Supplement

Theorem 1. *Consider model equations F containing endogenous variables V with bipartite graph $\mathcal{B}$. Suppose F is extended with equations F_+ containing endogenous variables in $V \cup V_+$, where V_+ contains endogenous variables that are added by the model extension.*[8]*. Let $\mathcal{B}_{\mathrm{ext}}$ be the bipartite graph associated with $F_{\mathrm{ext}} = F \cup F_+$ and $V_{\mathrm{ext}} = V \cup V_+$, and $\mathcal{B}_+$ the bipartite graph associated with the extension F_+ and V_+. If $\mathcal{B}$ and $\mathcal{B}_+$ both have a perfect matching then:*

(i) *$\mathcal{B}_{\mathrm{ext}}$ has a perfect matching,*
(ii) *ancestral relations in $\mathrm{CO}(\mathcal{B})$ are also present in $\mathrm{CO}(\mathcal{B}_{\mathrm{ext}})$,*
(iii) *d-connections in $\mathrm{MO}(\mathcal{B})$ are also present in $\mathrm{MO}(\mathcal{B}_{\mathrm{ext}})$.*

Proof. The causal ordering graph $\mathrm{CO}(\mathcal{B})$ is constructed from a perfect matching M for the bipartite graph $\mathcal{B} = \langle V, F, E \rangle$. Let M_+ be a perfect matching for $\mathcal{B}_+$. Note that $M_{\mathrm{ext}} = M \cup M_+$ is a perfect matching for $\mathcal{B}_{\mathrm{ext}} = \langle V \cup V_+, F \cup F_+, E_{\mathrm{ext}} \rangle$. Following the causal ordering algorithm for $\mathcal{B}, M$ and $\mathcal{B}_{\mathrm{ext}}, M_{\mathrm{ext}}$, we note that clusters in $\mathrm{CO}(\mathcal{B})$ are fully contained in clusters in $\mathrm{CO}(\mathcal{B}_{\mathrm{ext}})$. Therefore edges in $\mathrm{MO}(\mathcal{B})$ are preserved in $\mathrm{MO}(\mathcal{B}_{\mathrm{ext}})$. □

Theorem 2. *Let F, F_+, F_{ext}, V, V_+, V_{ext}, $\mathcal{B}$, $\mathcal{B}_+$, and $\mathcal{B}_{\mathrm{ext}}$ be as in Theorem 1. If $\mathcal{B}$ and $\mathcal{B}_+$ both have perfect matchings and no vertex in V_+ is adjacent to a vertex in F in $\mathcal{B}_{\mathrm{ext}}$ then:*[9]

(i) *absent ancestral relations in $\mathrm{CO}(\mathcal{B})$ are also absent in $\mathrm{CO}(\mathcal{B}_{\mathrm{ext}})$,*
(ii) *d-separations in $\mathrm{MO}(\mathcal{B})$ are also present in $\mathrm{MO}(\mathcal{B}_{\mathrm{ext}})$.*

Proof. Since $\mathcal{B}$ and $\mathcal{B}_+$ both have perfect matchings the results of Theorem 2 hold. Let $\mathcal{G}(\mathcal{B}, M)$, and $\mathcal{G}(\mathcal{B}_{\mathrm{ext}}, M_{\mathrm{ext}})$ be as in the proof of Theorem 1. Note that in M_{ext} vertices in F_+ are matched to vertices in V_+ and therefore edges between $f_+ \in F_+$ and $v \in \mathrm{adj}_{\mathcal{B}_{\mathrm{ext}}}(F_+) \setminus V_+$ are oriented as $(f_+ \leftarrow v)$ in $\mathcal{G}(\mathcal{B}_{\mathrm{ext}}, M_{\mathrm{ext}})$. By assumption, we therefore have that vertices in V_+ are non-ancestors of vertices in $V \cup F$ in $\mathcal{G}(\mathcal{B}_{\mathrm{ext}}, M_{\mathrm{ext}})$. Since $M \subseteq M_{\mathrm{ext}}$ we know that the same directed edges between vertices in V, F, the exogenous variables in F, and the parameters in F appear in both $\mathcal{G}(\mathcal{B}, M)$ and $\mathcal{G}(\mathcal{B}_{\mathrm{ext}}, M_{\mathrm{ext}})$. By steps (ii) and (iii) of the construction of the causal ordering graph the edges and clusters in $\mathcal{G}$ are also edges and clusters in the causal ordering graph of the extended model. The desired result follows by construction of the Markov ordering graph. □

Proposition 1. *Consider a dynamical model for endogenous variables V and a dynamical model extension for additional endogenous variables V_+. Let F and $F_{\mathrm{ext}} = F \cup F_+$ be the equilibrium equations of the original and extended model respectively, and assume that the associated bipartite graphs $\mathcal{B}$ and $\mathcal{B}_{\mathrm{ext}}$ both have perfect matchings. If all variables in $V \cup V_+$ are self-regulating then d-connections in $\mathrm{MO}(\mathcal{B})$ are also present in $\mathrm{MO}(\mathcal{B}_{\mathrm{ext}})$.*

Proof. Recall that the equilibrium equation constructed from the derivative of a variable i is labelled f_i according to the natural labelling. Throughout this proof variable vertices v_i will be associated with equilibrium equations f_i through the natural labelling. When a variable in $v_i \in V \cup V_+$ is self-regulating then it can be matched to its equilibrium equation f_i. This means that we can simply apply Theorem 1 to conclude that d-connections in $\mathrm{MO}(\mathcal{B})$ are also present in $\mathrm{MO}(\mathcal{B}_{\mathrm{ext}})$. □

[8] Recall that V_+ may also contain parameters or exogenous variables that appear in F.

[9] Note that V_+ is adjacent to F when one of the exogenous random variables or parameters in F becomes an endogenous variable in the model extension.