those cases where the base of the segment is greater than triple its height–just as HERO indicates at the end of *Metrica* I, 31. It thus appears that the scholiast has annotated his text of the *Geometrica* on the basis of a text of the *Metrica* and that the attribution to ARCHIMEDES in I, 32 has since dropped out of the tradition, as represented by the unique manuscript now extant. Just as in III, 17 and 23 (where excerpts of proofs from *SC II*, 3 and 4 are given), so also in I, 32 HERO must have reproduced a proof in explication of a theorem ascribed explicitly to ARCHIMEDES.

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# *Einstein and Hilbert: Two Months in the History of General Relativity*

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## 1. Introduction

The vast majority of current English language textbooks on relativity theory treat, either explicitly or implicitly, the field equations of general relativity as EINSTEIN's equations.[1] By contrast, PAULI[2] credits HILBERT as being co-discover of the field equations.[3] GUTH[4] has claimed that HILBERT deserves no credit since he knew of EINSTEIN's formulation of the field equations before proceeding to an after-the-fact derivation from a variational principle. GUTH's claims have in turn been disputed by MEHRA.[5] Questions about the priority of discoveries are often among the least interesting and least important issues in the history of science, and our main purpose here is to illuminate the development of EINSTEIN's ideas rather than to engage in priority disputes. It turns out, however, that for the crucial period of 1915, these two matters are intimately linked.

EINSTEIN corresponded regularly with MICHELE BESSO, PAUL EHRENFEST, ERWIN FREUNDLICH, H.A. LORENTZ, and ARNOLD SOMMERFELD. But during the period from late October to late November of 1915, the correspondence virtually ceases – in the EINSTEIN Papers at Princeton there are no letters from

[1] See, for example, C. MØLLER, *The Theory of Relativity* (Oxford: Clarendon Press, 1952); R. C. TOLMAN, *Relativity, Thermodynamics, and Cosmology* (Oxford: Clarendon Press, 1962); S. WEINBERG, *Gravitation and Cosmology* (New York: John Wiley, 1972); and R. ADLER, M. BAZIN & M. SCHIFFER, *Introduction to General Relativity* (New York: McGraw Hill, 1975).

[2] W. PAULI, *Theory of Relativity* (New York: Pergamon Press, 1958), footnote 277, p. 145.

[3] See also H. WEYL, "Zu David Hilberts siebzigstem Geburtstag." *Die Naturwissenschaften*, **20** (1932), 57–58, "50 Jahre Relativitätstheorie," *ibid*, **38** (1951), 73–83; P. JORDAN, *Schwerkraft und Weltall* (Braunschweig: F. Vieweg und Sohn, 1955), Secs. 13–14; and C. REID, *Hilbert* (New York: Springer-Verlag, 1970), p. 141.

[4] E. GUTH, "Contribution to the History of Einstein's Geomatry as a Branch of Physics," in M. CARMELI (ed.), *Relativity* (New York: Plenum Pres, 1970).

[5] J. MEHRA, "Einstein, Hilbert, and the Theory of Gravitation," in J. MEHRA (ed.), *The Physicist's Conception of Nature* (Dordrecht-Holland: D. Reidel, 1973).

EINSTEIN to any of these figures for the period in question.[6] The reason for this hiatus is given by EINSTEIN in a dramatic letter to SOMMERFELD, dated November 28, 1915:

> Dear Sommerfeld:
>
> Don't be angry with me for only today answering your friendly and interesting letter. But last month I had one of the most exciting, most strenuous times of my life, also one of the most rewarding. I could not concentrate on writing.
>
> I recognized, namely, that my former gravitational field equations were completely unfounded![7]

After he had succeeded in formulating the "correct" equations, there was a flood of letters from EINSTEIN to his friends, explaining the details of what he regarded as "the most priceless find I have made in my life."[8] The main outline of the route by which EINSTEIN arrived at this "most priceless find" can be discerned from the four papers he published in the Proceedings of the Berlin Academy during November of 1915. But additional light is shed by the surviving postcards and letters of the correspondence between EINSTEIN and HILBERT during this month.[9] This correspondence links the matter of EINSTEIN's progress with the issue of priority. But before turning to the details, some brief remarks about the respective approaches of EINSTEIN and HILBERT are in order.

EINSTEIN had been working intensively on gravitational theory since 1912. Besides the field equations which he proposed in 1913 (see Section 2 below), he had developed and published much of the framework of general relativity, including the ideas that gravitational effects require a tensor theory, that these effects determine a non-Euclidean geometry, that this metric role of gravitation results in a red shift and in the bending of light passing near a massive body. Gravitation had become a lively field of research during these years, with detailed theories proposed by ABRAHAM, MIE, NORDSTRÖM, and many others.[10] But among these many theories, there is little evidence that before

[6] Nor are there any letters to W. DÄLLENBACH, M. GROSSMANN or L. HOPF.

[7] A. HERMANN (ed.), *Albert Einstein/Arnold Sommerfeld Briefwechsel* (Basel: Schwabe and Co., 1968), p. 32. Our translations of the EINSTEIN-SOMMERFELD correspondence are taken from ROGER and HELGA STUEWER's translation of this book. We are grateful to them for their permission to use their unpublished translation.

[8] From a letter to SOMMERFELD, dated December 9, 1915: *Einstein/Sommerfeld Briefwechsel*, p. 37.

[9] The mail service, which was very efficient by current standards, allowed EINSTEIN and HILBERT to be in almost instantaneous contact – a letter mailed one day by EINSTEIN in Berlin or by HILBERT in Göttingen would reach the other the following day. As will be seen from our discussion below, the nature of the correspondence tempts one to speculate that the history of gravitational theory might have been significantly different if the mail delivery had been less efficient.

[10] See M. ABRAHAM, "Una nuova teoria della gravitazione," *Nuovo Cimento*, **4** (1912), 459–481; G. MIE, "Grundlagen einer Theorie der Materie," *Annalen der Physik* **37** (1912), 511–534; **39** (1912), 1–40; **40** (1913), 1–66; G. NORDSTRÖM, "Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips," *Annalen der Physik*, **42** (1913), 533–554.



1916 EINSTEIN's ideas were afforded any special prominence, despite the prominence of their author.

While we do not know exactly when HILBERT became interested in gravitational theory, we do know that by mid-1915 he was well acquainted with EINSTEIN's work, for that summer EINSTEIN addressed a seminar in Göttingen, HILBERT attended and EINSTEIN reported to SOMMERFELD that "In Göttingen I was extremely pleased that everything was understood in detail."[11] HILBERT was also familiar with GUSTAV MIE's theories and was at work on combining MIE's approach to a theory of matter with some of EINSTEIN's ideas on gravitation (see Section 5 below). HILBERT's aim was not simply synthetic; it was, rather, to illustrate the power of the axiomatic method in the domain of physics.[12]

## 2. The Einstein-Grossmann theory

In a paper of 1913, written jointly with his friend and collaborator MARCEL GROSSMANN, EINSTEIN had proposed the following field equations for gravitation:[13]

$$\Lambda^{ij} = \chi T^{ij}$$

$$\Lambda^{ij} \equiv 1/\sqrt{-g}\,(g^{km}\sqrt{-g}\,g^{ij}{}_{,m})_{,k} - g^{km} g_{nr} g^{in}{}_{,k} g^{jr}{}_{,m} + \tfrac{1}{2} g^{ki} g^{mj} g_{nr,k} g^{nr}{}_{,m} - \tfrac{1}{4} g^{ij} g^{km} g_{nr,k} g^{nr}{}_{,m} \tag{1}$$

where the $T^{ij}$ are the contravariant components of the energy-momentum tensor for "matter fields", the $g^{ij}$ and the $g_{ij}$ are respectively the contravariant and covariant components of the metric tensor, and $g$ is the determinant of $g_{ij}$. In public, EINSTEIN continued to maintain this theory until November of 1915; but his private attitude lacked the wholehearted commitment that was so characteristic of his approach to physics. EINSTEIN's ambivalence is evident in a letter to SOMMERFELD, dated July 15, 1915. On the one hand, EINSTEIN is concerned to make it clear that the theory of which (1) is the core is *his* theory:

> Grossmann will never claim to be co-discoverer. He only helped me to orient myself in the mathematical literature, but contributed nothing materially to the results.[14]

[11] *Einstein/Sommerfeld Briefwechsel*, p. 30.

[12] For more details, see L. PYENSON, *The Göttingen Reception of General Relativity*, Ph. D. thesis, Johns Hopkins University, 1974.

[13] A. EINSTEIN & M. GROSSMANN, "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation," *Zeitschrift für Mathematik und Physik* **62** (1913), 225–261. EINSTEIN's notation has been modernized. In particular, ordinary derivatives are denoted by a comma and covariant derivatives by a semi-colon. Latin indices run from 1 to 4, and the EINSTEIN summation convention on repeated indices is used throughout.

[14] *Einstein/Sommerfeld Briefwechsel*, p. 30. We shall discuss EINSTEIN's seemingly ungenerous treatment of GROSSMANN's contribution in another paper on the EINSTEIN-GROSSMANN collaboration during 1913–1914.

On the other hand, EINSTEIN is obviously less than enthusiastic about SOMMERFELD's plan to include the papers on "general relativity" in a new edition of *The Principle of Relativity*.

> I should like to see this little volume printed without including anything on the general theory of relativity, because none of the presentations to date are complete.... Should you insist upon including the gen[eral] theory of rel[ativity] in the new edition, it does not matter to me.[15]

EINSTEIN's lack of enthusiasm was due in large measure to the fact that the equations(1) are not generally covariant. Despite the considerations which EINSTEIN thought to require him to abandon general covariance,[16] he abandoned it "with a heavy heart"[17] and a nagging sense that something was not right.[18] Still, it was not until sometime in late 1915 that EINSTEIN concluded that the EINSTEIN-GROSSMANN theory was wrong. When exactly did EINSTEIN reach this conclusion? And what were the grounds on which it was based?

In a postcard to HILBERT, dated November 7, 1915,[19] EINSTEIN states that he had known for "about four weeks" that the demonstation of the field equations (1) which he had attempted in 1914 in a paper published in the Proceedings of the Berlin Academy[20] was "delusive." There is some independent confirmation of this dating in a letter to LORENTZ dated October 12, 1915,[21] in which EINSTEIN expressed reservations about his attempt to limit the choice of the Hamiltonian function that, in conjunction with a variational principle, was to yield the equations (1). And this is the reason for abandoning equations (1) that EINSTEIN gave in his paper of November 4, 1915, which was the first of four papers to appear in the Proceedings of the Berlin Academy for that month.[22]

In his paper of 1914, EINSTEIN had attempted to derive a characterization of the tensor density $\mathscr{G}_{ij}$ in a field equation of the form

[15] *Einstein/Sommerfeld Briefwechsel*, p. 30.

[16] See the Appendix to the "Entwurf" paper (Ref. 13).

[17] A. EINSTEIN, "Zur allgemeinen Relativitätstheorie," *Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, **44** (1915), p. 778.

[18] EINSTEIN's uneasiness about using non-generally covariant equations can be seen in his papers, "Zum gegenwärtigen Stande des Gravitationsproblems," *Physikalische Zeitschrift*, **14** (1913), esp. pp. 1257–1258, and "Prinzipielles zur verallgemeinerten Relativitätstheorie und Gravitationstheorie," *ibid*, **15** (1914), esp. pp. 176–178. See also EINSTEIN's correspondence with EHRENFEST and LORENTZ, EINSTEIN Papers, Princeton University, microfilm reels I.B. 1, nos. 9 and 16. We discuss these matters in detail in our paper "Lost in the Tensors: Einstein's Struggles with Covariance Principles from 1912 to 1916," to appear in *Studies in History and Philosophy of Science*.

[19] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 13.

[20] A. EINSTEIN, "Die formale Grundlage der allgemeinen Relativitätstheorie," *Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, **41** (1914), 1030–1085.

[21] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 16.

[22] Ref. 17.



$$\mathscr{G}_{ij}=\chi\mathscr{T}_{ij} \tag{1'}$$

where $\mathscr{G}_{ij}$ represents the gravitational field itself and where $\mathscr{T}_{ij}\equiv\sqrt{-g}\,T_{ij}$. The derivation begins with a version of the principle of relativity; namely, the requirement that the field equations be covariant under arbitrary linear coordinate transformations. From this requirement, EINSTEIN argues that $\mathscr{G}_{ij}$ must have the form

$$\mathscr{G}_{ij}=\frac{\partial(H\sqrt{-g})}{\partial g^{ij}}-\left(\frac{\partial(H\sqrt{-g})}{\partial g^{ij}{}_{,k}}\right)_{,k} \tag{2}$$

where $H$ is a function of $g^{ij}$ and of $g^{ij}{}_{,k}$. He then argues that $H$ is determined up to a constant factor by the further requirement that it be a homogeneous function of the second degree of the $g^{ij}{}_{,k}$. It was this derivation that EINSTEIN now found delusive.

As a matter of logic, perhaps, the failure of this derivation was not by itself a compelling reason for abandoning the theory, for the fact that one attempt to justify the theory fails does not show that no other justification is possible, much less that the theory is incorrect. But for EINSTEIN, who characteristically sought to found gravitational theory on some sweeping principle, his inability in this case to found his field equations on the principle of relativity may have been sufficient to undermine his confidence in the equations themselves. Thus, in his paper of November 4, 1915 he wrote:

> For this reason, I completely lost confidence in my earlier equations and sought a new method by which to restrict the possibilities in a natural way.[23]

There were, however, other considerations which may have been even more important cases of this loss of confidence. In correspondence EINSTEIN cites the consideration about the choice of the Hamiltonian in conjunction with other more compelling factors. In the letter to SOMMERFELD of November 28, EINSTEIN lists three reasons for his loss of confidence in the methods and results of the EINSTEIN-GROSSMANN theory:

> 1) I proved that the gravitational field for a uniform rotating system does not satisfy the field equations.
> 2) The motion of the perihelion of Mercury yielded 18″ instead of 45″ per century.
> 3) The covariance requirement in my paper of last year did not yield the Hamiltonian function $H$. It permits, if appropriately generalized, an arbitrary $H$.[24]

The same reasons are listed in a letter to LORENTZ dated January 1, 1916;[25] the only difference is that the order of points 1) and 2) is inverted.

Point 2) was obviously a serious blow since from very early on EINSTEIN

[23] Ref. 17, p. 778.

[24] *Einstein/Sommerfeld Briefwechsel*, pp. 32–33.

[25] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 16.

had believed that the precession of the perihelion of Mercury was a general relativistic effect. In order to understand point 1), it is necessary to take note of two facts. First, in his paper of 1914 in the Proceedings of the Berlin Academy and in a joint paper with GROSSMANN published in that same year,[26] EINSTEIN argued that although the equations (1) are not valid in all coordinate systems, they are valid in any system satisfying the coordinate condition

$$B_m \equiv (\sqrt{-g}\, g^{jk} g_{mn} g^{ni}{}_{,k})_{,i,j} = 0. \tag{3}$$

Coordinate systems satisfying (3) are called "adapted" and coordinate transformations between two such systems are called "justified." Second, EINSTEIN thought that because the justified transformations can be non-linear they therefore include transformations between stationary and rotating systems and, as a result, that the equations (1) fulfill a "principle of equivalence" for rotating and non-rotating systems. To BESSO he wrote in March 1914:

> From this [the covariance of the field equations (1) under justified coordinate transformation], it follows that there are transformations representing accelerations of a very varied nature which transform the equations into themselves (for example, rotations) so that the equivalence hypothesis is vouchsafed in its original form.[27]

Such an equivalence between rotating and non-rotating systems was thought to be a necessary condition for a resolution of "MACH's Paradox."[28] Indeed, in the paper EINSTEIN presented to the Berlin Academy on November 4, 1915, the single "application" of the new field equations was to demonstrate that they were covariant under transformation to a uniformly rotating coordinate system.[29]

Thus, it may be that 1) and 2) were the main reasons for EINSTEIN's rejection of the EINSTEIN-GROSSMANN theory and that 3) served chiefly as the seed of doubt. EINSTEIN's postcard to HILBERT on November 7 raises the intriguing possibility that HILBERT was the sower of the seed. EINSTEIN there reports that SOMMERFELD has written to him that HILBERT has voiced objections to the paper of 1914 in the Proceedings of the Berlin Academy. Since SOMMERFELD's letter to EINSTEIN has been lost, one can only speculate about the objections SOMMERFELD reported to EINSTEIN, but we do know that the main tool HILBERT used in his investigation of gravitation and electromagnetism was the variational principle and that one of his main desiderata on the

[26] A. EINSTEIN & M. GROSSMANN, "Kovarianzeigenschaften der Feldgleichungen der auf die verallgemeinerte Relativitätstheorie gegründeten Gravitationstheorie," *Zeitschrift für Mathematik und Physik*, **63** (1914), 215–225.

[27] P. SPEZIALI (ed.), *Albert Einstein/Michele Besso Correspondance* (Paris: Hermann, 1972), p. 53.

[28] The importance of "MACH's Paradox" for EINSTEIN's thinking about gravitation during this period can be seen from his papers "Bases Physiques d'une Théorie de la Gravitation," *Archives des Sciences Physiques et Naturelles* **37** (1914), 5–12, and "Zum Relativitäts-Problem," *Scientia* **15** (1914), 337–348. See also Ref. 53 below.

[29] Ref. 17, Sec. 4.



Hamiltonian function was that it be an invariant under arbitrary coordinate transformations (see Section 5 below). We also know the content of at least one of HILBERT's criticisms of EINSTEIN's paper of 1914 for the Berlin Academy. HILBERT pointed out that in deriving (2), EINSTEIN had used the incorrect relation

$$\Delta g^{ij}{}_{,k} = (\Delta g^{ij})_{,k} \tag{4}$$

where $\Delta g^{ij}{}_{,k} \equiv g'^{ij}{}_{,k} - g^{ij}{}_{,k}$ expresses the difference between the components of $g^{ij}{}_{,k}$ in primed and unprimed coordinate systems which are "infinitesimally close". EINSTEIN acknowledged his error in a letter to HILBERT dated March 30, 1916.[30]

By the beginning of November 1915, then, EINSTEIN was hard at work developing new field equations. His old foundational principles were not sufficient for carrying out this task; what was required was a new commitment to the principle of covariance, and, within the framework provided by that principle, considerable mathematical ingenuity. As we shall see, the success of EINSTEIN's project may also have required help from HILBERT.

## 3. Einstein's steps towards the field equations

In his postcard of November 7, 1915, EINSTEIN tells HILBERT that he is sending by the same post the proofs of his first paper for November, and he closes by saying: "I am curious whether you will be well disposed towards this solution." HILBERT would certainly not have been well disposed towards EINSTEIN's solution, for the new field equations, though in a sense an advance over the old EINSTEIN-GROSSMANN equations, are still not generally covariant. In that paper EINSTEIN splits the RICCI tensor $G_{im}$ into two parts,[31]

$$\begin{aligned} G_{im} &= R_{im} + S_{im}, \\ R_{im} &\equiv \left\{ {l \atop im} \right\}_{,l} + \left\{ {n \atop il} \right\} \left\{ {l \atop nm} \right\}, \\ S_{im} &\equiv \left\{ {l \atop il} \right\}_{,m} - \left\{ {n \atop im} \right\} \left\{ {l \atop nl} \right\}, \end{aligned} \tag{5}$$

where the $\left\{ {l \atop im} \right\}$ are the CHRISTOFFEL symbols of the second kind. The new field equations are

$$R_{im} = \chi T_{im}. \tag{6}$$

The left hand side of (6) is not a tensor, so the equations (6) are not generally covariant. However, they are covariant for coordinate transformations satisfying

[30] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 13.

[31] The current convention is to use $R_{im}$ for the RICCI tensor and $G_{im}$ for the EINSTEIN tensor. We shall use EINSTEIN's original notation.

$$\frac{\partial(x'_1, \ldots, x'_4)}{\partial(x_1, \ldots, x_4)} = 1 \tag{7}$$

since then $R_{im}$ transforms like a tensor.

An advantage of imposing the condition (7) is that $g$ is an invariant and thus, according to EINSTEIN, the distinction between tensors and tensor densities "falls away."[32] In his paper of 1914 for the Berlin Academy, EINSTEIN had taken the law governing $T^{ij}$, that is

$$T^{ij}{}_{;j} = K^i \tag{8}$$

where the $K^i$ are the components of the external force, and rewritten it in the form

$$\mathscr{T}^j_{i,j} = \tfrac{1}{2} g^{km} g_{mj,i} \mathscr{T}^j_k + \mathscr{K}_i \tag{9}$$

where $\mathscr{T}^j_i \equiv \sqrt{-g}\, T^j_i$ and $\mathscr{K}_i \equiv \sqrt{-g}\, K_i$. EINSTEIN's new line of thought seems to have been that since the distinction between tensors and tensor-densities falls away, (9) can be construed as

$$T^j_{i,j} = \tfrac{1}{2} g^{km} g_{mj,i} T^j_k + K_i. \tag{10}$$

When $T^j_i$ signifies the energy tensor for the entire matter field, EINSTEIN claims $K_i$ must vanish and (10) reduces to

$$\begin{aligned} T^j_{i,j} &= \tfrac{1}{2} g^{km} g_{mj,i} T^j_k \\ &= -\Gamma^k_{ij} T^j_k \end{aligned} \tag{11}$$

where the $\Gamma^k_{ij} = -\left\{ {k \atop ij} \right\}$ are the components of the connection.[33] EINSTEIN then shows that an ordinary conservation law for the combined matter and gravitational fields is satisfied in the sense that

$$(T^j_i + t^j_i)_{,j} = 0 \tag{12}$$

where $t^j_i \equiv \tfrac{1}{2}\chi(\delta^j_i g^{mn} \Gamma^k_{mr} \Gamma^r_{nk} - g^{mn} \Gamma^k_{mi} \Gamma^j_{nk})$ is the "energy tensor" for the gravitational field. (Throughout this period EINSTEIN kept distinct the stress energy tensor for matter, including electromagnetic radiation, $T^j_i$, and the stress-energy "tensor" for the gravitational field, $t^j_i$. The latter, of course, is not strictly a tensor at all, as EINSTEIN knew. Thus, in "accelerated" coordinate systems $t^j_i$ may be non-zero, but since $\Gamma^k_{mr}$ can always be made to vanish at a point by means of an appropriate choice of coordinates, $t^j_i$ will vanish in those coordinates. But a tensor equal to zero in one coordinate system is equal to zero in all. See below, p. 304.)

Using (11) and (12), EINSTEIN derives

$$(g^{ij}(\log\sqrt{-g})_{,j})_{,i} = \chi \sum_i T^i_i \tag{13}$$

[32] Ref. 17, p. 779.

[33] Ref. 17, pp. 782–783. We use the notion of a connection. EINSTEIN did not.



from which he concludes that the coordinate system cannot be so chosen that $\sqrt{-g} = \text{constant}$ since then the energy scalar $T \equiv \sum_i T^i_i$, for the matter fields would vanish.

EINSTEIN's procedure is incoherent. Equation (10) is equivalent to the condition $T^j_{i;j} = 0$ with $\mathscr{K}_i = (\log\sqrt{-g})_{,j} T^j_i$. But for arbitrary $T^j_i$, $K_i$ vanishes if and only if $\sqrt{-g}$ is a constant.

EINSTEIN did not remain content for very long with this strange theory. On November 11 he submitted a second paper to the Berlin Academy;[34] in this paper the field equations are now taken to be

$$G_{im} = \chi T_{im}. \tag{14}$$

The next day he sent a postcard to HILBERT announcing that he had finally achieved generally covariant field equations.[35] The equations (14) are certainly generally covariant, but they appear to involve a fatal difficulty. Since we now have general covariance, we are free to choose the coordinate system so that $\sqrt{-g} = \text{constant}$. Then $S_{im}$ vanishes and the field equations (14) reduce to the previous equations (6). Further, equation (11) and its consequences are now valid for the chosen coordinates, so by (13) we have the seemingly absurd result that $T = 0$.

Over the past three years, EINSTEIN had used imagined difficulties to justify the rejection of general covariance. Now, faced with a genuine difficulty, he proposes to maintain general covariance and argue that the difficulty is only apparent. After noting that in the usual electromagnetic theory, the energy scalar for the electromagnetic field is zero, he states:

> There are not just a few who hope that matter can be reduced to a purely electromagnetic phenomenon which, however, would have to follow a theory more general than Maxwell's electrodynamics. Now let us suppose for the moment that in such a general electrodynamics the scalar of the energy would also vanish! Would this result prove that matter could not be constructed with the help of this theory? I believe that this question can be answered in the negative, for it is entirely possible that gravitational fields are an essential ingredient in "matter" ... Then $\sum_i T^i_i$ can seem to be positive, whereas in reality only $\sum_i (T^i_i + t^i_i)$ is positive while $\sum_i T^i_i$ vanishes everywhere.[36]

Now EINSTEIN's faith in general covariance is so strong that in order to achieve it he is prepared to risk a highly speculative hypothesis about the constitution of matter. There is a related difficulty with equations (14), one that EINSTEIN did not note at the time, but which can be dealt with by the same device. In conjunction with the conservation law for the total matter field

[34] A. EINSTEIN, "Zur allgemeinen Relativitätstheorie (Nachtrag)," *Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin* **46** (1915), 799–801.

[35] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 13.

[36] Ref. 34, pp. 799–800.

$$T^{ij}{}_{;j}=0 \tag{15}$$

and the identity

$$(G^{ij}-\tfrac{1}{2}g^{ij}G)_{;j}=0 \tag{16}$$

where $G$ is the curvature scalar, (14) entails that $T$=constant, *i.e.*, inhomogeneous matter fields are not allowed.[37]

Two further points about the equations (14) should be mentioned. First, they do yield the correct equations for the free space case:

$$G_{im}=0. \tag{17}$$

Thus, although equations (14) are not "correct", they can be used to give a correct prediction for the free-space case; in particular, they can be used to predict the advance of the perihelion of Mercury. Sometime between the 11th and the 18th of November EINSTEIN succeeded in showing that (17) explains the anomalous advance of Mercury's perihelion, and on the 18th he communicated his results to the Berlin Academy.[38] Obviously excited, on the same day he wrote a postcard to HILBERT reporting his success. The card closes with the remark that "Until now no gravitational theory has succeeded in doing this."[39]

The second point about the equations (14) is that EINSTEIN and GROSSMANN had considered them in 1913 but had rejected them on the erroneous grounds that they do not have the right Newtonian limit.[40] Of course, as we have just seen, the equations (14) involve grave difficulties, but it is a reasonable conjecture that had they not been rejected out of hand in 1913, the final solution would have emerged long before the end of 1915.

## 4. The final field equations and the correspondence with Hilbert

On November 25 EINSTEIN submitted his fourth paper in a single month to the Berlin Academy.[41] It contained the now familiar field equations of general relativity in the form

$$G_{im}=\chi(T_{im}-\tfrac{1}{2}g_{im}T). \tag{18}$$

EINSTEIN's paper makes no reference to HILBERT. How did EINSTEIN arrive at these equations? What role, if any, did HILBERT play in the transition from (14) to (18)? And what are the merits of the conflicting claims of priority?

[37] EINSTEIN does not refer to the identity (16) in any of his papers of 1915. An important implication of this fact will be discussed below.

[38] A. EINSTEIN, "Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie," *Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, **47** (1915), 831–839.

[39] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 13.

[40] See reference 13, esp. p. 257. This matter is discussed in our paper "Lost in the Tensors: Einstein's Struggles with Covariance Principles from 1912 to 1916" (Ref. 18).

[41] A. EINSTEIN, "Die Feldgleichungen der Gravitation," *Sitzungsberichte der Königlichen Preussischen Akademie der Wissenschaften zu Berlin*, **48** (1915), 844–847.



A number of clues are contained in the extant EINSTEIN-HILBERT correspondence for November. As already noted, the beginning of this correspondence on November 7 is marked by a postcard in which EINSTEIN announces both his rejection of the EINSTEIN-GROSSMANN theory and his new approach as elaborated in his paper of November 4 for the Berlin Academy. Five days later EINSTEIN wrote to HILBERT again to tell him that he had at last found generally covariant field equations – that is, the equations (14) – but these equations are not on the card. In this card of November 12, EINSTEIN also thanks HILBERT for his "kind letter" which was presumably sent and received between the 7th and the 12th but which is not in the EINSTEIN Papers. HILBERT replied on the 14th with a long message written on two separate postcards. HILBERT briefly describes something of his own new theory, emphasizing that the electromagnetic field equations are a consequence of the gravitational equations and, in a postscript, that his theory is "wholly distinct" from EINSTEIN's theory. It is unclear whether he was referring to EINSTEIN's theory of November 4 (equations (6)), or to EINSTEIN's theory of November 11 (equations (14)), or to both. HILBERT also extended an invitation to EINSTEIN to come to Göttingen to hear a lecture which HILBERT was to deliver to the Mathematische Gesellschaft on November 23. Several things should be noted about these twin cards. First, HILBERT does not actually give his field equations. Second, mathematical expressions from HILBERT's theory are used without any explanation, suggesting that there was perhaps other technical correspondence between the two, on the context of which HILBERT could rely. Finally, the tone of the cards is very cordial – HILBERT urges EINSTEIN to come to Göttingen a day before the lecture and pass the night at HILBERT's home.[42]

The next communication is an undated card from EINSTEIN, almost certainly sent on November 15.[43] In it EINSTEIN expresses great interest in HILBERT's investigations on a "bridge between gravitation and electromagnetism"; but he declines HILBERT's invitation on the grounds of fatigue and a stomach ailment. Perhaps this was only a polite excuse, or perhaps EINSTEIN recovered rapidly, for in the next three days he completed his calculation of the advance of Mercury's perihelion. Most significantly, EINSTEIN asked HILBERT to send him proofs of his lecture.[44]

We surmise that HILBERT subsequently sent some details of his lecture, for on the 18th EINSTEIN wrote another card to HILBERT in which he says that, as

[42] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 13. The printed version of HILBERT's lecture (Ref. 47) gives 20 November 1915 as the date of the session to which it was delivered. But HILBERT, writing on Sunday, November 14, invites EINSTEIN to attend his lecture on "Dienstag," presumably Tuesday, November 23. We have no explanation for this discrepancy.

HILBERT's invitation to EINSTEIN to spend the night at his home was not extraordinary, and we are informed by G. KREISEL that HILBERT's wife was well known for her hospitality.

[43] EINSTEIN Papers, Princeton University, microfilm reel I.B. 1, no. 13. The postcard is simply dated "Monday"; the card of November 12 is dated "Friday, November 12".

[44] "Schicken Sie mir bitte, wenn möglich, ein Korrektur-exemplar Ihrer Untersuchung, um meiner Ungeduld entgegenzukommen."

far as he can see, HILBERT's system agrees with the one he has found within the last week and has communicated to the Berlin Academy[45] – that is, the theory of November 11 (equations (14)). Now the field equations of HILBERT's printed lecture are formally equivalent to EINSTEIN's equations (18) of November 25 and not to EINSTEIN's equations (14) of November 11, which suggests that either EINSTEIN was mistaken in claiming the equivalence of the systems, or that whatever HILBERT sent to EINSTEIN was quite different from the published version of the lecture. As will be seen in the next section, however, neither of these explanations need be correct. Given the mathematical techniques which formed the basis of HILBERT's approach, the latter explanation seems so improbable that it can be dismissed from further attention. Moreover, we will argue that in EINSTEIN's eyes HILBERT's published equations may very well have been equivalent to those of EINSTEIN's paper of November 11.

In this same card of November 18, EINSTEIN remarks that he had known about the equations (14) "for three years" but that he and GROSSMANN had rejected them on the grounds that in the Newtonian limit they were not compatible with "NEWTON's Law", meaning NEWTON's or more properly POISSON's field equation. Finally, EINSTEIN informed HILBERT that he had succeeded in explaining the advance of the perihelion of Mercury from general relativity alone without the aid of any subsidiary hypotheses. The next day, November 19, HILBERT sent his congratulations and remarked cheerfully that if he could calculate as quickly as EINSTEIN, he would be able to explain why the electron in the hydrogen atom does not radiate.[46] He also asked EINSTEIN to keep him abreast of further developments.

In summary, the extant correspondence in the EINSTEIN Papers shows, first, that there was an intensive exchange between EINSTEIN and HILBERT on the subject of gravitational theory in the month of November 1915; second, that on November 15 EINSTEIN asked for proofs of HILBERT's lecture, and since EINSTEIN wrote HILBERT three days later that their systems were in agreement, presumably that he received something in response to his request; third, that to judge from the language of the correspondence, the two men were on good terms, although their mutual cordiality may have covered a deeper hostility of temper.

## 5. Einstein's view of Hilbert's theory

How did EINSTEIN interpret HILBERT's field equations? And why did he think that his equations (14) and HILBERT's equations were in agreement? In the printed version of his lecture,[47] HILBERT uses the variational principle $\delta H\sqrt{-g}\,d^4x=0$, and the postulate that $H=G+L$, where $L$ may depend upon

[45] "Das von Ihnen gegebene System stimmt – soweit ich sehe – genau mit dem überein was ich in den letzten Wochen gefunden und der Akademie überreicht habe."

[46] EINSTEIN Papers, Princeton University, microfilm reel I.B.1, no. 13.

[47] D. HILBERT, "Die Grundlagen der Physik," *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen* (1915), 395–407.



the metric potentials but not their derivatives, to obtain the field equations

$$G_{im}-\tfrac{1}{2}g_{im}G=-(1/\sqrt{-g})(\partial(\sqrt{-g}L)/\partial g^{im}). \tag{19}$$

If we define $T_{im}$ by the right hand side of (19), then (19) assumes the form

$$G_{im}-\tfrac{1}{2}g_{im}G=T_{im}, \tag{19'}$$

which is formally equivalent to (18) when units are chosen so that $\chi=1$.

However, (19′) is a misleading representation of HILBERT's equations. GUSTAV MIE had developed an influential electrical theory of matter, and proposed to make gravitation a consequence of electrodynamics.[48] HILBERT's aim was to combine MIE's theory of matter with EINSTEIN's theory of gravitation without, however, wholly adopting MIE's point of view. Instead, HILBERT attempted to make electrodynamics a consequence of gravitation, but he did incorporate a restriction on the energy-momentum tensor which was in keeping with MIE's approach; namely, no matter field quantities of a non-electromagnetic nature are to be used in $T_{im}$. Thus HILBERT specifies that $L$ is to be constructed exclusively from the electromagnetic potentials, the derivatives of the electromagnetic potentials, and the metric potentials. As a result, the $T_{im}$ appearing in (19′) is not the same as that EINSTEIN intended for the equations (18). To emphasize the difference, let us represent HILBERT's equations not by (19′) but by

$$G_{im}-\tfrac{1}{2}g_{im}G=T^{EM}_{im}. \tag{19''}$$

Now if one assumes, as EINSTEIN did at the point at which he advocated (14), that the energy scalar associated with $T^{EM}_{im}$ vanishes, then contracting both sides of (19″) with $g^{im}$ leads to

$$G-2G=T^{EM}=0. \tag{20}$$

So $G=0$ and (19″) reduces to

$$G_{im}=T^{EM}_{im}, \tag{21}$$

and (21) is equivalent to (14) under the choice of units $\chi=1$ and the assumption that $T^{EM}_{im}$ is the complete energy tensor for matter fields, an assumption in harmony with EINSTEIN's speculation that "matter can be reduced to purely electromagnetic phenomena."

This is the best explanation we can offer for EINSTEIN's belief that his equations (14) were equivalent to HILBERT's equations. Needless to say, there is a sense in which EINSTEIN was mistaken; for even though the equations (19″) reduce to the equations (21) which under the special assumptions mentioned are in turn equivalent to (14), the equations (19″) are not themselves equivalent to (14). Most importantly, the equations (19″) entail the covariant

[48] See G. MIE, "Grundlagen einer Theorie der Materie," *Annalen der Physik*, **37** (1912), 511–534; **39** (1912), 1–40; **40** (1913), 1–66. MIE's avowed purpose was to develop a theory in which the electromagnetic field quantities "completely suffice to describe all the phenomena of the material world" (**37** (1912), p. 513).

conservation law (15) but the equations (14) do not. This does not contradict our explanation, for, as will be seen shortly, EINSTEIN did not realize at the time in question that the covariant divergence of $G_{im}-\frac{1}{2}g_{im}G$ vanishes identically.

## 6. The rationale for Einstein's final field equation

Granted that there was some justification for EINSTEIN's belief in the equivalence of his equations (14) and HILBERT's equations (19″), it is still quite possible that the $\frac{1}{2}g_{im}G$ term in HILBERT's equations suggested to EINSTEIN that the equations (14) should be modified to (18). Of course, even if he had not known about HILBERT's equations, EINSTEIN had an independent reason for introducing the $g_{im}T$ term which appears on the right hand side of (18) —namely, it allows him to escape the consequence that $T=0$ and the consequent need for a speculative supporting hypothesis. But independently of HILBERT's equations, did EINSTEIN have a reason for introducing the coefficient $-\frac{1}{2}$ for this term?

To the modern reader, the most obvious conjecture is that the $-\frac{1}{2}$ factor is needed to make the covariant conservation law (15) a consequence of the field equations. However, this was certainly *not* EINSTEIN's reason, for in his paper of November 25 for the Berlin Academy he did not realize that the equations (18) entail (15), and in fact, the latter relation is postulated separately.[49] Nor is the $-\frac{1}{2}$ factor needed to get the conservation law in the form (12). To derive (12), EINSTEIN specializes to a coordinate system where $\sqrt{-g}=1$. It follows that in such a coordinate system

$$g_{im}g^{im}{}_{,n}=-(\log\sqrt{-g})_{,n}=0. \tag{22}$$

Then on the way to (12), EINSTEIN multiplies both sides of (18) by $g^{im}{}_{,n}$ and sums over $i$ and $m$. As a consequence of (22), the term involving $g_{im}T$ drops out.

The only place where the factor $-\frac{1}{2}$ appears to play an essential role is in the derivation of the relation

$$g^{ij}{}_{,i,j}-\chi(T+t)=0 \tag{23}$$

(where $t$ is $\sum_i t_i^i$), which follows from (18) with the help of the coordinate condition $\sqrt{-g}=1$. EINSTEIN emphasizes that in (23) the energy tensors for gravitation and matter appear symmetrically, which had not been true for the two previous attempts at field equations. This symmetry did hold, however, in the version of the theory in 1913–1914.[50] This symmetry is the only explicit reason EINSTEIN gives for the $\frac{1}{2}g_{im}T$ term in equations (14): "The considerations that have induced me to introduce the second term on the right-hand side," he writes, "will become clear from the following reflection ...",[51] and then he derives equation (23). From the point of view of hindsight, this is not a very impressive argument for the factor $-\frac{1}{2}$. As SCHRÖDINGER pointed out in 1918, the so-called energy tensor $t_i^j$ for the gravitational field is not a tensor at all and can be made to vanish at any point by means of a suitable coordinate transformation, whatever the nature of the gravitational field; and conversely, even in a pseudo-Euclidean space-time (no gravitational field) its components can be made non-vanishing by the choice of a non-Euclidean coordinate system.[52] But, in 1915, EINSTEIN was convinced that $T_i^j$ and $t_i^j$ should be treated as though they were on an equal footing.

EINSTEIN's argument for the extra term in (18) may have been *post hoc*. He gives no argument to show that the requirement that, in suitable coordinates, $T$ and $t$ be treated equally in relation to $g^{ij}$ and its derivatives is uniquely satisfied by equation (18). His paper of 1916 on the foundations of general relativity[53] does suggest such an argument. There, EINSTEIN begins by taking the equations for free space to be given by (17). Under the condition that $\sqrt{-g}=1$, he shows that (17) can be rewritten as

$$(g^{mk}\Gamma^n_{ik})_{,n}=\chi(t_i^m-\tfrac{1}{2}\delta_i^m t). \tag{24}$$

When matter fields are present, equal treatment demands that (24) be replaced by

$$(g^{mk}\Gamma^n_{ik})_{,n}=\chi[(t_i^m+T_i^m)-\tfrac{1}{2}\delta_i^m(t+T)], \tag{25}$$

which is equivalent to

$$R_{im}=\chi(T_{im}-\tfrac{1}{2}g_{im}T), \tag{26}$$

and (26) is the result of (18) under the condition $\sqrt{-g}=1$. Unfortunately, nothing we can find in either the published papers from 1915 or in the correspondence reveals whether or not this is the route by which EINSTEIN made the transition from (14) to (18) at the end of November 1915.

## 7. Aftermath

After his lecture in Göttingen during the summer of 1915, EINSTEIN wrote to SOMMERFELD: "I am very enthusiastic about Hilbert; an impressive man! I

[49] See Ref. 41, p. 845. H. WEYL is certainly correct in saying that EINSTEIN "recognized only later" that the term $\frac{1}{2}g_{im}T$ "is required as a result of the energy-momentum theorem" (see *Space, Time, and Matter* (New York: Dover, 1958), p. 239). But WEYL's remark is misleading in that it has suggested to some that EINSTEIN added the term $\frac{1}{2}g_{im}T$ in order to satisfy the "energy-momentum theorem". This is the implication, for example, of P. SPEZIALLI's remark that "Dans le premier mémoire d'Einstein le term $\frac{1}{2}g_{\mu\nu}T$ manquait. Mais Einstein reconnut ensuite la présence de ce terme est exigée par les théorèmes de conservation."; *Einstein/Besso Correspondence*, p. 66. Again, PYENSON claims that EINSTEIN introduced the additional term because he realized that the covariant derivative of the "matter tensor" vanishes; *cf. The Göttingen Reception of General Relativity, op cit.* (Ref. 12).

[50] See, for example, A. EINSTEIN, "Zum gegenwärtigen Stande des Gravitationsproblems," *Physikalische Zeitschrift* **14** (1913), p. 1258, eq. (7b).

[51] Ref. 41, p. 846.

[52] E. SCHRÖDINGER, "Die Energiekomponenten des Gravitationsfeldes," *Physikalische Zeitschrift* **19** (1918), 4–7.

[53] A. EINSTEIN, "Die Grundlage der allgemeinen Relativitätstheorie," *Annalen der Physik* **49** (1916), 769–822.

am very curious about your opinion."[54] This enthusiasm was apparently reciprocated by HILBERT who recommended EINSTEIN for the third Bolyai Prize in 1915 for "the high mathematical spirit behind all his achievement."[55] However, the mutual admiration was neither uniform nor unqualified. In an extraordinary letter dated December 20, 1915, EINSTEIN wrote to HILBERT:

> I want to take this opportunity to say something to you which is important to me.
> There has been a certain spell of coolness between us, the cause of which I do not want to analyze. I have, to be sure, struggled against any resentment, and with complete success. I think of you once again with untroubled friendliness, and I ask you to try to think of me in the same way. It is really a shame when two such real fellows, whose work has taken them above this shabby world, give one another no pleasure.[56]

While other explanations are certainly possible, it seems most plausible that the "spell of coolness" resulted from the events surrounding the publication of the field equations. EINSTEIN knew that HILBERT was at work on gravitation, and after having struggled with the problem for so long, EINSTEIN was surely anxious that the final solution be his. Concern for the priority of the discovery of the field equations was almost certainly part of the motivation for EINSTEIN's correspondence with HILBERT during November of 1915 – there is simply no other plausible explanation either for the content of the correspondence or for the fact that it so suddenly and completely displaced EINSTEIN's regular correspondence with those friends who had hitherto served as sounding boards for his struggles with general relativity. The friendly tone of the November correspondence cited above in Section 4, all of which occurred before EINSTEIN's publication of the "correct" field equations on November 25, EINSTEIN's remark on December 20 that "I think of you *once again* with untroubled friendliness," the likelihood that HILBERT did communicate his field equations before the 25th of November, and the fact that in his paper of November 25 EINSTEIN makes no mention of HILBERT's work, together suggest that some hard feelings may have developed over the authorship of the field equations. If that is so, the dispute never became public.

Whatever the source of the coolness, it is a reasonable guess that EINSTEIN's letter of December 20 was prompted in part by the tone and content of the printed version of HILBERT's lecture. In this paper – the proofs of which had probably reached EINSTEIN by the 20th of December[57] – HILBERT refers to all of EINSTEIN's papers for November, including the fourth paper in which the equations (18) are presented. HILBERT makes no claim to be the discoverer of the field equations; on the contrary, his presentation makes it appear that he is simply combining and unifying the insights of EINSTEIN and MIE. And he refers to EINSTEIN's final creation as a "noble theory."[58] Whether or not EINSTEIN interpreted these features as marks of generosity, he could hardly have failed to have seen them as signs of good will, and given what we know of EINSTEIN's character, he could not have failed to respond in kind.

[54] *Einstein/Sommerfeld Briefwechsel*, p. 30.

[55] Quoted in C. REID, *Hilbert*, p. 142.

[56] EINSTEIN Papers, Princeton University, microfilm reel I.B.1, no. 13. "Bei dieser Gelegenheit drängt es mich dazu, Ihnen noch etwas zu sagen, was mir wichtiger ist.

"Es ist zwischen uns eine gewisse Verstimmung gewesen, deren Ursache ich nicht analysieren will. Gegen das damit verbundene Gefühl der Bitterkeit habe ich gekämpft, und zwar mit vollständigem Erfolge. Ich gedenke Ihrer wieder in ungetrübter Freundlichkeit, und bitte Sie, dasselbe bis mir zu versuchen. Es ist objecktiv [?] schade, wenn sich zwei wirkliche Kerle, die sich aus dieser schäbigen Welt etwas herausgearbeitet haben, nicht gegenseitig zur Freude gereichen."



## 8. Conclusions

From the internal evidence of the published papers and from EINSTEIN's correspondence with HILBERT, it seems probable that HILBERT had derived his field equations and communicated at least an outline of his results to EINSTEIN sometime before November 18, a time at which EINSTEIN still believed in the equations (14). But whether HILBERT's field equations are equivalent to those EINSTEIN announced on November 25 is a question of some nicety. Given the *Ansatz* that all matter is reducible to electromagnetic phenomena, HILBERT's equation may be understood to imply EINSTEIN's final field equations (18). It is likely that in EINSTEIN's eyes, however, the equations were quite different; for, we have argued, EINSTEIN seems to have understood HILBERT's equations as implying the vanishing of the trace of the stress-energy tensor, and hence as equivalent, not to equations (18) but to equations (21) which lack the additional term $\frac{1}{2} g_{im} G$. HILBERT's equations cannot have suggested to EINSTEIN the content of his own equations (18), but they may well have suggested their form.

This leads to an issue which, for us at least, is more important than the issue of priority – namely HILBERT's influence on EINSTEIN's development. The evidence of the correspondence raises the possibility that HILBERT was in part responsible for heightening EINSTEIN's discontent with the EINSTEIN-GROSSMANN theory, and it suggests, but does not prove, that HILBERT's results determined EINSTEIN's transition from (14) to (18).

Whether one chooses to call the field equations (18) and (19′) EINSTEIN's equations, or HILBERT's equations, or the EINSTEIN-HILBERT equations, or the HILBERT-EINSTEIN equations, it seems to us proper to call the general theory of relativity EINSTEIN's theory. HILBERT's recognition of EINSTEIN's undeniable authorship of both the general framework and the central ideas of the theory may well account for the fact that HILBERT never claimed credit for the general theory of relativity.

[57] Apparently, they had not reached him by the 9th of December, for on that date he wrote to SOMMERFELD that "As far as I know Hilbert's theory, it uses for electrodynamic events a point of departure which – apart from the treatment of the gravitational field – connects closely with Mie's." *Einstein/Sommerfeld Briefwechsel*, p. 37.

[58] Ref. 47, p. 405.

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