ and $u_{h_0}(\omega) = 0$ for all $\omega$. Also, $a[h_1(\omega)] = a[h_0(\omega)] = 0$ for all $\omega$. It follows that $v_{h_1} = 1, v_{h_0} = 0, a[h_1] = a[h_0] = 0$. So $V(h_1) = 1$ and $V(h_0) = 0$. For each $\omega_0 \in \Omega$, define $S_{h_1 - h_0,\{\omega_0\}}$ as in (14). Then

$$U(S_{h_1 - h_0,\{\omega_0\}}) = \frac{1}{\ell} > 0.$$ 

This makes $h_1 - h_0$ a numeraire with $U(h_1 - h_0) = 1$. The probability on $\Omega$ constructed from $\mathcal{K}_0$ is $P$ (by design.) For each prize $p$, $u_p(\omega) = p$ for all $\omega$ and $w_p = 0$, so $V(p) = p$. It follows that the state-dependent utility (which is actually state-independent in this example) is $V(\omega,p) = p$ for all $\omega$ and $p$. The integral of $V(\omega,p)$ with respect to $h(\omega)$ is then $u_h(\omega)$, and the mean of this is $v_h$. But $V(h) = v_h + w_h$. A horse-lottery $h$ with $v_h \neq 0$ is the following: Let $h(\omega)$ be the uniform distribution on $(0,1)$ for all $\omega$. Then $a[h(\omega)] = 1$ for all $\omega$ and $w_h = 1$. The state-dependent utility representation fails for this $h$.

In light of Example 16, we will be able to interpret $V(\omega,r)$, defined in (15), as the expected value of $V(\omega,p)$ with respect to the probability $r$ over $\mathcal{P}$ only when $r$ is simple. We return to this issue in Section 6.

Nevertheless, we can identify conditions under which we are able to interpret $V(h)$ as an expected value of $V(\omega,h)$ with respect to a probability $P$ over $\Omega$, regardless of whether or not there exist $\omega$ for which $h(\omega)$ is non-simple.

5.2.3. Finite State Spaces and Simple Lotteries

We note that [2] restrict attention to the case in which $\Omega$ is finite and lotteries are simple. In such cases, we can find a numeraire and interpret a representing function as an expected value of state-dependent utility.

Let $U$ be a standard-linear function that represents $\mathcal{T}$. Let the elements of $\Omega$ be $\omega_1, \ldots, \omega_m$.

**Lemma 18.** Let $\mathcal{R}$ be a set of lotteries. Assume that $\mathcal{H} = \mathcal{R}^\Omega$ and that that $\Sigma = 2^\Omega$. There exists a numeraire.
Proof. Let 0 be the zero element of $K_0$. Let $N$ be the subset of $\Omega$ that is the union of all null singletons. Then $N$ is null and $N^C$ is non-null, and every non-empty subset of $N^C$ is non-null. For each $\omega_0 \in N^C$, there exists $X_{\omega_0} \in K_0$ such that $X_{\omega_0}(\omega) = 0(\omega)$ for all $\omega \neq \omega_0$ and $-(X_{\omega_0} \sim 0)$. For each $\omega \in \Omega$, define

$$X(\omega) = \begin{cases} 0(\omega) & \text{if } \omega \in N, \\ X_\omega(\omega) & \text{if } \omega \in N^C \text{ and } 0 < X_\omega, \\ -X_\omega(\omega) & \text{if } \omega \in N^C \text{ and } X_\omega < 0. \end{cases}$$

Define $P$ on $\Sigma$ as in the statement of Lemma 17. That $P$ is additive follows as in the proof of Lemma 17. Since $P(\{\omega\}) \geq 0$ for every $\omega$, $P$ is non-negative. Then $P$ is a finitely-additive probability as in the proof of Lemma 17, and $P(B) > 0$ for every non-null $B$ by construction. This makes $X$ a numeraire.

An expected utility interpretation for a representing function $U$ is available in the case in which each element of $R$ is a simple lottery. That is, each lottery involves only finitely many distinct prizes. Example 16 below illustrates why we make such an assumption. We do not assume that the set of prizes is finite, only that each lottery involves only finitely many prizes. Since, we are also assuming that $\Omega$ is finite, each horse-lottery involves only finitely many prizes.

Suppose that $h(\omega)$ assigns prize $p_j$ with (standard) probability $a_j$ for $j = 1, \ldots, n$. It is straightforward to show that

$$V(\omega, h) = \sum_{j=1}^{n} a_j V(\omega, p_j). \quad (16)$$

Let $h$ be a general element of $H$. For each $\omega$, let $p_1, \ldots, p_n$ be all of the distinct prizes that show up in $h$. Express each $h(\omega_k) = \sum_{j=1}^{n} a_{k,j} p_j$, where all $a_{k,j}$ are standard. Then

$$V(h) = \sum_{k=1}^{\ell} \sum_{j=1}^{n} a_{k,j} V(\omega_j, p_k) P(\{\omega_k\}). \quad (17)$$

Equation (17) says that $V(\cdot)$ is an expected state-dependent utility on $H$ when all lotteries are simple.
5.2.4. General State Spaces and Simple Lotteries

In general, there may be no expected-utility interpretation for $V$ if the state space $\Omega$ is infinite. The following example is like Example 16, with all lotteries simple, but the state space is infinite, and there is no expected-utility interpretation for the representing function.

Example 17. Let $\Omega = \{1, 2, 3, \ldots\}$, let $P(\{\omega\}) = 2^{-\omega}$ for each $\omega$. Let $\mathcal{P}$ be the open interval $[0, 1]$. For each horse-lottery $h$, define

$$a(h) = \lim_{\omega \to \infty} \omega h(\omega)[(0, 1/\omega)],$$

if the limit exists. Let $\mathcal{H}$ be the set of horse-lotteries $h$ such that:

- for every $\omega \in \Omega$, $h(\omega)$ is a simple probability over $\mathcal{P}$, and
- $a(h)$ exists and is finite.

For example, if for all but finitely many $\omega$, $h(\omega)$ is a discrete uniform distribution over $\omega$ distinct elements of $\mathcal{P}$, exactly $k$ of which are less than $1/\omega$, then $a(h) = k$.

It is straightforward that

$$a[\alpha h_1 + (1 - \alpha)h_2] = \alpha a(h_1) + (1 - \alpha)a(h_2),$$

for all $h_1, h_2 \in \mathcal{H}$ and all $\alpha \in [0, 1]$. It follows that $\mathcal{H}$ is a mixture set. For each $h \in \mathcal{H}$, define

$$V(h) = P \left[ \int_{\mathcal{P}} p h(\cdot)(dp) \right] + a(h).$$

That is, for each $h \in \mathcal{H}$, compute $V(h)$ as follows:

1. For each $\omega$ find the countably-additive mean of $p$ with respect to $h(\omega)$, and call the result $u_h(\omega)$.
2. Compute the countably-additive mean of $u_h(\cdot)$, and call it $v_h$.
3. Set $V(h) = v_h + a(h)$.

Define a full trading system on $\mathcal{K}_0$ as follows: For $X = \alpha(h_2 - h_1)$ with $\alpha > 0$, define $U(X) = \alpha[V(h_2) - V(h_1)]$. That $U$ represents a full trading system follows from Lemma 11. In addition, the preferences are Archimedean because $V$ is standard valued.
A numeraire exists, for example the difference between constant horse-lotteries \( h_1 = 1 \) and \( h_0 = 0 \) is \( X = 1 - 0 \), which satisfies \( U(S_{X,B}) = P(B) \) for each \( B \subseteq \Omega \). Because every state is non-null, we can compute \( V(\omega, h) \) for each \( h \) as in (15). For example, \( V(\omega, p) = p \) for each \( p \in \mathcal{P} \).

Now, consider the following element of \( \mathcal{H} \): For each \( \omega \), \( h(\omega) \) is the discrete uniform distribution over the numbers \( k/(\omega + 1) \) for \( k = 1, \ldots, \omega \). Then \( v_h = 0.5 \) and \( a(h) = 1 \), so that \( V(h) = 1.5 \). Also \( U(S_{h-h_0,\{\omega\}}) = 2^{-\omega-1} \), so that \( V(\omega, h) = 0.5 \) by (15). Clearly, \( V(h) \) is not equal to the expected value of \( V(\omega, h) \) with respect to a probability on \( \Omega \).

The problem that arises in Example 17 is that \( V(h) \) is allowed to assume a value far outside of the range of values \( \{V(\omega, h) : \omega \in \Omega\} \). If the trading system satisfies a form of dominance, that we introduce next, a numeraire and an expected utility interpretation for \( V \) both exist.

**Definition 19.** Let \( \mathcal{T} \) be a full trading system on a set \( \mathcal{H} \) of horse lotteries. Let \( N \) be the union of all null events. Let \( h_1, h_2 \in \mathcal{H} \), and let \( X = h_2 - h_1 \). If \( 0 \prec X\bigr|\{\omega\} \) for all \( \omega \in N^C \), we say that \( h_2 \) dominates \( h_1 \) in preference (or \( p \)-dominates \( h_1 \)), denoted \( h_1 \prec_{p-Dom} h_2 \). Call \( \mathcal{T} \) \( p \)-coherent if

- \( N \) is null and \( N^C \neq \emptyset \), and
- \( h_1 \prec_{p-Dom} h_2 \) implies \( h_1 \prec h_2 \).

The proof of the following result is virtually identical to the proof of Lemma 18.

**Proposition 6.** Let \( \mathcal{R} \) be a set of lotteries. Assume that \( \mathcal{H} = \mathcal{R}^\Omega \), that that \( \Sigma \) contains \( 2^{N^C} \), and that the full trading system is \( p \)-coherent. There is a numeraire.

As in the case of finite state spaces, we can define a state-dependent utility function in the \( p \)-coherent case. Let \( X = h_1 - h_0 \in \mathcal{K}_0 \) be a numeraire. Assume that \( V \) has been normalized so that \( V(h_1) = 1 \) and \( V(h_0) = 0 \). This makes \( U(0) = 0 \) and \( U(X) = 1 \). For each \( \omega_0 \in N^C \) and each \( Y \in \mathcal{K}_0 \), define \( S_{Y,\{\omega_0\}} \) as in (14). Then define \( V(\omega, h) \) as in (15) for \( \omega \in N^C \). The state-dependent utility of a horse-lottery \( h \in \mathcal{H} \) in state \( \omega \) is \( V(\omega, h) \). It is straightforward to show that \( V(\omega, h) \) has the same form as in (16) when \( h(\omega) \) is simple.

We conclude this section by showing that \( U \) acts as an expected value.
Lemma 19. Let $\mathcal{T}$ be a $p$-coherent full trading system on $\mathcal{K}_0$, and let $U$ be a standard-linear function that represents $\mathcal{T}$. There exists a standard-linear space $W$ that contains all standard constants and all indicators of elements of $\Sigma$ such that $U$ acts as an expected state-dependent utility on $W$ with respect to the probability $P$ defined in Lemma 17.

Proof. First, extend the domain of definition of $V(\omega,Y)$ to include null $\omega$. Set $V(\omega,Y) = 0$ for $\omega \in N$ and all $Y$. Let $X$ be a numeraire. If $B \in \Sigma$, $S_{X,B} \in \mathcal{K}_0$, $P(B) = U(S_{X,B})$, and $V(\omega,S_{X,B}) = I_B(\omega)$. Let $W$ be the set of all functions on $\Omega$ of the form $V(\cdot,Y)$ for $Y \in \mathcal{K}_0$, which is a standard-linear space that contains all indicators of elements of $\Sigma$. It follows easily from the fact that $U$ is standard-linear that $V(\omega,\cdot)$ is standard linear as a function of its second argument for each $\omega$. It follows from $p$-coherence that $V(\omega,Y_1) \leq V(\omega,Y_2)$ for all $\omega$ implies that $U(Y_1) \leq U(Y_2)$. Since $V(\omega,X) = 1$ for all $\omega$, $W$ contains all standard constants and $U(1) = U(V(\cdot,X)) = U(X) = 1$. According to Definition 15, we see that $U$ acts as an expected value on $W$ with respect to $P$. \hfill \Box

5.2.5. State-independent Utility

The assumption that the set $N$ of null states is a null event may seem heavy-handed. Many other developments of expected-utility representations, e.g. [2], make a different heavy-handed assumption, namely that the utilities of prizes and lotteries are the same in all states. This assumption can be expressed in terms of the preference relation, without reference to utility.

Axiom 2 (State-independence). Let $r_1, r_2 \in \mathcal{R}$ be lotteries. Let $h_1, h_2$ be horse-lotteries and let $B$ be a non-null event. Suppose that $h_1(\omega) = h_2(\omega)$ for all $\omega \in B^c$, and that $h_j(\omega) = r_j$ for all $\omega \in B$ and $j = 1, 2$. Then $h_1 < h_2$ if and only if $r_1 < r_2$.

We have not assumed Axiom 2, but if one does, then a numeraire exists. For the results of this section, we take the liberty of treating a horse-lottery full trading system $\mathcal{T}$ as being on either $\mathcal{H}$ or the corresponding $\mathcal{K}_0$, whichever is more convenient.

Lemma 20. Let $\Sigma$ be a field of subsets of $\Omega$. Let $\mathcal{T}$ be a full trading system that satisfies Axiom 2 on $\mathcal{H}$. Assume that there are two lotteries $r_1, r_2 \in \mathcal{R}$ such that $r_1 \prec r_2$. Let $X = r_2 - r_1$, and assume that, for all $B \in \Sigma$, $S_{X,B} \in \mathcal{K}_0$. Then $X$ is a numeraire on $\Sigma$. 39
Proof. We need to show that \( 0 \prec X \mid B \) for every non-null \( B \in \Sigma \). Let \( B \in \Sigma \) be non-null. Define

\[
\begin{align*}
    h_1(\omega) &= r_1, & \text{for all } \omega, \\
    h_2(\omega) &= \begin{cases} 
        r_2 & \text{if } \omega \in B, \\
        r_1 & \text{if } \omega \in B^C.
    \end{cases}
\end{align*}
\]

Axiom 2 says that \( h_1 \prec h_2 \). Since \( h_1(\omega) = h_2(\omega) \) for \( \omega \in B^C \), we have \( 0 \prec h_2 - h_1 \mid B \). Since \( S_{X,B} = h_2 - h_1 \), it follows that \( 0 \prec X \mid B \).

Axiom 2 also allows a proof that a representing function has an expected-utility interpretation when lotteries are all simple. We still need a condition like \( p \)-coherence (Definition 19.) We take the liberty of using the same terminology and notation as in Definition 19 in this slightly different setting.

**Definition 20.** Assume the conditions of Lemma 20. If \( h_1(\omega) \prec h_2(\omega) \) for all \( \omega \) in the complement of a null event, we say that \( h_2 \) dominates \( h_1 \) in preference (or \( p \)-dominates \( h_1 \)), denoted \( h_1 \prec_{p-Dom} h_2 \). Call \( T \) \( p \)-coherent if \( h_1 \prec_{p-Dom} h_2 \) implies \( h_1 \prec h_2 \).

**Lemma 21.** Assume the conditions of Lemma 20, and assume that \( T \) is \( p \)-coherent. If a standard-linear function \( U \) represents \( T \) and is scaled so that \( U(X) = 1 \), then \( U \) acts as an expected value of utility.

Proof. Let \( X \) be the numeraire constructed in the proof of Lemma 20. Let \( U \) be standard-linear function that represents \( T \) on \( \mathcal{K}_0 \) and has been rescaled so that \( U(X) = 1 \). The resulting probability \( P \) on \( \Omega \) is given by Lemma 17. In analogy to \( V(\omega,p) \), define \( U(\omega,Y) = U(Y(\omega)) \) for each \( Y \in \mathcal{K}_0 \). Note that \( U(\omega,X) = 1 \) for all \( \omega \) and \( U(\omega,S_{X,B}) = I_B(\omega) \) for all \( B \in \Sigma \). Also, if \( Y = \alpha(h_2 - h_1) \) with \( \alpha > 0 \) and \( h_1, h_2 \in \mathcal{H} \), then \( U(\omega,Y) = \alpha[V(\omega,h_2) - V(\omega,h_1)] \).

Let \( W \) be the linear span of all functions on \( \omega \) of the form \( U(\omega,Y) \) for \( Y \in \mathcal{K}_0 \). Since \( U(\omega,Y - Z) = U(\omega,Y) - U(\omega,Z) \), all that remains to show is that: for all \( Y \in \mathcal{K}_0 \), \( 0 \leq U(\omega,Y) \) implies \( 0 \leq U(Y) \). Let \( Y \in \mathcal{K}_0 \) be such that \( 0 \leq U(\omega,Y) \). Express \( Y = \alpha(h_2 - h_1) \). Then \( V(\omega,h_1) < V(\omega,h_2) \) for all \( \omega \), and \( h_1 \prec_{p-Dom} h_2 \). Hence \( h_1 \prec h_2 \) and \( 0 \leq U(Y) \).

5.3. The Standard Part of a Representation

In this section, we assume that \( T = (\prec, \prec) \) is a coherent full trading system on a linear space \( \mathcal{O} \) of standard-valued functions that includes all constants with dominance defined as weak dominance (see Definition 7.)
Let $U$ represent the coherent order $\preceq$. Assume that $U(c) = c$ for each standard constant $c$. The standard part of $U$ is a coherent prevision on $\mathcal{O}$ in the following sense. The proof of Lemma 22 appears in Appendix A.7 of the supplemental article [7].

**Lemma 22.** For each $X \in \mathcal{O}$ and each nonempty $E \subseteq \Omega$, define $P(X|E) = \mathbb{R}[U(XE)/U(E)]$, where we have used the name of $E$ to stand for its indicator. Let $X_1, \ldots, X_n \in \mathcal{O}$, let $E_1, \ldots, E_n$ be nonempty subsets of $\Omega$, let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, and let $c_1, \ldots, c_n \in \mathbb{R}$ be such that $\alpha_j > 0$ for all $j$ such that $P(X_j|E_j) = \infty$, $\alpha_j < 0$ for all $j$ such that $P(X_j|E_j) = -\infty$, and $c_j = P(X_j|E_j)$ for all $j$ such that $P(X_j|E_j)$ is finite. Let $E = \bigcup_{j=1}^n E_j$. Then

$$\sup_{\omega \in E} \sum_{j=1}^n \alpha_j E_j(\omega)[X_j(\omega) - c_j] \geq 0. \quad (18)$$

Equation (18) expresses the generalization of the coherence condition of de Finetti needed to handle infinite previsions. That condition is that no finite linear combination of gambles based on $P$ is uniformly negative on the set $E$ where at least one of the gambles gets settled. (There is some controversy over whether the sup in (18) should be over $\omega \in E$ or over $\omega \in \Omega$. Although we made one of the two choices in the statement of Lemma 22, the proof is virtually identical for the other choice.) The use of arbitrary $c_j$ when $P(X_j|E_j)$ is infinite is to prevent the value that changes hands in a trade from being infinite. The conditions on the $\alpha_j$ that correspond to infinite previsions are needed so that an agent never gives away a random variable with positive-infinite prevision and never accepts a random variable with negative-infinite prevision. These conditions express the fact that “$P(X_j) = \infty$” means that the agent is willing to pay any finite amount to buy $X_j$ but is not willing to sell $X_j$. Similarly, $P(X_j) = -\infty$ means that the agent is willing to sell $X_j$ for any finite amount but is not willing to buy $X_j$.

6. Discussion

The major contributions of this paper are

- a systematic representation of preference relations regardless of how strong is the form of dominance that one wishes to respect,

- the use of nonstandard models of the reals to represent non-Archimedean preferences,
• an expected utility interpretation for the nonstandard representation in the random-variable case, and

• an expected state-dependent utility interpretation for the nonstandard representation in the lottery case when all lotteries are simple.

In the lottery case, there are two ways in which a representing function $V$ can be interpreted as an expected state-dependent utility:

1. For each $h \in H$, $V(h)$ is an expected value of $V(\omega, h)$ with respect to the probability on $\Omega$ from Lemma 17.

2. For each $h \in H$ and each $\omega \in \Omega$, $V(\omega, h)$ is an expected value of $V(\omega, p)$ with respect to the probability $h(\omega)$ on $P$.

Lemma 19 gives us 1 above if the preferences satisfy a dominance condition that we did not require in the lottery case. If all lotteries in $\mathcal{R}$ are simple, we get 2 above. “What are the weakest conditions under which we get both 1 and 2 above with general lotteries?” remains an open question.

In the random-variable case some, but not all, of the examples of non-Archimedean preferences arise from the use of weak dominance in the definition of coherence. For example, it is impossible to have an Archimedean preference structure that respects weak dominance on the set of random variables defined on an uncountable state space.

Weak dominance is the weakest of the three dominance concepts in Definition 7. Weak dominance is the same as the form of dominance used to define inadmissibility in statistical decision theory. The strongest of the three dominance concepts is the one used in de Finetti's theory, namely uniform dominance. Strict dominance is intermediate to the other two. Since dominance is used to prevent calling a trading system coherent, the stronger the dominance condition, the weaker the sense of coherence, i.e., the easier it is to call a trading system coherent. Since some of our results use the weakest form of dominance, those results use the strongest form of coherence. We gave examples to show that some coherent trading systems (using weak dominance) could be represented neither by standard-valued functions nor by lexicographic probability. The same could be done for strict dominance.

*Example 18 (Modification of Example 4).* Let $\Omega = \mathbb{Z}^+$, the positive standard integers. This time, assume that all of the singleton indicators $I_{(n)}$ are indifferent to 0. That is, $I_{(n)} \sim 0$ for every standard integer $n$. Introduce a new random variable $X(n) = 1/n$, and require $0 \prec X$. This preference respects
strict dominance but not weak dominance. We already showed in Example 4 that every lexicographic probability \( \{P_\alpha\}_{\alpha \in \mathbb{R}} \) would have \( P_\alpha(\{n\}) = 0 \) for all \( \alpha \) and all \( n \). This would imply that \( P_\alpha(X) \leq 1/n \) for all \( n \) and \( \alpha \), hence \( P_\alpha(X) = 0 \) for all \( \alpha \), and we cannot get \( 0 \prec X \).

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References


Supplementary Material for “When No Price Is Right”

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Abstract

This supplement contains five appendices to the main paper. Appendix A contains the lengthier proofs of the results in the main paper. Appendix B contains an overview of binary relations along with some technical results that are used in our proofs. Appendix C contains an overview of nonstandard models of the reals. Appendix D contains the technical details of closing a partial preference under the assumptions made in the main paper. Appendix E contains the technical details about the equivalence of binary relations on the sets $\mathcal{H}$ and $\mathcal{K}_0$ introduced in Section 2.2 of the main paper.

Appendix A. Lengthy Proofs

This section contains the lengthier proofs of the results in the main paper. We start with a couple of technical results that are used in the proofs.

Lemma 23. Assume that $\prec$ and $\triangleleft$ are codirectional and affine. If $X \prec Y$ and $Z \triangleleft W$, then $X + Z \prec Y + W$.

Proof. From affine, we conclude that $X + Z \prec Y + Z$ and $Y + Z \triangleleft Y + W$. From codirectional, we conclude $X + Z \prec Y + W$. $\square$

Lemma 24. If $\sim$ and $\prec$ are affine and $\prec$ is defined on the equivalence classes of $\sim$, then $[\alpha X + Z]_\sim = \alpha[X]_\sim + Z$.

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Proof. Note first that $\alpha[X] \sim Z = \{\alpha X' + Z : X' \sim X\}$. For all $Y,W$, $Y \sim W$ if and only if there is $F \sim 0$ such that $Y = W + F$. So $\alpha[X] \sim Z = \{\alpha X + Z + F : F \sim 0\} = [\alpha X + Z]_\sim$. \hfill\Box

Appendix A.1. Proof of Lemma 8

Let $C_*$ be the partition that results from coarsening $C$ by combining each pair of elements $([\alpha Q_1 + Z]_C, [\alpha Q_2 + Z]_C)$ with $\alpha \neq 0$ standard and $Z \in \mathcal{X}$ into $[\alpha Q_1 + Z]_C \cup [\alpha Q_2 + Z]_C$. This is the same as extending $\sim$ to $\sim_*$ defined by $X \sim_*, Y$ if there exist standard $\alpha$ and $Z \in \mathcal{X}$ such that $X,Y \in [\alpha Q_1 + Z]_C \cup [\alpha Q_2 + Z]_C$. (Note that we allow $\alpha = 0$ in this last expression in order to capture cases of $X \sim Y$.) Let $\mathcal{F}_T$ be the fair trades according to $T_*$. Each element of $\mathcal{F}_T$ has the form $T + \alpha (Q_1 - Q_2)$ where $\alpha$ is standard and $T \in \mathcal{F}_T$. Hence, the equivalence classes that make up $C_*$ have the form

$$[X]_{C_*} = [X]_C + \{\alpha (Q_1 - Q_2) : \alpha \in \mathbb{R}\}, \quad (A.1)$$

where the representation in (A.1) is not unique. Also, $\mathcal{F}_T \subseteq \mathcal{F}_{T_*}$, so every fair trade under $\mathcal{T}$ is fair under $\mathcal{T}_*$.

Define $\prec_*$ on $C_*$ as follows:

$$[X]_{C_*} \prec_* [Y]_{C_*} \text{ if there exist } X' \in [X]_{C_*} \text{ and } Y' \in [Y]_{C_*} \text{ such that } [X']_{C} \prec [Y']_{C_*}. \quad (A.2)$$

Note that this is equivalent to the existence of standard $\alpha$ such that

$$[X - Y]_{C} \prec [\alpha (Q_1 - Q_2)]_{C_*}. \quad (A.2)$$

We see that $\prec_*$ extends $\prec$ since $\alpha = 0$ is allowed in (A.2).

Define $\triangleleft_*$ on $C_*$ as follows:

$[X]_{C_*} \triangleleft_* [Y]_{C_*}$ if all of the following are true:

- There exists standard $\alpha$ such that

$$[X - Y]_{C} \triangleleft [\alpha (Q_1 - Q_2)]_{C_*}, \quad (A.3)$$

- $[X]_{C_*} \neq [Y]_{C_*}$, and

- $[X]_{C_*} \not\prec_* [Y]_{C_*}$.  

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Next, we prove that all acceptable trades under $T$ remain acceptable under $T_*$. First, recall that $\sim_*$ and $\prec_*$ are extensions of $\sim$ and $\prec$ respectively. This means that all fair and strictly preferred trades under $T$ remain respectively fair and strictly preferred under $T_*$. From (A.3), we see that every semi-preferred trade either remains semi-preferred or becomes fair or strictly preferred under $T_*$. Hence, all acceptable trades under $T$ remain acceptable under $T_*$. 

Next, we prove that $\triangleright_*$ is asymmetric. To do this, we show that if (A.3) holds and there exists standard $\beta$ such that 

$$[Y - X]_c \triangleright [\beta(Q_1 - Q_2)]_c,$$

then $[X]_c \sim_* [Y]_c$. Since $\triangleright$ is affine,

$$[-\beta(Q_1 - Q_2)]_c \triangleright [X - Y]_c \triangleright [\alpha(Q_1 - Q_2)]_c.$$ 

In words, the agent (using $T$) was willing to pay $-\beta(Q_1 - Q_2)$ to get $X - Y$ and was willing to pay $X - Y$ to get $\alpha(Q_1 - Q_2)$. As we showed above, these trades are still acceptable using $T_*$. In addition, under $T_*$, both $-\beta(Q_1 - Q_2)$ and $\alpha(Q_1 - Q_2)$ are in $[0]_c$. So, $X - Y \in [0]_c$.

Next, we prove that $\prec_*$ is a strict partial order. To see that $\prec_*$ is asymmetric, assume that $[X]_c \prec_* [Y]_c$. Then there exists standard $\alpha$ such that (A.2) holds. Assume, to the contrary, that $[Y]_c \prec_* [X]_c$. Then there exists standard $\beta$ such that

$$[Y - X]_c \prec [\beta(Q_1 - Q_2)]_c.$$ 

Use the fact that $\prec$ is affine to conclude

$$[X - Y - \alpha(Q_1 - Q_2)]_c \prec [0]_c \prec [X - Y + \beta(Q_1 - Q_2)]_c,$$

hence $[0]_c \prec [(\alpha + \beta)(Q_1 - Q_2)]_c$, which is false for all $\alpha, \beta$.

To see that $\prec_*$ is transitive, let $[X]_c \prec_* [Y]_c$ and $[Y]_c \prec_* [Z]_c$. Let $\alpha, \beta$ be such that (A.2) holds and

$$[Y - Z]_c \prec [\beta(Q_1 - Q_2)]_c.$$ 

Since $\prec$ is affine, we have

$$[X - Y + Y - Z]_c \prec [(\alpha + \beta)(Q_1 - Q_2)]_c,$$
which is (A.2) for checking if \([X]_{C_0} \prec_s [Z]_{C_0}\).

Next, we prove that \(C_0\) is affine. Let \(X \in [Y]_{C_0}\). We need to show that, for all standard \(\beta\) and \(Z \in \mathcal{X}\), \(\beta X + Z \in [\beta Y + Z]_{C_0}\). Let \(\alpha\) be such that

\[
[X - Y]_C = [\alpha(Q_1 - Q_2)]_C.
\]

Since \(C\) is affine, we have

\[
\beta X - \beta \alpha(Q_1 - Q_2) + Z \in [\beta Y + Z]_C,
\]

which implies that \(\beta X + Z \in [\beta Y + Z]_{C_0}\).

Next, we show that \(\prec_s\) is affine. Let \([X]_{C_0} \prec_s [Y]_{C_0}\). We need to show that, for all standard \(\beta\) and \(Z \in \mathcal{X}\), \([\beta X + Z]_{C_0} \prec_s [\beta Y + Z]_{C_0}\). Let \(\alpha\) be such that

\[
[X - Y]_C \prec [\alpha(Q_1 - Q_2)]_C.
\]

Since \(\prec\) is affine, we have

\[
[\beta X - \beta \alpha(Q_1 - Q_2) + Z]_C \prec [\beta Y + Z]_C,
\]

which implies that \([\beta X + Z]_{C_0} \prec_s [\beta Y + Z]_{C_0}\).

Next, we show that \(\prec_s\) is affine. Let \([X]_{C_0} \prec_s [Y]_{C_0}\). We need to show that, for all standard \(\beta\) and \(Z \in \mathcal{X}\), \([\beta X + Z]_{C_0} \prec_s [\beta Y + Z]_{C_0}\). Let \(\alpha\) be such that

\[
[X - Y]_C \prec [\alpha(Q_1 - Q_2)]_C.
\]

Since \(\prec\) is affine, we have

\[
[\beta X - \beta \alpha(Q_1 - Q_2) + Z]_C \prec [\beta Y + Z]_C,
\]

which implies (A.3). We still need to prove that the other two bullets hold. Since both \(\prec_s\) and \(\sim_s\) are affine, if either of the other two bullets fail for the pair \((\beta X + Z, \beta Y + Z)\), then the corresponding bullet also fails for the pair \((X, Y)\). This contradicts \([X]_{C_0} \prec_s [Y]_{C_0}\).

Next, we prove that \(\prec_s\) and \(\prec_s\) are codirectional. For the first property, assume that \([X]_{C_0} \prec_s [Y]_{C_0}\) and \([Y]_{C_0} \prec_s [Z]_{C_0}\). Then, there exist standard \(\alpha\) and \(\beta\) such that \([X - Y]_{C_0} \prec_s [\alpha(Q_1 - Q_2)]_{C_0}\) and \([Y - Z]_{C_0} \prec_s [\beta(Q_1 - Q_2)]_{C_0}\). Since \(\prec\) is affine, \([X - Z]_{C_0} \prec_s [(\alpha + \beta)(Q_1 - Q_2)]_{C_0}\), and (A.3) holds for \(X\) and \(Z\). We need only prove that \([X]_{C_0} \neq [Z]_{C_0}\). Assume, to the contrary, that \([X]_{C_0} = [Z]_{C_0}\). Then either \([Z]_{C_0} = [Y]_{C_0}\) or \([Z]_{C_0} \prec_s [Y]_{C_0}\), both of which are contradictions to \([Y]_{C_0} \prec_s [Z]_{C_0}\). For the second property, assume that...
\[X]_C. ∼_d [Y]_C, and [Y]_C. ≺_d [Z]_C. \text{Then there exist standard } \alpha, \beta \text{ such that } [X - Y]_C \prec [\alpha(Q_1 - Q_2)]_C \text{ and } [Y - Z]_C \prec [\beta(Q_1 - Q_2)]_C. \text{ By Lemma 23, we have } [X - Z]_C \prec [\alpha + \beta)(Q_1 - Q_2)]_C \text{ so that } [X]_C. ≺_d [Z]_C. \text{ A similar argument works for the third property.}

If \(T\) is coherent, that \(T\) is coherent follows from Proposition 4 because \(T\) extends \(T\).

Appendix A.2. Proof of Lemma 9

Since \(C\) is the only partition in the proof, we will let \([X]\) denote the element of \(C\) that contains \(X\) for each \(X \in \mathcal{X}\). We begin by extending the strict partial order \(\prec\) on \(C\) to \(\prec^*\) that includes the comparisons \([\alpha X + Z]\prec^* [\alpha Y + Z]\) for all \(\alpha > 0\) and all \(Z \in \mathcal{X}\).

We will use the representations of strict partial orders on \(C\) as a subset of \(C^2\) in the proof. (See Definition 21 in Section Appendix B.1.) Let \(A\) be the set that represents \(\prec\).

Let \(\prec^*\) be represented by the set \(A^* = A \cup B\), where

\[
B_{\alpha, D^-} = \{[Z] \in C : [Z] \preceq [\alpha X + D]\},
B_{\alpha, D^+} = \{[W] \in C : [\alpha Y + D] \preceq [W]\},
B = \bigcup_{D \in \mathcal{X}, \alpha > 0} (B_{\alpha, D^-} \times B_{\alpha, D^+}).
\]

Since \(A^*\) contains \(A\) as a subset, \(\prec^*\) extends \(\prec\). By construction (\(\alpha = 1\) and \(D = 0\)) \(([X],[Y]) \in B\). The bulk of the proof consists of proving that \(\prec^*\) is an affine strict partial order.

Since \(A\) contains neither \(([X],[Y])\) nor \(([Y],[X])\) and \(\prec\) is affine, \(A\) also contains no pair of the form \(([\alpha X + D],[\alpha Y + D])\) or \(([\alpha Y + D],[\alpha X + D])\) for \(D \in \mathcal{X}\) and \(\alpha > 0\). Other consequences of being affine are implications of the form “if \([G] \preceq [Y + D]\), then \([G - D] \preceq [Y]\).” That is, we can add or subtract the same thing from either side of a \(\preceq\) relationship. In fact, \([\alpha G - D] = \alpha [G] - D\) for all \(\alpha > 0\), \(G\) and \(D\).

First, we show that \(B_{\alpha, D^-} \cap B_{\alpha, D^+} = \emptyset\) for each \(D \in \mathcal{X}\) and \(\alpha > 0\). In order for \([G] \in B_{\alpha, D^-} \cap B_{\alpha, D^+}\), we must have

\([\alpha Y + D] \preceq [G] \preceq [\alpha X + D]\),

which contradicts that all of the relations in (12) fail. It follows that \(B\) contains no pairs with both coordinates the same.
If a pair \([G, [H]]\) \(\in B\), then there exist \(\alpha > 0\) and \(D\) such that

\[
[G] \preceq [\alpha X + D] \text{ and } [\alpha Y + D] \preceq [H].
\]  
(A.4)

To see that \(\prec^*\) is asymmetric, assume that \(([U], [V]) \in A'\). We need to show that \(([V], [U]) \notin A'\). There are two cases: either \(([U], [V]) \in A\) or \(([U], [V]) \in B\).

**Case A:** \(([U], [V]) \in A\).

Then \(([V], [U]) \notin A\). Assume, to the contrary, that \(([V], [U]) \in B\). Then \([U] \prec [V]\) and there are \(\alpha > 0\) and \(D \in \mathcal{X}\) such that \([V] \preceq [\alpha X + D]\) and \([\alpha Y + D] \preceq [U]\). It follows

\[
[\alpha Y + D] \preceq [U] \prec [V] \preceq [\alpha X + D],
\]

which implies \([Y] \prec [X]\), a contradiction.

**Case B:** \(([U], [V]) \in B\).

Then, there are \(\alpha > 0\) and \(D\) such that \([U] \preceq [\alpha X + D]\) and \([\alpha Y + D] \preceq [V]\). We need to show that \(([V], [U]) \notin A'\). Assume, to the contrary, that \(([V], [U]) \in A'\). As we just proved (switching the names \(U\) and \(V\)) \(([V], [U]) \in A\) would contradict \(([U], [V]) \in B\). So, \(([V], [U]) \in B\). That is, there is \(\alpha' > 0\) and \(D'\) such that \([V] \preceq [\alpha' X + D']\) and \([\alpha' Y + D'] \preceq [U]\). It follows that

\[
[\alpha Y + D] \preceq [V] \preceq [\alpha' X + D'],
[\alpha' Y + D'] \preceq [U] \preceq [\alpha X + D].
\]

Because \(\preceq\) is affine and transitive, we have

\[
[\alpha Y - \alpha' X] \preceq [0] \preceq [\alpha X + D - \alpha' Y + D'],
\]

from which it follows that

\[
[(\alpha + \alpha') (Y - X)] \preceq [0].
\]

which is a contradiction because \(\alpha' + \alpha > 0\) and \(\neg ([Y] \preceq [X])\).

To see that \(\prec^*\) is transitive, we need to show that, if \(([U], [V]) \in A'\) and \(([V], [R]) \in A'\), then \(([U], [R]) \in A'\). Assume that \(([U], [V]) \in A'\) and \(([V], [R]) \in A'\). The rest of the proof depends on precisely where in \(A'\) the pairs \(([U], [V])\) and \(([V], [R])\) lie.

**Case 1:** Both \(([U], [V])\) and \(([V], [R])\) are in \(A\).

By transitivity of \(\prec\), \(([U], [R]) \in A \subseteq A'\).
Case 2: \(([U],[V]) \in A\) and \(([V],[R]) \in B\).

Then \([U] \prec [V]\), and there are \(\alpha > 0\) and \(D \in \mathcal{X}\) such that \([U] \prec [V] \preceq [\alpha X + D]\) and \([\alpha Y + D] \preceq [R]\). We then have \([U] \in B_{\alpha,D-}\) and \([R] \in B_{\alpha,D+}\), so \((([U],[R]) \in B \subseteq A'\).

Case 3: \(([U],[V]) \in B\) and \(([V],[R]) \in A\).

Then \([V] \prec [R]\), and there are \(\alpha > 0\) and \(D \in \mathcal{X}\) such that \([U] \preceq [\alpha X + D]\) and \([\alpha Y + D] \preceq [V] \prec [R]\). We then have \([U] \in B_{\alpha,D-}\) and \([R] \in B_{\alpha,D+}\), so \((([U],[R]) \in B \subseteq A'\).

Case 4: Both \(([U],[V])\) and \(([V],[R])\) are in \(B\).

Then there are \(\alpha, \alpha' > 0\) and \(D, D' \in \mathcal{X}\) such that \([U] \preceq [\alpha X + D]\), \([\alpha Y + D] \preceq [V] \preceq [\alpha' X + D']\), and \([\alpha' Y + D'] \preceq [R]\). Since \(\preceq\) is transitive, \([\alpha' Y + D'] \preceq [\alpha Y + D]\). Since \(\preceq\) is affine, it follows that \([0] \preceq [D' - D - \alpha Y + \alpha' X]\).

Combining this with \([\alpha' Y + D'] \preceq [R]\) yields

\[
([\alpha + \alpha'] Y - \alpha' X + D] \preceq [R].
\] (A.5)

Let \(\alpha^* = \alpha + \alpha'\), and \(D^* = D - \alpha' X\). Since \(\alpha X + D = \alpha^* X + D^*\), use (A.5) to conclude

\[
[U] \preceq [\alpha^* X + D^*], \\
[\alpha^* Y + D^*] \preceq [R].
\]

This means that \([U] \in B_{\alpha^*,D^*-}\) and \([R] \in B_{\alpha^*,D^*}\) so that \([U] \prec^* [R]\).

Next, we prove that \(\prec^*\) is affine. Suppose that \([Z] \prec^* [W]\). Let \(\alpha > 0\) and \(D \in \mathcal{X}\). If \(([Z],[W]) \in A\), then \(([\alpha Z + D],[\alpha W + D]) \in A \subseteq A'\). If \(([Z],[W]) \in B\), then there are \(D' \in \mathcal{X}\) and \(\alpha' > 0\) such that \([Z] \in B_{\alpha',D'-}\) and \([W] \in B_{\alpha',D'+}\). Then \([\alpha Z + D] \in B_{\alpha',\alpha'(D'-D)-}\) and \([\alpha W + D] \in B_{\alpha',\alpha'(D'+D)+}\). Since \(\alpha \alpha' > 0\) and \(\alpha D' + D \in \mathcal{X}\), \(([\alpha Z + D],[\alpha W + D]) \in B \subseteq A'\).

Next, we define \(\triangleleft^*\) to be the binary relation represented by the set \(A_{\triangleleft} \setminus A'\), where \(A_{\triangleleft}\) is the set that represents \(\triangleleft\) according to Definition 21 in Section Appendix B.1.

Next, we prove that \(\triangleleft^*\) is affine and asymmetric. Since both \(A_{\triangleleft}\) and \(A'\) are affine, it follows easily that \(A_{\triangleleft} \setminus A'\) is affine. A subset of a set that represents an asymmetric relation also represents an asymmetric relation, so \(\triangleleft^*\) is asymmetric.

Next, we prove that \(\prec^*\) and \(\triangleleft^*\) are codirectional. For the first property, assume that \([U] \prec^* [V]\) and \([V] \triangleleft^* [W]\). Then \([U] \triangleleft [V]\) and \([V] \triangleleft [W]\) so either \([U] \triangleleft [W]\) or \([U] \prec [W]\). It follows that either \([U] \prec^* [W]\) or \([U] \prec^* [W]\). For the second property, assume that \([U] \prec^* [V]\) and \([V] \prec [W]\). Then,
[U] \prec [V] and there are standard \( \alpha \) and \( D \in \mathcal{X} \) such that \( [V] \preceq [\alpha X + D] \) and \( [\alpha Y + D] \prec [W] \). It follows that \([U] \preceq [\alpha X + D] \), so that \([U] \prec^* [W] \).

Finally, if \( \mathcal{T} \) is coherent then \( \mathcal{T}^* \) is coherent according to Proposition 4.

Appendix A.3. Proof of Theorem 2

The proof is by transfinite induction. Let \( \Gamma \) be the cardinality of the set of all pairs \((X, Y)\) of distinct elements of \( \mathcal{X} \). Well-order this set of pairs as \( \{(X_\gamma, Y_\gamma)\}_{\gamma=1}^\Gamma \). The induction hypothesis is

**Induction hypothesis:** Let \( \gamma \geq 0 \) be an ordinal. There exists a (coherent) partial trading system \( \mathcal{T}_\gamma = (\sim_\gamma, \prec_\gamma, \prec_\gamma) \) that extends \( \mathcal{T} \) in such a way that for each \( \lambda \leq \gamma \) exactly one of the following holds:

\[
X_\lambda \sim_\gamma Y_\lambda, \text{ or } X_\lambda \prec_\gamma Y_\lambda, \text{ or } Y_\lambda \prec_\gamma X_\lambda.
\]

For each \( \gamma \) for which the induction hypothesis holds, define \( \preceq_\lambda \) by

\[
X \preceq_\lambda Y \text{ if and only if } (X \sim_\lambda Y) \lor (X \prec_\lambda Y) \lor (X \prec_\lambda Y),
\]

and note that \( \preceq_\lambda \) is a preorder whose associated equivalence relation (from Lemma 28) is \( \sim_\lambda \). Also, \( \preceq_\lambda \) extends \( \preceq_0 \) if \( \mathcal{T}_\lambda \) extends \( \mathcal{T} \).

First, we need to establish the induction hypothesis for \( \gamma = 0 \). Since \( \prec \) and \( \sim \) are extensions of themselves, we can set \( \mathcal{T}_0 = \mathcal{T} \). Since there are no \( \lambda < 0 \), the induction hypothesis is trivially satisfied for \( \gamma = 0 \).

Next, let \( \gamma \geq 0 \) be an ordinal and assume that the induction hypothesis holds for all \( \lambda \leq \gamma \). If one of the following holds

\[
X_{\gamma+1} \sim_\gamma Y_{\gamma+1} \text{ or } X_{\gamma+1} \prec_\gamma Y_{\gamma+1} \text{ or } Y_{\gamma+1} \prec_\gamma X_{\gamma+1}, \quad (A.6)
\]

the induction hypothesis holds for \( \gamma + 1 \) with \( \mathcal{T}_{\gamma+1} = \mathcal{T}_\gamma \). If none of the three relations in (A.6) holds, extend \( \mathcal{T}_\gamma \) to \( \mathcal{T}_{\gamma+1} \) by doing the following:

- If \( X_{\gamma+1} \prec_\gamma Y_{\gamma+1} \), either apply Lemma 8 to include the comparison \( X_{\gamma+1} \sim_{\gamma+1} Y_{\gamma+1} \), or apply Lemma 9 to include either of the two comparisons: \( X_{\gamma+1} \prec_{\gamma+1} Y_{\gamma+1} \) or \( Y_{\gamma+1} \prec_{\gamma+1} X_{\gamma+1} \).

- If \( X_{\gamma+1} \prec_\gamma Y_{\gamma+1} \), either apply Lemma 8 to include the comparison \( X_{\gamma+1} \sim_{\gamma+1} Y_{\gamma+1} \), or apply Lemma 9 to include the comparison \( X_{\gamma+1} \prec_{\gamma+1} Y_{\gamma+1} \).
• If $Y_{\gamma+1} \preceq_{\gamma} X_{\gamma+1}$, either apply Lemma 8 to include the comparison $Y_{\gamma+1} \sim_{\gamma+1} X_{\gamma+1}$, or apply Lemma 9 to include the comparison $Y_{\gamma+1} \prec_{\gamma+1} X_{\gamma+1}$.

In all cases above, the induction hypothesis holds for $\gamma + 1$. This establishes the induction hypothesis for all successor ordinals.

If $\gamma$ is a limit ordinal, define $T_\gamma$ as follows: First, define $T_{\gamma-} = (\sim_{\gamma-}, \prec_{\gamma-}, \preceq_{\gamma-})$ by

- $X \sim_{\gamma-} Y$ if there is $\lambda < \gamma$ such that $X \sim_\lambda Y$.
- $X \prec_{\gamma-} Y$ if there is $\lambda < \gamma$ such that $X \prec_\lambda Y$.
- $X \preceq_{\gamma-} Y$ if there is $\lambda < \gamma$ such that $X \preceq_\lambda Y$.
- $X \prec_{\gamma-} Y$ if $(X \preceq_{\gamma-} Y) \land (X \sim_{\gamma-} Y) \lor (X \prec_{\gamma-} Y)$.

To see that $\sim_{\gamma-}$ is an affine equivalence relation, note that $\sim_{\gamma-}$ is trivially reflexive and symmetric. To verify transitivity, assume that $X \sim_{\gamma-} Y$ and $Y \sim_{\gamma-} Z$. Let $\lambda_1$ and $\lambda_2$ be such that $X \sim_{\lambda_1} Y$ and $Y \sim_{\lambda_2} Z$. Let $\lambda = \max\{\lambda_1, \lambda_2\}$. Since $\sim_\lambda$ extends both $\sim_{\lambda_1}$ and $\sim_{\lambda_2}$, we have $X \sim_\lambda Z$ and $X \sim_{\gamma-} Z$. To see that $\sim_{\gamma-}$ is affine, assume $X \sim_{\gamma-} Y$, $\alpha > 0$ and $Z \in X$. Then, there is $\lambda < \gamma$ such that $X \sim_\lambda Y$ and $\alpha X + Z \sim_\lambda \alpha Y + Z$ since $\sim_\lambda$ is affine. So $\alpha X + Z \sim_{\gamma-} \alpha Y + Z$. Clearly, $\sim_{\gamma-}$ extends all $\sim_\lambda$ for $\lambda < \gamma$.

We prove next that $\prec_{\gamma-}$ is an affine strict partial order that extends $\prec_0$. To see that $\prec_{\gamma-}$ is asymmetric, assume that $X \prec_{\gamma-} Y$ so that there is $\lambda < \gamma$ such that $X \prec_\lambda Y$. If, to the contrary, $Y \prec_{\gamma-} X$, there would be $\lambda' < \gamma$ such that $Y \prec_{\lambda'} X$. In such a case, let $\lambda^* = \max\{\lambda, \lambda'\}$. Since $\prec_{\lambda^*}$ extends both $\prec_\lambda$ and $\prec_{\lambda'}$, we would have a contradiction to $\prec_{\lambda^*}$ being asymmetric.

To verify transitivity, assume that $X \prec_{\gamma-} Y$ and $Y \prec_{\gamma-} Z$. Let $\lambda_1$ and $\lambda_2$ be such that $X \prec_{\lambda_1} Y$ and $Y \prec_{\lambda_2} Z$. Let $\lambda = \max\{\lambda_1, \lambda_2\}$. Since $\prec_\lambda$ extends both $\prec_{\lambda_1}$ and $\prec_{\lambda_2}$, we have $X \prec_\lambda Z$ and $X \prec_{\gamma-} Z$. Clearly, $\prec_{\gamma-}$ extends $\prec_\lambda$ for all $\lambda < \gamma$. The proof that $\prec_{\gamma-}$ is affine is similar to the proof that $\sim_{\gamma-}$ is affine.

Next, we show that $\preceq_{\gamma-}$ is an affine preorder with $\sim_{\gamma-}$ as its associated equivalence relation as in Lemma 28. Clearly, $\preceq_{\gamma-}$ is reflexive. To verify transitivity, assume that $X \preceq_{\gamma-} Y$ and $Y \preceq_{\gamma-} Z$. Let $\lambda_1$ and $\lambda_2$ be such that $X \preceq_{\lambda_1} Y$ and $Y \preceq_{\lambda_2} Z$. Let $\lambda = \max\{\lambda_1, \lambda_2\}$. Since $\preceq_\lambda$ extends both $\preceq_{\lambda_1}$ and $\preceq_{\lambda_2}$, we have $X \preceq_\lambda Z$ and $X \preceq_{\gamma-} Z$. The equivalence relation $\rho$ associated with $\preceq_{\gamma-}$ from Lemma 28 is $X \rho Y$ if and only if $(X \preceq_{\gamma-} Y) \lor (Y \preceq_{\gamma-} X)$. 

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If \( X \rho Y \), then there is \( \lambda < \gamma \) such that \( X \preceq_{\lambda} Y \), which implies that \( Y \preceq_{\lambda} X \) since each \( \preceq_{\lambda} \) is a preorder. Since the equivalence relation associated with \( \preceq_{\lambda} \) is \( \sim_{\lambda} \), it follows that \( X \sim_{\lambda} Y \) and \( X \sim_{\gamma-} Y \). If \( X \sim_{\gamma-} Y \), there is \( \lambda < \gamma \) such that \( X \sim_{\lambda} Y \), and so \( X \preceq_{\lambda} Y \) and \( Y \preceq_{\lambda} X \), which imply that \( X \preceq_{\gamma-} Y \) and \( Y \preceq_{\gamma-} X \) hence \( X \rho Y \). The proof that \( \preceq_{\gamma-} \) is affine is similar to the proof that \( \sim_{\gamma-} \) is affine.

Next, we show that \( \preceq_{\gamma-} \) is an affine asymmetric and codirectional with \( \sim_{\gamma-} \). For asymmetry, let \( X \preceq_{\gamma-} Y \), and assume to the contrary that \( Y \preceq_{\gamma-} X \). It follows that \( X \preceq_{\gamma-} Y \) and \( Y \preceq_{\gamma-} X \), which implies \( X \sim_{\gamma-} Y \), a contradiction to \( X \preceq_{\gamma-} Y \). To see that \( \preceq_{\gamma-} \) and \( \sim_{\gamma-} \) are codirectional, first assume \( X \preceq_{\gamma-} Y \) and \( Y \preceq_{\gamma-} Z \). Then \( X \preceq_{\gamma-} Z \). If, to the contrary, \( X \sim_{\gamma-} Z \), then there is a \( \lambda \) such that \( X \sim_{\lambda} Z \), \( X \preceq_{\lambda} Y \) and \( Y \preceq_{\lambda} Z \). It would then be true that \( Z \preceq_{\lambda} Y \) and \( Y \sim_{\lambda} Z \), which implies \( Y \sim_{\gamma-} Z \), a contradiction to \( Y \preceq_{\gamma-} Z \). So \( \sim(X \sim_{\gamma-} Z) \). If \( \sim(X \sim_{\gamma-} Z) \), then \( X \preceq_{\gamma-} Z \), otherwise \( X \prec_{\gamma-} Y \). Second, assume that \( X \prec_{\gamma-} Y \) and \( Y \prec_{\gamma-} Z \). Then \( X \preceq_{\gamma-} Y \), and there is \( \lambda \) such that both \( X \preceq_{\lambda} Y \) and \( Y \preceq_{\lambda} Z \), so that \( X \prec_{\lambda} Z \). This implies \( X \prec_{\gamma-} Y \). The proof for the third defining property of codirectional is similar. To see that \( \preceq_{\gamma-} \) is affine, first note that \( \preceq_{\gamma-} \), \( \sim_{\gamma-} \), and \( \prec_{\gamma-} \) are affine. These facts entail that for \( \rho \) being either \( \prec_{\gamma-} \) or \( \sim_{\gamma-} \), \( X \rho Y \) if and only if \( \alpha X + Z \rho \alpha Y + Z \) for all \( \alpha > 0 \) and \( Z \in \mathcal{X} \).

Finally, treat the subscript \( \gamma- \) as if it were \( \gamma-1 \) and use the argument for the successor case to find \( \mathcal{T}_\gamma \) which satisfies the induction hypothesis for \( \gamma \).

If \( \mathcal{T} \) is coherent, then every extension in the transfinite induction is coherent because the various lemmas applied all preserve coherence.

**Appendix A.4. Proof of Lemma 13**

Define \( U'(X) = U(X) \) for \( X \in \mathcal{Y} \). We need to define \( U'(X) \) for those \( X \in \mathcal{Z} \setminus \mathcal{Y} \). We have

\[
\mathcal{Z} \setminus \mathcal{Y} = \{ \alpha Z + X : X \in \mathcal{Y}, \alpha \in \mathbb{R} \setminus \{0\} \}.
\]  
(A.7)

It is straightforward to show that the representation of elements of \( \mathcal{Z} \setminus \mathcal{Y} \) in (A.7) is unique. After we choose the value \( U'(Z) \), we will make \( U' \) standard-linear by defining

\[
U'(\alpha Z + X) = \alpha U'(Z) + U(X),
\]  
(A.8)

for \( X \in \mathcal{Y} \) and standard \( \alpha \). Then, we will prove that the results satisfy (13). If there is \( X \in \mathcal{Y} \) such that \( Z \sim X \), choose \( U'(Z) = U(X) \) and then use
(A.8) to extend $U$ to $U'$ on $Z$ so that it is standard-linear and represents $T$ on $Z$. For the remainder of the proof, assume that there is no $X \in \mathcal{Y}$ such that $X \sim Z$.

We need to choose $U'(Z)$ so that (13) holds. Define

$$
\mathcal{L} = \{U(X) : X \in \mathcal{Y}, X \prec Z\},
$$
$$
\mathcal{U} = \{U(X) : X \in \mathcal{Y}, Z \prec X\}.
$$

Every value of $U(X)$ for $X \in \mathcal{Y}$ is in $\mathcal{L} \cup \mathcal{U}$ because there are no $X$ such that $X \sim Z$. Transitivity of $\prec$ guarantees that $u_1 < u_2$ for all $u_1 \in \mathcal{L}$ and $u_2 \in \mathcal{U}$.

There are several cases (and subcases) to handle:

(a) Both $\mathcal{L}$ and $\mathcal{U}$ are nonempty, and

(a)(i) there is $x \in \mathbb{IR}$ such that $u_1 < x < u_2$ for all $u_1 \in \mathcal{L}$ and $u_2 \in \mathcal{U}$, or

(a)(ii) there is no $x \in \mathbb{IR}$ such that $u_1 < x < u_2$ for all $u_1 \in \mathcal{L}$ and $u_2 \in \mathcal{U}$.

(b) $\mathcal{L} = \emptyset$ and $\mathcal{U} \neq \emptyset$, and

(b)(i) there is $x \in \mathbb{IR}$ such that $x < u$ for all $u \in \mathcal{U}$, or

(b)(ii) there is no $x \in \mathbb{IR}$ such that $x < u$ for all $u \in \mathcal{U}$.

(c) $\mathcal{U} = \emptyset$ and $\mathcal{L} \neq \emptyset$, and

(c)(i) there is $x \in \mathbb{IR}$ such that $x > u$ for all $u \in \mathcal{L}$, or

(c)(ii) there is no $x \in \mathbb{IR}$ such that $x > u$ for all $u \in \mathcal{L}$.

In cases (a)(i), (b)(i), and (c)(i), let $U'(Z) = x$ and $\mathbb{IR}' = \mathbb{IR}$.

In case (a)(ii), apply Lemma 31 (in Section Appendix C.2) to find an extension $\mathbb{IR}'$ of $\mathbb{IR}$ and $x \in \mathbb{IR}'$ such that $u_1 < x < u_2$ for all $u_1 \in \mathcal{L}$ and all $u_2 \in \mathcal{U}$, and set $U'(Z) = x$.

In case (b)(ii), apply Lemma 31 to find an extension $\mathbb{IR}'$ of $\mathbb{IR}$ and $x \in \mathbb{IR}'$

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\textsuperscript{1}The claim that every value of $U(X)$ for $X \in \mathcal{Y}$ is in $\mathcal{L} \cup \mathcal{U}$ might seem to rule out cases (a)(i), (b)(i), and (c)(i) because all mixtures of the form $\alpha U(X^1) + [1 - \alpha U(X^2)]$ are in $\mathcal{L} \cup \mathcal{U}$ if $\alpha \in \mathbb{R}$. However, there may be nonstandard mixtures of elements of $\mathcal{L} \cup \mathcal{U}$ that are in $\mathbb{IR}$, but not in $\mathcal{L} \cup \mathcal{U}$. 

---
such that $x < U(X)$ for all $X \in \mathcal{U}$, and set $U'(Z) = x$.

In case (c)(ii), apply Lemma 31 to find an extension $^*\mathbb{R}'$ of $^*\mathbb{R}$ and $x \in ^*\mathbb{R}'$ such that $x > U(X)$ for all $X \in \mathcal{L}$, and set $U'(Z) = x$.

We must now show that (13) holds. Let $X, Y \in \mathcal{Z}$ be represented as in (A.7) by

$$
\begin{align*}
X &= q(X)Z + X', \\
Y &= q(Y)Z + Y',
\end{align*}
$$

with $X', Y' \in \mathcal{Y}$ and $q(X), q(Y) \in \mathbb{R}$. Then

$$
\begin{align*}
U'(X) &= xq(X) + U(X'), \\
U'(Y) &= xq(Y) + U(Y').
\end{align*}
$$

It suffices to show that

$$
[q(X) - q(Y)]Z \prec Y' - X' \text{ if and only if } [q(X) - q(Y)]x < U(Y' - X'). \quad (A.9)
$$

If $q(X) = q(Y)$, then (A.9) holds because $U$ is standard-linear and represents $\preceq$ on $\mathcal{Y}$. Otherwise, set

$$
W = \frac{Y' - X'}{q(X) - q(Y)}.
$$

For the “if” part of (A.9), assume that $[q(X) - q(Y)]x < U(Y' - X')$. If $q(X) > q(Y)$, then $x < U(W) \in \mathcal{U}$ and $Z < W$, which implies $[q(X) - q(Y)]Z < Y' - X'$. Similarly, if $q(X) < q(Y)$, then $x > U(W) \in \mathcal{L}$ and $W < Z$, which implies $[q(X) - q(Y)]Z < Y' - X'$.

For the “only if” part of (A.9), assume that $[q(X) - q(Y)]Z < Y' - X'$. If $q(X) > q(Y)$, then $Z < W$, and $x < U(W) \in \mathcal{U}$, which implies $[q(X) - q(Y)]x < U(Y' - X')$. Similarly, if $q(X) < q(Y)$, then $W < Z$, and $x > U(W) \in \mathcal{L}$, which implies $[q(X) - q(Y)]x < U(Y' - X')$.

### Appendix A.5. Proof of Theorem 3

Let $\preceq$ be the weak order that corresponds to $\mathcal{T}$. Let $\mathcal{X}_0$ be the singleton set $\{0\}$, where $0$ stands for the additive identity element of the linear space $\mathcal{X}$. Let $^*\mathbb{R}_0 = \mathbb{R}$. Define $U_0$ on $\mathcal{X}_0$ as $U_0(0) = 0$. This function is standard-linear on $\mathcal{X}_0$ and $^*\mathbb{R}_0$-valued, and it represents $\preceq$ on $\mathcal{X}_0$.

The proof for the whole space $\mathcal{X}$ will proceed by transfinite induction on $\mathcal{X} \setminus \mathcal{X}_0$. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a well-ordering of the elements of $\mathcal{X} \setminus \mathcal{X}_0$. 

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**Induction hypothesis:** Let $\lambda \in \Lambda$ be an ordinal. There is a linear space $\mathcal{X}_\lambda$ that contains $\{X_\gamma\}_{\gamma \leq \lambda}$, a nonstandard model $^{*}\mathbb{R}_\lambda$ of the reals, and a standard-linear function $U_\lambda : \mathcal{X}_\lambda \to ^{*}\mathbb{R}_\lambda$ that represents $\preceq$ on $\mathcal{X}_\lambda$.

First, note that the induction hypothesis holds for $\lambda = 0$. Next, we deal with an arbitrary successor ordinal $\gamma$. Assume that the induction hypothesis holds for $\lambda = \gamma - 1$. We must prove that the induction hypothesis holds for $\lambda = \gamma$. Apply Lemma 13 with $Y = X_{\gamma - 1}$, $Z = X_\gamma$, $U = U_{\gamma - 1}$, and $^{*}\mathbb{R} = ^{*}\mathbb{R}_{\gamma - 1}$. Then $\mathcal{X}_\gamma$ is the $Z$ in Lemma 13. Let $U_\gamma$ and $^{*}\mathbb{R}_\gamma$ be respectively the $U'$ and $^{*}\mathbb{R}'$ that result from Lemma 13.

Finally, we prove that the induction hypothesis holds for each limit ordinal $\lambda$. We start by creating objects to play the roles of $\mathcal{Y}$, $U$, and $^{*}\mathbb{R}$ in Lemma 13. Define

$$^{*}\mathbb{R}_{<\lambda} = \bigcup_{\gamma < \lambda}^{*}\mathbb{R}_\gamma,$$

$$\mathcal{X}_{<\lambda} = \{X_\gamma\}_{\gamma < \lambda} = \bigcup_{\gamma < \lambda} \mathcal{X}_\gamma.$$

For $X \in \mathcal{X}_{<\lambda}$, let $U_{<\lambda}(X) = U_\gamma(X)$, where $\gamma$ is the first ordinal such that $X \in \mathcal{X}_\gamma$. Each such $\gamma$ is strictly less than $\lambda$. This makes $U_{<\lambda} : \mathcal{X}_{<\lambda} \to ^{*}\mathbb{R}_{<\lambda}$. To see that $U_{<\lambda}$ is standard-linear and represents $\preceq$ on $\mathcal{X}_{<\lambda}$, let $X^1, X^2 \in \mathcal{X}_{<\lambda}$. Let $\gamma$ be the first ordinal for which both $X^1$ and $X^2$ are in $\mathcal{X}_\gamma$. Then $\gamma < \lambda$ and $U_{<\lambda}(X^j) = U_\gamma(X^j)$ for $j = 1, 2$. It follows that $X^1 \prec X^2$ if and only if $U_{\gamma}(X^1) < U_{\gamma}(X^2)$ if and only if $U_{<\lambda}(X^1) < U_{<\lambda}(X^2)$. Also, every linear combination $\alpha X^1 + \beta X^2$ is in $\mathcal{X}_\gamma$ with

$$U_{<\lambda}(\alpha X^1 + \beta X^2) = U_{\gamma}(\alpha X^1 + \beta X^2) = \alpha U_{\gamma}(X^1) + \beta U_{\gamma}(X^2) = \alpha U_{<\lambda}(X^1) + \beta U_{<\lambda}(X^2).$$

To complete the proof, apply Lemma 13 with $\mathcal{Y} = \mathcal{X}_{<\lambda}$, $^{*}\mathbb{R} = ^{*}\mathbb{R}_{<\lambda}$, $Z = X_\lambda$, and $U = U_{<\lambda}$.

**Appendix A.6. Proof of Theorem 4**

For a full trading system, recall that $X \preceq Y$ means $(X \sim Y) \lor (X \prec Y)$. For the “only if” direction, assume that $\mathcal{T}$ is a (coherent) partial trading system. Let $\mathcal{A}$ be the set of all (coherent) full trading systems $\mathcal{T}^* = (\sim_{\mathcal{T}^*}, \prec_{\mathcal{T}^*})$ that are extensions of $\mathcal{T}$ created by the method of Theorem 2. For each $\mathcal{T}^* \in \mathcal{A}$, let $\preceq_{\mathcal{T}^*}$ be the corresponding weak order. We will prove that
\( \mathcal{T} \) is the type-2 consensus relation \( \preceq_{\mathcal{A}} = (\sim_{\mathcal{A}}, \prec_{\mathcal{A}}, \ll_{\mathcal{A}}) \) of \( \{ \preceq_{\mathcal{T}^*: \mathcal{T}^* \in \mathcal{A}} \} \). By construction of the extensions, we know that \( X \sim Y \) implies \( X \sim_{\mathcal{A}} Y \), and \( X \prec Y \) implies \( X \prec_{\mathcal{A}} Y \). If \( X \ll Y \), then \( X \preceq_{\mathcal{T}^*} Y \) for every \( \mathcal{T}^* \in \mathcal{A} \), but there are some extensions for which \( X \sim_{\mathcal{T}^*} Y \) and others for which \( X \prec_{\mathcal{T}^*} Y \). Hence, neither \( X \sim_{\mathcal{A}} Y \) nor \( X \prec_{\mathcal{A}} Y \). It follows that \( X \ll_{\mathcal{A}} Y \). We need to prove the reverse implications for all three relations. Assume to the contrary, that there is an instance of \( X \sim_{\mathcal{A}} Y \) but none of \( X \sim Y \), \( X \prec Y \), or \( X \ll Y \) are true. There are three cases to consider: (i) \( Y \prec X \), (ii) \( Y \ll X \), or (iii) \( X \ll Y \). Case (i) contradicts \( X \sim_{\mathcal{A}} Y \) because the extensions all preserve strict preferences. In cases (ii) and (iii) consider well-orderings in the proof of Theorem 2 that start with \((X_1, Y_1) = (X, Y)\). It follows that all three of the relations (A.6) fail at \( \gamma = 0 \), and the choice \( Y \ll X \) is available for \( \mathcal{T}^1 \), which contradicts \( X \sim_{\mathcal{A}} Y \).

For the “if” direction, assume that there is a collection \( \mathcal{A} \) of (coherent) full trading systems whose type-2 consensus relation is \( \mathcal{T} = (\sim, \prec, \ll) \). For each \( \mathcal{T}^* = (\sim^*, \prec^*, \ll^*) \in \mathcal{A} \), let \( \sim^* \) be the corresponding weak order. Let \( A_{\sim^*} \), \( A_{\prec^*} \), \( A_{\ll^*} \), and \( A_{\ll} \) be the subsets of \( \mathcal{X}^2 \) that respectively define \( \preceq^*, \prec^*, \sim^* \), and \( \ll^* \) in the manner of Definition 21 in Section Appendix B.1. Then the sets

\[
A_\prec = \bigcap_{\sim^* \in \mathcal{A}} A_{\prec^*},
\]

\[
A_\sim = \bigcap_{\sim^* \in \mathcal{A}} A_{\sim^*},
\]

\[
A_\ll = \bigcap_{\ll^* \in \mathcal{A}} \left( A_{\ll^*} \setminus (A_\prec \cup A_\sim) \right),
\]

respectively define \( \prec \), \( \sim \), and \( \ll \). Both \( \sim \) and \( \prec \) are affine because the property of being affine is inherited by intersections of the corresponding subsets of \( \mathcal{X}^2 \). To see that \( \ll \) is affine, first note that \( \preceq^* \), \( \sim^* \), and \( \prec^* \) are affine. These facts entail that for \( \rho \) being either \( \prec \) or \( \sim \), we have \( X\rho Y \) if and only if \( \alpha X + Z\rho Y + Z \) for all \( \alpha > 0 \) and \( Z \in \mathcal{X} \).

Use Lemma 29 to show that \((\sim, \prec, \ll)\) is a partial trading system. (If each \( \mathcal{T}^* \) is a coherent full trading system, then each \( \prec^* \) extends dominance. It follows that \( \ll \) extends dominance. By Proposition 4, \( \mathcal{T} \) is coherent.)

**Appendix A.7. Proof of Lemma 22**

All \( j \) such that \( \alpha_j = 0 \) can be dropped from the sum in (18) without changing the value of that sum. Note also that \( P(-X|G) = -P(X|G) \) for
all $X \in \mathcal{O}$ and nonempty $G$. It follows that we can replace each term in (18) for which $\alpha_j < 0$ by $-\alpha_j E_j(-X_j + c_j)$ without changing the sum in (18). So, assume that $\alpha_j > 0$ for all $j$. For each $j$ such that $P(X_j|E_j)$ is infinite, $\alpha_j(X_j - c_j)$ is a preferred trade under $\succeq$. For each $j$ such that $U(X_j|E_j)/U(E_j)$ is standard, $c_j = U(X_j|E_j)/U(E_j)$ and $\alpha_j E_j(X_j - c_j)$ is fair under $\succeq$. For each $j$ such that $U(X_j) > P(X_j)$, $\alpha_j(X_j - c_j)$ is a preferred trade under $\succeq$. If each $j = 1, \ldots, n$ falls into one of the three categories above, then (18) holds because the sum is either a fair trade or a preferred trade.

For the remainder of the proof, assume that there is at least one $j$ such that $P(X_j)$ is finite and strictly greater than $U(X_j)$ (obviously by an infinitesimal amount). Let $m \geq 1$ be the number of such $j$, and assume that they are $j = 1, \ldots, m$ without loss of generality. Suppose, to the contrary, that (18) fails. Then, there is $d > 0$ such that the left-hand side of (18) is less than $-2d$. We will find a contradiction by showing that a preferred trade under $\succeq$ is dominated by 0. For each $j \in \{1, \ldots, m\}$, let $X'_j = X_j + d/(m\alpha_j)$. For $j > m$, let $X'_j = X_j$. Then for each $j$, $\alpha_j E_j(X'_j - c_j)$ is either a fair trade or a preferred trade. Also,

$$\sum_{j=1}^{n} \alpha_j E_j[X'_j(\omega) - c_j] = \sum_{j=1}^{n} \alpha_j E_j[X_j(\omega) - c_j] + d < -d,$$

for all $\omega \in E$. Since $\sum_{j=1}^{n} \alpha_j(X'_j - c_j)$ is a preferred trade, this contradicts coherence.

Appendix B. Binary Relations

It is common to represent preferences amongst elements of a set of options by means of an order relation. In general, order relations on a set are binary relations that satisfy certain properties.

Appendix B.1. Definitions

There are two common ways to express binary relations.

Definition 21. A binary relation $R$ on a set $\mathcal{Y}$ is a subset $A_R$ of $\mathcal{Y}^2$, where $(x, y) \in A_R$ is denoted $xRy$. If $A_R = \emptyset$, the relation $R$ is called empty.

Example 1. If $\mathcal{Y} = \mathbb{R}$, we can let $R$ be $<$ so that $A_< = \{(x, y) : x < y\}$. Another example is $R$ being $=$, so that $A_\equiv = \{(x, y) : x = y\}$. 

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There are a few properties that we use to define the various types of binary relations that we use in this paper.

**Definition 22.** A binary relation $R$ on a set $\mathcal{Y}$ is called **transitive** if, for all $x, y, z \in \mathcal{Y}$, $(xRy) \land (yRz)$ implies $xRz$.

A binary relation $R$ on a set $\mathcal{Y}$ is called **reflexive** if, for all $x \in \mathcal{Y}$, $xRx$.

A binary relation $R$ on a set $\mathcal{Y}$ is called **symmetric** if, for all $x, y \in \mathcal{Y}$, $xRy$ implies $yRx$.

A binary relation $R$ on a set $\mathcal{Y}$ is called **asymmetric** if, for all $x, y \in \mathcal{Y}$, $xRy$ implies $\neg(yRx)$.

Next, we define the major types of binary relations that we will use.

**Definition 23.** A binary relation $\preceq$ on a set $\mathcal{Y}$ is a **preorder** if $\preceq$ is reflexive, and transitive.

If $\preceq$ is a preorder that also satisfies

$$
\forall x, y \in \mathcal{Y}, (x \preceq y) \lor (y \preceq x),
$$

then we call $\preceq$ a **weak order**.

A binary relation $\sim$ on $\mathcal{Y}$ is an **equivalence relation** if $\sim$ is reflexive, symmetric, and transitive.

Let $\sim$ be an equivalence relation on $\mathcal{Y}$. For each $x \in \mathcal{Y}$, we call $[x] = \{y \in \mathcal{Y} : x \sim y\}$ an **equivalence class**. We use the symbol $\mathcal{Y}/\sim$ to denote the set of equivalence classes.

A binary relation $\prec$ on a set $\mathcal{Y}$ is called a **strict partial order** if $\prec$ is asymmetric and transitive.

Let $R$ be a binary relation and let $\sim$ be an equivalence relation on the same set $\mathcal{Y}$. We say that $R$ is defined on the equivalence classes of $\sim$ if, for all $x, y \in \mathcal{Y},$

$$xRy \text{ if and only if } x'Ry' \text{ for all } x' \in [x] \text{ and all } y' \in [y]. \quad (B.1)$$

Let $\prec$ be a strict partial order, and let $\triangleleft$ be an asymmetric relation. We say that $\prec$ and $\triangleleft$ are **codirectional** if they satisfy the following:

- If $x \triangleleft y$ and $y \triangleleft z$, then either $x \triangleleft z$ or $x \prec z$.
- If $x \prec y$ and $y \prec z$, then $x \prec z$.
- If $x \prec y$ and $y \triangleleft z$, then $x \prec z$. 

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The definition of “defined on the equivalence classes” might seem counterintuitive. Intuition would suggest that such a binary relation should be a relation between the equivalence classes themselves. Lemma 25 below will show that the intuition is equivalent to the definition.

Example 2. For each of the three dominance concepts in Definition 7, there is a binary relation \( \prec \) defined by \( Y \prec X \) if and only if \( X \) dominates \( Y \). Each of the three binary relations defined this way can easily be seen to be a strict partial order.

Appendix B.2. Properties of Binary Relations

Some results are easier to state and/or prove in terms of the functions in a set \( \mathcal{X} \), while others are more naturally stated in terms of the equivalence classes that arise from an equivalence relation on \( \mathcal{X} \). There are a number of equivalent ways to express order relations, and the following general results establish these equivalences, allowing us to choose the most appropriate setup for stating or proving each of our main results later.

Lemma 25. Let \( \sim \) be an equivalence relation on a set \( \mathcal{Y} \). A binary relation \( R \) on \( \mathcal{Y} \) is defined on the equivalence classes of \( \sim \) if and only if there is a binary relation \( R' \) on \( \mathcal{C} = \mathcal{Y} / \sim \) such that

\[ xRy \text{ if and only if } [x]R'[y]. \]  

(B.2)

Proof. For the “only if” direction, suppose that \( R \) is defined on the equivalence classes of \( \sim \). Define \( R' \) on \( \mathcal{C} \) by (B.2). Then (B.1) ensures that \( R' \) is well defined.

For the “if” direction, suppose that \( R' \) is a binary relation on \( \mathcal{C} \) such that (B.2) holds. Let \( x' \in [x] \) and \( y' \in [y] \). Then \( [x'] = [x] \) and \( [y'] = [y] \), so \( x'R'y' \), and \( R \) satisfies (B.1). \( \square \)

In this paper, whenever we refer to a binary relation \( R \) on a set \( \mathcal{Y} \) that is defined on the equivalence classes of an equivalence relation \( \sim \), we will use the same symbol \( R \) to stand for the relation on \( \mathcal{Y} \) and the relation on \( \mathcal{Y} / \sim \).

Equivalence relations and partitions are intimately related.

Lemma 26. A collection \( \mathcal{C} = \{C_\beta\}_{\beta \in \mathbb{N}} \) of subsets of a set \( \mathcal{Y} \) is a partition of \( \mathcal{Y} \) if and only if there is an equivalence relation \( \sim \) on \( \mathcal{Y} \) such that \( \mathcal{C} = \mathcal{Y} / \sim \).
Proof. For the “if” direction, assume that ∼ is an equivalence relation whose equivalence classes are \( C \). For each \( x \in Y \), \( x \in [x] \) and \([x]\) is one of the \( C_\beta \), so every element of \( Y \) is in at least one equivalence class. All that remains of the “if” direction is to show that the equivalence classes are disjoint. If \( y \in [x] \), then \( x \sim y \) and \( x \in [y] \). By transitivity of \( \sim \), it follows that \([x] = [y] \). If \( y \in [x] \cap [z] \), then \([x] = [y] = [z] \), and the equivalence classes are disjoint.

For the “only if” direction, assume that \( C \) is a partition of \( Y \), and define \( \sim \) by

\[
x \sim y \text{ if and only if } y \in [x].
\]

That \( \sim \) is reflexive and symmetric are trivial. To see that \( \sim \) is transitive, let \( x \sim y \) and \( y \sim z \). Then \( x, y, z \) are all in the same \( C_\beta \), hence \( x \sim z \).

Here are some simple examples of partitions.

Example 3. For an arbitrary set \( Y \), the finest partition is the collection of singleton sets, \( C = \{\{y\}, y \in Y\} \). The coarsest partition is the collection \( C = \{Y\} \) consisting of only the whole set \( Y \). It is easy to see that every binary relation on \( Y \) is defined on the equivalence classes of the finest partition.

Lemma 27. Let \( \sim \) be an equivalence relation on \( Y \). Let \( \prec \) and \( \triangleleft \) be respectively a strict partial order on \( Y \) and an asymmetric relation on \( Y \) that are codirectional and defined on the equivalence classes of \( \sim \). Define \( x \preceq y \) by \((x \sim y) \lor (x \prec y) \lor (x \triangleleft y)\). Then \( \preceq \) is a preorder that is defined on the equivalence classes of \( \sim \).

Proof. That \( x \preceq x \) follows from \( x \sim x \). Suppose that \( x \preceq y \) and \( y \preceq z \). We need to show that \( x \preceq z \). If either \( x \sim y \) or \( y \sim z \) (or both,) then \( x \preceq z \) follows either by transitivity of \( \sim \) or by the fact that \( \prec \) and \( \triangleleft \) are defined on the equivalence classes of \( \sim \). If \((x \prec y) \land (y \triangleleft z)\) or \((x \triangleleft y) \land (y \prec z)\) or \((x \prec y) \land (y \triangleleft z)\), then \( x \preceq z \) follows by codirectionality of \( \prec \) and \( \triangleleft \). Finally, if \( x \prec y \) and \( y \prec z \), then \( x \preceq z \) follows by transitivity of \( \prec \). That \( \preceq \) is defined on the equivalence classes of \( \sim \), follows easily from the fact that both \( \prec \) and \( \triangleleft \) are defined on the equivalence classes of \( \sim \).

Lemma 28. Let \( \preceq \) be a preorder on \( Y \). Define two new relations on \( Y \) by

\[
x \sim y \text{ if and only if } (x \preceq y) \land (y \preceq x),
\]

\[
x \prec y \text{ if and only if } (x \preceq y) \land (\neg[y \preceq x]).
\]

Then \( \sim \) is an equivalence relation and \( \prec \) is a strict partial order that is defined on the equivalence classes of \( \sim \). Also, \( x \preceq y \) if and only if \((x \prec y) \lor (x \sim y)\).
Proof. That $\sim$ is reflexive and symmetric are straightforward. That $\prec$ is asymmetric is built into its definition, and transitivity follows from transitivity of $\preceq$. To see that $\prec$ is defined on the equivalence classes of $\sim$, let $x, y \in Y$. We need to prove (B.1). For the “if” direction of (B.1), let $x' \sim x$ and $y' \sim y$ be such that $x' \prec y'$. Then $x \preceq x' \preceq y' \preceq y$, so $x \preceq y$. To see that $\neg(y \preceq x)$, assume to the contrary that $y \preceq x$. Then $y' \preceq y \preceq x \preceq x'$, which implies $y' \preceq x'$, a contradiction to $x' \prec y'$. The “only if” direction of (B.1), is immediate.

For the “if” direction of the final claim, assume that $(x \prec y) \lor (x \sim y)$. If $x \prec y$, then $x \preceq y$. If $x \sim y$, then $x \preceq y$. The “only if” direction is immediate from the definitions of $\sim$ and $\prec$.

Corollary 1. Assume the conditions of Lemma 27. Let $\preceq$ be the preorder that results from Lemma 27. The strict partial order that results from Lemma 28 is not $\prec$ from Lemma 27, but rather the following: $x \prec_\ast y$ if and only if $(x \prec y) \lor (x \sim y)$.

Appendix B.3. Consensus

There is a natural concept of consensus amongst a collection of binary relations.

Definition 24. Let $\mathcal{R} = \{R_\alpha\}_{\alpha \in \mathbb{N}}$ be a set of binary relations on a set $Y$ with $R_\alpha$ corresponding to the subset $A_{R_\alpha}$ of $Y^2$. Let $A = \bigcap_{\alpha \in \mathbb{N}} A_{R_\alpha}$. We call the relation that corresponds to $A$ the simple consensus relation of $\mathcal{R}$.

For a collection $\mathcal{R} = \{\preceq_\alpha\}_{\alpha \in \mathbb{N}}$ of weak orders, there are two ways to construct consensus binary relations from $\mathcal{R}$.

Definition 25. Let $\mathcal{R} = \{\sim_\alpha\}_{\alpha \in \mathbb{N}}$ be a collection of weak orders on a set $Y$. The simple consensus relation of $\mathcal{R}$ is also called the type-1 consensus relation. The type-2 consensus relation $(\sim, \prec, \triangleleft)$ is defined as follows. First, let $A_1$ be the subset of $Y^2$ that corresponds to the type-1 consensus relation. Second, for each $\alpha \in \mathbb{N}$, let $\sim_\alpha$ and $\prec_\alpha$ be respectively the equivalence relation and strict partial order that correspond to $\preceq_\alpha$ by Lemma 28. Third, define $\mathcal{R}_\sim = \{\sim_\alpha\}_{\alpha \in \mathbb{N}}$ and $\mathcal{R}_\prec = \{\prec_\alpha\}_{\alpha \in \mathbb{N}}$. Fourth, let $\sim$ and $\prec$ be respectively the simple consensus relations of $\mathcal{R}_\sim$ and $\mathcal{R}_\prec$ with corresponding subsets $A_\sim$ and $A_\prec$ of $Y^2$. Finally, let $A_\triangleleft = A_1 \setminus (A_\sim \cup A_\prec)$ and define $\triangleleft$ to be the binary relation that corresponds to $A_\triangleleft$.

It is straightforward to show that the type-1 consensus relation of a collection of weak orders is a preorder and is not empty. It is also straightforward
to show that the set $A_1$ in Definition 25 contains the set $A_\sim \cup A_\prec$. We do not use the type-1 consensus in this paper. Instead, we use the type-2 consensus because it better matches the way that we define our preference relations. The following result helps to see why this is true.

**Lemma 29.** Let $\mathcal{R} = \{\leq_\alpha\}_{\alpha \in \mathbb{R}}$ be a set of weak orders on a set $\mathcal{Y}$. Let $(\sim, \prec, \triangleleft)$ be the type-2 consensus as in Definition 25. Then $\sim$ is an equivalence relation, $\prec$ is a strict partial order, and $\triangleleft$ is asymmetric. Also, $\prec$ and $\triangleleft$ are codirectional and are defined on the equivalence classes of $\sim$.

**Proof.** Because $(x, x) \in A_\sim$ for all $x \in \mathcal{Y}$, $\sim$ is not empty. If $\prec$ is empty, then $\prec$ trivially a strict partial order that is defined on the equivalence classes of $\sim$. Similarly, if $\triangleleft$ is empty, then $\triangleleft$ is trivially asymmetric, defined on the equivalence classes of $\sim$ and codirectional with $\prec$. For the rest of the proof, assume that $\prec$ and $\triangleleft$ are not empty.

First, show that $\sim$ is an equivalence relation. Reflexivity holds because $(x, x) \in A_\sim$ for all $x \in \mathcal{Y}$, and $\sim$ is symmetric. Similarly, if $\prec$ is empty, then $\prec$ trivially a strict partial order that is defined on the equivalence classes of $\sim$. Similarly, if $\triangleleft$ is empty, then $\triangleleft$ is trivially asymmetric, defined on the equivalence classes of $\sim$ and codirectional with $\prec$. For the rest of the proof, assume that $\sim$ and $\triangleleft$ are not empty.

Second, show that $\prec$ is a strict partial order. If $(x, y) \in A_\prec$, then $(x, y) \in A_\prec \alpha$ for all $\alpha$, so no $A_\prec \alpha$ contains $(y, x)$ and neither does $A_\prec$. This shows that $\prec$ is asymmetric. If $(x, y), (y, z) \in A_\prec$, then $(x, z) \in A_\prec \alpha$ for all $\alpha$ and $(x, z) \in A_\prec$ so $\sim$ is transitive.

Third, show that $\prec$ is defined on the equivalence classes of $\sim$. Let $x, y \in \mathcal{Y}$. For the “if” direction of (B.1), let $x' \sim x$ and $y' \sim y$ be such that $x' \prec y'$. Then $x' \prec_\alpha y'$, $x \leq_\alpha x'$, and $y' \leq_\alpha y$ for all $\alpha$. So, for all $\alpha$,

$$x \leq_\alpha x' \leq_\alpha y' \leq_\alpha y,$$

so $x \leq y$. To see that $\neg (y \leq x)$, assume to the contrary that $y \leq x$. Then $y \leq_\alpha x$ for all $\alpha$ and

$$y' \leq_\alpha y \leq_\alpha x \leq_\alpha x',$$

for all $\alpha$, which implies $y' \leq x'$, a contradiction to $x' \prec y'$. The “only if” direction of (B.1) is immediate.

Fourth, show that $\triangleleft$ is asymmetric and codirectional with $\prec$. If $(x, y) \in A_\triangleleft$, we need to show that $(y, x) \notin A_\triangleleft$. Assume, to the contrary, that $(y, x) \in A_\triangleleft$. Then

$$\{(x, y), (y, x)\} \subseteq A_\triangleleft \cup A_\sim \cup A_\prec,$$
for all $\alpha$. This implies that $x \sim_\alpha y$ for all $\alpha$, and $(x, y) \in A_\sim$, which contradicts $(x, y) \in A_\triangleleft$. This shows that $\triangleleft$ is asymmetric. To see that $\triangleleft$ and $\prec$ are codirectional, first, assume that $x \prec y$ and $y \prec z$. Then $x \preceq_\alpha z$ for all $\alpha$. It can't be that $x \sim_\alpha y \sim_\alpha z$ for all $\alpha$. So either $x \prec_\alpha z$ for all $\alpha$ (in which case $x \prec z$) or $x \sim z$. Second, assume that $x \prec y$ and $y \prec z$. Then $x \prec_\alpha z$ for all $\alpha$ and $x \prec z$. The proof when $x \prec y$ and $y \prec z$ is similar. So, $\triangleleft$ and $\prec$ are codirectional.

Finally, show that $\triangleleft$ is defined on the equivalence classes of $\sim$. Let $x, y \in Y$. For the “if” direction of (B.1), let $x' \sim x$ and $y' \sim y$ be such that $x' \triangleleft y'$. Then, as in the similar proof for $\prec$, we have, $x \preceq y$. Also, $\neg(x' \sim y')$, and $\neg(x \sim y)$ because $\sim$ is an equivalence relation. In addition $\neg(x' \prec y')$. Since $\prec$ is defined on the equivalence classes of $\sim$, it must be that $\neg(x \prec y)$, so $x \triangleleft y$. The “only if” direction of (B.1) is immediate.

The following example shows how the type-1 and type-2 consensus relations can differ.

**Example 4.** Let $Y = \mathbb{R}^2$. For each $p \in [0, 1/2]$, define the weak order $\preceq_p$ by

$$(x_1, x_2) \preceq_p (y_1, y_2) \text{ if and only if } p(y_1 - x_1) + (1 - p)(y_2 - x_2) \geq 0.$$ 

Let $\preceq_1$ and $\preceq_2$ be respectively the type-1 and type-2 consensus relations. Because $1 - 2p \geq 0$ for all $p \in [0, 1/2]$, we have $(1, 0) \preceq_1 (0, 1)$. Because $1 - 2p > 0$ for $p < 1/2$, it is false that $(1, 0) \sim_2 (0, 1)$. Because $1 - 2(1/2) = 0$, it is false that $(1, 0) \prec_2 (0, 1)$. Hence it is false that $(1, 0) \preceq_2 (0, 1)$.

One of our results (Theorem 4) says that a preference relation satisfies the conditions that we define in Section 2 if and only if it is a type-2 consensus relation based on a collection of weak orders that also satisfy the conditions defined in Section 2.

**Appendix C. Overview of Nonstandard Models**

This section is intended to give only examples and an intuitive overview of the concept of nonstandard models of the reals. Those needing a more thorough understanding should read one of the many treatments such as [1].

A nonstandard model of the reals is an embedding of the real numbers $\mathbb{R}$ into a larger set $\star\mathbb{R}$ that preserves many of the familiar properties of the reals (e.g., being a linearly ordered algebraic field) while introducing others that are convenient for certain analyses (e.g., “infinite” numbers that obey
the usual rules of arithmetic). In particular, a nonstandard model \( \mathcal{F} \) of the reals should be non-Archimedean in one of the many equivalent senses, such as the following: There exist \( x, y \in \mathcal{F} \) with \( x < y \) such that \( nx < y \) for every standard integer \( n \). For those with more knowledge of nonstandard models, our analysis will be entirely external. The formal meaning of “external” is not important here, but it includes the ability to refer to subsets of the standard reals (the familiar \( \mathbb{R} \)) as subsets of \( {}^*\mathbb{R} \). The cost of an external analysis includes, among other things, the inability to carry theorems and proofs back and forth between the standard and nonstandard models. The external approach also requires us to distinguish between standard and nonstandard notions of finite, infinite, and countable. The main thing that we gain from the external approach is the non-Archimedean nature of \( {}^*\mathbb{R} \) as opposed to \( \mathbb{R} \). One manifestation of a non-Archimedean property is the non-existence of suprema and/or infima for certain bounded external subsets of \( {}^*\mathbb{R} \).

We will describe a popular class of nonstandard models of the reals known as ultraproducts. They rely on the concept of ultrafilter.

Appendix C.1. Ultrafilters

Definition 26. Let \( \mathcal{Z} \) be a set. A nonempty subset \( \mathcal{U} \) of \( 2^{\mathcal{Z}} \) is called an ultrafilter on \( \mathcal{Z} \) if it has the following properties:

- \((A \in \mathcal{U}) \land (A \subseteq B) \) implies \( B \in \mathcal{U} \).
- \((A \in \mathcal{U}) \land (B \in \mathcal{U}) \) implies \( A \cap B \in \mathcal{U} \).
- For each \( A \subseteq \mathcal{Z} \) either \( A \in \mathcal{U} \) or \( A^C \in \mathcal{U} \) but not both.

The simplest example of an ultrafilter is to let \( z_0 \in \mathcal{Z} \) and define

\[
\mathcal{U} = \{ A \subseteq \mathcal{Z} : z_0 \in A \}.
\]

Such an ultrafilter is called principal. All other ultrafilters are called non-principal.

Ultrafilters on \( \mathcal{Z} \) are equivalent to \( 0 - 1 \)-valued probabilities on \( \mathcal{Z} \). It is straightforward to show that a (possibly finitely-additive) probability \( P \) defined on \( 2^{\mathcal{Z}} \) takes only the values 0 and 1 if and only if \( \{ A \subseteq \mathcal{Z} : P(A) = 1 \} \) is an ultrafilter. Principal ultrafilters correspond to countably-additive probabilities, while non-principal ultrafilters correspond to merely finitely-additive probabilities. The existence of ultrafilters for general sets \( \mathcal{Z} \) depends on the axiom of choice. Theorem 7.1 of [2] theorem gives a simple condition
that insures that a subset of $2^\mathbb{Z}$ can be extended to an ultrafilter. That theorem relies on the following concept.

**Definition 27.** A nonempty collection $\mathcal{F}$ of subsets of a set $\mathbb{Z}$ has the *finite intersection property* if the intersection of every nonempty finite collection of elements of $\mathcal{F}$ is nonempty.

Theorem 7.1 of [2] then says that if $\mathcal{F}$ has the finite intersection property, then there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$.

**Example 5.** Let $\mathbb{Z} = \mathbb{Z}_+$, the positive integers. Let $\mathcal{F}$ be the collection of all subsets of the form \{n, n + 1, n + 2, \ldots\}. There are many ultrafilters that contain $\mathcal{F}$. All such ultrafilters are non-principal, and they all correspond to merely finitely-additive probabilities. The reason is that, if $\mathcal{U} \supseteq \mathcal{F}$ and $n_0 \in \mathbb{Z}_+$, \{n_0, n_0 + 1, \ldots\} $\in \mathcal{U}$, so $P(\{n_0\}) = 1$.

**Appendix C.2. Ultraproducts**

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{Z}$. Denote elements of $\mathbb{R}^\mathbb{Z}$ as $x = (x_1, x_2, \ldots)$. Define a binary relation of $\mathbb{R}^\mathbb{Z}$ by

$$x _\mathcal{U} y \text{ if and only if } \{n : x_n = y_n\} \in \mathcal{U}.$$ 

It is easy to see that $_\mathcal{U}$ is an equivalence relation because the intersection of finitely many elements of $\mathcal{U}$ is in $\mathcal{U}$. Let $*_\mathbb{R}_\mathcal{U} = \mathbb{R}^{\mathbb{Z}} / _\mathcal{U}$, the set of equivalence classes corresponding to $_\mathcal{U}$. The embedding of $\mathbb{R}$ into $*_\mathbb{R}_\mathcal{U}$ is merely $f_\mathcal{U}(a) = [(a, a, a, \ldots)]_\mathcal{U}$, for each $a \in \mathbb{R}$. The extension of $\leq$ to $*_\mathbb{R}$ is $[x]_\mathcal{U} \leq [y]_\mathcal{U}$ if $\{n : x_n \leq y_n\} \in \mathcal{U}$. This $\leq$ is a linear order on $*_\mathbb{R}$.

**Example 6.** Let $*_\mathbb{R}_\mathcal{U}$ be an ultraproduct nonstandard model as above. Let $x = (1, 1/2, 1/3, \ldots)$, i.e, $x_n = 1/n$ for each standard integer $n$. It is easy to check that $[x]_\mathcal{U} < [f_\mathcal{U}(y)]_\mathcal{U}$ for every strictly positive standard real $y$. Just note that each set of the form $\{n : x_n < y\}$ has the form $\{m, m+1, m+2, \ldots\}$ where $m$ is the first integer such that $m > 1/y$. Although it is an abuse of notation, it should cause no confusion to use the same symbol to denote a real number and its image under $f_\mathcal{U}$. Similarly, we can refer to the equivalence class that contains a sequence $x \in \mathbb{R}^\mathbb{Z}$ by the symbol $x$ when it is clear that we mean that $x \in *_\mathbb{R}$.

A real-valued function $g$ on $\mathbb{R}$ can be extended to $*_\mathbb{R}$ by

$$g([x]_\mathcal{U}) = [(g(x_1), g(x_2), \ldots)]_\mathcal{U}.$$
If the domain of $g$ is a proper subset $A$ of $\mathbb{R}$, the domain of the extension of $g$ to $\mathbb{R}^*$ is $\{[x]_U : \text{all coordinates of } x \text{ are in } A\}$.

A nonstandard $z$ such that $|z| < y$ for every positive standard real $y$ is called *infinitesimal*. A nonstandard $z$ such that $|z| > y$ for every positive standard real $y$ is called *externally infinite*. A nonstandard that is not externally infinite is called *externally finite*. The infinitesimals and standard reals are externally finite, as are hybrid nonstandards such as $1 + x$, where $x$ is the infinitesimal defined in the previous paragraph.

**Lemma 30.** Each externally finite nonstandard $x$ has a nearest standard real $\mathbb{R}(x)$.

**Proof.** If $x$ is itself standard real, no other standard real is closer to $x$, so $\mathbb{R}(x) = x$. For the remainder of the proof, assume that $x$ is not itself standard real. Let $L = \{z \in \mathbb{R} : z < x\}$ and $U = \{z \in \mathbb{R} : x < z\}$. Then both $L$ and $U$ are nonempty, $L \cap U = \emptyset$, and $L \cup U = \mathbb{R}$. Also, each element of $U$ is an upper bound for $L$ and each element of $L$ is a lower bound for $U$. Since both $L$ and $U$ are sets of standard reals, $\text{sup } L = \text{inf } U$, and we call the common value $x'$, which is standard real. We now show that $\mathbb{R}(x) = x'$ is the nearest standard real to $x$. Let $z < x'$ so that $z < (z + x')/2 < x'$. Since $(z + x')/2$ is closer to $x$ than $z$, no standard real less than $x'$ is closer to $x$. Similarly, let $y > x'$ so that $(y + x')/2$ is closer to $x$ than $y$, so no $y < x'$ is closer to $x$. \(\square\)

We call $\mathbb{R}(x)$ the *standard part* of $x$. Note that $x$ and $x + z$ have the same standard part if and only if $x - z$ is infinitesimal. For convenience, we say that the standard part of an externally infinite $x$ is infinite and express the fact as $\mathbb{R}(x) = \infty$ or $\mathbb{R}(x) = -\infty$ as appropriate.

An example of a bounded external set of nonstandards that has no supremum or infimum is the following.

**Example 7.** Let $x_0$ be a standard real, and let $A = \{x : \mathbb{R}(x) = x_0\}$. It is clear that, for every $x \in A$, $x + z \in A$ for every positive infinitesimal $z$, so no element of $A$ is an upper bound for $A$. Similarly, no element of $A$ is a lower bound for $A$. Hence, every upper bound $y$ for $A$ has standard part $\mathbb{R}(y) > x_0$. For every such $y$, $(x_0 + \mathbb{R}(y))/2$ is an upper bound for $A$ that is smaller than $y$. Similarly, for every lower bound $z$ for $A$, $(x_0 + \mathbb{R}(z))/2$ is a larger lower bound for $A$. Hence, $A$ has neither a greatest lower bound nor a least upper bound.
The externally finite nonstandards have upper and lower bounds that are externally infinite, but there is neither a least upper bound nor a greatest lower bound.

The following lemma is used at key steps in the proof of our representation either when a supremum/infimum would be useful but doesn’t exist or when an existing supremum/infimum would be inappropriate to represent a preference.

**Lemma 31.** Let $\mathcal{F}$ be a standard or nonstandard model of the reals. Let $B_1$ and $B_2$ be (possibly external) subsets of $\mathcal{F}$ such that, $b_1 < b_2$ for all $b_1 \in B_1$ and all $b_2 \in B_2$. Let $\mathcal{N}$ be the positive integers in $\mathcal{F}$. Let $\mathcal{U}$ be a non-principal ultrafilter of subsets of $\mathcal{N}$, and let $\ast\mathcal{F}$ be the corresponding ultraproduct nonstandard model. Then

(i) $\mathcal{F}$ is naturally embedded in $\ast\mathcal{F}$, and

(ii) there exist elements $x-, x, x+ \in \ast\mathcal{F}$ such that $x- < b_1 < x < b_2 < x+$ for all $b_1 \in B_1$ and all $b_2 \in B_2$.

**Proof.** Claim (i) is straightforward. For claim (ii), let $\{a_{n,1}\}_{n \in \mathcal{N}}$ be elements of $B_1$ such that $a_{n+1,1} \geq a_{n,1}$ for all $n$ and, for every $b_1 \in B_1$, there is $n \in \mathcal{N}$ with $b_1 \leq a_{n,1}$. Also let $\{a_{n,2}\}_{n \in \mathcal{N}}$ be elements of $B_2$ such that $a_{n+1,2} \leq a_{n,2}$ for all $n$ and, for every $b_2 \in B_2$, there is $n \in \mathcal{N}$ with $b_2 \geq a_{n,2}$.

Let $a_n = (a_{n,1} + a_{n,2})/2x \in \ast\mathcal{F}$ be the equivalence class containing $\{a_n\}_{n \in \mathcal{N}}$. For each $b_1 \in B_1$, $b_1 < a_n$ for all but finitely many $n$, so $b_1 < x$ for all $b_1 \in B_1$. Similarly, for each $b_2 \in B_2$, $b_2 > a_n$ for all but finitely many $n$, so $x < b_2$ for all $b_2 \in B_2$. Let $x+ \in \ast\mathcal{F}$ be the equivalence class containing $\{n\}_{n \in \mathcal{N}}$.

For each element $b$ of $\mathcal{F}$, there is $n \in \mathcal{F}$ such that $b < n$, so $b < x$ for all $b \in B$. Similarly, let $x-$ be the equivalence class containing $\{-n\}_{n \in \mathcal{N}}$. For each element $b$ of $\mathcal{F}$, there is $n \in \mathcal{F}$ such that $b < -n$, so $b < x-$ for all $b \in B$.

The reader is reminded that, in the statement of Lemma 31, the set $\mathcal{N}$ of integers is the set of integers in the model $\mathcal{F}$ which might itself be nonstandard. Hence, $\mathcal{N}$ might be much larger than the set of standard integers.

**Appendix C.3. Nonstandard Models of the Reals**

In this paper, when we refer to a nonstandard model of the reals, we will refer to what results from applying Lemma 31 successively an arbitrary number of times, where the first application takes $\mathcal{F} = \mathbb{R}$.
The standard real numbers $\mathbb{R}$ are a linearly ordered algebraic field, and each application of Lemma 31 produces a $^*\mathcal{F}$ which is also a linearly ordered algebraic field that contains the previous $\mathcal{F}$ from which it was constructed. However, each resulting $^*\mathcal{F}$ has non-Archimedean structure.

To be specific, let $\Gamma$ be an ordinal. Let $\mathcal{F}_0 = \mathbb{R}$. For each successor $\gamma \leq \Gamma$, let $\mathcal{F}_\gamma$ be the result of applying Lemma 31 to $\mathcal{F}_{\gamma - 1}$. For each limit $\gamma \leq \Gamma$ (if any), let $\mathcal{F}_\gamma = \bigcup_{\delta < \gamma} \mathcal{F}_\delta$. It is straightforward to show that $\mathcal{F}_\gamma$ is a linearly ordered algebraic field when $\gamma$ is a limit ordinal.

Appendix C.4. Some Notes About Infinity

The use of the symbol $\infty$ to stand in for “larger than every standard number” has a long history, and rarely causes trouble when discussing standard reals. Certain conventions allow some arithmetic with $\infty$. For example,

- for all finite $x$, $\infty + x = \infty$, and
- for all finite, non-zero $x$, $x\infty$ equals $\pm\infty$, with the sign matching that of $x$.

However, there is no place for standard infinity in a nonstandard model of the reals. Externally infinite, but internally finite, nonstandards replace standard infinity, and they require no special conventions to allow internally finitary arithmetic. Whenever we need to represent something in a nonstandard model that is larger than every number in the model, we will appeal to Lemma 31 which essentially iterates the ultraproduction construction to produce a larger nonstandard model that contains internally finite numbers to represent what we need. Of course, the symbol $\infty$ could be introduced to stand for “larger than every number in the nonstandard model.”

Appendix C.5. Countable Additivity

Nonstandard models of the reals are designed to have all of the finitary properties of the reals along with non-Archimedean structure. One must be careful not to expect everything that one knows about infinite sets of standard reals to apply to nonstandards. We already saw examples of bounded sets with no suprema or infima. (See Example 7.) A related property is that externally countable sums do not behave the same in nonstandard arithmetic as they do in standard arithmetic. For example, when the sequence of finite partial sums of a standard positive countable sequence is bounded, one can define the sum of the entire sequence to be the supremum of the partial sums. The same is not always possible for nonstandards.

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Example 8. Let $\epsilon > 0$ be infinitesimal, and let $x_n = \epsilon/2^n$ for each standard positive integer $n$. One might think that that $\sum_{n=1}^{\infty} x_n = \epsilon$. However, we show next that, despite the fact that the sequence of finite partial sums is bounded, there is no least upper bound. Clearly, each finite partial sum $y_m = \sum_{n=1}^{m} x_n < \epsilon$ for every standard positive integer $m$, so that $\epsilon$ is an upper bound on the finite partial sums. Let $w$ be an arbitrary finite upper bound on the finite partial sums. We will show that there is a smaller number than $w$ that is also an upper bound. Let $z = \epsilon^2$ so that $z < x_n$ for every standard positive integer $n$. It follows that, for each standard positive integer $m$,

\[
\begin{align*}
y_m + x_{m+1} &< w, \\
y_m + z &< w, \\
y_m &< w - z.
\end{align*}
\]

Hence $w - z$ is also a smaller upper bound on the sequence of finite partial sums.

Another, even less intuitive, example helps to explain why some common nonstandard-valued probabilities are not countably additive.

Example 9. Let $\epsilon > 0$ be infinitesimal, and let $\Omega$ be a countable set. We can assign each element of $\Omega$ probability $\epsilon$, but there is no way to choose $\epsilon$ to make the probability countably additive. To see this, note that sum of $\epsilon$ once for each element of $\Omega$ (if such a sum exists) must be larger than $n\epsilon$ for every standard integer $n$. First, note that $\sqrt{\epsilon} > n\epsilon$ for all $n$ because $1/\sqrt{\epsilon}$ is larger than every finite standard. So the sequence of finite partial sums of the probabilities is bounded above by an infinitesimal. Next, we show that the sequence of finite partial sums has no least upper bound. Let $\gamma$ be an upper bound. Then $\gamma > n\epsilon$ for every $n$, so $\gamma > 2n\epsilon$ for every $n$ and $\gamma/2$ is also an upper bound. But $\gamma/2 < \gamma$.

The remainder of this section is devoted to exploring the extent to which one can make sense of the sum of externally countably many elements of a nonstandard model of the reals. The short answer is “hardly ever.”

First, we note that the sequence of finite partial sums $\{y_m = \sum_{k=1}^{m} x_k\}_{m=1}^{\infty}$ of an externally countable sequence $\{x_k\}_{k=1}^{\infty}$ in a nonstandard model of the reals converges to an element $x$ of the same model if and only if $\{|y_m - x|\}_{m=1}^{\infty}$ converges to 0. So, our results will be stated in terms of externally countable sequences converging to 0. Since we are not interested in whether constant
sequences converge (they do), we restrict attention to sequences in which every element is strictly positive. To say that a strictly positive externally countable sequence \( \{w_k\}_{k=1}^{\infty} \) converges to 0 means that, for every \( \epsilon > 0 \) in the model, there is a standard integer \( N_\epsilon \) such that for all \( k > N_\epsilon \), \( w_k < \epsilon \). To show that such a sequence fails to converge, we need to find an \( \epsilon > 0 \) in the model such that \( w_k > \epsilon \) for all \( k \).

The first step is the following, which applies only to the first ultraproduct model constructed at the start of Section Appendix C.2.

**Lemma 32.** Let \( F = IR \) in Lemma 31, and let \( \ast F \) be the resulting ultraproduct model. No externally countable sequence of strictly positive elements of \( \ast F \) converges to 0.

**Proof.** Let \( \{t_k\}_{k=1}^{\infty} \) be an externally countable sequence of strictly positive elements of \( \ast F \). Represent each \( t_k = \left[ (y_{k,1}, y_{k,2}, \ldots) \right]_U \), where \( U \) is the appropriate ultrafilter. Since each \( t_k > 0 \), we can assume without loss of generality that \( y_{k,n} > 0 \) for all \( k, n \). Construct another strictly positive nonstandard \( u = \left[ (v_1, v_2, \ldots) \right]_U \) as follows. Let \( 0 < v_1 < y_{1,1} \), and for each \( n > 1 \), let \( 0 < v_n < \min\{y_{1,n}, \ldots, y_{n,n}\} \). For each \( k \),

\[
\{n : 0 < v_n < y_{k,n}\} = \{k, k+1, \ldots\} \in U.
\]

Hence, \( 0 < u < t_k \) for all \( k \). \( \square \)

The next step is to extend the conclusion of Lemma 32 to some further applications of Lemma 31.

**Lemma 33.** Let \( F \) be a nonstandard model of the reals such that no externally countable sequence of strictly positive elements of \( F \) converges to 0. Apply Lemma 31 to create \( \ast F \). Then no externally countable sequence of strictly positive elements of \( \ast F \) converges to 0.

**Proof.** Let \( \mathcal{N} \) be the integers in \( F \), and let \( \{x_k\}_{k=1}^{\infty} \) be an externally countable sequence of strictly positive elements of \( \ast F \). We will find an \( \epsilon > 0 \) in \( \ast F \) such that \( x_k > \epsilon \) for all \( k \). Represent each \( x_k \) as the equivalence class that contains \( \{y_{k,n}\}_{n \in \mathcal{N}} \). For each \( n \in \mathcal{N} \), \( \{y_{k,n}\}_{k=1}^{\infty} \) is an externally countable sequence of strictly positive elements of \( F \), hence there exists \( \epsilon_n > 0 \) in \( F \) such that \( y_{k,n} > \epsilon_n \) for all \( k \). Define \( \epsilon \) to be the equivalence class that contains \( \{\epsilon_n\}_{n \in \mathcal{N}} \), an element of \( \ast F \). By construction, \( x_k > \epsilon \) for all \( k \). \( \square \)
The next natural step would be to show a similar result for the union of countably many applications of Lemma 31. Oddly enough, something different occurs.

**Lemma 34.** Let $F_0$ be a standard or nonstandard model of the reals. For each standard integer $n = 1, 2, \ldots$, let $F_n$ be the result of applying Lemma 31 to $F = F_{n-1}$. Let $F_\infty = \bigcup_{n=0}^{\infty} F_n$. There exist externally countable sequences of strictly positive elements of $F_\infty$ that converge to 0.

*Proof.* For each $n = 1, 2, \ldots$, let $x_n \in F_n$ be a strictly positive infinitesimal that is introduced in the $n$th application of Lemma 31. That is, $0 < x_n < x$ for all $x > 0$ in $F_{n-1}$. To see that $\{x_n\}_{n=1}^{\infty}$ converges to 0 in $F_\infty$, let $\epsilon > 0$ be in $F_\infty$. Then $\epsilon \in F_k$ for some $k \in \{0, 1, \ldots\}$. For every $n > k$, $x_n < \epsilon$. \qed

We end with another negative result.

**Lemma 35.** Let $F$ be a nonstandard model of the reals, and let $\ast F$ be the result of applying Lemma 31 to $F$. If $\{x_n\}_{n=1}^{\infty}$ is an externally countable sequence of elements of $F$ that converges to 0 in $F$, then it does not converge to an element of $\ast F$.

*Proof.* Since the new infinitesimals in $\ast F$ are strictly between 0 and every positive element of $F$ (including every $x_n$), the sequence cannot converge to 0 or to a negative value. Also, for every new infinitesimal $\epsilon \in \ast F$, $2\epsilon$ is strictly between $\epsilon$ and every $x_n$, so the sequence cannot converge to a new infinitesimal. Finally, let $x \in \ast F$ be strictly greater than every new infinitesimal. Then $x \geq y$ for some $y \in F$. Since the sequence converges to 0 in $F$, it eventually gets strictly below $y/2$, hence it cannot converge $x > y/2$. \qed

Recall that convergence of an externally countable sum to an element $x$ of a nonstandard model of the reals requires convergence to 0 of the difference between $x$ and the finite partial sums. Since convergence to 0 of a countable sequence in nonstandard models of the reals is such an ethereal concept, we will assume only that probabilities are finitely additive. Furthermore, the non-Archimedean nature of the preferences that we model leads to utilities being nonstandard-valued. Integrating a nonstandard-valued function with respect to countably-additive probability runs into the same problems we just exhibited for nonstandard-valued probabilities. The limits required to define the integral of a non-simple non-standard-valued function will not
exist in general. Hence there is nothing to be gained by assuming that the lottery distributions that go into the definition of horse-lottery are countably additive. The expectation functional with respect to a countably-additive probability, when applied to nonstandard-valued utility functions will not behave in a countably-additive fashion.

Appendix D. Closing Under the Assumptions

It is straightforward to close a subset $\mathcal{J}$ of $\mathcal{G}^2$ under Assumptions 1 and 2.

Proposition 1. • Let $\mathcal{J} \subseteq \mathcal{G}^2$. A set $\mathcal{J}'$ includes $\mathcal{J}$ and satisfies Assumption 1 if and only if $\mathcal{J} \cup \{(X, X) : X \in \mathcal{G}\} \subseteq \mathcal{J}'$.

• Let $\mathcal{J} \subseteq \mathcal{G}^2$ satisfy Assumption 1. Let $\mathcal{J}'$ be the set of all pairs of convex combinations of the form (2) for all finite $n$ and all $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathcal{J}$. Then $\mathcal{J}'$ includes $\mathcal{J}$ and satisfies Assumptions 1 and 2. Also, every subset of $\mathcal{G}^2$ that includes $\mathcal{J}$ and satisfies Assumptions 1 and 2 includes $\mathcal{J}'$ as a subset.

Appendix D.1. Assumption 3

Closing a set $\mathcal{J}$ under Assumption 3 proceeds differently in the lottery and random-variable cases. The lottery case makes use of the finitely-additive Jordan decomposition of signed measures.

Proposition 2 (Dunford and Schwartz, 1957, Theorem III.1.8, p. 98). Let $\mu$ be a bounded finitely-additive signed measure on a field $\Sigma$ of subsets of a set $\mathcal{P}$. For each $E \in \Sigma$, define

$$
\mu^+(E) = \sup_{F \subseteq E} \mu(F),
$$

$$
\mu^-(E) = -\inf_{F \subseteq E} \mu(F),
$$

where the sup and inf are over elements of $\Sigma$ that are subsets of $E$. Then $\mu^+$ and $\mu^-$ are finitely-additive measures on $\Sigma$ and $\mu(E) = \mu^+(E) - \mu^-(E)$ for all $E \in \Sigma$.

The key to closing a subset $\mathcal{J}$ of $\mathcal{H}^2$ under Assumption 3 is being able to recognize which elements $(X, Y)$ of $\mathcal{H}^2$ can be expressed as (D.1) below.
Lemma 36. Let $\mathcal{H}$ be a set of horse lotteries, and let $\mathcal{J} \subseteq \mathcal{H}^2$ satisfy Assumptions 1 and 2. For each $(X,Y) \in \mathcal{J}$ and each $\omega \in \Omega$, let $\mu_{X,Y,\omega}$ denote the bounded finitely-additive signed measure $X(\omega) - Y(\omega)$, and define

$$
\beta_{X,Y,\omega} = \mu^{+}_{X,Y,\omega}(\mathcal{P}),
\alpha_{X,Y} = \sup_{\omega} \beta_{X,Y,\omega}.
$$

Then there exist $\alpha \in (0,1)$ and $X', Y', Z \in \mathcal{R}^{\Omega}$ such that

$$
(X,Y) = (\alpha X' + [1 - \alpha] Z, \alpha Y' + [1 - \alpha] Z), \tag{D.1}
$$

if and only if $\alpha_{X,Y} < 1$.

Proof. First, notice that $\mu_{X,Y,\omega}(\mathcal{P}) = 0$ for all $\omega$ so that

$$
\mu^{+}_{X,Y,\omega}(\mathcal{P}) = \mu^{-}_{X,Y,\omega}(\mathcal{P}),
$$

for all $\omega$. Also, by Proposition 2,

$$
X(\omega) - \mu^{+}_{X,Y,\omega} = Y(\omega) - \mu^{-}_{X,Y,\omega}, \tag{D.2}
$$

is a finitely-additive measure (non-negative) that gives measure $1 - \beta_{X,Y,\omega}$ to $\mathcal{P}$.

For the “only if” part, assume that there exist $\alpha \in (0,1)$ and $X', Y', Z \in \mathcal{R}^{\Omega}$ such that (D.1) holds. It follows that $\beta_{X,Y,\omega} \leq \alpha$ for all $\omega$, and $\alpha_{X,Y} \leq \alpha < 1$.

For the “if” part, assume that $\alpha_{X,Y} < 1$, and let $\alpha \in [\alpha_{X,Y}, 1)$. Define

$$
Z(\omega) = \frac{1}{1 - \beta_{X,Y,\omega}} [X(\omega) - \mu^{+}_{X,Y,\omega}],
$$

$$
X'(\omega) = \frac{1}{\alpha} [X(\omega) - (1 - \alpha) Z(\omega)],
$$

$$
Y'(\omega) = \frac{1}{\alpha} [Y(\omega) - (1 - \alpha) Z(\omega)].
$$

From (D.2), it follows that $Z \in \mathcal{R}^{\Omega}$. Because $1 - \alpha \leq 1 - \beta_{X,Y,\omega}$ for all $\omega$, both $X'$ and $Y'$ are in $\mathcal{R}^{\Omega}$. From (D.2) we also see that (D.1) holds. \qed

The $X',Y',Z$ that are constructed in the proof of Lemma 36 are just one of many triples that could satisfy (D.1), and there is no guarantee that any of the three are in $\mathcal{H}$. With that in mind, let $X,Y \in \mathcal{H}$ and define

$$
\mathcal{H}_{X,Y} = \{ (X',Y',Z,\alpha) \in (\mathcal{R}^{\Omega})^3 \times [\alpha_{X,Y}, 1] : \text{ such that } X = \alpha X' + (1 - \alpha) Z, Y = \alpha Y' + (1 - \alpha) Z \}.
$$

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Because we allow $\alpha = 1$, $(X, Y, Z, 1) \in \mathcal{H}_{X,Y}$ for all $X, Y, Z \in \mathcal{H}$. Lemma 36 tells us that $\mathcal{H}_{X,Y}$ contains 4-tuples with $\alpha < 1$ if and only if $\alpha_{X,Y} < 1$. We can think of $\mathcal{H}_{X,Y}$ as the set of \textit{unmixings} of $(X, Y)$. That is, each element of $\mathcal{H}_{X,Y}$ has the components that appear on the right-hand side of (D.1) when representing $X$ and $Y$ as the type of mixture that appears in Assumption 3.

The following set collects the first two coordinates of those elements of the $\mathcal{H}_{X,Y}$ sets that involve only elements of $\mathcal{H}$. These are the unmixings that need to be added to $\mathcal{J}$ in order to satisfy Assumption 3.

$$\mathcal{J}^* = \{(X', Y') : \exists (X, Y) \in \mathcal{J}, Z \in \mathcal{H}, \alpha \in [\alpha_{X,Y}, 1] \text{ such that } (X', Y', Z, \alpha) \in \mathcal{H}_{X,Y}\}.$$ (D.3)

Note that $\mathcal{J} \subseteq \mathcal{J}^*$ because $\alpha = 1$ is allowed in (D.3). The following result is useful in closing a set $\mathcal{J}$ under Assumption 3.

\textbf{Lemma 37.} If $\mathcal{J}$ satisfies Assumptions 1 and 2, then $\mathcal{J}^*$ is a mixture set.

\textit{Proof.} The mixing in $\mathcal{J}^*$ is as follows. If $(X'_1, Y'_1), (X'_2, Y'_2) \in \mathcal{J}^*$ and $\alpha \in [0, 1]$, then

$$\alpha(X'_1, Y'_1) + (1 - \alpha)(X'_2, Y'_2) = (\alpha X'_1 + (1 - \alpha)X'_2, \alpha Y'_1 + (1 - \alpha)Y'_2).$$ (D.4)

We need to show that the right-hand side of (D.4) is in $\mathcal{J}^*$. If $\alpha \in \{0, 1\}$, the result is trivial, so assume that $0 < \alpha < 1$. For $j = 1, 2$, there exist $(X_j, Y_j) \in \mathcal{J}, Z_j \in \mathcal{H},$ and $\alpha_j \in [\alpha_{X_j,Y_j}, 1]$ such that

$$(X'_j, Y'_j, Z_j, \alpha_j) \in \mathcal{H}_{X_j,Y_j.}$$

It follows that the right-hand side of (D.4) can be written as

$$\begin{align*}
\alpha\alpha_1 X_1 + (1 - \alpha)\alpha_2 X_2 + \alpha[1 - \alpha_1]Z_1 + [1 - \alpha][1 - \alpha_2]Z_2, \\
\alpha\alpha_1 Y_1 + [1 - \alpha]\alpha_2 Y_2 + \alpha[1 - \alpha_1]Z_1 + [1 - \alpha][1 - \alpha_2]Z_2.
\end{align*}$$ (D.5)

Let

$$\begin{align*}
\beta &= \alpha\alpha_1 + (1 - \alpha)\alpha_2, \\
X' &= \frac{\alpha\alpha_1}{\beta} X_1 + \frac{(1 - \alpha)\alpha_2}{\beta} X_2, \\
Y' &= \frac{\alpha\alpha_1}{\beta} Y_1 + \frac{(1 - \alpha)\alpha_2}{\beta} Y_2, \\
Z &= \frac{\alpha(1 - \alpha_1)}{1 - \beta} Z_1 + \frac{(1 - \alpha)(1 - \alpha_2)}{1 - \beta} Z_2.
\end{align*}$$
Then $X', Y', Z \in \mathcal{H}$ and (D.5) equals $(\beta X' + [1 - \beta]Z, \beta Y' + [1 - \beta]Z)$. Lemma 36 implies that $\beta \geq \alpha X_1 + [1 - \alpha]X_2, \alpha Y_1 + [1 - \alpha]Y_2$, hence (D.5) is an element of $\mathcal{J}^*$.

Lemma 38. Let $\mathcal{J}$ be a set that satisfies Assumptions 1 and 2. Define

$$ \mathcal{J}' = \{(\alpha X + [1 - \alpha]Z, \alpha Y + [1 - \alpha]Z) : (X, Y) \in \mathcal{J}^*, Z \in \mathcal{H}, \alpha \in (0, 1]\}. $$

(D.6)

Then $\mathcal{J} \subseteq \mathcal{J}'$, and $\mathcal{J}'$ satisfies Assumptions 1–3.

Proof. Since $\alpha = 1$ is allowed in (D.6), we have $\mathcal{J}^* \subseteq \mathcal{J}'$, and we already knew that $\mathcal{J} \subseteq \mathcal{J}^*$. This implies that $\mathcal{J}'$ satisfies Assumption 3. Because $\mathcal{J} \subseteq \mathcal{J}'$, $\mathcal{J}'$ inherits Assumption 1 from $\mathcal{J}$. For Assumption 2, assume that $(X_1, Y_1), (X_2, Y_2) \in \mathcal{J}'$ and $\alpha \in (0, 1)$. We need to show that $(\alpha X_1 + [1 - \alpha]X_2, \alpha Y_1 + [1 - \alpha]Y_2) \in \mathcal{J}'$. From the definition of $\mathcal{J}'$, there exist $\beta_1, \beta_2 \in (0, 1], (X_1', Y_1'), (X_2', Y_2') \in \mathcal{J}^*$, and $Z_1, Z_2 \in \mathcal{H}$ such that $X_j = \beta_j X_j' + (1 - \beta_j)Z_j$ and $Y_j = \beta_j Y_j' + (1 - \beta_j)Z_j$, for $j = 1, 2$. It follows that

$$ \alpha X_1 + (1 - \alpha)X_2 = \gamma X^* + (1 - \gamma)Z^*, $$

$$ \alpha Y_1 + (1 - \alpha)Y_2 = \gamma Y^* + (1 - \gamma)Z^*, $$

where

$$ \gamma = \alpha \beta_1 + (1 - \alpha)\beta_2, $$

$$ X^* = \frac{\alpha \beta_1}{\gamma} X_1 + \frac{(1 - \alpha)\beta_2}{\gamma} X_2, $$

$$ Y^* = \frac{\alpha \beta_1}{\gamma} Y_1 + \frac{(1 - \alpha)\beta_2}{\gamma} Y_2, $$

$$ Z^* = \frac{\alpha (1 - \beta_1)}{1 - \gamma} Z_1 + \frac{(1 - \alpha)(1 - \beta_2)}{1 - \gamma} Z_2. $$

Since $Z_1, Z_2 \in \mathcal{H}$, we have $Z^* \in \mathcal{H}$. From Lemma 37, $(X^*, Y^*) \in \mathcal{J}^*$. It follows that $(\alpha X_1 + [1 - \alpha]X_2, \alpha Y_1 + [1 - \alpha]Y_2) \in \mathcal{J}'$. □

Appendix D.2. Coherence

The following result says that every coherent partial trading system can be extended so that its strict partial order extends dominance and its $\preceq$ relation extends $\leq$. All of the new acceptable trades are at least as large as existing acceptable trades, so an agent with a coherent partial trading system can comfortably accept every trade of $X$ for $Y$ so long as $X \leq Y$.
Lemma 39. Let $\mathcal{T} = (\mathcal{C}, \prec, \triangleleft)$ be an almost coherent partial trading system. There exists a coherent partial trading system $\mathcal{T}^\ast = (\mathcal{C}, \prec^\ast, \triangleleft^\ast)$ such that $\prec^\ast$ extends both $\prec$ and $\prec_{\text{dom}}$. Also $\triangleleft^\ast$ extends $\leq$. Finally, every element of $\mathcal{V}_{\mathcal{T}^\ast} \setminus \mathcal{V}_{\mathcal{T}}$ is at least large as some element of $\mathcal{V}_{\mathcal{T}}$.

Proof. Since $\mathcal{C}$ is the only partition in this proof, equivalence classes will not include a subscript to identify the corresponding equivalence relation. Define $[X] \triangleleft^\ast [Y]$ if there exists $Z \geq 0$ such that $X + Z \triangleleft [Y]$. Define $[X] \prec^\ast [Y]$ if $[X] \triangleleft^\ast [Y]$ and at least one of the following holds:

- $[X] \prec [Y]$,
- there is $Z$ such that $[0] \prec_{\text{dom}} [Z]$ and $X + Z \triangleleft [Y]$.

Define $[X] \triangleleft^\ast [Y]$ if

$$( [X] \triangleleft^\ast [Y] ) \land \neg \{ ([X] \prec^\ast [Y]) \lor ([X] = [Y]) \}.$$  

The following are straightforward from the definitions above:

- $\triangleleft^\ast$ is asymmetric.
- $\triangleleft^\ast$ extends $\prec$ and $\prec_{\text{dom}}$.
- $\triangleleft^\ast$ extends $\leq$ and $\leq$.
- If $[X] \triangleleft^\ast [Y]$, then either $[X] \prec [Y]$ or there is $Z \geq 0$ such that $0 \bowtie Z$ and $X + Z \triangleleft [Y]$.

First, we show that $\triangleleft^\ast$ is affine. Assume that $[X] \triangleleft^\ast [Y]$, $\alpha > 0$ and $W \in \mathcal{X}$. We need to show that

$$[\alpha X + W] \triangleleft^\ast [\alpha Y + W].$$  

(D.7)

Let $Z \geq 0$ be such that $X + Z \triangleleft [Y]$. Since $\leq$ is affine and $\alpha > 0$, we have $[\alpha X + \alpha Z + W] \leq [\alpha Y + W]$. Since $\alpha Z \geq 0$, (D.7) holds.

Next, we show that $\triangleleft^\ast$ is a preorder on $\mathcal{C}$. Clearly, $\leq$ is reflexive. To see that $\triangleleft^\ast$ is transitive, assume that $[U] \triangleleft^\ast [V]$ and $[V] \triangleleft^\ast [W]$. Then there exist $Z_1, Z_2 \geq 0$ such that $[U + Z_1] \triangleleft [V]$ and $[V + Z_2] \triangleleft [W]$. Since $\leq$ is affine, we have $[U + Z_1 + Z_2] \leq [V + Z_2]$. Since $\leq$ is transitive, we have $[U + Z_1 + Z_2] \leq [W]$. Since $Z_1 + Z_2 \geq 0$, we have $[U] \triangleleft^\ast [W]$.
Next, we show that \( \prec^* \) is affine. Assume that \([X] \prec^* [Y] \), \( \alpha > 0 \) and \( W \in \mathcal{X} \). We need to show that (D.7) holds. If \([X] \prec [Y] \), then holds (D.7) because \( \prec \) is affine. If there is \( Z \) such that \( 0 \prec_{\text{Dom}} Z \) and \([X + Z] \prec [Y]\), then \([\alpha X + \alpha Z + W] \prec [\alpha Y + W] \) because \( \prec \) is affine. Since \( 0 \prec_{\text{Dom}} \alpha Z \), (D.7) holds.

Next, we show that \( \prec^* \) is a strict partial order on \( \mathcal{C} \). To see that \( \prec^* \) is asymmetric, assume that \([X] \prec^* [Y] \). Assume, to the contrary, that \([Y] \prec^* [X] \). Both of these \( \prec^* \) relations coming from the first bullet in the definition of \( \prec^* \) contradicts that \( \prec \) is asymmetric. If \([X] \prec^* [Y] \) comes from the first bullet and \([Y] \prec^* [X] \) comes from the second bullet, then there is \( Z \) such that \( 0 \prec_{\text{Dom}} Z \) and \([Y + Z] \preceq [X]\). It follows that \([Y + Z] \prec [Y] \) and \([Y] \prec_{\text{Dom}} [Y + Z] \) a contradiction. A similar contradiction arises if \([X] \prec^* [Y] \) comes from the second bullet and \([Y] \prec^* [X] \) comes from the first bullet. If both relations come from the second bullet, then there are \( Z_1, Z_2 \) such \( 0 \prec_{\text{Dom}} Z_j \) for \( j = 1, 2 \), \([X + Z_1] \prec [Y] \) and \([Y + Z_2] \prec [X] \). It follows that \([X + Z_1 + Z_2] \prec [X] \prec_{\text{Dom}} [X + Z_1 + Z_2] \), a contradiction. To see that \( \prec^* \) is transitive, assume that \([X] \prec^* [Y] \) and \([Y] \prec^* [W] \). There are four cases depending on which bullet in the definition of \( \prec^* \) leads to each of the relations:

**Case 1:** \([X] \prec [Y] \) and \([Y] \prec [W] \). In this case \([X] \prec [W] \) because \( \prec \) is transitive, and \([X] \prec^* [W] \).

**Case 2:** \([X] \prec [Y] \) and there exists \( 0 \prec_{\text{Dom}} Z \) such that \([Y + Z] \prec [W] \). In this case \([X + Z] \prec [Y + Z] \) because \( \prec \) is affine. Then \([X + Z] \prec [W] \) because \( \prec \) is transitive. It follows that \([X] \prec^* [W] \) by the second bullet in the definition.

**Case 3:** There exists \( 0 \prec_{\text{Dom}} Z \) such that \([X + Z] \prec [Y] \), and \([Y] \prec [W] \). In this case \([Y + Z] \prec [W] \) because \( \prec \) is transitive, and \([X] \prec^* [W] \) by the second bullet in the definition.

**Case 4:** There exist \( 0 \prec_{\text{Dom}} Z_1 \) and \( 0 \prec_{\text{Dom}} Z_2 \) such that \([X + Z_1] \prec [Y] \), and \([Y + Z_2] \prec [W] \). In this case \([X + Z_1 + Z_2] \prec [Y + Z_2] \) because \( \prec \) is affine. It follows that \([X + Z_1 + Z_2] \prec [W] \) because \( \prec \) is transitive. Since \( 0 \prec_{\text{Dom}} Z_1 + Z_2 \), we have \([X] \prec^* [W] \) by the second bullet in the definition.

Next, we prove that every element of \( \mathcal{V}_T \setminus \mathcal{V}_T \) is at least as large as some element of \( \mathcal{V}_T \). Each \( W \in \mathcal{V}_T \setminus \mathcal{V}_T \) can be written as \( W' + D \) where \( W' \in \mathcal{V}_T \) and \( D = \sum_{j=1}^n \alpha_j (Y_j - X_j) \) for some finite \( n \), \( \alpha_1, \ldots, \alpha_n > 0 \) and \( (X_j, Y_j) \) for \( j = 1, \ldots, n \), where for each \( j \) there is \( Z_j \geq 0 \) such that \([Z_j] \leq [Y_j - X_j] \). In
Hence, \( W = W' + D' + \sum_{j=1}^{n} \alpha_j Z_j \) with \( W' + D' \in \mathcal{V}_T \) and \( \sum_{j=1}^{n} \alpha_j Z_j \geq 0 \).

Next, we prove that \( \triangleleft \) and \( \triangleleft^* \) are codirectional. For the first property, assume that \( [U] \triangleleft^* [V] \) and \( [V] \triangleleft^* [W] \). It follows that \( \neg(([U] = [W]) \). There are four cases.

**Case (a):** \([U] \triangleleft [V] \) and \([V] \triangleleft [W] \). Since \( \triangleleft \) and \( \triangleleft \) are codirectional, it follows that either \([U] \triangleleft [W] \) or \([U] \triangleleft [W] \). If \([U] \triangleleft [W] \), then either \([U] \triangleleft^* [V] \) or \([U] \triangleleft^* [W] \). If \([U] \triangleleft [W] \), then \([U] \triangleleft^* [W] \).

**Case (b):** \([U] \triangleleft [V] \) and there exists \( Z \geq 0 \) such that \([V + Z] \preceq [W] \). In this case, \([U + Z] \triangleleft [V + Z] \preceq [W] \). Since \( \triangleleft \) and \( \triangleleft \) are codirectional, we have either \([U + Z] \triangleleft [W] \) or \([U + Z] \triangleleft [W] \). Hence, either \([U] \triangleleft^* [W] \) or \([U] \triangleleft^* [W] \).

**Case (c):** There exist \( Z \geq 0 \) such that \([U + Z] \preceq [V] \) and \([V] \triangleleft [W] \). This case is similar to Case (b).

**Case (d):** There exist \( Z_1, Z_2 \geq 0 \) such that \([U + Z_1] \preceq [V] \) and \([V + Z_2] \preceq [W] \). In this case, \([U + Z_1 + Z_2] \preceq [V + Z_2] \preceq [W] \) and \([U + Z_1 + Z_2] \preceq [W] \) with \( Z_1 + Z_2 \geq 0 \). So \([U] \preceq^* [W] \), and either \([U] \triangleleft^* [W] \) or \([U] \triangleleft^* [W] \).

For the second property, assume that \([U] \triangleleft^* [V] \) and \([V] \triangleleft^* [W] \). Then, there are four cases.

**Case (i):** \([U] \triangleleft [V] \) and \([V] \triangleleft [W] \). In this case, \([U] \triangleleft [W] \) because \( \triangleleft \) extends \( \triangleleft \).

**Case (ii):** \([U] \triangleleft [V] \) and there exists \( Z \) such that \([0] \triangleleft_{\text{Dom}} [Z] \) and \([V + Z] \preceq [W] \). In this case, \([U + Z] \triangleleft [V + Z] \preceq [W] \), so \([U] \triangleleft^* [W] \).

**Case (iii):** There exist \( Z \) such that \([0] \triangleleft_{\text{Dom}} [Z] \) and \([U + Z] \preceq [V] \), and \([V] \triangleleft [W] \). This case is similar to Case (ii).

**Case (iv):** There exist \( Z_1, Z_2 \) such that \([0] \triangleleft_{\text{Dom}} [Z_j] \) for \( j = 1, 2 \), \([U + Z_1] \preceq [V] \), and \([V + Z_2] \preceq [W] \). In this case, \([U + Z_1 + Z_2] \preceq [V + Z_2] \preceq [W] \) with \([0] \triangleleft_{\text{Dom}} [Z_1 + Z_2] \). So \([U] \triangleleft^* [W] \).

The proof of the third property is similar to that of the second property.

**Appendix E. Technical Results About Linearizing a Mixture Set**

The following are useful results about some of the types of binary relations.
Lemma 40. Let $\prec$ be a transitive relation that satisfies the independence axiom on a set $\mathcal{X}$. Then $(x_1 \prec x_2) \land (x_3 \prec x_4)$ implies $\delta x_1 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_4$ for all $\delta \in (0, 1)$.

Proof. Let $\delta \in (0, 1)$. Apply the independence axiom to $x_1 \prec x_2$ by mixing $x_3$ into both sides with weight $1 - \delta$ to get

$$\delta x_1 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_3. \quad (E.1)$$

Apply the independence axiom to $x_3 \prec x_4$ by mixing $x_2$ into both sides with weight $\delta$ to get

$$\delta x_2 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_4. \quad (E.2)$$

Use transitivity of $\prec$ with (E.1) and (E.2) to get the desired result. \qed

Lemma 41. Let $\prec$ and $\triangleleft$ be codirectional relations that satisfy the independence axiom on a set $\mathcal{X}$ with $\prec$ a strict partial order and $\triangleleft$ asymmetric. Then, for each $\delta \in (0, 1)$, $(x_1 \triangleleft x_2) \land (x_3 \triangleleft x_4)$ implies either $\delta x_1 + (1 - \delta)x_3 \triangleleft \delta x_2 + (1 - \delta)x_4$ or $\delta x_1 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_4$.

Proof. Let $\delta \in (0, 1)$. Apply the independence axiom to $x_1 \triangleleft x_2$ by mixing $x_3$ into both sides with weight $1 - \delta$ to get

$$\delta x_1 + (1 - \delta)x_3 \triangleleft \delta x_2 + (1 - \delta)x_3. \quad (E.3)$$

Apply the independence axiom to $x_3 \triangleleft x_4$ by mixing $x_2$ into both sides with weight $\delta$ to get

$$\delta x_2 + (1 - \delta)x_3 \triangleleft \delta x_2 + (1 - \delta)x_4. \quad (E.4)$$

Use codirectionality with $\triangleleft$ with (E.3) and (E.4) to get the desired result. \qed

Lemma 42. Let $\prec$ and $\triangleleft$ be codirectional relations that satisfy the independence axiom on a set $\mathcal{X}$ with $\prec$ a strict partial order and $\triangleleft$ asymmetric. Then, for each $\delta \in (0, 1)$, $(x_1 \prec x_2) \land (x_3 \triangleleft x_4)$ implies $\delta x_1 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_4$.

Proof. Let $\delta \in (0, 1)$. Apply the independence axiom to $x_1 \prec x_2$ by mixing $x_3$ into both sides with weight $(1 - \delta)$ to get

$$\delta x_1 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_3. \quad (E.5)$$
Apply the independence axiom to $x_3 \prec x_4$ by mixing $x_2$ into both sides with weight $\delta$ to get

$$\delta x_2 + (1 - \delta)x_3 \prec \delta x_2 + (1 - \delta)x_4.$$  \hspace{1cm} (E.6)

Use codirectionality with $\prec$ with (E.5) and (E.6) to get the desired result.

As a special case, $\alpha(h_1 - h_2)\rho'0$ if and only if $h_1 \rho h_2$. Also, $k_1 \rho' k_2$ if and only if $-k_2 \rho' - k_1$.

We could also start with an affine relation $\rho'$ on $K_0$, and then derive a relation $\rho$ on $H$ so that $h_1 \rho h_2$ if and only if $\alpha(h_1 - h_2)\rho'0$ for all standard $\alpha > 0$. To be precise, if $\rho'$ is an affine relation on $K_0$, we can define $h_1 \rho h_2$ to mean that there is a standard $\alpha > 0$ such that $\alpha(h_1 - h_2)\rho'0$. Then $\rho$ is well-defined and satisfies the independence axiom. To see that $\rho$ is well-defined, note that $\rho'$ being affine implies that $\alpha(h_1 - h_2)\rho'0$ for one $\alpha > 0$ if and only if $\alpha(h_1 - h_2)\rho'0$ for all $\alpha > 0$. That $\rho$ satisfies the independence axiom is straightforward because of (4).

To summarize, if we start either with $\rho$ that satisfies the independence axiom on $H$ or $\rho'$ that is affine on $K_0$, we can derive the other, and the two relations satisfy the following: There is standard $\alpha > 0$ such that $\alpha(h_1 - h_2)\rho'0$ if and only if $h_1 \rho h_2$. We will apply this idea to three types of binary relations on $H$:

- an equivalence relation $\sim$ which tells us which horse lotteries the agent is willing to trade in both directions.
- a strict partial order $\prec$ which tells us which horse lotteries the agent is willing to trade in only one direction, and
- an asymmetric relation $\triangleleft$ (codirectional with $\prec$) which tells us which horse lotteries the agent holds is willing to trade in only one direction, but might be willing to trade in both directions upon further reflection.

We show next that $\rho$ and $\rho'$ inherit each other’s characteristics of reflexivity, symmetry, asymmetry, transitivity, and/or codirectionality.

**Reflexivity.** We have $h\rho h$ for all $h \in H$ if and only if $0\rho'0$ if and only if $k\rho'k$ for all $k \in K_0$.

**Symmetry.** We have for all $h_1, h_2 \in H$ $h_1 \rho h_2$ implies $h_2 \rho h_1$ if and only if for all $h_1, h_2 \in H$ $h_1 - h_2 \rho'0$ implies $h_2 - h_1 \rho'0$ if and only if for all $h_1, h_2 \in H$ and standard $\alpha > 0$ $\alpha(h_1 - h_2)\rho'0$ implies $\alpha(h_2 - h_1)\rho'0$ if and only if for all
$k \in \mathcal{K}$ $k \rho'0$ implies $-k\rho0$ if and only if for all $k_1, k_2 \in \mathcal{K}$ $k_1 - k_2 \rho0$ implies $k_2 - k_1 \rho0$ if and only if for all $k_1, k_2 \in \mathcal{K}$ $k_1 \rho' k_2$ implies $k_2 \rho' k_1$.

**Asymmetry.** We have for all $h_1, h_2 \in \mathcal{H}$ $h_1 \rho h_2$ implies $-(h_2 \rho h_1)$ if and only if for all $h_1, h_2 \in \mathcal{H}$ $h_1 - h_2 \rho0$ implies $-(h_2 - h_1 \rho0)$ if and only if for all $h_1, h_2 \in \mathcal{H}$ and standard $\alpha > 0$ $\alpha(h_1 - h_2)\rho0$ implies $-[\alpha(h_2 - h_1)\rho0]$ if and only if for all $k \in \mathcal{K}$ $k \rho0$ implies $-(-k\rho0)$ if and only if for all $k_1, k_2 \in \mathcal{K}$ $k_1 - k_2 \rho0$ implies $-(k_2 - k_1 \rho0)$ if and only if for all $k_1, k_2 \in \mathcal{K}$ $k_1 \rho' k_2$ implies $-(k_2 \rho' k_1)$.

**Transitivity.** For one direction, assume that $\rho'$ is transitive. Then, for all $k_1, k_2, k_3 \in \mathcal{K}$, $(k_1 \rho' k_2) \land (k_2 \rho' k_3)$ implies $k_1 \rho' k_3$. Let $h_1, h_2, h_3 \in \mathcal{H}$ be such that $(h_1 \rho h_2) \land (h_2 \rho h_3)$. We need to show that $h_1 \rho h_3$, i.e., $(h_1 - h_3)\rho0$. Let $k_1 = h_1 - h_2$, $k_2 = 0$, and $k_3 = h_3 - h_2$. Then $(k_1 \rho k_2) \land (k_2 \rho k_3)$, which implies $k_1 \rho' k_3$, i.e., $h_1 - h_2 \rho h_3 - h_2$ hence $h_1 - h_3 \rho0$ because $\rho'$ is affine. For the other direction, assume that $\rho$ is transitive. Then, for all $h_1, h_2, h_3 \in \mathcal{H}$, $(h_1 \rho h_2) \land (h_2 \rho h_3)$ implies $h_1 \rho h_3$. Let $k_1, k_2, k_3 \in \mathcal{K}$ be such that $(k_1 \rho k_2) \land (k_2 \rho k_3)$. We need to show that $k_1 \rho' k_3$. Express $k_j = \alpha_j(h_j - h'_j)$ with $\alpha_j > 0$ and $h_j, h'_j \in \mathcal{H}$ for $j = 1, 2, 3$. We have

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} h_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} h'_2 \quad \rho \quad \frac{\alpha_2}{\alpha_1 + \alpha_2} h_2 + \frac{\alpha_1}{\alpha_1 + \alpha_2} h'_1, \quad (E.7)$$

$$\frac{\alpha_1}{\alpha_2 + \alpha_3} h_2 + \frac{\alpha_3}{\alpha_2 + \alpha_3} h'_3 \quad \rho \quad \frac{\alpha_3}{\alpha_2 + \alpha_3} h_3 + \frac{\alpha_2}{\alpha_2 + \alpha_3} h'_2. \quad (E.8)$$

We need to show that

$$\frac{\alpha_1}{\alpha_1 + \alpha_3} h_1 + \frac{\alpha_3}{\alpha_1 + \alpha_3} h'_3 \quad \rho \quad \frac{\alpha_3}{\alpha_1 + \alpha_3} h_3 + \frac{\alpha_1}{\alpha_1 + \alpha_3} h'_1. \quad (E.9)$$

Apply Lemma 40 to mix the two sides of (E.8) into the corresponding sides of (E.7) with weights $\delta = (\alpha_1 + \alpha_2)/(\alpha_1 + 2\alpha_2 + \alpha_3)$ and $1 - \delta$. The result is

$$\frac{\alpha_1 h_1 + \alpha_2 h'_2}{\alpha_1 + 2\alpha_2 + \alpha_3} + \frac{\alpha_2 h_2 + \alpha_3 h'_3}{\alpha_1 + 2\alpha_2 + \alpha_3} \rho \frac{\alpha_2 h_2 + \alpha_1 h'_1}{\alpha_1 + 2\alpha_2 + \alpha_3} + \frac{\alpha_3 h_3 + \alpha_2 h'_2}{\alpha_1 + 2\alpha_2 + \alpha_3}.$$  

Apply the independence axiom to remove the common terms involving $h_2$ and $h'_2$, and the result is (E.9).

**Codirectionality.** For one direction, assume that $\prec$ and $\prec'$ are codirectional on $\mathcal{H}$ with $\prec$ a strict partial order and $\prec'$ asymmetric. Then, for all $h_1, h_2, h_3 \in \mathcal{H}$, we have all three of the following:

- $(h_1 \prec h_2) \land (h_2 \prec h_3)$ implies $(h_1 \prec h_3) \lor (h_1 \prec h_3), \quad (E.9)$
• \((h_1 \prec h_2) \land (h_2 \prec h_3)\) implies \(h_1 \prec h_3\), and

• \((h_1 \prec h_2) \land (h_2 \preceq h_3)\) implies \(h_1 \prec h_3\).

We need to prove the three analogous properties for \(\prec'\) and \(\preceq'\). Represent \(k_j = \alpha_j(h_j - h'_j)\) for \(j = 1, 2, 3\) as in the transitivity proof. First, assume that \((k_1 \prec' k_2) \land (k_2 \prec' k_3)\). Then, we have

\[
\frac{\alpha_1}{\alpha_1 + \alpha_2} h_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2} h'_2 < \frac{\alpha_2}{\alpha_1 + \alpha_2} h_2 + \frac{\alpha_1}{\alpha_1 + \alpha_2} h'_1, \quad \text{(E.10)}
\]

\[
\frac{\alpha_2}{\alpha_2 + \alpha_3} h_2 + \frac{\alpha_3}{\alpha_2 + \alpha_3} h'_3 < \frac{\alpha_3}{\alpha_2 + \alpha_3} h_3 + \frac{\alpha_2}{\alpha_2 + \alpha_3} h'_2. \quad \text{(E.11)}
\]

We need to prove that either

\[
\frac{\alpha_1}{\alpha_1 + \alpha_3} h_1 + \frac{\alpha_3}{\alpha_1 + \alpha_3} h'_3 < \frac{\alpha_3}{\alpha_1 + \alpha_3} h_3 + \frac{\alpha_1}{\alpha_1 + \alpha_3} h'_1, \quad \text{or}
\]

\[
\frac{\alpha_1}{\alpha_1 + \alpha_3} h_1 + \frac{\alpha_3}{\alpha_1 + \alpha_3} h'_3 > \frac{\alpha_3}{\alpha_1 + \alpha_3} h_3 + \frac{\alpha_1}{\alpha_1 + \alpha_3} h'_1.
\]

Apply Lemma 41 to mix the two sides of (E.11) into the corresponding sides of (E.10) with weights \(\delta = (\alpha_1 + \alpha_2)/(\alpha_1 + 2\alpha_2 + \alpha_3)\) and \(1 - \delta\). The result is either

\[
\frac{\alpha_1 h_1 + \alpha_2 h'_2}{\alpha_1 + 2\alpha_2 + \alpha_3} + \frac{\alpha_2 h_2 + \alpha_3 h'_3}{\alpha_1 + 2\alpha_2 + \alpha_3} < \frac{\alpha_2 h_2 + \alpha_1 h'_1}{\alpha_1 + 2\alpha_2 + \alpha_3} + \frac{\alpha_3 h_3 + \alpha_2 h'_2}{\alpha_1 + 2\alpha_2 + \alpha_3},
\]

or

\[
\frac{\alpha_1 h_1 + \alpha_2 h'_2}{\alpha_1 + 2\alpha_2 + \alpha_3} + \frac{\alpha_2 h_2 + \alpha_3 h'_3}{\alpha_1 + 2\alpha_2 + \alpha_3} > \frac{\alpha_2 h_2 + \alpha_1 h'_1}{\alpha_1 + 2\alpha_2 + \alpha_3} + \frac{\alpha_3 h_3 + \alpha_2 h'_2}{\alpha_1 + 2\alpha_2 + \alpha_3}.
\]

Apply the independence axiom to remove the common terms involving \(h_2\) and \(h'_2\) from both formulas above, and the result is the desired conclusion. Second, assume that \((k_1 \prec' k_2) \land (k_2 \preceq' k_3)\). Then, we have (E.10) and

\[
\frac{\alpha_2}{\alpha_2 + \alpha_3} h_2 + \frac{\alpha_3}{\alpha_2 + \alpha_3} h'_3 < \frac{\alpha_3}{\alpha_2 + \alpha_3} h_3 + \frac{\alpha_2}{\alpha_2 + \alpha_3} h'_2. \quad \text{(E.13)}
\]

We need to prove that

\[
\frac{\alpha_1}{\alpha_1 + \alpha_3} h_1 + \frac{\alpha_3}{\alpha_1 + \alpha_3} h'_3 < \frac{\alpha_3}{\alpha_1 + \alpha_3} h_3 + \frac{\alpha_1}{\alpha_1 + \alpha_3} h'_1.
\]

Apply Lemma 42 to mix the two sides of (E.13) into the corresponding sides of (E.10) with weights \(\delta = (\alpha_1 + \alpha_2)/(\alpha_1 + 2\alpha_2 + \alpha_3)\) and \(1 - \delta\). The
result is (E.12). Apply the independence axiom to remove the common terms involving $h_2$ and $h'_2$ from (E.12), and the result is the desired conclusion. Third, assume that $(k_1 \prec' k_2) \land (k_2 \prec' k_3)$. The proof is similar to the “Second” case.

For the other direction, assume that $\prec'$ and $\prec'$ are codirectional on $\mathcal{K}_0$ with $\prec'$ a strict partial order and $\prec'$ asymmetric. Then, for all $k_1, k_2, k_3 \in \mathcal{K}_0$, we have all three of the following:

- $(k_1 \prec' k_2) \land (k_2 \prec' k_3)$ implies $(k_1 \prec' k_3) \lor (k_1 \prec' k_3)$,
- $(k_1 \prec' k_2) \land (k_2 \prec' k_3)$ implies $k_1 \prec' k_3$, and
- $(k_1 \prec' k_2) \land (k_2 \prec' k_3)$ implies $k_1 \prec' k_3$.

We need to prove the three analogous properties for $\prec$ and $\prec$. First, assume $(h_1 \prec h_2) \land (h_2 \prec h_3)$. Then, we have $h_1 - h_2 \prec' 0$ and $0 \prec' (h_3 - h_2)$. It follows that $(h_1 - h_2 \prec' h_3 - h_2) \lor (h_1 - h_2 \prec' h_3 - h_2)$ which implies (by affine) that $(h_1 - h_3 \prec' 0) \lor (h_1 - h_3 \prec' 0)$ which implies $(h_1 \prec h_3) \lor (h_1 \prec h_3)$, the desired result. Second, assume $(h_1 \prec h_2) \land (h_2 \prec h_3)$. Then, we have $h_1 - h_2 \prec' 0$ and $0 \prec' (h_3 - h_2)$. It follows that $h_1 - h_2 \prec' h_3 - h_2$, which implies (by affine) that $h_1 - h_3 \prec' 0$, which implies $h_1 \prec h_3$, the desired result. Third, assume $(h_1 \prec h_2) \land (h_2 \prec h_3)$. Then, we have $h_1 - h_2 \prec' 0$ and $0 \prec' (h_3 - h_2)$. It follows that $h_1 - h_2 \prec' h_3 - h_2$, which implies $h_1 - h_3 \prec' 0$, which implies $h_1 \prec h_3$, the desired result.

References
