Finite Additivity, Complete Additivity, and the Comparative Principle

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Abstract

In the longstanding foundational debate whether to require that probability is countably additive, in addition to being finitely additive, those who resist the added condition raise two concerns that we take up in this paper.

- *Existence*: Settings where no countably additive probability exists though finitely additive probabilities do.
- *Complete Additivity*: Where reasons for countable additivity don't stop there. Those reasons entail complete additivity – the (measurable) union of probability 0 sets has probability 0, regardless the cardinality of that union. Then probability distributions are discrete, not continuous.

We use Easwaran's (2013) advocacy of the *Comparative* principle to illustrate these two concerns. Easwaran supports countable additivity, both for numerical probabilities and for finer, qualitative probabilities, by defending a condition he calls the *Comparative* principle [$\mathcal{C}$]. For numerical probabilities, principle $\mathcal{C}$ contrasts pairs, P_1 and P_2, defined over a common partition $\Pi = \{a_i: i \in I\}$ of measurable events. $\mathcal{C}$ requires that no P_1 may be pointwise dominated, i.e., no (finitely additive) probability P_2 exists such that for each $i \in I$, $P_2(a_i) > P_1(a_i)$. By design, the cardinality of Π is not limited in $\mathcal{C}$, which Easwaran asserts is important when arguing that the principle does not require more, or less, than that probability is countably additive. We agree that a numerical probability P satisfies principle $\mathcal{C}$ in all partitions just in case P is countably additive. However, we show that for numerical probabilities, by considering the size of the algebra of events to which probability is applied, principle $\mathcal{C}$ is subject to each of the two concerns, above.

Also, Easwaran considers principle $\mathcal{C}$ with non-numerical, qualitative probabilities, where a qualitative probability may be finer than an almost agreeing numerical probability P. A qualitative probability is regular if possible events are strictly more likely than impossible events. Easwaran motivates and illustrates regular qualitative probabilities using a continuous, almost agreeing quantitative probability that is uniform on the unit interval. We make explicit the conditions for applying principle $\mathcal{C}$ with qualitative probabilities and show that $\mathcal{C}$ restricts regular qualitative probabilities to those whose almost agreeing quantitative probabilities are completely additive. For instance, Easwaran's motivating example of a regular qualitative probability is precluded by principle $\mathcal{C}$.

1. Introduction and overview.

One of the continuing debates in the foundations of mathematical probability is whether to require that mathematical probability is countably additive.[1] That is, whether to require that the probability of a *countably* infinite union of pairwise disjoint events is the limit of the partial sums of the probabilities of the individual events. This is the conventional position, often taking Kolmogorov's (1956) work as the seminal theory – though Kolmogorov offers his equivalent version of countable additivity (his Axiom VI) merely as an "expedient" for working with infinite fields of events.

A rival theory, *finite additivity*, requires less: The probability of a finite union of pairwise disjoint events is the sum of the individual probabilities, with no further restrictions placed on the probability of an infinite union of events. (When a probability is finitely but not countably additive say it is *merely finitely additive*.) De Finetti's (1974) and Savage's (1954) theories offer well known examples of this position. Two reasons among others they give for not requiring countable additivity address *existence* and *complete additivity* of numerical probabilities.

Regarding existence, de Finetti (1974, p. 121) objects that countable additivity precludes a uniform probability distribution on a countable partition, but not on a finite partition and not on a continuum partition of the unit interval. By contrast, for each partition, regardless of its cardinality, there exists a finitely additivity probability that gives equal probability to each element of the partition. We extend this reasoning by investigating when, because of the algebraic structure of the space of possible events, imposing countable additivity rather than finite additivity precludes all probabilities. We refer to this as the issue of existence and argue it is a wider problem than is evidenced by de Finetti's example involving a uniform distribution on a denumerable partition.

[1] . In this paper we use the phrase "mathematical probability" and "numerical probability" to mean a probability function whose values are real-valued. This contrasts with a binary "qualitative probability relation" for which there may not exist an agreeing numerical probability.

Regarding *complete additivity*, a simple corollary to Ulam's (1930) theorem entails that, assuming the continuum is not greater than the least weakly inaccessible cardinal, each countably additive probability with sufficiently many measurable sets to create an Ulam-matrix is a discrete, completely additive probability.[2] No such restriction obtains with finite additivity where, e.g., a continuous countably additive probability on Borel measurable sets may be extended to a finitely additive probability on the powerset of those Borel sets. We use this idea to debate countable additivity versus finite additivity by investigating when imposing countable additivity brings with it added restrictions, e.g., that the (measurable) union of a set of events, each set having probability 0, has probability 0. We refer to this as the issue of complete additivity.

In his 2013 paper, *Why Countable Additivity?*, K. Easwaran's presents two defenses of countable additivity. He writes,

> I give two arguments that probability functions must satisfy countable additivity, which don't generalize to support full additivity. (p. 53)

and

> … my purpose in this article is to show that countable additivity is not merely an arbitrary stopping point on the way to full additivity. (p. 54)

We contest these assertions.

The first argument is a variant of de Finetti's *Book* where, contrary to de Finetti's account, infinitely many favorable bets may be placed in one round of the game. We consider this reason for countable additivity in section 2. We argue that this variant of the Book argument entails complete additivity when applied to de Finetti's original theory of <u>fair</u> bets, or when applied to <u>strictly favorable</u>, called-off bets for a countably additive probability that is not completely additive.

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[2] Let κ be an uncountable cardinal. A countably additive probability P is κ^+-additive if the measurable union of κ or fewer disjoint sets of probability 0 has probability 0. A countably additive probability P is completely additive if it is κ^+-additive for each uncountable cardinal κ.

The second reason Easwaran offers for countable additivity is based on the *Comparative* principle [$\mathscr{C}$], which is the central topic of this paper.

> *The Comparative principle* (with italics added): If Π is a partition of the sure event for two probability functions P_1 and P_2, then it is not the case that for every member *a* of Π, *a* is *strictly more likely* under P_2 than *a* is under P_1.

Easwaran credits Pruss (2014) for motivation. Easwaran argues that the Comparative principle is equivalent to countable additivity for numerical probabilities defined on a field of sets, and does not require stronger additivity conditions.

In section 3, we explore the issues of existence and complete additivity for principle $\mathscr{C}$ when applied to numerical probabilities. Concerning existence, in section 3.2, we adapt a result of M. Amer (1985b, Theorem 5) to show that if A is an infinite, free Boolean algebra then no numerical probability on A satisfies principle $\mathscr{C}$ in all denumerable partitions.[3] However, there exist purely finitely additive probabilities – an extreme form of merely finitely additive probabilities – that are defined for A. An example of such a Boolean algebra A for which no numerical probability satisfies principle $\mathscr{C}$ is the *Lindenbaum-Tarski* algebra for sentential logic. Thus, the existence problem is wider than de Finetti's concern about a uniform probability distribution on a countable partition.

We investigate the issue of complete additivity for numerical probabilities in section 3.3. Consider an uncountable state space, $\kappa = |\Omega| \geq \aleph_1$. When the field B of measurable events is sufficiently large to include the κ-many sets in an Ulam-matrix, then the only numerical probabilities that satisfy principle $\mathscr{C}$ are discrete (completely additive) distributions. Principle $\mathscr{C}$ evaluates a probability P *separately* for each partition Π and, so, appears to place no demands on P beyond finite additivity in each uncountable partition. That appearance is deceptive. When the sets in the Ulam-matrix are all measurable, however, there are further combinatorial constraints on the additivity of a countably additive probability that arise from satisfying principle $\mathscr{C}$, *simultaneously*, in infinitely many uncountable partitions. These constraints require that the countably additive

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[3] A Boolean algebra is *free* provided that is has a set of "independent" generators. In the *Lindenbaum-Tarski* algebra for sentential logic, the denumerable set of atomic sentence letters serve as such a set of generators.

probability P is completely additive. This restriction does not obtain for merely finitely additive probabilities defined on the same measurable space.

Easwaran also entertains non-numerical, qualitative versions of principle $\mathcal{C}$. A qualitative probability relation between events may be finer than an almost agreeing numerical probability P. Numerical probability P *almost agrees* with a qualitative relation of *is strictly more likely than*, provided that whenever P(E) > P(F), then E *is strictly more likely than* F, abbreviated E ⊳ F. When a numerical P almost agrees with a qualitative probability, it may be that E ⊳ F even though P(E) = P(F). Easwaran motivates this case by introducing qualitative probabilities that distinguish between *possible* and *impossible* P-null events. Consider two events E and F that are P-null: P(E) = P(F) = 0, but where E is possible under P whereas F is not. Qualitative probability can capture this distinction, allowing E ⊳ F. We call such qualitative probabilities *regular*.

In section 4, we investigate principle $\mathcal{C}$ for regular qualitative probabilities. We show principle $\mathcal{C}$ requires that each almost agreeing quantitative probability P is a discrete (completely additive) distribution. Thus, the qualitative principle $\mathcal{C}$ is restrictive with respect to countable additivity. It imposes more restrictive additivity conditions on an almost agreeing quantitative probability than countable additivity does. Hence, the qualitative principle $\mathcal{C}$ is not equivalent to countable additivity for regular, qualitative probabilities.

In a recent paper, Stewart and Nielsen (2021) investigate the Comparative principle for numerical probabilities P and introduce a condition (here called *SN-disintegrability*) that they prove is satisfied by P in a (positive) *denumerable* partition Π if and only if P satisfies the Comparative principle in Π:. We show that, regardless the cardinality of the state space Ω, if a finitely additive probability P satisfies SN-disintegrability in the (finest) partition by states $\{\omega\}$ of Ω, then P is completely additive. Thus, SN-disintegrability is more restrictive than the Comparative principle when used in uncountable partitions.

2. Infinitary *Books*.

Easwaran's first reason for countable additivity is a variant of de Finetti's *Book* argument that creates a sure loss for anyone betting with fair odds fixed by a merely finitely additive probability P. The strategy Easwaran uses to create the *Book* requires summing together the proceeds of denumerably many unconditional bets, each of which, on its own, is strictly favorable (i.e. each is strictly preferred to the alternative of abstaining from betting) according to the expectations fixed by P. If P is merely finitely additive, there exists a countably infinite set of favorable unconditional gambles whose infinite combination results in a (uniform) sure loss. That cannot happen if P is countably additive.

Next, we remind the reader of the consequences also of accepting the infinite combinations of fair bets, as Easwaran assumes it is appropriate to accept infinite combinations of favorable bets. (A bookie judges a bet fair if indifferent to holding either side of the wager, and indifferent to abstaining from that bet with consequence the status-quo.) De Finetti (1972, p. 91) objected to requiring infinite combinations of fair bets on the grounds that it

> "is circular, for only if we know that complete additivity holds can we think of extending the notion of combinations of fair bets to combinations of an *infinite* number of bets."[4]

De Finetti's criterion of sure-loss from betting – what he calls "incoherence$_1$" – requires a uniform loss from a finite combination of fair bets, regardless the cardinality of the set of events for which the bookie has offered fair odds. As de Finetti established, a set of fair odds for a (possibly infinite) set of events are coherent$_1$ *if and only if* there exists a finitely additive probability P where, for each event E, the fair betting odds on E versus not-E are P(E) : [1-P(E)].

When de Finetti's criterion of coherence$_1$ is modified to require avoiding sure loss from combining infinitely many fair bets, the result is a stronger additivity condition even than countable additivity. For instance, the countably additive uniform (Lebesgue) measure on [0,1] assigns probability 0 to each point $X = x$, $0 \leq x \leq 1$. The associated (real-valued) fair betting odds

[4] De Finetti uses 'complete' here to mean 'countable'. See (1972, pp. 84-85).

are extreme, 1 : 0 – for each x, $P(X \neq x) = 1$. So, a gambler makes \$1 for sure from the bookie against these fair odds by combining the continuum of individual bets, each at a wager of \$0 against \$1, that $X = x$. Regardless which is the realized value $X = x$, the gambler loses all but one bet, for a combined loss of \$0, payable to the bookie. The remaining bet is won, for a net gain of \$1 from the bookie. Thus, the criterion of avoiding a (uniform) sure loss from combining infinitely many <u>fair</u> gambles is more restrictive than requiring countable additivity. That criterion requires fair odds agreeing with a completely additive probability: the union of events each having 0 probability must have 0 probability.

Easwaran acknowledges this result. Regarding the infinitary version of de Finetti's *Book* argument that requires the bookie to accept an infinite set **A** of fair gambles, he writes,

> However, it also entails countable additivity, and even full additivity.
> Nothing in this argument requires the set **A** to be countable. (2013, p. 55)

We do not see why the bookie is obliged to accept an infinite set of strictly favorable gambles but not an infinite set of fair gambles.

A second concern with Easwaran's infinitary variant of the *Book* argument arises with infinite sets of strictly favorable conditional (*called-off*) bets. Let E be an event, associated with its indicator function $E(\omega) = 1$ if event E obtains in state ω,

and $E(\omega) = 0$ if event E does not obtain in state ω.

An unconditional bet by the bookie on an event E, at odds of $x : (1-x)$, with combined stake **S** has payoff to the bookie of $\mathbf{S}[E(\omega) - x]$ in state ω

and payoff to the gambler of $-\mathbf{S}[E(\omega) - x]$ in state ω.

A conditional (called-off) bet by the bookie on E, given an event F, at odds of $x : (1-x)$, with combined stake **S** has

payoff to the bookie of $F(\omega)(\mathbf{S}[E(\omega) - x])$ in state ω

and payoff to the gambler of $-F(\omega)(\mathbf{S}[E(\omega) - x])$ in state ω.

If event F obtains in state ω, the conditional bet has payoffs to the two players as in an unconditional bet. But if F fails to obtain in state ω, the conditional bet is called-off with the *status-quo* outcome, no payment from betting, for each player.

De Finetti's *Book* argument extends to the bookie's (possibly infinite) set of fair unconditional and conditional odds. These are coherent$_1$ (i.e., there is no finite set whose combination is uniformly dominated by abstaining) if and only if there is a finitely additive probability P that matches the bookie's unconditional odds – for betting on event E, the bookie's odds are P(E) : 1-P(E); and matches the bookie's conditional odds – for betting on E, called-off if event F fails, the bookie's conditional odds are P(E|F) : 1-P(E|F).

There is a second reason de Finetti gave, specifically relating to conditional bets, to justify the restriction that the gambler is limited to finite sets of gambles for creating a (uniform) sure loss for the bookie. That reason stems from non-conglomerability of conditional probabilities. As we explain, next, that reason applies to show that unless P is completely additive, there is an infinite set of strictly favorable bets, including conditional bets, that result in a uniform sure loss to the bookie.

Definition: A probability P is *non-conglomerable* in a partition $\Pi = \{h_i: i \in I\}$ provided there is a an event E such that $P(E) < infimum_{h_i \in \Pi} \{P(E|h_i)\}$.

Whenever the bookie's probability is non-conglomerable, it is straightforward for a gambler to create a uniform sure loss for the bookie using infinitely many bets, each of which is favorable for the bookie. Here is the recipe.

Let P be non-conglomerable in Π with $y = infimum_{h_i \in \Pi} \{P(E|h_i)\} > x = P(E)$, and let $0 < \varepsilon < (y - x)$. So, each of the following is a favorable bet from the perspective of the bookie's fair odds:

one unconditional favorable bet on E^c at odds of $[1 - (x + \varepsilon/4)] : (x + \varepsilon/4)$, with stake 1 unit;

$|\Pi|$-many conditional favorable bets on E, given h_i, each at odds of $(y - \varepsilon/4) : [1 - (y - \varepsilon/4)]$, with stake 1 unit.

Since Π is a partition, for each state $\omega \in \Omega$, all but one of these $|\Pi|$-many conditional bets is called off. Given ω, denote by h_i^{ω} the sole element of Π such that $h_i(\omega) = 1$. For all other elements of Π, $h_i(\omega) = 0$. Thus, regardless which state ω obtains, the bookie faces only two bets with non-zero payoffs, namely, the unconditional bet and one conditional bet. Their sum is a uniform, sure loss to the bookie.

$$(E^c(\omega) - [1 - (x + \varepsilon/4)]) + h_i^{\omega}(E(\omega) - (y - \varepsilon/4)) = x - y + \varepsilon/2 < -\varepsilon/2 < 0,$$

That is, the bookie's net payoff from all these bets is a uniform loss, regardless whether E or E^c obtains.

Schervish et al. (2017) show that, if P is a countably additive probability that is not κ^+-additive, then P is non-conglomerable in a (measurable) partition Π where $|\Pi| = \kappa$.[5] Thus, infinite combinations of strictly favorable unconditional and conditional bets, as assessed by a countably additive probability P that is not completely additive, lead to a uniform sure loss when those bets reflect the non-conglomerability of P.[6]

3. The Comparative principle for numerical probabilities.

[5] Their Theorem (2017, p. 291) uses Dubins' (1975) theory of conditional probability for controlling conditional probability given non-empty, P-null events. Dubins' theory of conditional probability is neutral in the debate over whether an unconditional finitely additive probability is countably additive. As noted by an anonymous referee, Easwaran (2019) favors Kolmogorov's rival theory of *regular conditional distributions*, rcd's, over Dubin's theory. However, rcd's are not defined unless both unconditional and conditional probability is countably additive. Hence, in the debate whether probability might be merely finitely additive, the question here is begged by appeal to rcd's. Moreover, not every countably additive unconditional probability admits rcd's. Concerning existence of rcd's, see Seidenfeld et al. (2001) p. 1614.

[6] Stern and Kadane (2015) investigate conditions for combining more than finitely many bets to assess sure loss, and when such combinations may be used to justify countable additivity. Schervish et al. (2014) show that if one wants to extend de Finetti's criterion of coherence to include a denumerable number of unconditional decisions without requiring countable additivity, then apply "coherence$_2$": avoid sure loss from a denumerable set of unconditional, finitely additive forecasts with losses subject to strictly proper scoring rules. That result does not extend, however, to coherence$_2$ for denumerably many conditional forecasts when probability is non-conglomerable.

The second reason Easwaran offers for countable additivity is formulated in terms of the *Comparative principle* [$\mathscr{C}$], which we repeat here for the reader's convenience.

> *The Comparative principle* (with italics added): If Π is a partition of the sure event for two probability functions P_1 and P_2, then it is not the case that for every member *a* of Π, *a* is *strictly more likely* under P_2 than *a* is under P_1.

Easwaran intends the Comparative principle to apply more broadly than only to numerical probabilities, as he acknowledges that being "*strictly more likely*" may include comparisons with non-numerical probabilities, e.g., non-Archimedean qualitative probabilities. Following Easwaran's notation, let the expression $(P_1, E_1) \succ (P_2, E_2)$ mean that event E_1 is *strictly more likely* under probability P_1 than is event E_2 under probability P_2.

For numerical probabilities P_1 and P_2, Easwaran requires that

- if $P_1(E_1) > P_2(E_2)$ then $(P_1, E_1) \succ (P_2, E_2)$.

For the converse he considers two cases:

- In what he calls an ***unambiguous*** case (p. 60), if $(P_1, E_1) \succ (P_2, E_2)$ then $P_1(E_1) > P_2(E_2)$. We consider this case, below.
- Otherwise, it may be that being *strictly more likely* is not defined by numerical probabilities alone; so that, in particular, it is allowed that $P_1(E_1) = P_2(E_2)$ but $(P_1, E_1) \succ (P_2, E_2)$. We consider this case in section 4.

3.1 Structural assumptions for numerical probabilities.

First, we investigate the unambiguous case of the Comparative principle for numerical probabilities in the setting of an atomic, measurable space $<\Omega, B>$, where set theoretic union and intersection are the infinitary Boolean operations in B. We adopt the formal theory of probability for a (finitely additive) measure space, $<\Omega, B, P>$, where P is a finitely additive probability defined on a field of sets B, with the sure-event Ω equal to a set of disjoint and mutually exhaustive possibilities, called “states”: $\Omega = \{\omega_i : i \in I\}$.[7] A set B is said to be *measurable* if $B \in \mathcal{B}$, i.e., if B is in the domain of the probability P. In this sub-section we assume that each state $\{\omega\}$ is measurable. That is, first, we deal with atomic fields, where the states are the atoms of the algebra $\mathcal{B}$. We relax these assumptions in section 3.2 where we investigate principle $\mathscr{C}$ for atomless, countable Boolean algebras whose infinitary operations need not be set theoretic union and intersection.

Definition: A numerical probability P is *finitely additive* if it satisfies these three axioms:

(i) For each measurable set B, $0 \leq P(B) \leq 1$.

(ii) $P(\Omega) = 1$.

(iii) If E and F are disjoint measurable sets, with $G = E \cup F$, then $P(G) = P(E) + P(F)$.

For probability P to be *countably additive* is to require also either of the following two axioms, which, given (i)-(iii), are equivalent provided that $\mathcal{B}$ is a field of sets.[8]

(iv-a) Let $\{A_i : i = 1, . . . \}$ be a denumerable sequence of measurable, pairwise disjoint sets, and assume that their union, A, is measurable. That is, assume $A_i \cap A_j = \emptyset$ if $i \neq j$, and $A = \cup_i A_i \in \mathcal{B}$.

P is countably additive$_1$ provided that $P(A) = \sum_i P(A_i)$ for each such sequence.

(iv-b) Let $\{B_i : i = 1, . . . \}$ be a decreasing (or, respectively increasing) denumerable sequence of measurable sets, and assume that their limit, B, is measurable. That is, $B_i \supseteq B_{i+1}$, and assume $B = \cap_i B_i$ (or respectively $B_i \subseteq B_{i+1}$ and $B = \cup_i B_i$) is measurable.

P is countably additive$_2$ provided that $P(B) = \lim_i P(B_i)$ for each such sequence.

When P is finitely but not countably additive, it is *merely finitely additive*.

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[7] It is commonplace to assume that B is a σ-field of sets, closed under countable unions and intersections. In order to discuss principle $\mathscr{C}$ broadly, we do not make this assumption. Also, we use capital letters ‘I’ and ‘J’ as index sets.

[8] See section 2 of Billingsley (1995).

Definition: $\Pi = \{a_j: j \in J\}$ is a measurable partition of Ω provided that: each element a_j of Π is measurable, the elements of Π are pairwise disjoint, and their union is Ω.

A numerical probability P_1 satisfies the unambiguous Comparative principle in partition $\Pi = \{a_j: j \in J\}$ when there is no finitely additive probability P_2 for Π where $P_2(a_j) > P_1(a_j)$ for each $j \in J$. (3.1)

A numerical probability P_1 satisfies the unambiguous Comparative principle if it does so for each measurable partition Π in the measurable space $<\Omega, B>$. (3.2)

Theorem (Easwaran). With respect to the a finitely additive numerical measure space, the unambiguous Comparative principle (3.2) is equivalent to countable additivity$_1$.

Easwaran writes,

> In order to unambiguously apply the Comparative Principle, we would need a case where $P_1(a) < P_2(a)$ for all a in A. But as mentioned above, if A is uncountable, then at least P_2 will violate finite additivity. Thus, the only finitely additive, non-negative, normalized functions that are ruled out as probabilities by the Comparative Principle are the ones that violate countable additivity. This principle does nothing to rule out violations of full additivity. So we have a second argument for countable additivity that fails to extend to full additivity. Countable additivity is not an arbitrary stopping point. (p. 60)

As Easwaran points out, it is evident that each finitely additive probability satisfies the unambiguous Comparative principle in each finite partition (by axiom (iii)), and in each uncountable partition – as no finitely additive probability P_2 is positive for uncountably many disjoint events. He emphasizes the point that the unambiguous Comparative principle applies in each (measurable) partition, regardless the cardinality of that partition. And, he asserts that it does not entail complete additivity (he calls it "full additivity") in an uncountable partition.

This reasoning does not address the two issues of existence and complete additivity:

- In section 3.2, we adapt Amer's (1985b) Theorem 5 about infinite, free Boolean algebras to show that only *purely finitely additive* probabilities exist on, e.g., the *Lindenbaum-Tarski* algebra of (equivalence classes of) sentential propositions. There are no probabilities for such Boolean algebras that satisfy principle $\mathcal{C}$. Existence is a problem for countably additive probabilities, but not for merely finitely additive probabilities.

- In section 3.3 we adapt Ulam's (1930) Theorem about measurable cardinals to show that Easwaran's reasoning about principle $\mathcal{C}$ neglects the combinatorics of infinitely many uncountable partitions. To impose countable additivity when the sets in an Ulam-matrix are measurable is to require complete additivity. Finitely additive probabilities bear no such restriction.

3.2 Numerical probabilities for infinite free Boolean algebras, principle $\mathcal{C}$, and existence.

Amer reports the following result:

Theorem (Amer, 1985b): Let A be an infinite, free Boolean algebra. There are no countably additive probabilities on A.

Example: The *Lindenbaum-Tarski* algebra L for sentential logic is a countable, free Boolean algebra with the denumerable set of sentence letters serving as a set of free generators. According to Amer's Theorem, L supports no countably additive probability. But L supports uncountably many merely finitely additive probabilities.[9]

Amer's proof uses countable additivity$_2$ (iv-b) applied with a monotone increasing, denumerably infinite sequence of A-measurable events whose infinitary join exists in A. But this method is not suited for applying principle $\mathcal{C}$, which contrasts finitely additive probabilities in denumerable partitions. We adapt Amer's reasoning using an algebraic version of (iv-a), countable additivity$_1$. Proposition 3.1, below, asserts that no numerical probability defined on A satisfies principle $\mathcal{C}$.

Also, our adaptation of Amer's method allows a modest improvement to the conclusion of his theorem by showing that each finitely additive probability defined on an infinite, free Boolean algebra is not only merely finitely additive, but is *purely finitely additive*. A purely finitely additive probability instantiates an extreme of failure of countable additivity, as we explain next.

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[9] There are $2^{\aleph_0}$ different, 2-valued, semantic models for L – where each model provides a truth value for each sentence letter in L Each semantic model induces a different 2-valued merely finitely additive probability P on L. (See Theorem 5 of Amer (1985a).) Hence, there are uncountably many merely finitely additive probabilities on L. These 2-valued merely finitely additive probabilities are *strongly finitely* additive, as well. That is, for each such P there exists an L-measurable, denumerable partition $\Pi = \{a_1, a_2, \ldots\}$ where $\sum_i P(a_i) = 0$. As explained in Appendix A, necessary for this result is that Π forms a partition in L because L is <u>not</u> a σ-algebra.

Yoshida and Hewitt (1952) establish a decomposition for each finitely additive probability P, defined on a field of events, that facilitates a characterization of the extent to which P is not countably additive.

Definition: A probability is <u>*purely finitely additive*</u> if the only non-negative countably additive set function Q that satisfies $P \geq Q \geq 0$ is $Q \equiv 0$.

The following equivalent condition of a purely finitely additive probability is helpful in proving Proposition 3.1.

Theorem (Schervish et al, 1984, p. 208):

> A probability P defined on a field F is purely finitely additive if and only if, for each $\varepsilon > 0$ there exists a denumerable partition $\Pi = \{h_1, h_2, \ldots\}$ such that $\sum_i P(h_i) < \varepsilon$.◊

Theorem (Yoshida and Hewitt, 1952):

> For each finitely additive probability P defined on a field of events F, there exist $P_{\boldsymbol{C}}$ (a countably additive probability defined on F), $P_{\boldsymbol{D}}$ (a purely finitely additive probability defined on F), and two real numbers $\alpha_P \geq 0$, $\beta_P \geq 0$ (with $\alpha_P + \beta_P = 1$), where $P = \alpha_P P_{\boldsymbol{C}} + \beta_P P_{\boldsymbol{D}}$.
>
> The numbers α_P and β_P are unique. If $\alpha_P \neq 0$, $P_{\boldsymbol{C}}$ is unique. Likewise, if $\beta_P \neq 0$, $P_{\boldsymbol{D}}$ is unique.◊

Thus, the magnitude of β_P in the Yoshida and Hewitt decomposition provides a characterization of the extent to which P is not countably additive. Also, it provides a convenient index of the extent to which a finitely additive probability P_1 fails principle $\mathscr{C}$, as we explain, next.

Let P_1 be a finitely additive probability. Let $\boldsymbol{\Pi} = \{\Pi_i\ (i \in I)\}$, where partition $\Pi_i = \{h_{i1}, h_{i2}, \ldots\}$ is denumerable and P_1-measurable. Define $p_{1ij} = P_1(h_{ij})$, for $j = 1, 2, \ldots$.

Let $\mathcal{P}_i$ be the class of finitely additive probabilities P_2 for which Π_i is a measurable partition and where, for each $j = 1, 2, \ldots$, $p_{2ij} \geq p_{1ij}$.

Define the extent to which P_1 fails principle $\mathscr{C}$ in partition Π_i as

$$\mathscr{C}(P_1, \Pi_i) = \text{supremum}_{P_2 \in \mathcal{P}_i} \sum_j (p_{2ij} - p_{1ij}).$$

Define the extent to which P_1 fails principle $\mathscr{C}$ as:

$$\mathscr{C}(P_1) = \textit{supremum}_{\Pi_i \in \boldsymbol{\Pi}}\ \textit{supremum}_{P_2 \in \mathcal{P}_i} \sum_j (p_{2ij} - p_{1ij}).$$

Lemma 3.1: $\mathscr{C}(P_1) = \beta_{P_1}$.

Proof: Immediate from the fact that if $a_{\mathrm{P}_1} = x$, then for each Π_l, $\sum_j p_{1ij} \geq x$, and for each $\varepsilon > 0$, there exists a denumerable partition Π_l where $\sum_j p_{1ij} < x + \varepsilon$. And then
$supremum_{\Pi_i \in \boldsymbol{\Pi}}\ supremum_{\mathrm{P}_2 \in \mathcal{P}_i} \sum_j (p_{2ij} - p_{1ij}) = (1\text{-}x)$.◊**Lemma 3.1**
So, $\mathcal{C}(\mathrm{P}_1) = 1$ if and only if P_1 is purely finitely additive.

Proposition 3.1: Let A be an infinite, free Boolean algebra. Each finitely additive probability P on A fails principle $\mathcal{C}$ maximally, $\mathcal{C}(\mathrm{P}) = 1$. That is, each finitely additive probability P on A is purely finitely additive.

Proof: See Appendix A.

Thus, principle $\mathcal{C}$ makes it impossible to apply numerical probabilities on such a Boolean algebra A. But, there are uncountably many purely finitely additive probabilities on A.

3.3 Ulam-matrices and complete additivity.

Let $<\Omega$, B, P$>$ be a countably additive infinite measure space with $|\Omega| = \kappa$ where $\kappa = \lambda^+$ is a successor cardinal: hence, κ is a regular, uncountable cardinal. An Ulam-matrix M of κ-many rows and λ-many columns, with entries $B_{\alpha\beta}$ ($\alpha < \kappa$, $\beta < \lambda$) that are B-measurable subsets of S, satisfies:

(i) for each β, if $\alpha \neq \alpha'$, then $B_{\alpha\beta} \cap B_{\alpha'\beta} = \emptyset$, i.e. within each column sets are pairwise disjoint,

(ii) for each α, $|S - \cup_\beta B_{\alpha\beta}| \leq \lambda$, i.e., for each row, the union of sets in that row leaves off at most λ-many elements of S.[10]

The following is a simple consequence of Ulam's (1930) Theorem.

Proposition 3.2: For B a sufficiently large subset of the powerset of Ω to contain an Ulam-matrix, if P is countably additive it is a discrete (completely additive) probability, i.e., where, for some countable set $C \in$ B, P(C) = 1.◊

[10] See Jech (2002) Lemma 27.8 for existence of an Ulam-matrix. See Pfeffer and Prikry (1989) for existence of "small" σ-algebras that contain an Ulam-matrix.

Proof. Argue indirectly. Let $\kappa = |\Omega|$ be the least cardinal that carries a countably additive probability P where, for each $\omega \in \Omega$, $P(\{\omega\}) = 0$. Then P is κ-additive.[11] Suppose there exists a B-measurable, co-countable subset $S \subseteq \Omega$, where $|\Omega - S| \leq \aleph_0$, $P(S) > 0$, and for each $\omega \in S$, $P(\{\omega\}) = 0$. So, $|S| = \kappa$. Build an Ulam-matrix M of κ-many rows and λ-many columns, with entries $B_{\alpha\beta}$ ($\alpha < \kappa$, $\beta < \lambda$) that are B-measurable subsets of S.

Under these circumstances, for each row α, $P(S - \cup_\beta B_{\alpha\beta}) = 0$. And then, as there are only λ-many columns, for at least one β in that row α, $P(B_{\alpha\beta}) > 0$. By the pigeon hole principle, for at least one column β, for κ-many values of α, $P(B_{\alpha\beta}) > 0$. But, as these are pairwise disjoint sets, some finite partial sums of these probabilities exceeds 1, yielding a measurable set with probability greater than 1.◊ Proposition 3.2

Hence, if <Ω,B,P> is a finitely additive measure space where $|\Omega|$ is an infinite successor cardinal, where P satisfies the Comparative principle, and where B is large enough to include an Ulam-matrix of measurable sets, then P is completely additive. Easwaran asserts that principle $\mathscr{C}$ places no additional additivity constraints on P apart from countable additivity. That reasoning fails to account for the combinatorics of satisfying $\mathscr{C}$ in infinitely many uncountable partitions simultaneously. Then, for an algebra, B, that contains an Ulam-matrix, being countably additive entails being completely additive.

4. On non-numerical interpretations of "strictly more likely."

On pp. 59-60 of his (2013) article, Easwaran suggests that the Comparative principle does not require more (or less) than countable additivity even when outside the unambiguous case. That is, Easwaran entertains situations where the 4-term, qualitative relation:

Event E_1 subject to numerical probability P_1

<u>is strictly more likely than</u>

event E_2 subject to numerical probability P_2

[11] See Jech (2002) Lemma 27.3.

is finer than the comparison given merely by the quantitative probabilities, $P_1(E_1)$ and $P_2(E_2)$, for these events. That is, he allows:

$$P_1(E_1) = P_2(E_2) \text{ but } (P_1, E_1) \succ (P_2, E_2).$$

As we understand his text, Easwaran applies the Comparative principle with this 4-term qualitative relation in all partitions. But what conditions apply to the relation, $\succ$, as it is used in the Comparative principle, to contrast (P_1, E_1) with (P_2, E_2) when outside the "unambiguous" case?

To make progress answering this question, consider restricted instances of the relation $\succ$, where the same numerical probability appears in both terms, i.e., restrict comparisons to pairs involving a common numerical probability, P_i. Specifically, as a condition of adequacy relating numerical probabilities and qualitative probability, Easwaran requires that

$$P_i(E_1) > P_i(E_2) \text{ only if } (P_i, E_1) \succ (P_i, E_2). \qquad (4.1)$$

In the literature on qualitative probability, this is the requirement that numerical probability P_i ***almost agrees*** with $\succ$ restricted to such pairs.[12]

For convenience, next we introduce three abbreviations associated with the qualitative relation $\succ$ restricted to comparisons that involve a common qualitative probability:

$E \rhd_i F$ *if and only if* $(P_i, E_1) \succ (P_i, E_2)$.

$E \equiv_i F$ *if and only if* neither $E \rhd_i F$ nor $F \rhd_i E$.

$E \unrhd_i F$ *if and only if* either $E \rhd_i F$ or $E \equiv_i F$.

In order to satisfy Easwaran's sufficient condition (4.1), assume that a qualitative probability relation $\rhd$ admits an almost agreeing numerical probability, $P(\cdot)$. Also, assume that $\rhd$ satisfies de Finetti's four axioms for a finitely additive qualitative probability:

Axiom $Qual_1$: $\rhd$ is a weak order: $\unrhd$ is transitive and all pairs of events are comparable.

This axiom avoids $\succ$ being cyclic. Let $P(E) = P(F) = P(G)$. Without $Qual_1$, then it might be that $E \succ_i F \succ_i G \succ_i E$.

Axiom $Qual_2$: For each event E, $\Omega \unrhd E \unrhd \varnothing$.

[12] The unambiguous case (i.e., $P_1(E_1) > P_1(E_2)$ if and only if $P_1(E_1) \succ P_1(E_2)$) is the requirement that P_1 ***agrees*** with $\succ_1$. See Fishburn (1986) for an excellent review of the literature on agreeing and almost agreeing probabilities.

When $0 < P_i(E) < 1$, $Qual_2$ obtains from the requirement that P almost agrees with $\succ_i$.

Thus, $Qual_2$ adds the conditions that no event is qualitatively more probable than the sure event, and no event is qualitatively less probable than the impossible event.

Axiom $Qual_3$: $\Omega \rhd \emptyset$.

In the context of an almost agreeing numerical probability, $Qual_3$ is redundant as, for each numerical probability P, $1 = P(\Omega) > P(\emptyset) = 0$.

Axiom $Qual_4$: Whenever $E \cap G = F \cap G = \emptyset$, then $E \cup G \trianglerighteq F \cup G$ if and only if $E \trianglerighteq F$.

$Qual_4$ is the qualitative version of finite additivity.

A qualitative probability $\trianglerighteq$ is *regular* if, whenever $E \neq \emptyset$, then $E \rhd \emptyset$.

Of course, though a qualitative probability $\trianglerighteq$ is regular, each of its almost agreeing numerical probabilities P may fail to be regular, as when with an uncountable Ω, for each $\{\omega\} \rhd \emptyset$.[13]

What constrains the qualitative comparison $\succcurlyeq$ between the pair (P_1,E_1) and (P_2,E_2)? We propose using the two qualitative probabilities, $\trianglerighteq_1$ and $\trianglerighteq_2$ as follows:

$\succcurlyeq$ Axiom 1: The comparison $\succcurlyeq$ is a weak order, with derived asymmetric relation $\succ$, and derived symmetric relation $\approx$.

$\succcurlyeq$ Axiom 2: $\succcurlyeq$ extends each qualitative probability relation, $\trianglerighteq_i$ ($i = 1, 2$).

For instance, if $E_1 \trianglerighteq_1 E_2$ then $(P_1, E_1) \succcurlyeq (P_1, E_2)$.

$\succcurlyeq$ Axiom 3: Suppose that, restricted to a subalgebra $\mathcal{A} \subseteq \mathcal{B}$, $\trianglerighteq_1 = \trianglerighteq_2 = \trianglerighteq$.

Then there is a model of $\succcurlyeq$ where, for each pair E_1, E_2 of $\mathcal{A}$-measurable events, and for $i,j = 1, 2$,

$$(P_i, E_1) \succcurlyeq (P_j, E_2) \quad \textit{if and only if} \quad E_1 \trianglerighteq E_2. \tag{4.2}$$

$\succcurlyeq$ Axiom 3 requires that when two qualitative probabilities are identical on an algebra, it is consistent that a comparison of "is strictly more likely" between events from that algebra

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[13] A well accepted qualitative version of countable $additivity_2$ is the following axiom, due to Villegas (1964), intended for theories of <u>agreeing</u> quantitative probabilities:
Axiom $Qual_5$: For all measurable E, E_i (i =1, 2, …) and F, with $E_i \supseteq E_{i+1}$ and $\cap_i E_i = E$, if for all i $E_i \trianglerighteq F$, then $E \trianglerighteq F$.
Definition: $\trianglerighteq$ is a *countably additive qualitative probability* if it satisfies the five Qual axioms.
The following explains why we do not use axiom $Qual_5$ for Easwaran's program.
Remark: Let <Ω,B,P> be a non-regular, countably additive measure space where there exists a nested sequence of P-non-null measurable events whose intersection is empty: $P(E_i) > 0$ (i =1, 2, …), $E_i \supseteq E_{i+1}$ and $\cap_i E_i = \emptyset$, and where also there exists a $\mathcal{B}$-measurable non-empty P-null event F: $F \neq \emptyset$ but $P(F) = 0$.
Let $\trianglerighteq$ be a regular qualitative probability for which P is an almost agreeing probability.
Then $\trianglerighteq$ is not countably additive as it fails axiom $Qual_5$.
Proof: Since P almost agrees with $\trianglerighteq$, for i =1, 2, …, $E_i \rhd F$. But by regularity, $F \rhd \emptyset$.◊ **Remark**.

depends solely on the events being compared, and does not depend on which qualitative probability is matched with the events in the comparison.

Next we investigate principle $\mathscr{C}$ when the relation $\succcurlyeq$ is not *unambiguous* and qualitative probabilities are regular. In what follows, we assume each qualitative probabilitiy $\trianglerighteq$ satisfies the four de Finetti axioms, and we assume that $\succcurlyeq$ comparisons satisfy the three axioms, above.

An important illustration that Easwaran offers for demonstrating being outside the *unambiguous* case is where $P_1(E) = P_2(E) = 0$, but where E is possible under P_1 while E is impossible under P_2. For instance, let the state space Ω = {H, T, E}, corresponding to the outcome of a flip of a coin landing exclusively and exhaustively, either H: heads-up, T: tails-up, or E: on-edge. Let P_1 be a numerical (non-regular) fair-coin probability: $P_1(H) = P_1(T) = 1/2$, and so $P_1(E) = 0$, but where E is a possible outcome. Let P_2 be the regular numerical fair-coin probability $P_2(H) = P_2(T) = 1/2$, so $P_2(E) = 0$, but where E is not a possible outcome. Specifically, let $P_2(\cdot) = P_1(\cdot|\text{not-E})$. Then, in order to respect the qualitative difference between possible and impossible probability 0 events, $P_1(E) = P_2(E) = 0$, but $(P_1,E) \succ (P_2,E)$. Of course, then principle $\mathscr{C}$ requires that at least one of, $(P_2,H) \succcurlyeq (P_1,H)$, or $(P_2,T) \succcurlyeq (P_1,T)$ if P_2 is not dominated by P_1.

Below, Proposition 4.1 establishes that, if a (finitely additive) regular qualitative probability $\trianglerighteq$ satisfies principle $\mathscr{C}$, that is, if there is no partition where another qualitative probability pointwise $\succ$-dominates $\trianglerighteq$, then if a countably additive numerical probability P almost agrees with $\trianglerighteq$, P is a discrete distribution. Thus, outside the unambiguous case, principle $\mathscr{C}$ restricts qualitative probabilities to those with only completely additive almost agreeing numerical probabilities.

To motivate Proposition 4.1 consider the following.

Heuristic Example: Consider the numerical measure space $< \Omega, \mathrm{B}, P_0 >$, where:

$\Omega = \{H,T\} \times \{y: 0 < y \leq 1\}$.

B is the product space of subsets of {H,T} and Lebesgue measurable subsets of (0,1].

P_0 is the product measure of the fair coin probability, P(H) = P(T) = ½, and the uniform Lebesgue measure over Lebesgue measurable subsets of (0,1]

Define the two numerical probabilities $P_1(\cdot) = P_0(\cdot|H)$ and $P_2(\cdot) = P_0(\cdot|T)$. So, under P_1 the outcome T is not only P_1-null but is not possible, whereas under P_2 the outcome H is P_2-null and not possible. For each interval $[a,b] = \{y: a \le y \le b\}$ $P_1([a,b]) = P_2([a,b]) = P_0([a,b])$.
All three quantitative probabilities are the same continuous probability on the subalgebra of the Lebesgue measurable sets in (0,1].

Partially define a finitely additive qualitative probability $\trianglerighteq_0$ on B that is finer than its almost agreeing numerical probability P_0, as follows. Treat as *unambiguous* two aspects of P_0 when specifying $\trianglerighteq_0$. Abbreviate (H x (0,1]) as H, (T x (0,1]) as T, and ({H,T}, [a,b]) as [a,b].

1. As $P_0(H) = P_0(T)$, stipulate that $H \equiv_0 T$.
2. And for $0 < c, c' \le 1$, as both $P_0(y = c) = P_0(y = c') = 0$ and P is uniform on (0,1], stipulate that $(y=c) \equiv_0 (y=c')$.

But for values $d \le 0$ or $d > 1$, even though $P_0(y = c) = P_0(y = d) = 0$, allow $\trianglerighteq_0$ to distinguish the possible event $(y=c)$ from the impossible event $(y=d)$ by the qualitative relations,

$$(y=c) \vartriangleright_0 (y=d) \equiv_0 \emptyset.$$

Thus, the unambiguous case does not apply as P_0 almost agrees but does not agree with the regular qualitative probability $\trianglerighteq_0$.

P_0, P_1 and P_2 are identical continuous numerical distributions on the subalgebra of the Lebesgue measurable subsets of (0,1]. Preserve that agreement by letting $\trianglerighteq_0 = \trianglerighteq_1 = \trianglerighteq_2$ over this subalgebra.

$$[a,b] \trianglerighteq_0 [c,d] \text{ \textit{if and only if} } [a,b] \trianglerighteq_i [c,d] \ (i = 1,2).$$

Then $\trianglerighteq_1 = \trianglerighteq_2$ on the subalgebra of the Lebesgue measurable subsets of (0,1].
By $\succcurlyeq$ Axiom 3 there is a model of $\succcurlyeq$ where,

$$\text{for all } d, \qquad (P_1, (y=d)) \approx (P_2, (y=d)). \tag{4.3}$$

That is, the event $(y=d)$ is just as likely under P_1 as it is under P_2.
Define two 1-1 functions

f_1: (0,1] ↔ (0,.5] by f_1: $y = y/2$

and f_2: (0,1] ↔ (.5,1] by f_2: $y = y/2 + ½$.

Use these to create an uncountable partition $\Pi_1 = \{a_y: 0 < y \leq 1\}$ of Ω where

$$a_y = \{(H,y), (T,f_1(y)), (T,f_2(y))\}.$$

Observe that

$$\begin{aligned}
(P_1, a_y) &\approx (P_1, (H, y)) && \text{since} \approx \text{agrees with } \trianglerighteq_1\\
&\approx (P_1, (y)) && \text{since} \approx \text{agrees with } \trianglerighteq_1\\
&\approx (P_2, (y)) && \text{by (4.3)}\\
&\approx (P_2, (T, y)) && \text{since} \approx \text{agrees with } \trianglerighteq_2\\
&\approx (P_2, (T,f_1(y))) && \text{since} \approx \text{agrees with } \trianglerighteq_2\\
&\prec (P_2, \{(T,f_1(y)), (T,f_2(y))\}). && \text{by Axiom Qual}_4\\
&\approx (P_2, a_y) && \text{since} \approx \text{agrees with } \trianglerighteq_2
\end{aligned}$$

And since $\preccurlyeq$ is a weak order, by iterated applications of transitivity,

$$(P_1, a_y) \prec (P_2, a_y).$$

So, for each element a_y of the partition Π_1, qualitative probability $\trianglerighteq_2$ pointwise $\succ$-dominates qualitative probability $\trianglerighteq_1$. $\trianglerighteq_1$ fails principle $\mathcal{C}$. Reverse the roles of P_1 and P_2 and then $\trianglerighteq_1$ pointwise $\succ$-dominates $\trianglerighteq_2$ in the partition $\Pi_2 = \{b_y: 0 < y \leq 1\}$ of Ω where $b_y = \{(T,y), (H,f_1(y)), (H,f_2(y))\}$. Thus, neither qualitative probability, $\trianglerighteq_1$ nor $\trianglerighteq_2$ satisfies principle $\mathcal{C}$.

Proposition 4.1 If countably additive P almost agrees with the regular qualitative probability $\trianglerighteq$, and if $\trianglerighteq$ satisfies principle $\mathcal{C}$, then P is a discrete (completely additive) distribution.

Proof: Argue indirectly. With $0 \leq z < 1$, let $P = zP_d + (1-z)P_c$, where P_d is a discrete probability, and P_c is a continuous probability. Let $D = \{\omega: P_d(\omega) > 0\}$. So, as $|D| \leq \aleph_0$, then $P_c(D) = 0$ and $P_c(D^c) = 1$. Let X be a real-valued random variable defined on states in D^c. And let $y = CDF_c(X = x)$, the Cumulative Distribution Function associated with P_c, defined for states in D^c. Then y is uniform U[0, 1] under the continuous conditional probability $P_c(\ |\ D^c)$. Apply the reasoning in the *Heuristic Example*, as follows. Let P_0 be defined on $\{H,T\} \times D^c$, where P_0 is the product measure of the fair coin probability on {H,T} and the continuous probability $P_c(\ |\ D^c)$. Define $P_1(\cdot) = P_0(\cdot|H)$ and $P_2(\cdot) = P_0(\cdot|T)$. Define the partitions Π_1 and Π_2 as in the *Heuristic Example*. Then each of P_1 and P_2 is inconsistent with principle $\mathcal{C}$.◊ Prop 4.1

Thus, in order to satisfy the Comparative principle $\mathscr{C}$ with a regular, qualitative probability, $\trianglerighteq$, each of its almost agreeing numerical probabilities P is completely additive. This establishes that, subject to the (de Finetti) four axioms for a qualitative probability, and the three axioms on $\succcurlyeq$ for the qualitative comparisons, principle $\mathscr{C}$ imposes additional restrictions, beyond what countable additivity requires, outside the unambiguous case.[14]

5 The Comparative Principle and Stewart and Nielsen's (2021) SN-disintegrability.

In a recent publication, Stewart and Nielsen (2021) ask how the (unambiguous) Comparative principle is related to each of two concepts that often arise in foundational debates over additivity of numerical probability: conglomerability and disintegrability of conditional probabilities. These authors summarize their findings, as follows:

> "The result that we present in the next section relates the comparative principle to the classical probabilistic concepts of conglomerability and disintegrability in a precise way. We will show that the comparative principle is a strict strengthening of conglomerability and equivalent to disintegrability." (p. 501)[15]

Stewart and Nielsen propose a condition – here we call it *SN-*disintegrability[16], (5.1). which is the subject of their result, discussed below.

- A finitely additive probability P is *SN-disintegrable* in a partition $\Pi = \{a_i : i \in I\}$, if

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[14] Note that in the *Heuristic Example*, and generally in the proof of Proposition 4.1, $\succcurlyeq$-comparability of (P_1,E_1) and (P_2,E_2) when $P_1 \neq P_2$, is used with small sets E_1 and E_2 that are either singletons or doubletons. Thus $\succcurlyeq$ Axiom 1, that $\succcurlyeq$ is a weak-order with $\succcurlyeq$-comparability for all pairs (P_1, E_1) and (P_2, E_2), is not needed for these results.

[15] The authors repeat and expand this assertion, as follows.

> "Because the comparative principle is equivalent to disintegrability and implies conglomerability, reasons for and against mandating disintegrability and conglomerability are seen to have direct bearing on the status of the comparative principle. Since the comparative principle is strictly stronger than conglomerability for a single partition, and all proponents of relaxing countable additivity have reconciled themselves to failures of conglomerability, for such probabilists, the dialectical force of the argument for countable additivity on the basis of the comparative principle is marginal. And, similarly, since disintegrability implies but is not implied by conglomerability, and our result establishes that disintegrability is equivalent to the comparative principle, it should not be surprising if opponents of mandating countable additivity accept violations of the comparative principle with equanimity." (p. 501)

[16] In Appendix B, we distinguish "disintegrability" and "S-N disintegrability" as the two are not equivalent, even in a positive, denumerable partition.

$$\forall B \in \mathcal{B} \;\; P(B) = \sum_{i \in I} P(B|a_i)P(a_i) \quad (5.1)$$

Call a partition Π *positive* if each of its elements has positive probability. Stewart and Nielsen's result establishes that, in a (positive) denumerable partition Π, a finitely additive probability P is SN-disintegrable in Π if and only if P satisfies the (unambiguous) Comparative Principle in Π. Stewart and Nielsen base their claim of equivalence between the Comparative principle and SN-disintegrability on this result.

However, this equivalence does not extend beyond denumerable partitions. Specifically, as we show below, when a probability P is SN-distintegrable in a partition Π then P is $|\Pi|^+$-additive in the sub-algebra generated by Π. So, if probability P satisfies SN-disintegrability in the finest partition by states, then P is completely additive. Since the (unambiguous) Comparative principle does not mandate more than countable additivity, the equivalence between SN-disintegrability and the Comparative principle is limited to countable partitions. That is, as noted previously, each finitely additive probability P satisfies the (unambiguous) Comparative principle in each uncountable partition. But only a discrete, countably additive probability satisfies SN-disintegrability in an uncountable partition. Here are the relevant details.

Theorem (Stewart-Nielsen, 2019): In a (positive) denumerable partition $\Pi = \{a_i: i = 1, \ldots\}$, a finitely additive probability P satisfies SN-disintegrability

$$\forall B \in \mathcal{B} \;\; P(B) = \sum_{i \in I} P(B|a_i)P(a_i)$$

if and only if, P satisfies the Comparative principle (2.1) in Π.

However,

Proposition 5.1: If P is SN-disintegrable in a partition Π, then P is $|\Pi|^+$-additive$_1$ in the sub-algebra generated by Π.

Proof: Let P be a finitely additive probability on a measurable space $\{\Omega, \mathcal{B}\}$. Let $\Pi = \{a_i: i \in \mathbf{I}\}$ be a (measurable) partition. Assume that (5.1) obtains: $\forall B \in \mathcal{B} \;\; P(B) = \sum_{i \in I} P(B|a_i)P(a_i)$, whenever the conditional probabilities $P(B|a_i)$ are well defined for each $i \in \mathbf{I}$.

For the special case, B = Ω, P(Ω|a_i) = 1 for each $i \in$ **I**. And this equality obtains in every theory of conditional probability that we know: Dubins (1975), Kolmogorov (1956), and Renyi (1970), to mention three prominent examples. Then P(Ω) $= 1 = \sum_{i\in \mathrm{I}}$ P(a_i). So, P is $|\Pi|^{+}$-additive$_1$ in the sub-algebra generated by Π. That is, if $\mathrm{A} \subseteq \Pi$ and P(a) = 0 for each $a \in \mathrm{A}$, then P($\cup$A) = 0.◊

Corollary 5.1: Let P be a finitely additive probability on a measurable space {Ω, B}. Let Π_Ω be the finest partition in B, the partition by the elements of Ω. If P is SN-disintegrable in Π_Ω, then P satisfies complete additivity.◊

Thus, considering the comparison with the (unambiguous) Comparative principle $\mathcal{C}$ in uncountable partitions – as is necessary in order to meet Easwaran's requirement that $\mathcal{C}$ does not entail complete additivity in uncountable partitions – SN-disintegrability is not equivalent to principle $\mathcal{C}$.

6. Summary of the principal observations regarding principle $\mathcal{C}$.

In his (2013), Easwaran offers the *unambiguous* Comparative principle $\mathcal{C}$ as a basis for countable additivity of numerical probability. The two are equivalent in finitely additive measure spaces. However, there are other challenges that arise when countable additivity is pitted against finite additivity.

When the set of measurable events is a countable, free Boolean algebra (e.g. the *Lindenbaum-Tarski* algebra for sentential logic), only purely finitely additive probabilities exist: no numerical probability satisfies the Comparative principle. And when the state space is uncountable, if the algebra of measurable events is sufficiently large to include the sets in an Ulam-matrix, only discrete (completely additive) probabilities satisfy the Comparative principle. For then the combinatorics of satisfying the Comparative principle simultaneously in infinitely many uncountable partitions brings with it demands for stronger additivity. By contrast, (merely) finitely additive probabilities are free from either of these consequences.

The Comparative principle for non-numerical probabilities is not fully defined in Easwaran's (2013) article. The version of $\mathcal{C}$ that we offer for regular qualitative probabilities, outside the *unambiguous* case (based on Easwaran's example), precludes countably additive continuous almost agreeing numerical probabilities. However, if regular qualitative probabilities are not required to satisfy $\mathcal{C}$, then a countably additive almost agree quantitative probability need not be completely additive. Thus, contrary to Easwaran's condition of adequacy for justifying countable additivity, principle $\mathcal{C}$ requires stronger additivity outside the *unambiguous* case.

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Appendix A.

Proposition 3.1: Let A be an infinite, free Boolean algebra. Each finitely additive probability P on A fails principle $\mathcal{C}$ maximally, $\mathcal{C}$(P) = 1. That is, each finitely additive probability P on A is purely finitely additive.

We establish the result for a subalgebra $A_\Gamma \subseteq A$ generated by $\Gamma = \{\gamma_1, \ldots\}$, a denumerable subset of generators of A. Without loss of generality, we use the Lindenbaum-Tarksi algebra L of sentential logic for this subalgebra. That is, up to isomorphism, L is the free Boolean algebra with countably many generators. Next, we summarize relevant details of L.

Let **L** be the first order sentential language with denumerably many proposition letters, *p*, which are the atomic propositions of **L**, $P = \{p_1, p_2, \ldots\}$. For convenience, let '**&**', 'v' and '¬' be the sentential operators in **L**, whose semantics are respectively the usual truth functions ‘and’, ‘or’, and ‘not’. Let ***WFF*** be the denumerable set of well formed formulas in **L**, which is the syntactic, recursive closure of the atomic propositions under the sentential operators.

Let $\equiv$ denote (semantic) logical equivalence, an equivalence relation between pairs of well formed formulas in **L**.

For $s \in$ **WFF**, let $\bar{s}$ be the equivalence class of its logically equivalent well formed formulas.

L is the Lindenbaum-Tarski algebra over **WFF**/$\equiv$.

L is a countable Boolean algebra, where, for s, $t \in$ **WFF**

the algebraic join $\bar{s} \vee \bar{t} = \overline{s \mathbf{v} t}$,

the algebraic meet $\bar{s} \wedge \bar{t} = \overline{s \mathbin{\&} t}$

the algebraic complement $\bar{s}' = \overline{\neg s}$.

For convenience, denote $\mathbf{T}$ = equivalence class of tautologies**,**

and $\bot$ = equivalence class of contradictions**.**

Define the (transitive) partial order $\preccurlyeq$ on L by $\bar{s} \preccurlyeq \bar{t}$ if s (semantically) entails t.

Note that $\preccurlyeq$ is a strict partial order; that is, if $\bar{s} \preccurlyeq \bar{t}$ and $\bar{t} \preccurlyeq \bar{s}$ then $\bar{s} = \bar{t}$.

Denote by $\bar{s} \prec \bar{t}$ the asymmetric, transitive relation, $\bar{s} \preccurlyeq \bar{t}$ and $\bar{s} \neq \bar{t}$.

$\mathbf{T}$ is the maximal element and $\bot$ is the minimal element of this strict partial order.

That is, $\bot \prec \mathbf{T}$ and if $\bot \neq \bar{s} \neq \mathbf{T}$ then $\bot \prec \bar{s} \prec \mathbf{T}$.

When neither $\bar{s} \preccurlyeq \bar{t}$ nor $\bar{t} \preccurlyeq \bar{s}$, say that $\bar{s}$ and $\bar{t}$ are ***independent***.

Observe that L is atomless: That is, consider $s \in$ **WFF** where $\bar{s} \neq \bot$**.** Let t = s & p, where ‘p’ does not appear among the atomic propositions in s. Then $\bot \prec \bar{t} \prec \bar{s}$. So $\bar{s}$ is not an atom of L.

L is (up to isomorphism) the countable, atomless Boolean algebra. Because L is a countable Boolean algebra, it is not a Boolean σ-algebra. (See Sikorski [1969, p. 66, (E)].) So we have to be careful about the existence of infinitary joins and infinitary meets within L. That is, an infinitary join is the least upper bound under $\preccurlyeq$ and the infinitary meet is the greatest lower bound under $\preccurlyeq$ of a (countable) set of elements of the Boolean algebra. These need not exist in L.

For $\bar{S} = \{\bar{s}_i \in \text{L}: i = 1, 2, \ldots\}$, say that $\bar{t} \in$L is the ***infinitary join*** of $\bar{S}$, written $\bar{t} = \bigvee \bar{S}$, provided that,

for each $\bar{s}_i \in \bar{S}$, $\bar{s}_i \preccurlyeq \bar{t}$, and

if also there exists $\overline{t'} \in$ L where, for each $\bar{s}_i \in \bar{S}$, $\bar{s}_i \preccurlyeq \overline{t'}$, then $\bar{t} \preccurlyeq \bar{t}'$.

The infinitary meet of $\bar{S}$ is defined similarly.

A (finitely additive) probability P on L satisfies:

(i) $0 \leq P(\bar{s}) \leq 1$

(ii) $P(\mathbf{T}) = 1$, $P(\bot) = 0$

(iii) $P(\bar{s} \vee \bar{t}) = P(\bar{s}) + P(\bar{t})$ whenever $\bar{s} \wedge \bar{t} = \bot$.

Definition: P is countably additive$_1$ on L provided that, whenever $\bar{S} = \{\bar{s}_i \in L\!: i = 1, 2, \ldots\}$ is a denumerable partition, i.e., satisfying

(i) $\bar{s}_i \wedge \bar{s}_j = \bot$ whenever $i \neq j$, and

(ii) where the infinitary join $\bar{t} = \bigvee\bar{S}$ exists,

then $P(\bar{t}) = \sum_i P(\bar{s}_i)$.

Proof of Proposition 3.1: Let P be a finitely additive probability on L. Let $\varepsilon > 0$. We show there exists a denumerable partition $\Psi = \{\psi_1, \psi_2, \ldots\}$ in L with $\sum_i P(\psi_i) < \varepsilon$.

Let $\Gamma = \{\gamma_1, \ldots\}$ be the set of the sentential generators of L: the set of (equivalence classes of the) atomic propositions.

Choose integer k that satisfies, $(1+\varepsilon)/\varepsilon < 2^k$; equivalently, $\frac{1/2^k}{1-1/2^k} < \varepsilon$.

For $j = 1, 2, \ldots$, define successive (disjoint) blocks b_j containing $j\times k$ many generators from Γ.

That is, $b_j = \{\gamma_{\frac{k(j-1)j}{2}+1}, \ldots, \gamma_{\frac{k(j+1)j}{2}}\}$.

Specifically, $b_1 = \{\gamma_1, \ldots, \gamma_k\}$, $b_2 = \{\gamma_{k+1}, \ldots, \gamma_{3k}\}$, $b_3 = \{\gamma_{3k+1}, \ldots, \gamma_{6k}\}$, etc.

The set of blocks partitions the set of generators in Γ.

Each block, b_j, generates $2^{j\times k}$ many Boolean elements β_m^j ($m = 1, \ldots, 2^{j\times k}$) of A_Γ of the form

$$\beta_m^j = \delta_{\frac{k(j-1)j}{2}+1} \wedge \ldots \wedge \delta_{\frac{k(j+1)j}{2}}$$

where $\delta_i = \gamma_i$ or $\delta_i = \gamma_i'$.

Note that, since the algebra A_Γ is free, each β_m^j satisfies: $\bot \prec \beta_m^j \prec \mathbf{T}$.

Trivially, for each block b_j, if $\beta_m^j \neq \beta_n^j$ then $\beta_m^j \wedge \beta_n^j = \bot$.

Equally evident, for each block b_j, $\mathbf{T} = \bigvee\{\beta_m^j\!: m = 1, \ldots, 2^{j\times k}\}$.

Because the generators are independent, the Boolean elements β_m^j β_n^k from different blocks b_j and b_k are also independent, i.e., neither $\beta_m^j \preccurlyeq \beta_n^k$ nor $\beta_n^k \preccurlyeq \beta_m^j$.

As P is finitely additive, then for each block b_j (j = 1, 2, …), there exists (at least) one Boolean element β_m^j with $P(\beta_m^j) \leq 1/2^{j \times k}$. For ease of notation, denote this element of A_Γ as β_j.

Define elements ψ_j of A_Γ as follows:

for $j = 1$, $\psi_1 = \beta_1$; and for $j \geq 2$, $\psi_j = \beta_j \wedge \psi_1' \wedge \ldots \wedge \psi_{j-1}'$

and let $\Psi = \{\psi_j: j = 1, \ldots\}$.

Claim: Ψ is a partition:

(i) $\psi_j \wedge \psi_k = \mathbf{0}$ whenever $j \neq k$. (So, also Ψ is an anti-chain.)

(ii) $\mathbf{T} = \bigvee\Psi$

Proof: (i) Immediate from the definition of the ψ_j.

(ii) Trivially, $\psi_j \preccurlyeq \mathbf{T}$. Next we show $\mathbf{T}$ is the least upper bound for Ψ.

By a simple induction, for each n = 1, 2, …, $\psi_1 \vee \ldots \vee \psi_n = \beta_1 \vee \ldots \vee \beta_n$. So, for each n, $\psi_1 \vee \ldots \vee \psi_n$ and $\beta_1 \vee \ldots \vee \beta_n$ share the same upper bounds in L. Argue indirectly. Let $\bar{s} \prec \mathbf{T}$ and suppose $\bar{s}$ is an upper bound for Ψ. Then (β_1 v … v β_{k-1}) entails $\bar{s}$. Let γ_k be an atomic proposition not appearing in s. So $\beta_k \not\preccurlyeq \bar{s}$, i.e., there is a truth assignment where $\mathbf{t}(\beta_k)$ = T and $\mathbf{t}(\bar{s})$ = F. Since the atomic propositions have independent truth assignments, there is a semantic model where, also, $\mathbf{t}(\beta_1$ v … v $\beta_{k-1})$ = F and $\mathbf{t}(\bar{s})$ = F. (If not, i.e., if $\mathbf{t}(\beta_1$ v … v $\beta_{k-1})$ = F entails $\mathbf{t}(s)$ = T, then (β_1 v … v β_{k-1})' entails $\bar{s}$. And then, as (β_1 v … v β_{k-1}) entails $\bar{s}$, $\mathbf{T} \preccurlyeq \bar{s}$.) Thus, ($\beta_1$ v … v β_k) $\not\preccurlyeq \bar{s}$. Therefore, $\mathbf{T} = \bigvee\Psi$.

It is evident that $P(\psi_j) \leq P(\beta_j)$. So, $P(\psi_j) \leq 1/2^{j \times k}$.

Then $\sum_j P(\psi_j) \leq \sum_j 1/2^{j \times k} = \frac{1/2^k}{1 - 1/2^k} < \varepsilon$, which establishes that P is purely finitely additive.◊**Prop. 3.3**

Note: What drives this result is the fact that $\mathbf{T} = \bigvee\Psi$.
Were A a σ-algebra then $\bigvee\Psi = (\psi_1 \vee \ldots \vee \psi_n \vee \ldots) \prec \mathbf{T}$.

Appendix B

We distinguish "disintegrability" and "S-N disintegrability" as the two are not equivalent, even in a positive, denumerable partition. We follow the approach in Dunford and Schwartz (1966, p. 112) for finitely additive integrals of bounded functions. Let <Ω, B, P> be a finitely additive measure space.

Definition. A probability P is *disintegrable* in a measurable partition P = $\{a_j: j \in J\}$ provided that

$$\forall B \in B \;\; P(B) = \int_P P(E|a)\, dP(P).$$

Regardless the cardinality of Ω, each finitely additive probability is disintegrable in the finest partition of its measure space, the partition Π_Ω by the elements of Ω. Let $|\Omega| = \aleph_0$ and let P be merely finitely additive with $P(\omega_i) > 0$, $i = 1, \ldots$. So Π_Ω is a positive, denumerable partition under P. P is disintegrable in Π_Ω. But since P is not countably additive, P is not S-N disintegrable in Π_Ω, nor does P satisfy principle $\mathscr{C}$ in Π_Ω. It is not the case that disintegrability entails S-N disintegrability, nor does disintegrability entail principle $\mathscr{C}$, even for positive, denumerable partitions.

In the converse direction, see Example 4.1 of Schervish et al. (2017, p. 297). That example instantiates an uncountable measure space, and uncountable measurable partition Π and a probability P that is countably additive but not $|\Pi|^+$-additive in P. Though P satisfies principle $\mathscr{C}$ in Π, P fails to be either SN-disintegrable or disintegrable in Π.