

INFINITE PREVISIONS AND FINITELY ADDITIVE EXPECTATIONS

BY M.J. SCHERVISH, TEDDY SEIDENFELD AND J.B. KADANE

Carnegie Mellon University

We give an extension of de Finetti's concept of coherence to unbounded (but real-valued) random variables that allows for gambling in the presence of infinite previsions. We present a finitely additive extension of the Daniell integral to unbounded random variables that we believe has advantages over Lebesgue-style integrals in the finitely additive setting. We also give a general version of the Fundamental Theorem of Prevision to deal with conditional previsions and unbounded random variables.

1. Introduction. De Finetti (1974) presented a theory of finitely additive probability in which the concept of *prevision* played the roles of both probability and expected value (or expectation). De Finetti's theory was motivated in two different, but equivalent, manners. First, he developed a theory of *coherent* gambling in which a bookie chooses fair prices for gambles while trying to avoid uniform sure loss. Second, he took a decision-theoretic approach in which an agent chooses previsions for random variables while being subject to a loss function. The agent tries to avoid choosing previsions such that an alternative choice could achieve uniformly smaller loss.

De Finetti developed his theory fairly completely for the case in which all random variables under consideration are bounded. He realized that unbounded random variables introduce interesting issues for his theory, but he did not pursue those issues very far. Crisma, Gigante, and Millosovich (1997) and Crisma and Gigante (2001) present one form of an extension of de Finetti's theory to unbounded random variables. This paper presents a number of extensions that are more in the spirit of de Finetti's original theory.

Section 2 gives a brief summary of de Finetti's theory for random variables with finite previsions and the notation that will be used in the rest of the paper. Section 3 gives our extension of coherence to infinite previsions and compares the extension to the existing extension of Crisma et al. (1997) and Crisma and Gigante (2001). Section 4 shows how the fair price

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and decision theoretic motivations of de Finetti's theory remain equivalent in the extension to unbounded random variables and to more general loss functions than de Finetti originally envisioned. Section 5 gives an introduction to finitely additive Daniell integrals along with their relation to finitely additive expectations and coherence. Section 6 characterizes the relationship between conditional and marginal previsions. Section 7 shows how to extend a collection of coherent previsions to an arbitrary larger collection.

2. De Finetti's Two Definitions of Coherence. Coherence of previsions, as de Finetti (1974, Chapter 3) formulates it, is the criterion that a rational decision maker tries to avoid making decisions that lead to uniformly larger loss than alternative available decisions.

Let Ω be a set. The elements of Ω will be called *states* and denoted ω . Subsets of Ω are called *events*. *Random variables* are real-valued functions with domain Ω , which we denote with capital letters. We take the liberty of identifying events with their indicator random variables. That is, if $B \subseteq \Omega$, we let $B(\omega)$ stand for the random variable that equals 1 for all $\omega \in B$ and equals 0 for all $\omega \notin B$. The decisions that de Finetti (1974) contemplated are the assignments of (conditional) previsions for random variables.

DEFINITION 1. Let X be a random variable and let B be a nonempty event. A *prevision* $P(X|B)$ for X given B is a fair price for buying and selling X under the condition that B occurs. That is, for all real α , the gamble that causes the agent to lose $\alpha B[X - P(X)]$ is considered fair (or acceptable). If $B = \Omega$, we can denote $P(X|\Omega) = P(X)$ and call it a *marginal prevision* for X .

A collection $\{P(X_i|B_i) : i \in I\}$ of such previsions is *coherent*₁ if, for every finite subset $\{i_1, \dots, i_n\} \subseteq I$ and all real $\alpha_1, \dots, \alpha_n$

$$(1) \quad \inf_{\omega} \sum_{j=1}^n \alpha_j B_{i_j}(\omega) [X_{i_j}(\omega) - P(X_{i_j})] \leq 0.$$

That is, the resulting fair gambles do not allow uniform sure loss.

A collection of forecasts is *coherent*₂ if, for every rival set $\{Q(X_i|B_i) : i \in I\}$ of previsions for the same random variables, every $\{i_1, \dots, i_n\} \subseteq I$, and all nonnegative $\alpha_1, \dots, \alpha_n$,

$$\inf_{\omega} \sum_{j=1}^n \alpha_j B_{i_j}(\omega) \left\{ [X_{i_j}(\omega) - P(X_{i_j})]^2 - [X_{i_j}(\omega) - Q(X_{i_j})]^2 \right\} \leq 0.$$

That is, no rival set of previsions leads to uniformly smaller squared-error loss.

In Definition 1, we interpreted the value of a gamble $\alpha B[X - P(X)]$ as the loss suffered by the agent who sets the previsions. De Finetti interprets the value of such a gamble as the income to the agent. Our choice was made in order that the concept of dominance would mean the same thing when discussing both gambles and loss functions, as in Schervish, Seidenfeld and Kadane (2013). The results that we state and prove here can be converted easily into the corresponding results using de Finetti's interpretation. For example, using de Finetti's interpretation, equation (1) would say that the supremum is nonnegative.

De Finetti (1974, pp. 88–89) proved that a decision maker who wishes to be both coherent₁ and coherent₂ must choose the same previsions for both purposes. Squared-error loss is a special case of a strictly proper scoring rule.

DEFINITION 2. A *scoring rule* for coherent previsions of a random variable X is a real-valued loss function g with two real arguments: a value of the random variable and a potential prevision q . Let \mathcal{P} be a collection of probability distributions that give finite prevision to X . We say that g is *proper for \mathcal{P}* if, for every probability $P \in \mathcal{P}$, $P[g(X, q)]$ is minimized (as a function of q) by $q = P(X)$. If, in addition, only the quantity $q = P(X)$ minimizes expected score, then the scoring rule is *strictly proper*.

Some authors reserve the qualification “strictly proper” for scoring rules that are designed to elicit an entire distribution, rather than just the mean of a distribution. (See Gneiting, 2011a, who calls the latter kind *strictly consistent*.) For the remainder of this paper, we follow the language of Definition 2, which matches the usage in Gneiting (2011b).

Schervish, et al. (2013) consider the class of all scoring rules of the form

$$(2) \quad g(x, q) = \begin{cases} \int_x^q (v - x) d\lambda(v) & \text{if } x \leq q, \\ \int_q^x (x - v) d\lambda(v) & \text{if } x > q, \end{cases}$$

where λ is a measure that is mutually absolutely continuous with Lebesgue measure and is finite on every bounded interval. They also show that all such scoring rules are strictly proper for the class of probability measures that give finite mean to $g(X, q)$ for at least one q . The form (2) is suggested by equation (4.3) of Savage (1971).

Theorem 1 in Section 4 generalizes the equivalence of coherence₁ and coherence₂ to the class of strictly proper scoring rules of the form (2).

3. Infinite Previsions. If one wishes to consider arbitrary sets of random variables, including unbounded random variables, then there will be

examples of random variables that cannot be assigned finite previsions or expectations. One cannot expect a bookie to offer to pay an infinite amount in exchange for a finite-valued random variable, no matter how likely it is to take large values. In particular, (1) makes no sense as a criterion for coherence if previsions are allowed to be infinite. Instead, we interpret an infinite prevision as an offer to accept one side or the other of a gamble, but not both sides. For example, $P(X) = \infty$ means that the bookie would pay an arbitrarily large amount c in exchange for receiving X , but she would not accept any finite amount in exchange for paying out X . We express that idea formally in Definition 3.

DEFINITION 3. Let X be a random variable, and let B be a nonempty event. To say that $P(X|B) = \infty$ means that every finite number can be the price to buy X , but no number can be the price to sell X , under the condition that B occurs. Similarly, to say that $P(X|B) = -\infty$ means that every finite number can be the price to sell X , but no number can be the price to buy X , under the condition that B occurs. The resulting gambles $\alpha B[X - c]$ that the agent is willing to accept, namely those with $\alpha \leq 0$ when $P(X|B) = \infty$ and those with $\alpha \geq 0$ when $P(X|B) = -\infty$ are called *acceptable*. We also call each finite sum of acceptable gambles *acceptable*.

We can now extend coherence₁ to collections that include infinite previsions.

DEFINITION 4. Let $\{P(X_i|B_i) : i \in I\}$ be a collection of previsions. The previsions are *coherent*₁ if,

$$(3) \quad \inf_{\omega} \sum_{j=1}^n \alpha_j B_{i_j}(\omega) [X_{i_j}(\omega) - c_j] \leq 0,$$

for all $\{i_1, \dots, i_n\} \subseteq I$, all real $\alpha_1, \dots, \alpha_n$ such that $\alpha_j \leq 0$ for all j with $P(X_{i_j}|B_{i_j}) = \infty$ and $\alpha_j \geq 0$ for all j with $P(X_{i_j}|B_{i_j}) = -\infty$, and all real c_1, \dots, c_n such that $c_j = P(X_{i_j}|B_{i_j})$ for each j such that $P(X_{i_j}|B_{i_j})$ is finite. That is, no acceptable gamble leads to uniform sure loss.

A necessary condition for $P(X) = \infty$ to be coherent₁ is that X is unbounded above, and a necessary condition for $P(X) = -\infty$ to be coherent₁ is that X is unbounded below. But there are unbounded random variables with coherent₁ finite previsions.

EXAMPLE 1. Let Ω be the integers, and suppose that $P(\{\omega = k\}) = 2^{-k}$ for each integer $k \geq 1$. This is a countably additive probability. Let $X(\omega) =$

$\sum_{k=1}^{\infty} k\{\omega = k\}$. The countably additive expectation of X is 2, which is a coherent₁ finite prevision for X . Actually, every number greater than or equal to 2 (including ∞) is a coherent₁ prevision for X . On the other hand, if $Y(\omega) = \sum_{k=1}^{\infty} 2^k\{\omega = k\}$, then the only coherent₁ prevision for Y is ∞ . Also, let $Z(\omega) = \sum_{k=1}^{\infty} -k\{\omega = -k\}$. The countably additive expectation of Z is 0, but we will set $P(Z) = -\infty$. Finally, let $V = Y + Z$. We cannot assign $P(V) = \infty - \infty$, however every extended real number is a possible coherent₁ prevision for V . The rest of this example shows that the stated previsions are coherent₁. The random variables to which we have assigned previsions are the indicators of the events $\{\omega = k\}$ for all integers k and the random variables X, Y, Z, V . The most general acceptable gamble is a linear combination of finitely many of the indicators and $X - x$ (with $x \geq 2$ and nonpositive coefficient if $P(X) = \infty$) together with $Y - y$ (with y real and nonpositive coefficient), $Z - z$ (with z real and nonnegative coefficient), and $V - v$ (with v real and coefficient whose sign is opposite to the sign of the prevision if $P(V)$ is infinite. That is, let k_1, \dots, k_n be integers, and let

$$W(\omega) = \sum_{j=1}^n \alpha_j [\{\omega = k_j\} - 2^{-k_j}] + \alpha_X[X - x] + \alpha_Y[Y - y] + \alpha_Z[Z - z] + \alpha_V[V - v],$$

where $x \geq 2$, y, z and v are finite, $\alpha_Y \leq 0$, $\alpha_Z \geq 0$, $\alpha_X \leq 0$ if $P(X) = \infty$, and the sign of α_V is opposite to the sign of $P(V)$ if $P(V)$ is infinite. Then all acceptable gambles are of the form of W . If $\alpha_Y < 0$ or if $\alpha_Z > 0$ or if $\alpha_X < 0$ or if $\alpha_V \neq 0$, then W clearly takes some nonnegative values so that the infimum is at least 0. The only cases not handled yet have $\alpha_X \geq 0$ and $\alpha_Y = \alpha_Z = \alpha_v = 0$. In this case,

$$W(\omega) = \sum_{j=1}^n \alpha_j [\{\omega = k_j\} - 2^{-k_j}] + \alpha_X[X - 2] - \alpha_X(c - 2).$$

Since $-\alpha_X(c - 2) \leq 0$, W is no greater than

$$W'(\omega) = \sum_{j=1}^n \alpha_j [\{\omega = k_j\} - 2^{-k_j}] + \alpha_X[X - 2].$$

Since $\sum_{\omega=1}^{\infty} W'(\omega)2^{-\omega} = 0$, $\inf_{\omega} W'(\omega) \leq 0$, hence $\sup_{\omega} W(\omega) \leq 0$, and the previsions are coherent₁.

An alternative definition of coherent infinite marginal previsions was presented by Crisma et al. (1997). We repeat their definition here, and then prove that it is equivalent to Definition 4 for marginal previsions.

DEFINITION 5. Let I be an index set, and let $\mathcal{D} = \{X_i : i \in I\}$ be a collection of random variables defined on Ω . Let P be an extended-real-valued function defined on \mathcal{D} . We say that P is *extended-coherent* if

$$(4) \quad \inf_{\omega \in \Omega} \sum_{j=1}^n \alpha_j X_{i_j}(\omega) \leq \sum_{j=1}^n \alpha_j P(X_{i_j}) \leq \sup_{\omega \in \Omega} \sum_{j=1}^n \alpha_j X_{i_j}(\omega),$$

for every finite integer n , all $i_1, \dots, i_n \in I$, and all real $\alpha_1, \dots, \alpha_n$ such that all infinite terms of the form $\alpha_j P(X_{i_j})$ have the same sign.

It is not clear how Definition 5 regulates the prices that a bookie is willing to pay/accept for various gambles with infinite previsions. Nevertheless, Definitions 5 and 4 are equivalent when applied to marginal previsions.

LEMMA 1. *A collection $\{P(X_i) : i \in I\}$ of marginal previsions is extended-coherent if and only if it is coherent₁.*

PROOF. For the “if” direction, assume that the previsions are coherent₁. Let n be a finite integer, let $i_1, \dots, i_n \in I$, and let $\alpha_1, \dots, \alpha_n$ be real numbers such that all infinite values of $\alpha_j P(X_{i_j})$ have the same sign. We need to show that (4) holds. First, suppose that all of the $P(X_{i_j})$ are finite. By coherence₁, we know that

$$(5) \quad \inf_{\omega \in \Omega} \sum_{j=1}^n \alpha_j [X_{i_j}(\omega) - P(X_{i_j})] \leq 0,$$

$$(6) \quad \inf_{\omega \in \Omega} \sum_{j=1}^n (-\alpha_j) [X_{i_j}(\omega) - P(X_{i_j})] \leq 0.$$

Inequality (5) implies the first inequality in (4), and (6) implies the second. Next, suppose that there are some infinite previsions among the $P(X_{i_j})$ and all of the corresponding $\alpha_j P(X_{i_j})$ have the same sign. If the common sign is negative, then $\sum_{j=1}^n \alpha_j P(X_{i_j}) = -\infty$, the second inequality in (4) is trivially satisfied, and the α_j corresponding to infinite previsions all have the correct signs to be used in acceptable gambles. It follows that for all real c_1, \dots, c_n such that $c_j = P(X_{i_j})$ whenever $P(X_{i_j})$ is finite,

$$\inf_{\omega \in \Omega} \sum_{j=1}^n \alpha_j [X_{i_j}(\omega) - c_j] \leq 0,$$

which implies the first inequality in (4). If the common sign of the $\alpha_j P(X_{i_j})$ is positive, then $\sum_{j=1}^n \alpha_j P(X_{i_j}) = \infty$, the first inequality in (4) is trivially

satisfied, and the α_j corresponding to infinite previsions all have the wrong signs to be used in acceptable gambles. It follows that for all real c_1, \dots, c_n such that $c_j = P(X_{i_j})$ whenever $P(X_{i_j})$ is finite,

$$\inf_{\omega \in \Omega} \sum_{j=1}^n (-\alpha_j)[X_{i_j}(\omega) - c_j] \leq 0,$$

which implies the second inequality in (4).

For the “only if” direction, assume that the previsions are extended-coherent. Let n be a finite integer, let $i_1, \dots, i_n \in I$, and let $\alpha_1, \dots, \alpha_n$ be real numbers such that all infinite values of $\alpha_j P(X_{i_j})$ are negative. We need to show that

$$(7) \quad \inf_{\omega \in \Omega} \sum_{j=1}^n \alpha_j [X_{i_j}(\omega) - c_j] \leq 0,$$

for all real c_1, \dots, c_n such that $c_j = P(X_{i_j})$ for all i such that $P(X_{i_j})$ is finite. If all of the $P(X_{i_j})$ are finite, then (7) follows from (4), so assume that at least one $P(X_{i_j})$ is infinite. It follows that $\sum_{j=1}^n \alpha_j P(X_{i_j}) = -\infty$, and (4) implies that $\inf_{\omega \in \Omega} \sum_{j=1}^n X_{i_j}(\omega) = -\infty$. Since, $\sum_{j=1}^n \alpha_j c_j$ is finite, no matter what c_j values are chosen, (7) follows. \square

Crisma and Gigante (2001) extend Definition 5 to conditional previsions. Their definition imposes conditions on coherence similar to those of Regazzini (1987) (both for bounded and for unbounded random variables) that are designed to regulate the extreme indeterminacy of conditional previsions given events with zero probability. We prefer to avoid such restrictions on the definition of coherence for reasons illustrated by the following example.

EXAMPLE 2. Let Ω be the positive integers, and let $P(\cdot)$ be a coherent₂ prevision that assigns probability 0 to every singleton $\{n\}$ with n a positive integer. Many such marginal previsions exist. Let

$$X(\omega) = \begin{cases} \omega & \text{if } 1 \leq \omega \leq 4, \\ 0 & \text{otherwise,} \end{cases}$$

and let $B = \{1, 2, 3, 4\}$ so that $P(B) = P(X) = 0$ and $BX = X$. The gamble

$$(8) \quad \alpha X = \alpha(X - 0) = \alpha B(X - 0)$$

is acceptable by all definitions of coherence. However, the restrictions that Regazzini (1987), Crisma and Gigante (2001), and others impose would say

that it is incoherent to assign $P(X|B)$ a value outside of the closed interval $[1, 4]$. That is, it is incoherent to offer the gamble $\alpha B(X - 0)$ because $0 \notin [1, 4]$. But, (8) shows that $\alpha B(X - 0)$ is already being offered without even contemplating what would be a coherent value for $P(X|B)$. Furthermore, for every real p , $-\alpha p B$ is also being offered unconditionally, so that the sum

$$(9) \quad \alpha B(X - 0) - \alpha p B = \alpha B(X - p)$$

is being offered for every real p . We think it is perfectly reasonable to assign $P(X|B)$ a value between 1 and 4 if one wishes, but we don't believe that it should be called incoherent to do otherwise. After all, the gambles that would be ruled out by such a declaration of incoherence are already being offered, as (9) illustrates.

The following lemma illustrates an intuitive property of infinite previsions.

LEMMA 2. *Let B be an event with $P(B) > 0$, and let X and Y be random variables with coherent₁ previsions $P(X|B)$ and $P(Y|B)$ respectively. If $X(\omega) \leq Y(\omega)$ for all $\omega \in B$, then $P(X|B) \leq P(Y|B)$.*

PROOF. Suppose, to the contrary, that $P(X|B) > P(Y|B)$. Then, in particular, $P(X|B) > -\infty$ and $P(Y|B) < \infty$. Coherence₁ implies that

$$(10) \quad \inf_{\omega} \{ \alpha B(\omega)[X(\omega) - c_X] + \beta B(\omega)[Y(\omega) - c_Y] + \gamma[B(\omega) - P(B)] \} \leq 0,$$

where $c_X = P(X|B)$ if $P(X|B)$ is finite, $c_Y = P(Y|B)$ if $P(Y|B)$ is finite, $\alpha \leq 0$ if $P(X|B) = \infty$, and $\beta \geq 0$ if $P(Y|B) = -\infty$. In (10), α , β , c_X and c_Y are otherwise unconstrained real numbers. Choose $c_X > c_Y$ if either is unconstrained. (If both are constrained, then $c_X > c_Y$ by assumption.) Let $\alpha = -1$, $\beta = 1$, and $\gamma = c_Y - c_X$. Then (10) becomes

$$\inf_{\omega} \{ -B(\omega)X(\omega) + B(\omega)Y(\omega) + (c_X - c_Y)P(B) \} \leq 0,$$

which is contradiction because $B(\omega)[Y(\omega) - X(\omega)] \geq 0$ for all ω , and $(c_X - c_Y)P(B) > 0$. \square

4. Extension of Coherence₂. In this section, we extend de Finetti's second coherence criterion in two ways: we include more general scoring rules, and we accommodate random variables with infinite previsions. For scoring rules of the form (2) we write

$$(11) \quad g(x, q) = \int_x^q (v - x) d\lambda(v),$$

where we use the convention that an integral whose limits are in the wrong order equals the negative of the integral with the limits in the correct order. It follows easily from (11) that, if a and b are real numbers, then

$$(12) \quad g(x, a) - g(x, b) = \lambda((a, b))[x - r(a, b, \lambda)],$$

where, for all a and b ,

$$(13) \quad r(a, b, \lambda) = \frac{\int_a^b v d\lambda(v)}{\lambda((a, b))}.$$

In equations (12) and (13), we used the same convention as above about integrals with limits in the wrong order. In particular, $\lambda((a, b)) = -\lambda((b, a))$ if $a > b$.

Equation 12 makes it clear that, if $P(X)$ is infinite, then $P[g(X, q)]$ is going to be infinite for all q . This is why Definition 2 includes the clause that $P(X)$ be finite before requiring that $P[g(X, q)]$ be minimized at $a = P(X)$. Nevertheless, strictly proper scoring rules can still be used to assess coherence in the spirit of coherence₂. The following is our generalization of coherence₂ that applies both with general scoring rules and with infinite previsions.

DEFINITION 6. Let \mathcal{C} be a class of strictly proper scoring rules. Let $\{(X_i, B_i) : i \in I\}$ be a collection of pairs each consisting of a random variable X_i and a nonempty event B_i with corresponding conditional forecasts $\{p_i : i \in I\}$. The forecasts are *coherent*₃ if, for every finite subset $\{i_j : j = 1, \dots, n\} \subseteq I$, every set of scoring rules $\{g_1, \dots, g_n\} \subseteq \mathcal{C}$, and every set $\{q_1, \dots, q_n\}$ of alternative forecasts,

$$\inf_{\omega} \sum_{j=1}^n B_{i_j}(\omega) [g_j(X_{i_j}(\omega), c_j) - g_j(X_{i_j}(\omega), q_j)] \leq 0,$$

where $c_j = p_{i_j}$ for all j such that p_{i_j} is finite, and c_j is finite and between q_j and p_{i_j} for all j such that p_{i_j} is infinite. That is, no finite rival set of forecasts can provide a uniformly smaller sum of scores than the original forecasts.

THEOREM 1. *Let \mathcal{C} be a class of scoring rules of the form (2). A collection of conditional previsions is coherent₁ if and only if it is coherent₃.*

PROOF. For the “only if” direction, assume that the conditional previsions are coherent₁. We want to show that no rival set of previsions provides

uniformly smaller sum of scores than the original previsions. That is, for each $(X_1, B_1), \dots, (X_n, B_n)$ with conditional previsions p_1, \dots, p_n and each set $\{g_1, \dots, g_n\} \subseteq \mathcal{C}$ of scoring rules and each set $\{q_1, \dots, q_n\}$ of rival previsions, we must show that

$$(14) \quad \inf_{\omega} \sum_{i=1}^n B_i(\omega) [g_i(X_i(\omega), c_i) - g_i(X_i(\omega), q_i)] \leq 0,$$

where $c_i = p_i$ for all i such that p_i is finite, and c_i is finite and between q_i and p_i for all i such that p_i is infinite. From (12),

$$(15) \quad \sum_{i=1}^n B_i [g_i(X_i, c_i) - g_i(X_i, q_i)] = \sum_{i=1}^n \lambda_i((c_i, q_i)) B_i [X_i - r(c_i, q_i, \lambda_i)].$$

Because $\lambda_i((c_i, q_i))$ and $r(c_i, q_i, \lambda_i) - c_i$ have the same sign,

$$\sum_{i=1}^n \lambda_i((c_i, q_i)) B_i [r(c_i, q_i, \lambda_i) - c_i] \geq 0.$$

Hence, the right-hand side of (15) is less than or equal to

$$(16) \quad \sum_{i=1}^n \lambda_i((c_i, q_i)) B_i [X_i - c_i].$$

The infimum of (15) is then less than or equal to the infimum of (16). By coherence₁,

$$\inf_{\omega} \sum_{i=1}^n \lambda_i((c_i, q_i)) B_i(\omega) [X_i(\omega) - c_i] \leq 0,$$

hence the rival previsions do not provide uniformly smaller sum of scores than the original previsions.

For the “if” direction, we prove the contrapositive. Assume that the conditional previsions are incoherent₁. If one of the previsions is the wrong value for a constant random variable, the result is trivial, so assume that each random variable takes at least two distinct values. We want to find a finite set of random variable/event pairs and corresponding conditional previsions together with a rival set of previsions such that they provide uniformly smaller sum of scores than the original set. That is, we need $(X_1, B_1), \dots, (X_n, B_n)$ with conditional previsions p_1, \dots, p_n a rival set of previsions q_1, \dots, q_n , and a set g_1, \dots, g_n of scoring rules from \mathcal{C} such that

$$(17) \quad \inf_{\omega} \sum_{i=1}^n B_i(\omega) [g_i(X_i(\omega), c_i) - g_i(X_i(\omega), q_i)] > 0,$$

where $c_i = p_i$ for all i such that p_i is finite, and c_i is finite and between q_i and p_i for all i such that p_i is infinite. In principle, some of the rival q_i could be infinite, but the rivals that we construct below will all be finite.

By incoherence₁, there exist $(X_1, B_1), \dots, (X_n, B_n)$ with conditional previsions p_1, \dots, p_n , $\alpha_1, \dots, \alpha_n$ and $\epsilon > 0$ such that

$$(18) \quad \inf_{\omega} \sum_{i=1}^n \alpha_i B_i(\omega) [X_i(\omega) - c_i] = \epsilon,$$

where c_i is finite for all i , $c_i = p_i$ for all i such that p_i is finite, and α_i has the opposite sign as p_i for all i such that p_i is infinite. Without loss of generality, we can assume that $\max_i |\alpha_i| = 1$. Let g be a scoring rule in \mathcal{C} with corresponding measure λ . Let $g_i = g$ for all i . Define

$$z_0 = \min \left\{ \min_{i=1, \dots, n} \{ \lambda((-\infty, c_i)) : \alpha_i < 0 \}, \min_{i=1, \dots, n} \{ \lambda((c_i, \infty)) : \alpha_i > 0 \} \right\},$$

so that $z_0 > 0$. For each $z \in (0, z_0)$, define $q_i(z)$ by the equation

$$\lambda((c_i, q_i(z))) = z\alpha_i,$$

which is possible because $\lambda((c_i, q))$ is continuous in q . Define

$$(19) \quad \ell(z) = \inf_{\omega} \sum_{i=1}^n B_i(\omega) [g(X_i(\omega), c_i) - g(X_i(\omega), q_i(z))].$$

By construction $c_i - r(c_i, q_i(z), \lambda)$, $c_i - q_i(z)$, and $r(c_i, q_i(z), \lambda) - q_i(z)$ have the opposite sign as α_i for each i , and $c_i - q_i(z)$ has the largest absolute value of the three differences. This means that, for all i such that p_i is infinite and all z , c_i is between $q_i(z)$ and p_i . The remainder of the proof consists of showing that there exists $z \in (0, z_0)$ such that $\ell(z) > 0$. The desired rival previsions are then $q_1(z), \dots, q_n(z)$. From (12),

$$(20) \quad \begin{aligned} \sum_{i=1}^n B_i [g(X_i, c_i) - g(X_i, q_i(z))] &= \sum_{i=1}^n \lambda((c_i, q_i(z))) B_i [X_i - r(c_i, q_i(z), \lambda)] \\ &= \sum_{i=1}^n z\alpha_i B_i [X_i - c_i] + \sum_{i=1}^n z\alpha_i B_i [c_i - r_i(c_i, q_i(z), \lambda)] \\ &\geq z\epsilon + z \sum_{i=1}^n \alpha_i B_i [c_i - q_i(z)]. \end{aligned}$$

Since, for all i , $q_i(0) = c_i$ and $q_i(z)$ is continuous in z , there exists $z_1 > 0$ such that, for all $z \in (0, z_1)$ and all ω ,

$$(21) \quad \sum_{i=1}^n \alpha_i B_i(\omega) [c_i - q_i(z)] > -\frac{\epsilon}{2}.$$

Choose $z \in (0, \min\{z_0, z_i\})$. Then combine (19), (20) and (21) to conclude that

$$\ell(z) \geq z \frac{\epsilon}{2} > 0,$$

which completes the proof. \square

5. Prevision and Expectation. The expectation of a random variable X defined on Ω is usually defined as the integral of X over the set Ω with respect to the underlying probability measure defined on subsets Ω . In the countably additive setting, such integrals can be defined (except for certain cases involving $\infty - \infty$) uniquely from a probability measure on Ω . Dunford and Schwartz (1958, Chapter III) give a detailed analysis of integration, with respect to finitely additive measures, that attempts to replicate the uniqueness of integrals. Their analysis requires additional assumptions if one wishes to integrate unbounded random variables.

An alternative to defining integrals with respect to specific measures is to define integrals as special types of linear functionals. Then measures can be constructed from the integrals. (The integral of the indicator of a set is the measure of the set.) This is the approach used in the study of the *Daniell integral*. (See Royden, 1968, Chapter 13. Regazzini, 1987 and Williams, 2007 take a similar approach for bounded random variables only.) DeFinetti's concept of prevision turns out to be a finitely additive generalization of the Daniell integral. (See Definition 7 below.) One major difference between the finitely additive Daniell integral and the theory developed by Dunford and Schwartz is that coherence₁ is the only assumption needed to define a finitely additive Daniell integral on an arbitrary space of random variables, including unbounded random variables and linear combinations of random variables with infinite integrals. Another difference is that multiple finitely additive Daniell integrals can lead to the same finitely additive probability distribution on Ω . Put another way, the finitely additive Daniell integral is not uniquely determined by its corresponding finitely additive probability.

In this paper, we take the approach of defining finitely additive expectations and probabilities in terms of finitely additive Daniell integrals, rather than integrals of the sort developed by Dunford and Schwartz (1958). This section is devoted to deriving and illustrating the properties of finitely additive Daniell integrals and their relation to coherent₁ previsions.

DEFINITION 7. Let \mathcal{L} be a linear space of real-valued functions defined on Ω that contains all constant functions, and let L be an extended-real-valued functional defined on \mathcal{L} . If $(X, Y \in \mathcal{L}$ and $X \leq Y)$ implies $L(X) \leq L(Y)$, we say that L is *nonnegative*. We call L an *extended-linear functional* on \mathcal{L} ,

if, for all real α, β and all $X, Y \in \mathcal{L}$,

$$(22) \quad L(\alpha X + \beta Y) = \alpha L(X) + \beta L(Y),$$

whenever the arithmetic on the right-hand side of (22) is well defined (i.e., not $\infty - \infty$) and where $0 \times \pm\infty = 0$. A nonnegative extended-linear functional is called a *finitely additive Daniell integral*. (See Schervish et al, 2008a.) If $L(1) = 1$, we say that L is *normalized*. A normalized finitely additive Daniell integral is called a *finitely additive expectation*.

The following simple result contains the primary justification for the names *finitely additive Daniell integral* and *finitely additive expectation*.

LEMMA 3. *Let L be a finitely additive Daniell integral on \mathcal{L} , where \mathcal{L} contains indicators of some subsets of Ω .*

1. *The restriction of L to those subsets of Ω whose indicators are in \mathcal{L} is a finitely additive measure. If $L(1) = 1$, then L is a finitely additive probability.*
2. *The restriction of L to the simple functions in \mathcal{L} matches the usual definition of integral of a simple function with respect to the measure in part 1.*

PROOF. 1. Because the constant 1 is in \mathcal{L} , the indicator of Ω itself is in \mathcal{L} . If A and B are disjoint, the indicator of $A \cup B$ is $A + B$. If A, B and $A \cup B$ are all in \mathcal{L} , then $L(A)$ and $L(B)$ are both nonnegative, and linearity give $L(A \cup B) = L(A) + L(B)$, and the sum is well defined. So L is a finitely additive measure. If $L(1) = 1$, nonnegativity implies that L , restricted to the set of indicators of events, is a finitely additive probability on Ω .

2. Let $X = \sum_{i=1}^n \alpha_i A_i$ be a simple function with each $A_i \in \mathcal{L}$, all A_i disjoint, and all α_i distinct. Then linearity gives $L(X) = \sum_{i=1}^n \alpha_i L(A_i)$, which is well defined and which matches the usual definition of the integral of X with respect to the measure L .

□

Finitely additive expectations are like integrals in several ways, e.g., the two parts of Lemma 3 as well as the linearity in Lemma 5 below. Another way in which finitely additive expectations are like integrals is continuity with respect to the uniform metric. The proof of the following result is trivial and omitted.

PROPOSITION 1. *Let X and Y be elements of a linear space \mathcal{L} of real-valued functions defined on Ω . Suppose that $\sup_{\omega} |X(\omega) - Y(\omega)| \leq \epsilon$. Then, for each finitely additive expectation L on \mathcal{L} , either $|L(X) - L(Y)| \leq \epsilon$ or $L(X)$ and $L(Y)$ equal the same infinite value.*

A special application of Proposition 1 is to a bounded function X . Every bounded function can be approximated arbitrarily closely by simple functions. So long as all of the approximating simple functions are in \mathcal{L} , their finitely additive expectations will be arbitrarily close to the expectation of X . For example, if \mathcal{L} is the set of functions that are measurable with respect to a σ -field, then every bounded function in \mathcal{L} is uniformly approximable by simple functions in \mathcal{L} . The expectations of all bounded random variables are then uniquely determined from the finitely additive probability on \mathcal{L} . Hence, when the finitely additive expectation defined here is restricted to bounded functions measurable with respect to a σ -field, it is the same as the definition of integral developed by Dunford and Schwartz (1958), and it is the same as the integral used by Dubins (1975) in his results about disintegrability.

On a linear space, coherent₁ previsions are the same as finitely additive expectations. One direction is simpler to prove than the other.

LEMMA 4. *Let L be a finitely additive expectation on a linear space $\mathcal{L} = \{X_i : i \in I\}$. Then L is a coherent₁ marginal prevision.*

PROOF. Suppose, to the contrary, that a finitely additive expectation L is incoherent₁ when used as a marginal prevision. From the definition of finitely additive expectation, the domain of L is a linear space that contains all constants. Incoherence₁ implies that there exist $i_1, \dots, i_n \in I$, real numbers $\alpha_1, \dots, \alpha_n$, and real numbers c_1, \dots, c_n such that α_j has the opposite sign as $L(X_{i_j})$ when $L(X_{i_j})$ is infinite, $c_j = L(X_{i_j})$ when $L(X_{i_j})$ is finite, and

$$\inf_{\omega} \sum_{j=1}^n \alpha_j [X_{i_j}(\omega) - c_j] > 0.$$

It follows that, there is $\epsilon > 0$ such that

$$\inf_{\omega} \sum_{j=1}^n \alpha_j X_{i_j}(\omega) = \epsilon + \sum_{j=1}^n \alpha_j c_j.$$

By nonnegativity and linearity of L ,

$$(23) \quad L \left(\sum_{j=1}^n \alpha_j X_{i_j} \right) \geq \epsilon + \sum_{j=1}^n \alpha_j c_j.$$

But

$$(24) \quad L \left(\sum_{j=1}^n \alpha_j X_{i_j} \right) = \sum_{j=1}^n \alpha_j L(X_{i_j}),$$

because the arithmetic on the right-hand side of (24) is well defined ($-\infty$ if infinite). A $-\infty$ value on the right-hand side of (24) would contradict (23) as would a finite value. \square

The other direction requires an additional result that is useful in its own right.

LEMMA 5. *Let $\mathcal{D} = \{X_i : i \in I\}$ be a set of random variables that contains all constants, and let Q be a coherent₁ prevision on \mathcal{D} . Let \mathcal{L} be that part of the linear span of \mathcal{D} that consists of linear combinations of the form $Y = \sum_{j=1}^n \alpha_j X_{i_j}$ where each $X_{i_j} \in \mathcal{D}$, and $p_Y = \sum_{j=1}^n \alpha_j Q(X_{i_j})$ is well-defined. Then $P(Y) = p_Y$ for all $Y \in \mathcal{L}$ is well-defined, and it is the unique coherent₁ extension of Q to \mathcal{L} .*

PROOF. We show first that P is well-defined, and then that it is the unique coherent₁ extension of Q to \mathcal{L} . Throughout the proof, we rely on Lemma 1 because the proof is slightly less cumbersome when worded in terms of extended-coherence. To see that P is well-defined, suppose to the contrary that $Y \in \mathcal{L}$ has two representations

$$(25) \quad Y = \sum_{j=1}^n \alpha_j X_{i_j} = \sum_{k=1}^m \beta_k X_{\ell_k},$$

such that

$$(26) \quad \sum_{j=1}^n \alpha_j Q(X_{i_j}) \neq \sum_{k=1}^m \beta_k Q(X_{\ell_k}),$$

where both sides of (26) are well-defined sums. That

$$(27) \quad 0 \equiv \sum_{j=1}^n \alpha_j X_{i_j} - \sum_{k=1}^m \beta_k X_{\ell_k}$$

is immediate from (25). Regardless of which sides (if any) of (26) are finite or infinite,

$$(28) \quad \sum_{j=1}^n \alpha_j Q(X_{i_j}) - \sum_{k=1}^m \beta_k Q(X_{\ell_k}) \neq 0,$$

and the sum on the left-hand side of (28) is well-defined. This contradicts extended-coherence when combined with (27).

That P is extended-coherent is immediate from Definition 5. It is also clear that P extends Q to \mathcal{L} . For each $Y \in \mathcal{L}$, to see that p_Y is the unique extended-coherent value for $P(Y)$, let $Y \in \mathcal{L}$ be represented as stated and let p be an extended real number such that

$$(29) \quad p \neq \sum_{j=1}^n \alpha_j Q(X_{i_j}).$$

We prove that setting $P(Y) = p$ is extended-incoherent. The sum

$$(30) \quad -p + \sum_{j=1}^n \alpha_j Q(X_{i_j})$$

is well-defined, regardless of whether or not either side or both sides of (29) are finite. Then $0 \equiv -Y + \sum_{j=1}^n \alpha_j X_{i_j}$ but (30) is not 0, which makes $P(Y) = p$ extended-incoherent when combined with existing previsions. \square

We can now prove the converse to Lemma 4.

LEMMA 6. *Let P be a coherent₁ marginal prevision on a linear space \mathcal{L} that contains all constants. Then P is a finitely additive expectation.*

PROOF. That $P(1) = 1$ is immediate from coherence₁. Lemma 2 tells us that $X \leq Y$ implies $P(X) \leq P(Y)$. That P is extended-linear follows from Lemma 5. \square

The ways in which coherent₁ previsions are more general than finitely additive expectations are the following:

- The domain of a coherent₁ prevision need not be a linear space.
- The domain of a coherent₁ prevision need not contain all constants.
- The definition of finitely additive expectation does not, as it stands, deal with conditional expectations.

It is trivial to extend the domain of a coherent₁ prevision to include all constants. To extend the domain of a coherent₁ prevision beyond what Lemma 5 provides is the subject of the Fundamental Theorem of Prevision (Theorem 2). The results in Section 6 give the tools needed to give meaning to finitely additive conditional expectation.

6. Relationship Between Conditional Previsions and Marginal Previsions. The relationship between marginal previsions and integrals is much more intuitive than the relationship between general conditional previsions and integrals. Even in the countably additive theory, conditional expectations are defined as Radon-Nikodym derivatives rather than as integrals. In the finitely additive theory used here, we defined finitely additive expectations (marginal previsions) as a type of integral (see Section 5). To a large extent, conditional previsions and marginal previsions determine each other. We make that statement precise in this section.

The results of this section fall into two categories:

- There are some relationships between conditional and marginal previsions that must hold in order for them to be jointly coherent₁. For example

$$P(XB) = P(B)P(X|B),$$

whenever the the product on the right is not 0 times an infinite value.

- There are some cases with $P(B) = 0$ in which $P(X|B)$ is completely unconstrained by other specified previsions.

The first result is that, when $P(B) > 0$, $P(X|B)$ is uniquely determined from $P(B)$ and $P(XB)$.

LEMMA 7. *Let $\mathcal{D} = \{(X_i, B_i) : i \in I\}$ be a collection of random variable/nonempty event pairs with coherent₁ conditional previsions $\{P(X_i|B_i) : i \in I\}$. Let (X, B) be a random variable/nonempty event pair such that (B, Ω) and (XB, Ω) are in \mathcal{D} and $P(B) > 0$. Then the only possible coherent₁ value for $P(X|B)$ is $P(XB)/P(B)$ (even if the numerator is infinite.)*

PROOF. Let $P(B) = q$. First, assume that $P(XB) = p$ is finite. We start by showing that the gamble $\alpha B(X - p/q)$ is acceptable for all real α . We know that $\alpha(XB - p)$ and $-(p/q)\alpha(B - q)$ are acceptable for all real α . Hence the following gamble is acceptable:

$$\alpha(XB - p) - (p/q)\alpha(B - q) = \alpha B(X - p/q).$$

To see that every value other than p/q is incoherent₁, suppose that $P(X|B) = r \neq p/q$. Then the following gamble would be acceptable for all real α :

$$\alpha(XB - p) - r\alpha(B - q) - \alpha B(X - r) = \alpha(-p + rq) > 0,$$

when α has the same sign as $-p + rq$. Next assume that $P(XB) = \infty$, as the $P(XB) = -\infty$ case is similar. First, we show that all gambles of the

form $\alpha B(X - c)$ with c real and $\alpha \leq 0$ are acceptable. For each $\alpha \leq 0$ and real c , let $c_1 = cq$ and $\beta = -\alpha c$. Then

$$\alpha B(X - c) = \alpha(BX - c_1) + \beta(B - q),$$

and the two gambles on the right are acceptable. To see that $P(X|B) < \infty$ is incoherent₁, let $\alpha > 0$, and let c be real. If $\alpha B(X - c)$ were acceptable, then the following gamble would be acceptable for all real d :

$$\alpha B(X - c) - \alpha(XB - d) + \alpha c(B - q) = -\alpha(d + cq).$$

Let $d < -cq$ to see that the infimum is positive. \square

When $P(B) = 0$, there may be multiple possible coherent₁ values of $P(X|B)$, but the set of possible values are determined from $P(XB)$.

LEMMA 8. *Let $\mathcal{D} = \{(X_i, B_i) : i \in I\}$ be a collection of random variable/nonempty event pairs with coherent₁ conditional previsions $\{P(X_i|B_i) : i \in I\}$. Let (X, B) be a random variable/nonempty event pair such that (B, Ω) and (XB, Ω) are in \mathcal{D} and $P(B) = 0$.*

1. *If $P(XB) \neq 0$, the only possible coherent₁ value for $P(X|B)$ is the infinite value with the same sign as $P(XB)$.*
2. *If $P(XB) = 0$, one can choose $P(X|B)$ to be any extended-real number c_X .*

PROOF. 1. Let c' have the same sign as $P(XB)$ if $P(XB)$ is infinite, and let $c' = P(XB)$ otherwise. To see that the specified infinite value is coherent₁, we will show that, for all real c and all α with the opposite sign as $P(XB)$, $\alpha B(X - c)$ is an acceptable gamble plus a positive number. For each such α and c ,

$$\alpha B(X - c) = \alpha(BX - c') - \alpha cB + \alpha c',$$

where $\alpha c' < 0$ and the other two gambles on the right are acceptable. To see that no other value is coherent₁, suppose that we try to set $P(X|B) = p$, where p is not the infinite value with the same sign as $P(XB)$. Then the following gamble is acceptable, where α has the opposite sign as $P(XB)$, and $c = p$ if p is finite:

$$-\alpha B(X - c) + \alpha(XB - c') - \alpha cB = -c'\alpha > 0.$$

Hence $P(X|B) = p$ is incoherent₁.

2. We show that all gambles of the form $\alpha_X B(X - c_X)$ with α_X and c_X real are acceptable. For each such α_X and c_X , let $\beta = -\alpha_X c_X$. Then

$$\alpha_X B(X - c_X) = \alpha_X XB + \beta B,$$

and the two gambles on the right are acceptable. □

We are now in a position to define finitely additive conditional expectation.

DEFINITION 8. Let \mathcal{L} be a linear space of random variables, and let \mathcal{B} be a collection of nonempty events that includes Ω . Suppose that $XB \in \mathcal{L}$ for every $X \in \mathcal{L}$ and $B \in \mathcal{B}$. Let L be a finitely additive expectation on \mathcal{L} . Let $L(\cdot|\cdot) : \mathcal{L} \times \mathcal{B} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be such that for each $B \in \mathcal{B}$, $L(\cdot|B)$ is a finitely additive expectation on \mathcal{L} with $L(XB|B) = L(X|B)$ for all $X \in \mathcal{L}$ and such that $\{L(X|B) : X \in \mathcal{L}, B \in \mathcal{B}\}$ is a coherent₁ conditional prevision. Then we call $L(\cdot|\cdot)$ a *finitely additive conditional expectation*.

The following result, whose proof is trivial, says that coherent₁ conditional prevision given an event of positive probability defines a finitely additive conditional expectation.

PROPOSITION 2. Let P be a coherent₁ prevision on a linear space \mathcal{L} containing all constants. Let $\mathcal{B} = \{B : P(B) > 0\}$. Assume that $XB \in \mathcal{L}$ for each $X \in \mathcal{L}$ and $B \in \mathcal{B}$. Define $L(X|B) = P(XB)/P(B)$ for all $X \in \mathcal{L}$ and $B \in \mathcal{B}$. Then $L(\cdot|\cdot)$ is a finitely additive conditional expectation.

In light of Lemma 8, it is clear that not every coherent₁ conditional prevision given an event of 0 probability defines a finitely additive conditional expectation. After we prove the Fundamental Theorem of Prevision, we can show (Lemma 10) that, if we start with a coherent₁ marginal prevision, then for each nonempty event with 0 probability, there exists a coherent₁ conditional prevision that defines a finitely additive conditional expectation.

7. The Fundamental Theorem of Prevision. Attempts to define a unique expectation from a probability often fail to provide an expectation for a random variable of the form $X - Y$ where both X and Y have infinite integral, especially when XY is identically 0. Finitely additive expectations for such random variables are guaranteed to exist, but they may not be unique. The tool for extending a coherent₁ collection of (conditional) previsions to a larger collection is the fundamental theorem of prevision.

De Finetti (1974) proved an elementary version of the fundamental theorem of prevision. That versions said that, if a collection of events has been assigned coherent₁ previsions, then for each additional event E , there is a nonempty closed interval such that one can coherently choose $P(E)$ to equal any number in that interval. Here, we prove a very general version that applies to unbounded random variables, infinite previsions, conditional previsions, and extensions to arbitrary collections of random variables.

The main step in the general version of the fundamental theorem is to start with a collection of coherent₁ conditional previsions and add one additional coherent₁ marginal prevision.

LEMMA 9. *Let $\mathcal{D} = \{(X_i, B_i) : i \in I\}$ be a collection of random variable/nonempty event pairs with coherent₁ conditional previsions $\{P(X_i|B_i) : i \in I\}$. Let X be a random variable. Then there exists a nonempty interval $[a, b]$ such that $P(X|\Omega) = p$ is coherent₁ with the previsions in \mathcal{D} if and only if $p \in [a, b]$.*

PROOF. Let \mathcal{C} be the set of acceptable gambles, and define

$$\begin{aligned} A &= \{f : \exists Y \in \mathcal{C} \text{ with } -Y + f \leq X\}, \\ B &= \{f : \exists Y \in \mathcal{C} \text{ with } Y + f \geq X\}, \\ a &= \sup A, \\ b &= \inf B. \end{aligned}$$

The first step is to show that $a \leq b$. If either $A = \emptyset$ or $B = \emptyset$, the inequality is trivial, so assume that neither set is empty. We need to show that for all $f^a \in A$ and $f^b \in B$, $f^a \leq f^b$. For all $f^a \in A$ and $f^b \in B$, there exist $Y^a, Y^b \in \mathcal{C}$ be such that

$$-Y^a + f^a \leq X \leq Y^b + f^b.$$

Hence

$$Y^a + Y^b \geq f^a - f^b.$$

Since $Y^a + Y^b \in \mathcal{C}$, $\inf_{\omega} [Y^a(\omega) + Y^b(\omega)] \leq 0$, it follows that $f^a - f^b \leq 0$, which completes the first step.

The second step is to show that every value in the interval $[a, b]$ is a coherent₁ value for $P(X|\Omega)$. Let $p \in [a, b]$. We need to show that, for every $Y \in \mathcal{C}$, $\inf_{\omega} \{Y(\omega) + \alpha[X(\omega) - c]\} \leq 0$, where $c = p$ if p is finite, and α has the opposite sign as p if p is infinite. Consider first, the case when p is finite. Suppose, to the contrary, that there exists $Y \in \mathcal{C}$ and α such that $\inf_{\omega} \{Y(\omega) + \alpha[X(\omega) - p]\} = \epsilon > 0$. Clearly $\alpha \neq 0$. If $\alpha > 0$, then $X \geq$

$p + (\epsilon/\alpha) - Y/\alpha$. Since $\alpha > 0$, $Y/\alpha \in \mathcal{C}$, so $p + (\epsilon/\alpha) \in A$ and $p + (\epsilon/\alpha) \leq a$, which contradicts $p \geq a$. Similarly, if $\alpha < 0$, then $X \leq p + (\epsilon/\alpha) - Y/\alpha$. Since $\alpha < 0$, $-Y/\alpha \in \mathcal{C}$, so $p + (\epsilon/\alpha) \in B$ and $p + (\epsilon/\alpha) \geq b$, which contradicts $p \leq b$. Next, consider the case in which $p = -\infty$ so that $\alpha > 0$, $a = -\infty$, and $A = \emptyset$. Then $Y/\alpha \in \mathcal{C}$, and $X \geq c + (\epsilon/\alpha) - Y/\alpha$, which means that $c + (\epsilon/\alpha) \in A$, which contradicts $A = \emptyset$. Finally, if $p = \infty$, then $\alpha < 0$, $b = \infty$, $B = \emptyset$, $-Y/\alpha \in \mathcal{C}$, $X \leq c + (\epsilon/\alpha) - Y/\alpha$, $c + (\epsilon/\alpha) \in B$, a contradiction.

The last step is to prove that every number outside of the interval $[a, b]$ is an incoherent₁ choice for $P(X|\Omega)$. First, assume that we choose $P(X|\Omega) = p < a$. Of necessity, $A \neq \emptyset$, $a > -\infty$, and for every $f \in A$, there exists $Y \in \mathcal{C}$ such that $-Y + f \leq X$. Let $\alpha > 0$, let $c = p$ if p is finite and $c < a$ if $p = -\infty$. Next, choose $f \in (c, a]$ and $Y \in \mathcal{C}$ such that $-Y + f \leq X$. Then $\alpha Y \in \mathcal{C}$, and

$$\alpha(X - c) + \alpha Y \geq \alpha(f - c) > 0,$$

showing that $P(X|\Omega) = p$ is incoherent₁. Finally, assume that we choose $P(X|\Omega) = p > b$. Of necessity, $B \neq \emptyset$, $b < \infty$, and for every $f \in B$, there exists $Y \in \mathcal{C}$ such that $Y + f \geq X$. Let $\alpha < 0$, let $c = p$ if p is finite and $c > b$ if $p = \infty$. Next, choose $f \in [b, c)$ and $Y \in \mathcal{C}$ such that $Y + f \geq X$. Then $-\alpha Y \in \mathcal{C}$, and

$$\alpha(X - c) - \alpha Y \geq \alpha(f - c) > 0,$$

showing that $P(X|\Omega) = p$ is incoherent₁. □

As an example of Lemma 9, we return to Example 1

EXAMPLE 3. In Example 1, Ω is the set of integers. We introduced a random variable $X = \sum_{k=1}^{\infty} k\{\omega = k\}$ after assigning previsions $P(\{\omega = k\}) = 2^{-k}$ for $k \geq 1$. We then proved (among other things) that every number in the interval $[2, \infty]$ could be chosen as a coherent₁ prevision for X . This fact actually follows from Lemma 9. Let $\mathcal{D} = \{(\{\omega = k\}, \Omega) : k \text{ an integer}\}$. The acceptable gambles have the form

$$Y(\omega) = \sum_{k=1}^{\infty} \alpha_k [\{\omega = k\} - 2^{-k}],$$

where only finitely many α_k are nonzero. In order for $-Y + f \leq X$ it is necessary and sufficient that

$$f \leq - \sum_{k=1}^{\infty} \alpha_k 2^{-k} + \inf_{k \geq 1} (k + \alpha_k) = 2 - \sum_{k=1}^{\infty} (\alpha_k + k) 2^{-k} + \inf_{k \geq 1} (k + \alpha_k).$$

Since $\sum_{k=1}^{\infty}(\alpha_k + k)2^{-k} \geq \inf_{k \geq 1}(k + \alpha_k)$, we have $f \leq 2$. We can get f as close as we want to 2 by choosing $\alpha_k = -k$ for all $1 \leq k \leq n$ and n large. This makes $a = 2$ in Lemma 9. Since $\sup_{\omega}[Y(\omega) + f]$ is finite for all acceptable gambles and all f , $Y + f \geq X$ is impossible. Hence, the set B in the proof of Lemma 9 is empty, and $b = \infty$.

We are now in position to prove the general extension theorem.

THEOREM 2. *Let $\mathcal{D} = \{(X_i, B_i) : i \in I\}$ be a collection of random variable/nonempty event pairs with coherent₁ conditional previsions $\{P(X_i|B_i) : i \in I\}$ and that contains all pairs of the form (c, Ω) with c a real number. Let \mathcal{F} be a set of random variable/nonempty event pairs that contains \mathcal{D} . Then there exists a coherent₁ extension P' of P to \mathcal{F} .*

PROOF. We prove the result by extending P one gamble at a time and applying transfinite induction to cover the whole set \mathcal{F} . Let $\mathcal{D}_0 = \mathcal{D}$ and $P_0 = P$. Well-order the set $\mathcal{F} \setminus \mathcal{D}$ as $\{(X_\gamma, B_\gamma) : 1 \leq \gamma \leq \Gamma\}$. For each $\gamma \leq \Gamma$, let $\mathcal{D}_\gamma = \mathcal{D} \cup \{(X_\beta, B_\beta) : 1 \leq \beta \leq \gamma\}$. For each successor ordinal $\gamma + 1$, assume that we have a coherent₁ extension P_γ to \mathcal{D}_γ . (The assumption is true by hypothesis when $\gamma = 0$.) We extend P_γ to $\mathcal{D}_{\gamma+1}$ as follows. Apply Lemma 9 with $\mathcal{D} = \mathcal{D}_\gamma$ and $X = B_{\gamma+1}$ to find a coherent₁ marginal prevision for $B_{\gamma+1}$. Let $\mathcal{D}' = \mathcal{D}_\gamma \cup \{(B_{\gamma+1}, \Omega)\}$. Apply Lemma 9 with $\mathcal{D} = \mathcal{D}'$ and $X = X_{\gamma+1}B_{\gamma+1}$ to find a coherent₁ marginal prevision for $X_{\gamma+1}B_{\gamma+1}$. Apply Lemma 7 or Lemma 8 to find a coherent₁ conditional prevision for $X_{\gamma+1}$ given $B_{\gamma+1}$. Set $P_{\gamma+1}(X_{\gamma+1}|B_{\gamma+1})$ equal to this coherent₁ conditional prevision. This completes the induction step for successor ordinals.

If γ is a limit ordinal, assume that we have a coherent₁ extension of P to P_β for all $\beta < \gamma$. Each $(X, B) \in \mathcal{D}_\gamma$ is either in \mathcal{D} or equals (X_β, B_β) for some $\beta < \gamma$. So we define $P_\gamma(X|B) = P_\beta(X|B)$. These previsions are coherent₁ because every finite collection appears in the induction at some $\beta < \gamma$. This completes the proof of the induction step. \square

Finally, we can prove the existence of finitely additive conditional expectations that agree with coherent₁ conditional previsions.

LEMMA 10. *Let \mathcal{D} be a collection of random variables that contains all constants with coherent₁ marginal previsions $P(\cdot)$. Let \mathcal{L} be the linear span of all functions of the form XB for $X \in \mathcal{D}$ and B a nonempty event. There exists a coherent₁ conditional prevision on the set $\{(X, B) : X \in \mathcal{L}, B \neq \emptyset\}$ such that $P(\cdot|B)$ is a finitely additive conditional expectation for each nonempty B .*

PROOF. Use Theorem 2 to extend P to $\{(X, \Omega) : X \in \mathcal{L}\}$, which includes all pairs of the form (B, Ω) with $B \neq \emptyset$. Lemma 7 and the first part of Lemma 8 specify all of the conditional previsions whose coherent₁ values are uniquely determined from the marginal previsions. The proof of the second part of Lemma 8 actually shows that $P(X|B)$ can simultaneously be set to arbitrary values for all (X, B) with $P(B) = P(XB) = 0$. The proof will be complete if we can choose values for all such $P(X|B)$ so that the resulting $P(\cdot|\cdot)$ is a finitely additive conditional expectation. The only restrictions that we have to obey are those caused by conditional previsions fixed by the first part of Lemma 8. For each B with $P(B) = 0$ and each $X \in \mathcal{L}$, let $X_B : B \rightarrow \mathbb{R}$ be defined by $X_B(\omega) = X(\omega)$ for $\omega \in B$. That is, X_B is X restricted to domain B . Let \mathcal{D}_B be the set of all (X_B, B) such that $X \in \mathcal{L}$ and either $P(X|B)$ is uniquely determined by the first part of Lemma 8 or X is constant on B . For constant X , set $Q(X_B|B)$ equal to that constant, and for all other X , let $Q(X_B|B) = P(X|B)$. This makes $Q(\cdot|B)$ a coherent₁ marginal prevision on \mathcal{D}_B with B as the state space. Use Theorem 2 with $\Omega = B$ and $\mathcal{D} = \mathcal{D}_B$ to extend Q to $\{(X_B, B) : X \in \mathcal{L}\}$, which makes $Q(\cdot|B)$ a finitely additive expectation on $\{X_B : X \in \mathcal{L}\}$. Define $P(X|B) = Q(X_B|B)$, and note that $P(XB|B) = P(X|B)$ and $P(\cdot|\cdot)$ is a finitely additive conditional expectation. \square

8. Discussion. In this paper we extend both of de Finetti's (1974) concepts of coherent prevision to unbounded random variables and infinite previsions. We define infinite prevision so that it makes sense in the gambling interpretation of prevision. We define how proper scoring rules can be used to score potentially infinite previsions. We extend the equivalence of the avoidance of sure loss from gambling and the nonexistence of uniformly smaller scores to a large class of strictly proper scoring rules and potentially infinite previsions. We use a finitely additive extension of the concept of Daniell integral to define finitely additive expectation on an arbitrary linear space of random variables, and show that it is equivalent to our extension of coherence to include infinite previsions. We give a version of the fundamental theorem of prevision that applies to unbounded random variables, infinite previsions, and conditional previsions. The fundamental theorem allows extension of a coherent conditional prevision from an arbitrary set of random variables to an arbitrary larger set of random variables, including the set of all random variables.

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MARK J. SCHERVISH AND JOSEPH B. KADANE
 DEPARTMENT OF STATISTICS
 CARNEGIE MELLON UNIVERSITY
 PITTSBURGH, PA 15213
 E-MAIL: mark@cmu.edu
 kadane@stat.cmu.edu

TEDDY SEIDENFELD
 DEPARTMENT OF STATISTICS
 AND DEPARTMENT OF PHILOSOPHY
 CARNEGIE MELLON UNIVERSITY
 PITTSBURGH, PA 15213
 E-MAIL: teddy@stat.cmu.edu