Supplement

A graphical illustration of the causal ordering algorithm applied to the equations of a cyclic model is provided in the first section. The second section contains proofs of the results in the main paper.

Causal ordering algorithm applied to a cyclic model

In this section we demonstrate how the causal ordering algorithm works on a set of equations for a cyclic model. The algorithm is also presented graphically. Consider the following equations for endogenous variables $\boldsymbol{X}$ and exogenous random variables $\boldsymbol{U}$:

$$f_1: \qquad g_1(X_{v_1}, U_{w_1}) = 0, \tag{13}$$
$$f_2: \qquad g_2(X_{v_2}, X_{v_1}, X_{v_4}, U_{w_2}) = 0, \tag{14}$$
$$f_3: \qquad g_3(X_{v_3}, X_{v_2}, U_{w_3}) = 0, \tag{15}$$
$$f_4: \qquad g_4(X_{v_4}, X_{v_3}, U_{w_4}) = 0, \tag{16}$$
$$f_5: \qquad g_5(X_{v_5}, X_{v_4}, U_{w_5}) = 0. \tag{17}$$

The associated bipartite graph in Figure 7a consists of variable vertices $V = \{v_1, \ldots, v_5\}$ and equation vertices $F = \{f_1, \ldots, f_5\}$. There is an edge between a variable vertex and an equation vertex whenever that variable appears in the equation. The associated bipartite graph has exactly two perfect matchings:

$$M_1 = \{(v_1 - f_1), (v_2 - f_2), (v_3 - f_3), (v_4 - f_4), (v_5 - f_5)\},$$
$$M_2 = \{(v_1 - f_1), (v_2 - f_3), (v_3 - f_4), (v_4 - f_2), (v_5 - f_5)\}.$$

Application of the first step of the causal ordering algorithm results either in the directed graph in Figure 7b or that in Figure 7c, depending on the choice of the perfect matching. The segmentation of vertices into strongly connected components, which takes place in the second step of the algorithm, results in the clusters $\{v_1\}$, $\{f_1\}$, $\{v_2, v_3, v_4, f_2, f_3, f_4\}$, $\{v_5\}$, and $\{f_5\}$. To construct the clusters of the causal ordering graph we add $S_i \cup M(S_i)$ to a cluster set $\mathcal{V}$ for each S_i in the segmentation. The segmentation of vertices into strongly connected components is displayed in Figures 7d and 7e. Notice that the segmentation in Figure 7d is the same as that in Figure 7e. It is known that the segmentation into strongly connected components is unique (i.e. it does not depend on the choice of the perfect matching), a result that can be found in Blom et al. [2] and Pothen et al. [14]. The cluster set $\mathcal{V}$ for the causal ordering graph in Figure 7f is constructed by merging clusters in the segmented graph whenever two clusters contain vertices that are matched and by adding exogenous variables as singleton clusters. The edge set $\mathcal{E}$ for the causal ordering graph is obtained by adding edges $(v \to C)$ from an endogenous vertex v to a cluster C, whenever $v \notin C$ and there is an edge from v to $f \in C$ in the directed graph. Finally, we also add edges from exogenous vertices to clusters that contain equations in which the corresponding exogenous random variables appear.

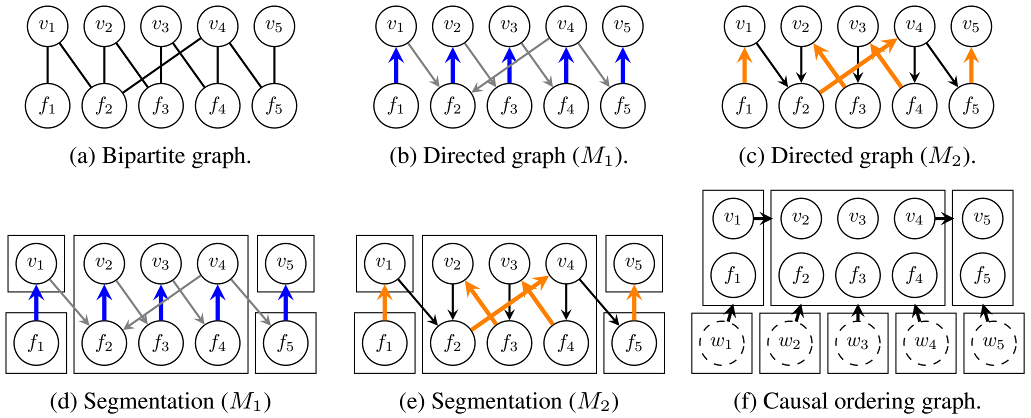
Proofs

Theorem 1. *Consider model equations F containing endogenous variables V with bipartite graph $\mathcal{B}$. Suppose F is extended with equations F_+ containing endogenous variables in $V \cup V_+$, where V_+ contains endogenous variables that are added by the model extension.*[7] *Let $\mathcal{B}_{\text{ext}}$ be the bipartite graph associated with $F_{\text{ext}} = F \cup F_+$ and $V_{\text{ext}} = V \cup V_+$, and $\mathcal{B}_+$ the bipartite graph associated with the extension F_+ and V_+, where variables in V appearing in F^+ are treated as exogenous variables (i.e. they are not added as vertices in $\mathcal{B}_+$). If $\mathcal{B}$ and $\mathcal{B}_+$ both have a perfect matching then:*

(i) *$\mathcal{B}_{\text{ext}}$ has a perfect matching,*
(ii) *ancestral relations in* $\text{CO}(\mathcal{B})$ *are also present in* $\text{CO}(\mathcal{B}_{\text{ext}})$,
(iii) *d-connections in* $\text{MO}(\mathcal{B})$ *are also present in* $\text{MO}(\mathcal{B}_{\text{ext}})$.

Proof. The causal ordering graph $\text{CO}(\mathcal{B})$ is constructed from a perfect matching M for the bipartite graph $\mathcal{B} = \langle V, F, E\rangle$. Let M_+ be a perfect matching for $\mathcal{B}_+$. Note that $M_{\text{ext}} = M \cup M_+$ is a perfect matching for $\mathcal{B}_{\text{ext}} = \langle V \cup V_+, F \cup F_+, E_{\text{ext}}\rangle$. Following the causal ordering algorithm for

[7] V_+ may also contain parameters or exogenous variables that appear in F and become endogenous in the extended model.



(a) Bipartite graph. (b) Directed graph (M_1). (c) Directed graph (M_2).

(d) Segmentation (M_1) (e) Segmentation (M_2) (f) Causal ordering graph.

Figure 7: Graphical illustration of the causal ordering algorithm that was described in Section 1.1. Figure 7a shows the bipartite graph that is associated with equations (13) to (17). Application of the first step of the causal ordering algorithm results in the directed graph in Figure 7b for perfect matching M_1 and that in Figure 7c for perfect matching M_2. Note that the blue and orange edges correspond to the edges in the perfect matchings M_1 and M_2, respectively. Figures 7d and 7e show that the segmentation into strongly connected components does not depend on the choice of perfect matching. Exogenous vertices and edges from these vertices to clusters were added to the causal ordering graph in Figure 7f.

$\mathcal{B}, M$ and $\mathcal{B}_{\text{ext}}, M_{\text{ext}}$, we note that clusters in $\text{CO}(\mathcal{B})$ are fully contained in clusters in $\text{CO}(\mathcal{B}_{\text{ext}})$. Therefore ancestral relations in $\text{CO}(\mathcal{B})$ are also present in $\text{CO}(\mathcal{B}_{\text{ext}})$. Furthermore, edges present in $\text{MO}(\mathcal{B})$ are also present in $\text{MO}(\mathcal{B}_{\text{ext}})$. □

Theorem 2. *Let F, F_+, F_{ext}, V, V_+, V_{ext}, $\mathcal{B}$, $\mathcal{B}_+$, and $\mathcal{B}_{\text{ext}}$ be as in Theorem 1. If $\mathcal{B}$ and $\mathcal{B}_+$ both have perfect matchings and no vertex in V_+ is adjacent to a vertex in F in $\mathcal{B}_{\text{ext}}$ then:*[8]

(i) ancestral relations absent in $\text{CO}(\mathcal{B})$ are also absent in $\text{CO}(\mathcal{B}_{\text{ext}})$,
(ii) d-connections absent in $\text{MO}(\mathcal{B})$ are also absent in $\text{MO}(\mathcal{B}_{\text{ext}})$.

Proof. Since $\mathcal{B}$ and $\mathcal{B}_+$ both have perfect matchings the results of Theorem 2 hold. Let $\mathcal{G}(\mathcal{B}, M)$, and $\mathcal{G}(\mathcal{B}_{\text{ext}}, M_{\text{ext}})$ be as in the proof of Theorem 1. Note that in M_{ext} vertices in F_+ are matched to vertices in V_+ and therefore edges between $f_+ \in F_+$ and $v \in \text{adj}_{\mathcal{B}_{\text{ext}}}(F_+) \setminus V_+$ are oriented as $(f_+ \leftarrow v)$ in $\mathcal{G}(\mathcal{B}_{\text{ext}}, M_{\text{ext}})$. By assumption, we therefore have that vertices in V_+ are non-ancestors of vertices in $V \cup F$ in $\mathcal{G}(\mathcal{B}_{\text{ext}}, M_{\text{ext}})$. Since $M \subseteq M_{\text{ext}}$ we know that the same directed edges between vertices in V and F appear in both $\mathcal{G}(\mathcal{B}, M)$ and $\mathcal{G}(\mathcal{B}_{\text{ext}}, M_{\text{ext}})$. Notice that the subgraph of $\mathcal{G}(\mathcal{B}_{\text{ext}}, M_{\text{ext}})$ induced by the vertices $V \cup F$ coincides with $\mathcal{G}(\mathcal{B}, M)$. The two statements follow from the construction of the causal and Markov ordering graphs. □

Lemma 1. *Consider a first-order dynamical model in canonical form for endogenous variables V and let F be the equilibrium equations of the model. If all variables in V are self-regulating then $\mathcal{B}$ has a perfect matching.*

Proof. Recall that the equilibrium equation constructed from the derivative of a variable i is labelled f_i according to the natural labelling. When a variable in $v_i \in V$ is self-regulating then it can be matched to its equilibrium equation f_i, and hence $\mathcal{B}$ has a perfect matching. □

[8] Note that V_+ is adjacent to F when one of the exogenous random variables or parameters in F becomes an endogenous variable in the model extension.