A Brief review of OLS regression

Since we use OLS regression for our results, we briefly review OLS estimators. We consider the following setup:

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e},$$

where $\mathbf{y}$ and $\mathbf{e}$ are $n \times 1$ vectors, $\mathbf{X}$ is an $n \times d$ matrix of observations, and β is the $d \times 1$ coefficient vector that we want to estimate. If $\mathbf{e} \perp\!\!\!\perp \mathbf{X}$ and $\mathbf{e} \sim \mathcal{N}(0, \sigma_e^2 \mathbf{I_n})$, where $\mathbf{I_n}$ is the $n \times n$ identity matrix, then the OLS estimate of β is

$$\begin{aligned}\widehat{\beta} &= (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{y} \\ &= \beta + (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{e},\end{aligned}$$

with $\mathbb{E}[\widehat{\beta}] = \beta$ and $\text{Var}(\widehat{\beta}) = \sigma_e^2 \mathbb{E}[(\mathbf{X}^\top \mathbf{X})^{-1}]$. If each row X_i of $\mathbf{X}$ is sampled from $X_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \mathbf{\Sigma})$, then the distribution of $(\mathbf{X}^\top \mathbf{X})^{-1}$ is an Inverse-Wishart distribution [Page Jr, 1984, Ch. 8]. Then the variance of $\widehat{\beta}$ is

$$\text{Var}(\widehat{\beta}) = \frac{\sigma_e^2 \mathbf{\Sigma}^{-1}}{n-d-1}. \tag{16}$$

B Covariance of $\widehat{a}$ and $\widehat{c}$

B.1 Frontdoor estimator

We prove that $\text{Cov}(\widehat{a}_f, \widehat{c}) = 0$ for the frontdoor estimator. The expressions for $\widehat{a}_f$ and $\widehat{c}$ are

$$\begin{aligned}\widehat{c} &= \frac{\sum x_i m_i}{\sum x_i^2} \qquad (17)\\ &= c + \frac{\sum x_i u_i^m}{\sum x_i^2} \\ \widehat{a}_f &= \frac{\sum x_i^2 \sum m_i y_i - \sum x_i m_i \sum x_i y_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2} \qquad (18)\\ &= a + \frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2},\end{aligned}$$

where $e_i = -\frac{b}{d} u_i^x + u_i^y$. Using the fact the (u^x, x) is bivariate normally distributed, we get

$$\begin{aligned}\mathbb{E}[e|x] &= \frac{b\sigma_{u_x}^2}{d(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)} x \\ &= Fx, \qquad (19)\end{aligned}$$

where $F = \frac{b\sigma_{u_x}^2}{d(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)}$. The covariance then is

$$\begin{aligned}\text{Cov}(\widehat{a}_f, \widehat{c}) &= \mathbb{E}[(\widehat{a}_f - a)(\widehat{c} - c)] \\ &= \mathbb{E}[\mathbb{E}[(\widehat{a}_f - a)(\widehat{c} - c)|x, m]] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum m_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\middle| x, m\right]\right] \\ &= \mathbb{E}\left[\left(\frac{\sum x_i^2 \sum m_i \mathbb{E}[e_i|x] - \sum m_i x_i \sum x_i \mathbb{E}[e_i|x]}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\left(\frac{\sum u_i^m x_i}{\sum x_i^2}\right)\right] \qquad (20)\\ &= \mathbb{E}\left[F\left(\frac{\sum x_i^2 \sum m_i x_i - \sum m_i x_i \sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\left(\frac{\sum u_i^m x_i}{\sum x_i^2}\right)\right] \\ &= 0,\end{aligned}$$

where in Eq. 20 we used the expression from Eq. 19. Also, in Eq. 20, we took u_i^m out of the conditional expectation because u_i^m is given x_i and m_i (because $u_i^m = m_i - cx_i$).

B.2 Combined estimator

We prove that $\text{Cov}(\widehat{a}_c, \widehat{c}) = 0$ for the combined estimator from Section 5. The expressions for $\widehat{a}_c$ and $\widehat{c}$ are

$$\begin{aligned}
\widehat{c} &= \frac{\sum x_i m_i}{\sum x_i^2} \qquad (21)\\
&= c + \frac{\sum x_i u_i^m}{\sum x_i^2}\\
\widehat{a}_c &= \frac{\sum w_i^2 \sum m_i y_i - \sum w_i m_i \sum w_i y_i}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2} \qquad (22)\\
&= a + \frac{\sum w_i^2 \sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}.
\end{aligned}$$

The covariance is

$$\begin{aligned}
\text{Cov}(\widehat{a}_c, \widehat{c}) &= \mathbb{E}[(\widehat{a}_c - a)(\widehat{c} - c)]\\
&= \mathbb{E}[\mathbb{E}[(\widehat{a}_c - a)(\widehat{c} - c)|x, m, w]]\\
&= \mathbb{E}\left[\left(\frac{\sum w_i^2 \sum m_i \mathbb{E}[u_i^y] - \sum m_i w_i \sum w_i \mathbb{E}[u_i^y]}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)\left(\frac{\sum u_i^m x_i}{\sum x_i^2}\right)\right] \qquad (23)\\
&= 0,
\end{aligned}$$

where in 23 we used the fact that $\mathbb{E}[u_i^y] = 0$.

C Unbiasedness of the estimators

C.1 Backdoor estimator

Recall that for the backdoor estimator, we take the coefficient of X in an OLS regression of Y on $\{X, W\}$. The outcome y_i can be written as

$$y_i = acx_i + bw_i + au_i^m + u_i^y.$$

The error term $au_i^m + u_i^y$ is independent of (x_i, w_i). In this case, the OLS estimator is unbiased. Therefore, $\mathbb{E}[\widehat{ac}_{\text{backdoor}}] = ac$.

C.2 Frontdoor estimator

For the frontdoor estimator, we first compute $\widehat{c}$ by taking the coefficient of X in an OLS regression of M on X. The mediator m_i can be written as

$$m_i = cx_i + u_i^m.$$

The error term u_i^m is independent of x_i. In this case, the OLS estimator is unbiased and hence, $\mathbb{E}[\widehat{c}] = c$.

We then compute $\widehat{a}_f$ by taking the coefficient of M in an OLS regression of Y on $\{M, X\}$. The outcome y_i can be written as

$$y_i = am_i + \frac{b}{d}x_i - \frac{b}{d}u_i^x + u^y.$$

In this case, the error term $-\frac{b}{d}u_i^x + u^y$ is correlated with x_i. The expression for $\widehat{a}$ is given in Eq. 18. The expectation $\mathbb{E}[\widehat{a}_f]$ is

$$\begin{aligned}
\mathbb{E}[\widehat{a}_f] &= a + \mathbb{E}\left[\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right] \\
&= a + \mathbb{E}\left[\mathbb{E}\left[\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\middle| x, m\right]\right] \\
&= a + \mathbb{E}\left[\frac{\sum x_i^2 \sum m_i \mathbb{E}[e_i|x] - \sum x_i m_i \sum x_i \mathbb{E}[e_i|x]}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right] \qquad (24) \\
&= a + \mathbb{E}\left[\frac{\sum x_i^2 \sum m_i (F x_i) - \sum x_i m_i \sum x_i (F x_i)}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right] \\
&= a,
\end{aligned}$$

where, in Eq. 24, the expression for $E[e_i|x]$ is taken from Eq. 19. Using the fact that $\text{Cov}(\widehat{a}_f, \widehat{c}) = 0$ (see proof in Appendix B.1), we can see that the frontdoor estimator is unbiased as

$$\begin{aligned}
\mathbb{E}[\widehat{a}_f \widehat{c}] &= \mathbb{E}[\widehat{a}_f]\mathbb{E}[\widehat{c}] + \text{Cov}(\widehat{a}_f, \widehat{c}) \\
&= ac.
\end{aligned}$$

C.3 Combined estimator

In the combined estimator, the expression for $\widehat{c}$ is the same as for the frontdoor estimator. Therefore, as shown in Appendix C.2, $\mathbb{E}[\widehat{c}] = c$. We compute $\widehat{a}$ by taking the coefficient of M in an OLS regression of Y on $\{M, W\}$. The outcome y_i can be written as

$$y_i = a m_i + b w_i + u_i^y.$$

The error term u_i^y is independent of (m_i, w_i). In this case, the OLS estimator is unbiased. Therefore, $\mathbb{E}[\widehat{a}_c] = a$. Using the fact that $\text{Cov}(\widehat{a}_c, \widehat{c}) = 0$ (see proof Appendix B.2), we can see that the combined estimator is unbiased as

$$\begin{aligned}
\mathbb{E}[\widehat{a}_c \widehat{c}] &= \mathbb{E}[\widehat{a}_c]\mathbb{E}[\widehat{c}] + \text{Cov}(\widehat{a}_c, \widehat{c}) \\
&= ac.
\end{aligned}$$

D Variance results for the frontdoor, backdoor, and combined estimators

D.1 Backdoor estimator

The outcome y_i can be written as

$$y_i = ac x_i + b w_i + a u_i^m + u_i^y.$$

We estimate the causal effect ac by taking the coefficient on X in an OLS regression of Y on $\{X, W\}$. Let $\mathbf{\Sigma} = \text{Cov}([X, W])$. Using Eq. 16, the finite sample variance of the backdoor estimator is

$$\begin{aligned}
\text{Var}(\widehat{ac})_{\text{backdoor}} &= \frac{\text{Var}(a u^m + u^y)\left(\mathbf{\Sigma}^{-1}\right)_{1,1}}{n-3} \\
&= \frac{a^2 \sigma_{u_m}^2 + \sigma_{u_y}^2}{(n-3)\sigma_{u_x}^2}.
\end{aligned}$$

OLS estimators are asymptotically normally for arbitrary error distributions (and hence, Gaussianity is not needed). Therefore, the asymptotic variance of the backdoor estimator is

$$\lim_{n\to\infty} \text{Var}(\sqrt{n}(\widehat{ac} - ac))_{\text{backdoor}} = \text{Var}(a u^m + u^y)\left(\mathbf{\Sigma}^{-1}\right)_{1,1} = \frac{a^2 \sigma_{u_m}^2 + \sigma_{u_y}^2}{\sigma_{u_x}^2}.$$

D.2 Frontdoor estimator

D.2.1 Variance of $\widehat{c}$

The regression of M on X can be written as $m_i = cx_i + u_i^m$. Let $\Sigma_c = \text{Var}(X)$. Using Eq. 16, $\text{Var}(\widehat{c})$ is

$$\text{Var}(\widehat{c}) = \frac{\text{Var}(u^m)(\Sigma_c^{-1})}{n-2} = \frac{\sigma_{u_m}^2}{(n-2)(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)}.$$

D.2.2 Variance of $\widehat{a}_f$

The regression of Y on $\{M, X\}$ can be written as $y_i = am_i + \frac{b}{d}x_i + e_i$, where $e_i = -\frac{b}{d}u_i^x + u_i^y$. In this case, the error e_i is not independent of the regressor x_i. Using the fact that (u^x, x) has a bivariate normal distribution, $\text{Var}(e|x)$ is

$$\begin{aligned}\text{Var}(e|x) &= \frac{b^2\sigma_{u_w}^2\sigma_{u_x}^2 + \sigma_{u_y}^2(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)}{(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)} \\ &:= V_e.\end{aligned} \tag{25}$$

Note that V_e is a constant and does not depend on x. From Eqs. 17 and 18, we know that

$$\begin{aligned}\widehat{c} &= c + \frac{\sum x_i u_i^m}{\sum x_i^2} \\ \widehat{a}_f &= a + \frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}.\end{aligned}$$

where $e_i = -\frac{b}{d}u_i^x + u_i^y$. Let

$$\begin{aligned}A &= \frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2} \\ C &= \frac{\sum x_i u_i^m}{\sum x_i^2}.\end{aligned}$$

First, we derive the expression for $\text{Var}(\widehat{a}_f)$ as follows,

$$\begin{aligned}\text{Var}(\widehat{a}_f) &= \text{Var}(a + A) \\ &= \text{Var}(A) \\ &= \text{Var}(\mathbb{E}[A|x,m]) + \mathbb{E}[\text{Var}(A|x,m)] \\ &= \text{Var}\left(\mathbb{E}\left[\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\middle| x, m\right]\right) + \mathbb{E}[\text{Var}(A|x,m)] \\ &= \text{Var}\left(\mathbb{E}\left[\frac{\sum x_i^2 \sum m_i \mathbb{E}[e_i|x] - \sum x_i m_i \sum x_i \mathbb{E}[e_i|x]}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right]\right) + \mathbb{E}[\text{Var}(A|x,m)] \\ &= \text{Var}\left(\mathbb{E}\left[\frac{\sum x_i^2 \sum m_i (Fx_i) - \sum x_i m_i \sum x_i (Fx_i)}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right]\right) + \mathbb{E}[\text{Var}(A|x,m)] \\ &= \mathbb{E}[\text{Var}(A|x,m)] \\ &= \mathbb{E}\left[\text{Var}\left(\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\middle| x, m\right)\right] \\ &= \mathbb{E}\left[\frac{1}{\left(\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2\right)^2}\text{Var}\left(\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i \middle| x, m\right)\right] \\ &= \mathbb{E}\left[\frac{1}{\left(\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2\right)^2}\text{Var}(e_i|x_i)\sum x_i^2\left(\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2\right)\right] \\ &= V_e\mathbb{E}\left[\frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right] \\ &= V_e\mathbb{E}[D],\end{aligned} \tag{26, 27}$$

where, in Eq. 26, we used the result from Eq. 19, and $D = \frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}$. Using the fact that D has the distribution of a marginal from an inverse Wishart-distributed matrix, that is, if the matrix $M \sim \mathcal{IW}(\text{Cov}([M, X])^{-1}, n)$, then $D = M_{1,1}$, in Eq. 27, we get

$$\begin{aligned}
\text{Var}(\widehat{a}_f) &= V_e \mathbb{E}\left[D\right] \\
&= V_e \frac{\text{Cov}([M, X])^{-1}_{1,1}}{n-2-1} \\
&= \frac{b^2 \sigma^2_{u_w} \sigma^2_{u_x} + \sigma^2_{u_y}(d^2 \sigma^2_{u_w} + \sigma^2_{u_x})}{(n-3)(d^2 \sigma^2_{u_w} + \sigma^2_{u_x}) \sigma^2_{u_m}},
\end{aligned}$$

where the expression for V_e is taken from Eq. 25.

D.2.3 Covariance of $\widehat{a}_f^2$ and $\widehat{c}^2$

We prove that $\text{Cov}(\widehat{a}_f^2, \widehat{c}^2) = \text{Var}(\widehat{a}_f)\text{Var}(\widehat{c})$. This covariance can be written as

$$\begin{aligned}
\text{Cov}(\widehat{a}_f^2, \widehat{c}^2) &= \mathbb{E}[(\widehat{a}_f^2 - \mathbb{E}[\widehat{a}_f^2])(\widehat{c}^2 - \mathbb{E}[\widehat{c^2}])] \\
&= \mathbb{E}[(\widehat{a}_f^2 - \text{Var}(\widehat{a}_f) - \mathbb{E}^2[\widehat{a}_f])(\widehat{c}^2 - \text{Var}(\widehat{c}) - \mathbb{E}^2[\widehat{c}])] \\
&= \mathbb{E}[\widehat{a}_f^2 \widehat{c}^2] - \text{Var}(\widehat{a}_f)\text{Var}(\widehat{c}) - a^2 \text{Var}(\widehat{c}) - c^2 \text{Var}(\widehat{a}_f) - a^2 c^2. \qquad (28)
\end{aligned}$$

We can write $\mathbb{E}[\widehat{a}_f^2 \widehat{c}^2]$ as

$$\begin{aligned}
\mathbb{E}[\widehat{a}_f^2 \widehat{c}^2] &= \mathbb{E}[(a + A)^2 (c + C)^2] \\
&= \mathbb{E}[a^2 c^2 + c^2 A^2 + a^2 C^2 + A^2 C^2 + 2aAC^2 + 2cCA^2] \\
&= a^2 c^2 + c^2 \text{Var}(\widehat{a}) + a^2 \text{Var}(\widehat{c}) + \mathbb{E}[A^2 C^2] + \mathbb{E}[2aAC^2] + \mathbb{E}[2cCA^2]. \qquad (29)
\end{aligned}$$

Substituting the result from Eq. 29 in Eq. 28, we get

$$\text{Cov}(\widehat{a}_f^2, \widehat{c}^2) = \mathbb{E}[A^2 C^2] + \mathbb{E}[2aAC^2] + \mathbb{E}[2cCA^2]. \qquad (30)$$

Now we expand each term in Eq. 30 separately. $\mathbb{E}[2aAC^2]$ is

$$\begin{aligned}
\mathbb{E}[2aAC^2] &= 2a\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2 \left(\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] \\
&= 2a\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2 \left(\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right) \middle| x, m\right]\right] \\
&= 2a\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2 \left(\frac{\sum x_i^2 \sum m_i \mathbb{E}[e_i|x] - \sum x_i m_i \sum x_i \mathbb{E}[e_i|x]}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] \qquad (31) \\
&= 2a\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2 \left(\frac{\sum x_i^2 \sum m_i (F x_i) - \sum x_i m_i \sum x_i (F x_i)}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] \\
&= 0, \qquad (32)
\end{aligned}$$

where, in Eq. 31, the expression for $\mathbb{E}[e|x]$ is taken from Eq. 19.

Next, we simplify $\mathbb{E}[2cCA^2]$ as

$$
\begin{aligned}
&\mathbb{E}[2cCA^2] \\
&= 2c\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)^2\right] \\
&= 2c\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\sum x_i^2 \sum m_i e_i - \sum x_i m_i \sum x_i e_i}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)^2 \middle| x, m\right]\right] \\
&= 2c\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\frac{(\sum x_i^2)^2(\sum m_i e_i)^2 + (\sum x_i m_i)^2(\sum x_i e_i)^2 - 2\sum x_i^2 \sum m_i e_i \sum x_i m_i \sum x_i e_i}{\left(\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2\right)^2} \middle| x, m\right]\right] \\
&= 2c\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\text{Var}(e|x)(\sum x_i^2)}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2} + \frac{F^2\left(2(\sum x_i^2)^2(\sum x_i m_i)^2 - 2(\sum x_i^2)^2(\sum x_i m_i)^2\right)}{\left(\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2\right)^2}\right)\right] \\
&= 2c\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\text{Var}(e|x)(\sum x_i^2)}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] \\
&= 2cV_e\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] \\
&= 2cV_e\mathbb{E}\left[(\widehat{c} - c)\left(\frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] \\
&= 2cV_e\left(\mathbb{E}\left[\widehat{c}\left(\frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right] - c\mathbb{E}\left[\left(\frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}\right)\right]\right) \\
&= 2cV_e\left(\mathbb{E}\left[\widehat{c}D\right] - c\mathbb{E}\left[D\right]\right), \qquad (33)
\end{aligned}
$$

where $D = \frac{\sum x_i^2}{\sum x_i^2 \sum m_i^2 - (\sum x_i m_i)^2}$. Using the fact that $\widehat{c}$ and D are independent of each other (see proof at the end of this section), we get

$$
\begin{aligned}
\mathbb{E}[\widehat{c}D] &= \mathbb{E}[\widehat{c}]\mathbb{E}[D] \\
&= c\mathbb{E}[D]. \qquad (34)
\end{aligned}
$$

Substituting the result from Eq. 34 in Eq. 33, we get

$$
\begin{aligned}
\mathbb{E}[2cCA^2] &= 2cV_e\left(c\mathbb{E}\left[D\right] - c\mathbb{E}\left[D\right]\right) \\
&= 0. \qquad (35)
\end{aligned}
$$

We proceed similarly to Eq. 33 to write $\mathbb{E}[A^2C^2]$ as

$$\mathbb{E}[A^2C^2] = V_e\mathbb{E}[C^2D].$$

Then we further simplify $\mathbb{E}[A^2C^2]$ as

$$
\begin{aligned}
\mathbb{E}[A^2C^2] &= V_e\mathbb{E}[C^2D] \\
&= V_e\mathbb{E}[(\widehat{c} - c)^2D] \\
&= V_e\left(\mathbb{E}\left[\widehat{c}^2\right]\mathbb{E}[D] - c^2\mathbb{E}[D]\right) \\
&= V_e\left(\text{Var}(\widehat{c})\mathbb{E}[D] + \mathbb{E}^2\left[\widehat{c}\right]\mathbb{E}[D] - c^2\mathbb{E}[D]\right) \\
&= V_e\text{Var}(\widehat{c})\mathbb{E}[D] \qquad (36) \\
&= V_e\text{Var}(\widehat{c})\frac{\text{Cov}([M, X])^{-1}_{1,1}}{n - 2 - 1} \\
&= \frac{V_e}{(n-3)\sigma^2_{u_m}}\text{Var}(\widehat{c}) \qquad (37) \\
&= \text{Var}(\widehat{a}_f)\text{Var}(\widehat{c}), \qquad (38)
\end{aligned}
$$

where, in Eq. 36, we used the fact that if the matrix $M \sim \mathcal{IW}(\text{Cov}([M, X])^{-1}, n)$, then $D = M_{1,1}$ (that is, D has the distribution of a marginal from an inverse Wishart-distributed matrix), and in Eq. 37, the expression for V_e is taken from Eq. 25.

Substituting the results from Eqs. 32, 35, and 38 in Eq. 30, we get

$$\text{Cov}(\widehat{a}_f^2, \widehat{c}^2) = \text{Var}(\widehat{a}_f)\text{Var}(\widehat{c}). \qquad (39)$$

Proof that $\widehat{c}$ and D are independent. Let $\mathbf{\Sigma}$ be the following sample covariance matrix:

$$\mathbf{\Sigma} = \frac{1}{n}\begin{bmatrix} \sum m_i^2 & \sum m_i x_i \\ \sum m_i x_i & \sum x_i^2 \end{bmatrix}.$$

The distribution of Σ is a Wishart distribution. That is, $\Sigma \sim \mathcal{W}(\text{Cov}([M, X]), n)$. Then $(\mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1})$ and $(\mathbf{\Sigma}_{2,1}, \mathbf{\Sigma}_{2,2})$ are independent [Eaton, 2007, Proposition 8.7]. We can see that

$$\begin{aligned}\mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1} &= \frac{\sum m_i^2 \sum x_i^2 - (\sum x_i m_i)^2}{\sum x_i^2} \\ &= \frac{1}{D}.\end{aligned}$$

Therefore, we get

$$\begin{aligned} &\frac{1}{D} \perp\!\!\!\perp \left(\sum x_i^2, \sum x_i m_i\right) \\ \implies\ & D \perp\!\!\!\perp \left(\sum x_i^2, \sum x_i m_i\right) \\ \implies\ & D \perp\!\!\!\perp \frac{\sum x_i m_i}{\sum x_i^2} \\ \implies\ & D \perp\!\!\!\perp \widehat{c}. \end{aligned}$$

D.2.4 Finite Sample Variance of $\widehat{a}_f\widehat{c}$

The variance of the product of two random variables can be written as

$$\begin{aligned}\text{Var}(\widehat{a}_f\widehat{c}) &= \text{Cov}(\widehat{a}_f^2, \widehat{c}^2) + (\text{Var}(\widehat{a}_f) + \mathbb{E}^2[\widehat{a}_f])(\text{Var}(\widehat{c}) + \mathbb{E}^2[\widehat{c}]) - (\text{Cov}(\widehat{a}_f, \widehat{c}) + \mathbb{E}[\widehat{a}_f]\mathbb{E}[\widehat{c}])^2 \quad (40) \\ &= \text{Cov}(\widehat{a}_f^2, \widehat{c}^2) + (\text{Var}(\widehat{a}_f) + a^2)(\text{Var}(\widehat{c}) + c^2) - (\text{Cov}(\widehat{a}_f, \widehat{c}) + ac)^2,\end{aligned}$$

where in Eq. 40 we used the facts that $\mathbb{E}[\widehat{a}_f] = a$, and $\mathbb{E}[\widehat{c}] = c$ (see Appendix C.2). Using the facts that $\text{Cov}(\widehat{a}_f^2, \widehat{c}^2) = \text{Var}(\widehat{a}_f)\text{Var}(\widehat{c})$ (from Eq. 38) and $\text{Cov}(\widehat{a}_f, \widehat{c}) = 0$ (from Appendix B.1), we get

$$\text{Var}(\widehat{a}_f\widehat{c}) = a^2\text{Var}(\widehat{c}) + c^2\text{Var}(\widehat{a}_f) + 2\text{Var}(\widehat{c})\text{Var}(\widehat{a}_f).$$

D.2.5 Asymptotic Variance of $\widehat{a}_f\widehat{c}$

Using asymptotic normality of OLS estimators, which does not require Gaussianity, we have

$$\begin{aligned} &\sqrt{n}\left(\begin{bmatrix}\widehat{a}_f \\ \widehat{c}\end{bmatrix} - \begin{bmatrix} a \\ c\end{bmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \lim_{n\to\infty}\begin{bmatrix}\text{Var}_\infty(\widehat{a}_f) & \text{Cov}(\sqrt{n}\widehat{a}_f, \widehat{c}) \\ \text{Cov}(\sqrt{n}\widehat{a}_f, \widehat{c}) & \text{Var}_\infty(\widehat{c})\end{bmatrix}\right) \\ \implies\ &\sqrt{n}\left(\begin{bmatrix}\widehat{a}_f \\ \widehat{c}\end{bmatrix} - \begin{bmatrix} a \\ c\end{bmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix}\text{Var}_\infty(\widehat{a}_f) & 0 \\ 0 & \text{Var}_\infty(\widehat{c})\end{bmatrix}\right),\end{aligned}$$

where $\text{Var}_\infty(\widehat{a}_f)$ and $\text{Var}_\infty(\widehat{c})$ are the asymptotic variances of $\widehat{a}_f$ and $\widehat{c}$, respectively. The expressions for asymptotic variances are

$$\begin{aligned}\text{Var}_\infty(\widehat{a}_f) &= V_e\text{Cov}([M, X])^{-1}_{1,1} = \frac{b^2\sigma_{u_w}^2\sigma_{u_x}^2 + \sigma_{u_y}^2(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)}{(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)\sigma_{u_m}^2} \\ \text{Var}_\infty(\widehat{c}) &= \text{Var}(u^m)(\Sigma_c^{-1}) = \frac{\sigma_{u_m}^2}{d^2\sigma_{u_w}^2 + \sigma_{u_x}^2}.\end{aligned}$$

In order to compute the asymptotic variance of $\widehat{a}_f\widehat{c}$, we use the Delta method:

$$\begin{aligned} &\sqrt{n}(\widehat{a}_f\widehat{c} - ac) \xrightarrow{d} \mathcal{N}\left(0, [c \quad a]\begin{bmatrix}\text{Var}_\infty(\widehat{a}_f) & 0 \\ 0 & \text{Var}_\infty(\widehat{c})\end{bmatrix}\begin{bmatrix} c \\ a\end{bmatrix}\right) \\ \implies\ &\sqrt{n}(\widehat{a}_f\widehat{c} - ac) \xrightarrow{d} \mathcal{N}\left(0, c^2\text{Var}_\infty(\widehat{a}_f) + a^2\text{Var}_\infty(\widehat{c})\right).\end{aligned}$$

D.3 Combined estimator

D.3.1 Finite sample variance of $\widehat{a}_c$

We can write the regression of Y on $\{M, W\}$ as $y_i = am_i + bw_i + u_i^y$. Let $\mathbf{\Sigma}_{a_c} = \text{Cov}([M, W])$. Using Eq. 16, we get

$$\text{Var}(\widehat{a}_c) = \frac{\text{Var}(u_i^y)(\mathbf{\Sigma}_{a_c}^{-1})_{1,1}}{n-3} = \frac{\sigma_{u_y}^2}{(n-3)(c^2\sigma_{u_x}^2 + \sigma_{u_m}^2)}.$$

D.3.2 Bounding the finite sample variance

We first compute the lower bound of the combined estimator. Since the estimator is unbiased (see Appendix B.2), we can apply the Cramer-Rao theorem to lower bound the finite sample variance.

Let the vector $\mathbf{s_i} = [x_i, y_i, w_i, m_i]$ denote the i^{th} sample. Since the data is multivariate Gaussian, the log-likelihood of the data is

$$\mathcal{LL} = -\frac{n}{2}\left[\log\left(\det \mathbf{\Sigma}\right) + \text{Tr}\left(\widehat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\right)\right],$$

where $\mathbf{\Sigma} = \text{Cov}([X, Y, W, M])$ and $\widehat{\mathbf{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{s_i}\mathbf{s_i}^\top$. Let $e = ac$ and $\widehat{e} = \widehat{a}_c c$. Since we want to lower bound the variance of $\widehat{e}$, we reparameterize the log-likelihood by replacing c with e/a to simplify calculations. Next, we compute the Fisher Information Matrix for the eight model parameters:

$$\mathbf{I} = -E\begin{bmatrix} \frac{\partial^2\mathcal{LL}}{\partial e^2} & \frac{\partial^2\mathcal{LL}}{\partial e\partial a} & \frac{\partial^2\mathcal{LL}}{\partial e\partial b} & \cdots & \frac{\partial^2\mathcal{LL}}{\partial e\partial \sigma_{u_y}} \\ \frac{\partial^2\mathcal{LL}}{\partial a\partial e} & \frac{\partial^2\mathcal{LL}}{\partial a^2} & \frac{\partial^2\mathcal{LL}}{\partial a\partial b} & \cdots & \frac{\partial^2\mathcal{LL}}{\partial a\partial \sigma_{u_y}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\mathcal{LL}}{\partial \sigma_{u_y}\partial e} & \frac{\partial^2\mathcal{LL}}{\partial \sigma_{u_y}\partial a} & \frac{\partial^2\mathcal{LL}}{\partial \sigma_{u_y}\partial b} & \cdots & \frac{\partial^2\mathcal{LL}}{\partial^2 \sigma_{u_y}} \end{bmatrix}.$$

Therefore, using the Cramer-Rao theorem, we have

$$\begin{aligned} \text{Var}(\widehat{e}) &= \text{Var}(\widehat{a}_c c) \\ &\geq (\mathbf{I}^{-1})_{1,1} \\ &= \frac{1}{n}\left(\frac{c^2\sigma_{u_y}^2}{c^2\sigma_{u_x}^2 + \sigma_{u_m}^2} + \frac{a^2\sigma_{u_m}^2}{d^2\sigma_{u_w}^2 + \sigma_{u_x}^2}\right). \end{aligned}$$

Next, we compute a finite sample upper bound for $\text{Cov}(\widehat{a}_c^2, \widehat{c}^2)$. We derive this in a similar manner as the frontdoor estimator in Appendix D.2.3. From Eqs. 21 and 22, we know that

$$\begin{aligned} \widehat{a}_c &= a + \frac{\sum w_i^2 \sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2} \\ \widehat{c} &= c + \frac{\sum x_i u_i^m}{\sum x_i^2}. \end{aligned}$$

Let

$$\begin{aligned} A &= \frac{\sum w_i^2 \sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2} \\ C &= \frac{\sum x_i u_i^m}{\sum x_i^2}. \end{aligned}$$

Then, similarly to Eq. 30, we get

$$\text{Cov}(\widehat{a}_c^2, \widehat{c}^2) = \mathbb{E}[A^2C^2] + \mathbb{E}[2aAC^2] + \mathbb{E}[2cCA^2]. \quad (41)$$

Now we simplify each term in Eq. 41 separately. $\mathbb{E}[2aAC^2]$ can be simplified as

$$\begin{aligned}
\mathbb{E}[2aAC^2] &= 2a\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2\left(\frac{\sum w_i^2\sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2\left(\frac{\sum w_i^2\sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)\middle| x, m, w\right]\right] \\
&= \mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^2\left(\frac{\sum w_i^2\sum m_i \mathbb{E}[u_i^y] - \sum w_i m_i \sum w_i \mathbb{E}[u_i^y]}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)\right] \qquad (42) \\
&= 0, \qquad (43)
\end{aligned}$$

where, in Eq. 42, we used the fact that $\mathbb{E}[u^y] = 0$.

Next, we simplify $\mathbb{E}[2cCA^2]$ as

$$\begin{aligned}
\mathbb{E}[2cCA^2] &= 2c\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\sum w_i^2\sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)^2\right] \\
&= 2c\mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\left(\frac{\sum w_i^2\sum m_i u_i^y - \sum w_i m_i \sum w_i u_i^y}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)^2\middle| x, m, w\right]\right] \\
&= 2c\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)\text{Var}(u^y)\left(\frac{\sum w_i^2}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}\right)\right] \\
&= 2c\sigma_{u_y}^2\mathbb{E}\left[CD\right], \qquad (44)
\end{aligned}$$

where $D = \frac{\sum w_i^2}{\sum w_i^2 \sum m_i^2 - (\sum w_i m_i)^2}$. We can upper bound the expression in Eq. 44 as

$$\begin{aligned}
\mathbb{E}[2cCA^2] &= 2c\sigma_{u_y}^2\mathbb{E}\left[CD\right] \\
&\leq 2|c|\sigma_{u_y}^2\mathbb{E}\left[CD\right] \qquad (45) \\
&\leq 2|c|\sigma_{u_y}^2\sqrt{\mathbb{E}[C^2]\mathbb{E}[D^2]} \\
&= 2|c|\sigma_{u_y}^2\sqrt{\text{Var}(\widehat{c})(\text{Var}(D) + \mathbb{E}[D]^2)} \qquad (46) \\
&= 2|c|\sigma_{u_y}^2\sqrt{\text{Var}(\widehat{c})\left(\frac{2\left[(\text{Cov}([M, W]))^{-1}_{1,1}\right]^2}{(n-2-1)^2(n-2-3)} + \left(\frac{(\text{Cov}([M, W]))^{-1}_{1,1}}{n-2-1}\right)^2\right)} \\
&= 2|c|\sigma_{u_y}^2\sqrt{\text{Var}(\widehat{c})}\sqrt{\frac{2}{(n-3)^2(n-5)(c^2\sigma_{u_x}^2 + \sigma_{u_m}^2)^2} + \frac{1}{(n-3)^2(c^2\sigma_{u_x}^2 + \sigma_{u_m}^2)^2}} \\
&= 2|c|\frac{\sigma_{u_y}^2}{(n-3)(c^2\sigma_{u_x}^2 + \sigma_{u_m}^2)}\sqrt{\text{Var}(\widehat{c})}\sqrt{\frac{n-3}{n-5}} \\
&= 2|c|\text{Var}(\widehat{a})\sqrt{\text{Var}(\widehat{c})}\sqrt{\frac{n-3}{n-5}}, \qquad (47)
\end{aligned}$$

where, in Eq. 45, we used the Cauchy–Schwarz inequality, and in Eq. 46, we used the fact that if the matrix $M \sim \mathcal{IW}(\text{Cov}([M, W])^{-1}, n)$, then $D = M_{1,1}$ (that is, D has the distribution of a marginal from an inverse Wishart-distributed matrix).

Similarly to Eq. 44, we simplify $\mathbb{E}[A^2C^2]$ as

$$\mathbb{E}[A^2C^2] = \sigma_{u_y}^2\mathbb{E}[C^2D]. \qquad (48)$$

The expression in Eq. 48 can be upper bounded using the Cauchy-Schwarz inequality as

$$\begin{aligned}
\mathbb{E}[A^2C^2] &= \sigma_{u_y}^2 \mathbb{E}[C^2 D] \\
&\leq \sigma_{u_y}^2 \sqrt{\mathbb{E}[C^4]\mathbb{E}[D^2]} \\
&= \sigma_{u_y}^2 \sqrt{\mathbb{E}[C^4](\text{Var}(D) + \mathbb{E}^2[D])} \\
&= \sigma_{u_y}^2 \sqrt{\mathbb{E}[C^4]\left(\frac{2\left[(\text{Cov}([M,W]))_{1,1}^{-1}\right]^2}{(n-2-1)^2(n-2-3)} + \left(\frac{(\text{Cov}([M,W]))_{1,1}^{-1}}{n-2-1}\right)^2\right)} \\
&= \frac{\sigma_{u_y}^2}{(n-3)(c^2\sigma_{u_x}^2 + \sigma_{u_m}^2)} \sqrt{\frac{n-3}{n-5}} \sqrt{\mathbb{E}[C^4]} \\
&= \text{Var}(\widehat{a}_c) \sqrt{\frac{n-3}{n-5}} \sqrt{\mathbb{E}[C^4]}.
\end{aligned} \tag{49}$$

We can simplify $E[C^4]$ as follows,

$$\begin{aligned}
E[C^4] &= \mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^4\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\sum x_i u_i^m}{\sum x_i^2}\right)^4 \middle| x\right]\right] \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\mathbb{E}\left[\left(\sum x_i u_i^m\right)^4 \middle| x\right]\right] \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\left\{\text{Var}\left(\left(\sum x_i u_i^m\right)^2 | x\right) + \mathbb{E}\left[\left(\sum x_i u_i^m\right)^2 | x\right]^2\right\}\right] \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\left\{\text{Var}\left(\left(\sum x_i u_i^m\right)^2 | x\right) + \sigma_{u_m}^4 \left(\sum x_i^2\right)^2\right\}\right] \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\left\{\text{Var}\left(\sigma_{u_m}^2 \sum x_i^2 \frac{(\sum x_i u_i^m)^2}{\sigma_{u_m}^2 \sum x_i^2}\middle| x\right) + \sigma_{u_m}^4 \left(\sum x_i^2\right)^2\right\}\right] \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\left\{\sigma_{u_m}^4 \left(\sum x_i^2\right)^2 \text{Var}\left(\frac{(\sum x_i u_i^m)^2}{\sigma_{u_m}^2 \sum x_i^2}\middle| x\right) + \sigma_{u_m}^4 \left(\sum x_i^2\right)^2\right\}\right] \quad (50) \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\left\{\sigma_{u_m}^4 \left(\sum x_i^2\right)^2 2 + \sigma_{u_m}^4 \left(\sum x_i^2\right)^2\right\}\right] \\
&= \mathbb{E}\left[\frac{1}{(\sum x_i^2)^4}\left\{3\sigma_{u_m}^4 \left(\sum x_i^2\right)^2\right\}\right] \\
&= 3\sigma_{u_m}^4 \mathbb{E}\left[\frac{1}{(\sum x_i^2)^2}\right] \\
&= 3\sigma_{u_m}^4 \left[\text{Var}\left(\frac{1}{\sum x_i^2}\right) + \mathbb{E}\left[\frac{1}{\sum x_i^2}\right]^2\right] \quad (51) \\
&= 3\sigma_{u_m}^4 \left[\frac{2}{(n-2)^2(n-4)(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)^2} + \frac{1}{(n-2)^2(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)^2}\right] \\
&= 3\frac{\sigma_{u_m}^4}{(n-2)^2(d^2\sigma_{u_w}^2 + \sigma_{u_x}^2)^2}\left[\frac{n-2}{n-4}\right] \\
&= 3\left(\text{Var}(\widehat{c}^2)\right)^2 \left[\frac{n-2}{n-4}\right],
\end{aligned} \tag{52}$$

where, in Eq. 50, we used the fact that $\frac{(\sum x_i u_i^m)^2}{\sigma_{u_m}^2 \sum x_i^2}\Big|x$ has a Chi-squared distribution, that is, $\frac{(\sum x_i u_i^m)^2}{\sigma_{u_m}^2 \sum x_i^2}\Big|x \sim \chi^2(1)$, and in Eq. 51, we used the fact that $\frac{1}{\sum x_i^2}$ has a scaled inverse Chi-squared distribution, that is, $\frac{1}{\sum x_i^2} \sim \text{Scale-inv-}\chi^2\left(n, \frac{(d^2\sigma_{u_w}^2+\sigma_{u_x}^2)^2}{n}\right)$.

Substituting the result from Eq. 52 in Eq. 49, we get

$$\mathbb{E}[A^2C^2] \leq \text{Var}(\widehat{a}_c)\text{Var}(\widehat{c})\sqrt{\frac{3(n-3)(n-2)}{(n-5)(n-4)}}. \tag{53}$$

Substituting the results from Eqs. 43, 47, and 53 in Eq. 41, we get

$$\text{Cov}(\widehat{a}_c^2, \widehat{c}^2) \leq \sqrt{\frac{n-3}{n-5}}\left(2|c|\text{Var}(\widehat{a}_c)\sqrt{\text{Var}(\widehat{c})} + \sqrt{3}\sqrt{\frac{n-2}{n-4}}\text{Var}(\widehat{a}_c)\text{Var}(\widehat{c})\right).$$

The variance of the product of two random variables can be written as

$$\begin{aligned}\text{Var}(\widehat{a}_c\widehat{c}) &= \text{Cov}(\widehat{a}_c^2, \widehat{c}^2) + (\text{Var}(\widehat{a}_c) + \mathbb{E}^2[\widehat{a}_c])(\text{Var}(\widehat{c}) + \mathbb{E}^2[\widehat{c}]) - (\text{Cov}(\widehat{a}_c, \widehat{c}) + \mathbb{E}[\widehat{a}_c]\mathbb{E}[\widehat{c}])^2 \\ &= \text{Cov}(\widehat{a}_c^2, \widehat{c}^2) + (\text{Var}(\widehat{a}_c) + a^2)(\text{Var}(\widehat{c}) + c^2) - (\text{Cov}(\widehat{a}_c, \widehat{c}) + ac)^2,\end{aligned}$$

where we used the facts that $\mathbb{E}[\widehat{a}_c] = a$, and $\mathbb{E}[\widehat{c}] = c$ (see Appendix C.3). Using the fact that $\text{Cov}(\widehat{a}_c, c) = 0$ (see Appendix B.2) and the upper bound for $\text{Cov}(\widehat{a}_c^2, \widehat{c}^2)$, we get

$$\begin{aligned}&\text{Var}(\widehat{a}_c\widehat{c}) \leq \\ &\quad c^2\text{Var}(\widehat{a}_c) + a^2\text{Var}(\widehat{c}) + \sqrt{\frac{n-3}{n-5}}\left(2|c|\text{Var}(\widehat{a}_c)\sqrt{\text{Var}(\widehat{c})} + \sqrt{3}\sqrt{\frac{n-2}{n-4}}\text{Var}(\widehat{a}_c)\text{Var}(\widehat{c})\right).\end{aligned}$$

D.3.3 Asymptotic variance

Using asymptotic normality of OLS estimators, which does not require Gaussianity, we have

$$\begin{aligned}&\sqrt{n}\left(\begin{bmatrix}\widehat{a}_c \\ \widehat{c}\end{bmatrix} - \begin{bmatrix}a \\ c\end{bmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \lim_{n\to\infty}\begin{bmatrix}\text{Var}_\infty(\widehat{a}_c) & \text{Cov}(\sqrt{n}\widehat{a}_c, \widehat{c}) \\ \text{Cov}(\sqrt{n}\widehat{a}_c, \widehat{c}) & \text{Var}_\infty(\widehat{c})\end{bmatrix}\right) \\ \implies &\sqrt{n}\left(\begin{bmatrix}\widehat{a}_c \\ \widehat{c}\end{bmatrix} - \begin{bmatrix}a \\ c\end{bmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix}\text{Var}_\infty(\widehat{a}_c) & 0 \\ 0 & \text{Var}_\infty(\widehat{c})\end{bmatrix}\right),\end{aligned}$$

where $\text{Var}_\infty(\widehat{a}_c)$ and $\text{Var}_\infty(\widehat{c})$ are the asymptotic variances of $\widehat{a}_c$ and $\widehat{c}$, respectively. The expressions for the asymptotic variances are

$$\begin{aligned}\text{Var}_\infty(\widehat{a}_c) &= \text{Var}(u_i^y)(\mathbf{\Sigma}_{a_c}^{-1})_{1,1} = \frac{\sigma_{u_y}^2}{c^2\sigma_{u_x}^2 + \sigma_{u_m}^2} \\ \text{Var}_\infty(\widehat{c}) &= \text{Var}(u^m)(\Sigma_c^{-1}) = \frac{\sigma_{u_m}^2}{d^2\sigma_{u_w}^2 + \sigma_{u_x}^2}.\end{aligned}$$

In order to compute the asymptotic variance of $\widehat{a}_c\widehat{c}$, we use the Delta method:

$$\begin{aligned}&\sqrt{n}(\widehat{a}_c\widehat{c} - ac) \xrightarrow{d} \mathcal{N}\left(0, [c \quad a]\begin{bmatrix}\text{Var}_\infty(\widehat{a}_c) & 0 \\ 0 & \text{Var}_\infty(\widehat{c})\end{bmatrix}\begin{bmatrix}c \\ a\end{bmatrix}\right) \\ \implies &\sqrt{n}(\widehat{a}_c\widehat{c} - ac) \xrightarrow{d} \mathcal{N}\left(0, c^2\text{Var}_\infty(\widehat{a}_c) + a^2\text{Var}_\infty(\widehat{c})\right).\end{aligned}$$

E Comparison of combined estimator with backdoor and frontdoor estimators

In this section, we provide more details on the comparison of the combined estimator presented in 5 to the backdoor and frontdoor estimators.

E.1 Comparison with the backdoor estimator

In Section 5.1, we made the claim that

$$\exists N, \text{s.t.}, \forall n > N, \text{Var}(\widehat{a}_c\widehat{c}) \leq \text{Var}(\widehat{ac})_{\text{backdoor}}.$$

In this case, by comparing Eqs. 5 and 12, we have

$$N = \frac{2\left(\sigma_{u_x}^4 F + d^2\sigma_{u_w}^2(\sigma_{u_m}^2 D + \sigma_{u_x}^2 F) + c^2\sigma_{u_x}^6\sqrt{c^2\sigma_{u_x}^6 D^2(F + 2\sqrt{3}\sigma_{u_m}^2)}\right)}{\sigma_{u_m}^2 D^2},$$

where $D = d^2\sigma_{u_w}^2 + \sigma_{u_x}^2$, $E = c^2\sigma_{u_x}^2 + \sigma_{u_m}^2$, and $F = E + (1+2\sqrt{3})\sigma_{u_m}^2$. Thus, for a large enough n, the combined estimator has lower variance than the backdoor estimator for all model parameter values.

E.2 Comparison with the frontdoor estimator

In Section 5.1, we made the claim that

$$\exists N, \text{s.t.}, \forall n > N, \text{Var}(\widehat{a}_c\widehat{c}) \leq \text{Var}(\widehat{a}_f\widehat{c}).$$

In this case, by comparing Eqs. 7 and 12, we have

$$N = \frac{2\left(\sigma_{u_m}^6 + 2\sqrt{3}c^2\sigma_{u_m}^4\sigma_{u_x}^2 - c^4\sigma_{u_m}^2\sigma_{u_x}^4 + \left(D + \sigma_{u_m}^4\sqrt{\sigma_{u_m}^4 + 4\sqrt{3}c^2\sigma_{u_m}^2\sigma_{u_x}^2 - 2c^4\sigma_{u_x}^4}\right)\right)}{D^2},$$

where $D = c^6\sigma_{u_x}^6$. Thus, for a large enough n, the combined estimator has lower variance than the frontdoor estimator for all model parameter values.

E.3 Combined estimator dominates the better of backdoor and frontdoor

In this section, we provide more details for the claim in Section 5.1 that the combined estimator can dominate the better of the backdoor and frontdoor estimators by an arbitrary amount. We show that the quantity

$$R = \frac{\min\{\text{Var}(\widehat{ac})_{\text{backdoor}}, \text{Var}(\widehat{a}_f\widehat{c})\}}{\text{Var}(\widehat{a}_c\widehat{c})}$$

is unbounded.

We do this by considering the case when $\text{Var}(\widehat{ac})_{\text{backdoor}} = \text{Var}(\widehat{a}_f\widehat{c})$. Note that

$$\begin{aligned} &\text{Var}(\widehat{ac})_{\text{backdoor}} = \text{Var}(\widehat{a}_f\widehat{c}) \\ \implies b = &\sqrt{\frac{-D(-a^2\sigma_{u_m}^4((n-2)d^2\sigma_{u_w}^2 + \sigma_{u_x}^2) + (-(n-2)d^2\sigma_{u_w}^2(\sigma_{u_m}^2 - c^2\sigma_{u_x}^2) + \sigma_{u_x}^2 E)\sigma_{u_y}^2)}{\sigma_{u_w}^2\sigma_{u_x}^4(2\sigma_{u_m}^2 + (n-2)c^2 D)}}, \end{aligned} \tag{54}$$

where $D = d^2\sigma_{u_w}^2 + \sigma_{u_x}^2$, and $E = (n-2)c^2\sigma_{u_x}^2 - (n-4)\sigma_{u_m}^2$. Hence, if the parameter b is set to the value given in Eq. 54, the backdoor and frontdoor estimators will have equal variance. We have to ensure that the value of b is real. b will be a real number if

$$|c| \leq \frac{\sigma_{u_m}}{\sigma_{u_x}}\sqrt{1 - \frac{2\sigma_{u_x}^2}{(n-2)D}}, \text{ and}$$
$$n > 2.$$

For the value of b in Eq. 54, the quantity R becomes

$$\begin{aligned} R &= \frac{\text{Var}(\widehat{ac})_{\text{backdoor}}}{\text{Var}(\widehat{a}_c\widehat{c})} \\ &\geq \frac{(n-2)DE(a^2\sigma_{u_m}^2 + \sigma_{u_y}^2)}{\sigma_{u_x}^2\left((n-3)a^2\sigma_{u_m}^2 E + \sigma_{u_y}^2\left(\sigma_{u_m}^2 + \sqrt{3}\sigma_{u_m}^2\left(r_1 r_2 + |c|(n-2)D\left(|c| + r_1\frac{\sigma_{u_m}}{\sqrt{(n-2)D}}\right)\right)\right)\right)}, \end{aligned}$$

where $D = d^2\sigma^2_{u_w} + \sigma^2_{u_x}$, $E = c^2\sigma^2_{u_x} + \sigma^2_{u_m}$, $r_1 = \sqrt{\frac{n-3}{n-5}}$ and $r_2 = \sqrt{\frac{n-2}{n-4}}$.

R does not depend on the parameter b. It is possible to set the other model parameters in a way that allows R to take any positive value. In particular, it can be seen that as $\sigma_{u_x} \to 0$, $R \to \infty$, which shows that R is unbounded.

F Combining Partially Observed Datasets

F.1 Cramer-Rao Lower Bound

We are interested in estimating the value of the product ac. Let $e = ac$. We reparameterize the likelihood in Eq. 15 by replacing c with e/a. This simplifies the calculations and improves numerical stability. Now, we have the following eight unknown model parameters: $\{e, a, b, d, \sigma^2_{u_w}, \sigma^2_{u_x}, \sigma^2_{u_m}, \sigma^2_{u_y}\}$.

In order to compute the variance of the estimate of parameter $e = ac$, we compute the Cramer-Rao variance lower bound. We first compute the Fisher information matrix (FIM) $\mathbf{I}$ for the eight model parameters:

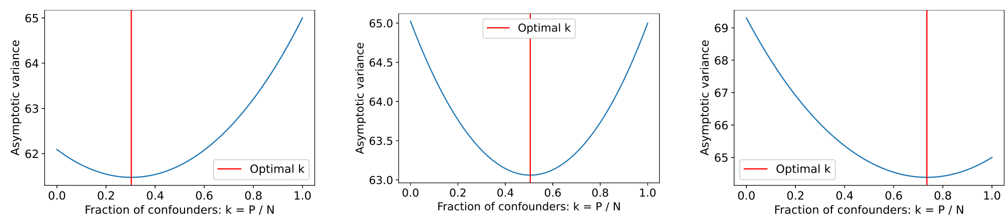
$$\mathbf{I} = -E \begin{bmatrix} \frac{\partial^2 \mathcal{LL}}{\partial e^2} & \frac{\partial^2 \mathcal{LL}}{\partial e \partial a} & \frac{\partial^2 \mathcal{LL}}{\partial e \partial b} & \cdots & \frac{\partial^2 \mathcal{LL}}{\partial e \partial \sigma_{u_y}} \\ \frac{\partial^2 \mathcal{LL}}{\partial a \partial e} & \frac{\partial^2 \mathcal{LL}}{\partial a^2} & \frac{\partial^2 \mathcal{LL}}{\partial a \partial b} & \cdots & \frac{\partial^2 \mathcal{LL}}{\partial a \partial \sigma_{u_y}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{LL}}{\partial \sigma_{u_y} \partial e} & \frac{\partial^2 \mathcal{LL}}{\partial \sigma_{u_y} \partial a} & \frac{\partial^2 \mathcal{LL}}{\partial \sigma_{u_y} \partial b} & \cdots & \frac{\partial^2 \mathcal{LL}}{\partial^2 \sigma_{u_y}} \end{bmatrix}$$

Let $\widehat{e}$ be the MLE. Since standard regularity conditions hold for our model (due to linearity and Gaussianity), the MLE is asymptotically normal. We can use the Cramer-Rao theorem to get the asymptotic variance of $\widehat{e}$. That is, for constant k, as $N \to \infty$, we have

$$\sqrt{N}(\widehat{e} - e) \xrightarrow{d} \mathcal{N}(0, V_e), \quad \text{and}$$
$$V_e = (\mathbf{I}^{-1})_{1,1}.$$

Below, we present the closed form expression for V_e. Let $V_e = \frac{X}{Y}$. Then

$$\begin{aligned} X =& (a^2\sigma^2_{u_m} + \sigma^2_{u_y})(-a^8 d^2(k-1)\sigma^2_{u_w}(\sigma^2_{u_m})^5(d^2\sigma^2_{u_w} + \sigma^2_{u_x})^2 + a^6(\sigma^2_{u_m})^3(d^2\sigma^2_{u_w} + \sigma^2_{u_x}) \\ & (b^2\sigma^2_{u_w}\sigma^2_{u_x}(c^2d^4(\sigma^2_{u_w})^2 + d^2\sigma^2_{u_w}(c^2(k+1)\sigma^2_{u_x} + (-2k^2+2k+1)\sigma^2_{u_m}) + \sigma^2_{u_x}(c^2k\sigma^2_{u_x} + \\ & \sigma^2_{u_m})) + \sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x})(c^2d^4(\sigma^2_{u_w})^2 + d^2\sigma^2_{u_w}(c^2(k+1)\sigma^2_{u_x} + (3-2k)\sigma^2_{u_m}) + \\ & k\sigma^2_{u_x}(c^2\sigma^2_{u_x} + \sigma^2_{u_m}))) - 4a^5bcd(k-1)k\sigma^2_{u_w}\sigma^2_{u_x}(\sigma^2_{u_m})^3(d^2\sigma^2_{u_w} + \sigma^2_{u_x}) \\ & (b^2\sigma^2_{u_w}\sigma^2_{u_x} + \sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x})) + a^4(\sigma^2_{u_m})^2(b^4(\sigma^2_{u_w})^2(\sigma^2_{u_x})^2(c^2d^4(\sigma^2_{u_w})^2 + \\ & d^2\sigma^2_{u_w}(2c^2(-k^2+k+1)\sigma^2_{u_x} + (k+1)\sigma^2_{u_m}) + \sigma^2_{u_x}(c^2(-2k^2+2k+1)\sigma^2_{u_x} + 2\sigma^2_{u_m})) - \\ & b^2\sigma^2_{u_w}\sigma^2_{u_x}\sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x})(2c^2d^4(k-2)(\sigma^2_{u_w})^2 + d^2\sigma^2_{u_w}(c^2(4k^2-3k-5)\sigma^2_{u_x} + 2 \\ & (k^2-2k-1)\sigma^2_{u_m}) + \sigma^2_{u_x}(c^2(4k^2-5k-1)\sigma^2_{u_x} - 2(k+1)\sigma^2_{u_m})) - (\sigma^2_{u_y})^2(d^2\sigma^2_{u_w} + \sigma^2_{u_x})^2 \\ & (c^2d^4(2k-3)(\sigma^2_{u_w})^2 + d^2\sigma^2_{u_w}(c^2(2k^2-k-3)\sigma^2_{u_x} + (k-3)\sigma^2_{u_m}) + k\sigma^2_{u_x}(c^2(2k-3)\sigma^2_{u_x} - \\ & 2\sigma^2_{u_m}))) - 4a^3bcd(k-1)k\sigma^2_{u_w}\sigma^2_{u_x}(\sigma^2_{u_m})^2\sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x})(b^2\sigma^2_{u_w}\sigma^2_{u_x} + \sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x})) + \\ & a^2\sigma^2_{u_m}(b^2\sigma^2_{u_w}\sigma^2_{u_x} + \sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x}))(b^4(\sigma^2_{u_w})^2(\sigma^2_{u_x})^2(c^2(d^2\sigma^2_{u_w} - (k-2)\sigma^2_{u_x}) + \sigma^2_{u_m}) + \\ & b^2\sigma^2_{u_w}\sigma^2_{u_x}\sigma^2_{u_y}(2c^2d^4(\sigma^2_{u_w})^2 + 2d^2\sigma^2_{u_w}(c^2(-k^2)\sigma^2_{u_x} + k(c^2\sigma^2_{u_x} + \sigma^2_{u_m}) + 2c^2\sigma^2_{u_x}) + \sigma^2_{u_x}(2c^2 \\ & (-k^2+k+1)\sigma^2_{u_x} + (k+1)\sigma^2_{u_m})) + (\sigma^2_{u_y})^2(d^2\sigma^2_{u_w} + \sigma^2_{u_x})(d^2\sigma^2_{u_w} + k\sigma^2_{u_x})(\sigma^2_{u_m} - c^2(2k-3) \\ & (d^2\sigma^2_{u_w} + \sigma^2_{u_x}))) + c^2(b^2\sigma^2_{u_w}\sigma^2_{u_x} + \sigma^2_{u_y}(d^2\sigma^2_{u_w} + \sigma^2_{u_x}))^2(b^4(\sigma^2_{u_w})^2(\sigma^2_{u_x})^2 + b^2\sigma^2_{u_w}\sigma^2_{u_x}\sigma^2_{u_y} \\ & (2d^2k\sigma^2_{u_w} + k\sigma^2_{u_x} + \sigma^2_{u_x}) + (\sigma^2_{u_y})^2(d^2\sigma^2_{u_w} + \sigma^2_{u_x})(d^2\sigma^2_{u_w} + k\sigma^2_{u_x}))), \end{aligned}$$



(a) **(b)** **(c)**

Figure 3: Cases where collecting a mix of confounders and mediators is better than collecting only confounders or mediators.

and

$$\begin{aligned}
Y =&(a^2\sigma^2_{u_m}(d^2\sigma^2_{u_w}+\sigma^2_{u_x})+b^2\sigma^2_{u_w}\sigma^2_{u_x}+\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x}))(a^6(-d^2)(k-1)\\
&\sigma^2_{u_w}(\sigma^2_{u_m})^4(d^2\sigma^2_{u_w}+\sigma^2_{u_x})^2+a^4(\sigma^2_{u_m})^2(d^2\sigma^2_{u_w}+\sigma^2_{u_x})(b^2\sigma^2_{u_w}\sigma^2_{u_x}(d^2k\sigma^2_{u_w}\\
&(c^2\sigma^2_{u_x}-2k\sigma^2_{u_m}+2\sigma^2_{u_m})+\sigma^2_{u_x}(c^2k\sigma^2_{u_x}+\sigma^2_{u_m}))+\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x})(d^2\sigma^2_{u_w}\\
&(c^2k\sigma^2_{u_x}-3k\sigma^2_{u_m}+3\sigma^2_{u_m})+k\sigma^2_{u_x}(c^2\sigma^2_{u_x}+\sigma^2_{u_m})))-4a^3bcd(k-1)k\sigma^2_{u_w}\sigma^2_{u_x}(\sigma^2_{u_m})^2\\
&(d^2\sigma^2_{u_w}+\sigma^2_{u_x})(b^2\sigma^2_{u_w}\sigma^2_{u_x}+\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x}))+a^2\sigma^2_{u_m}(b^4(\sigma^2_{u_w})^2(\sigma^2_{u_x})^2(\sigma^2_{u_x}\\
&(\sigma^2_{u_m}-2c^2(k-1)k\sigma^2_{u_x})-d^2(k-1)\sigma^2_{u_w}(2c^2k\sigma^2_{u_x}+\sigma^2_{u_m}))+2b^2\sigma^2_{u_w}\sigma^2_{u_x}\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x})\\
&(\sigma^2_{u_x}(\sigma^2_{u_m}-2c^2(k-1)k\sigma^2_{u_x})-2d^2(k-1)k\sigma^2_{u_w}(c^2\sigma^2_{u_x}+\sigma^2_{u_m}))+(\sigma^2_{u_y})^2(d^2\sigma^2_{u_w}+\sigma^2_{u_x})^2\\
&(-d^2(k-1)\sigma^2_{u_w}(2c^2k\sigma^2_{u_x}+3\sigma^2_{u_m})-k\sigma^2_{u_x}(2c^2(k-1)\sigma^2_{u_x}+(k-2)\sigma^2_{u_m})))-4abcd(k-1)\\
&k\sigma^2_{u_w}\sigma^2_{u_x}\sigma^2_{u_m}\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x})(b^2\sigma^2_{u_w}\sigma^2_{u_x}+\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x}))+b^6c^2k(\sigma^2_{u_w})^3(\sigma^2_{u_x})^4+b^4\\
&(\sigma^2_{u_w})^2(\sigma^2_{u_x})^2\sigma^2_{u_y}(d^2\sigma^2_{u_w}+\sigma^2_{u_x})(3c^2k\sigma^2_{u_x}-k\sigma^2_{u_m}+\sigma^2_{u_m})+b^2\sigma^2_{u_w}\sigma^2_{u_x}(\sigma^2_{u_y})^2(d^2\sigma^2_{u_w}+\sigma^2_{u_x})\\
&(d^2k\sigma^2_{u_w}(3c^2\sigma^2_{u_x}-2(k-1)\sigma^2_{u_m})+\sigma^2_{u_x}(3c^2k\sigma^2_{u_x}-k^2\sigma^2_{u_m}+\sigma^2_{u_m}))+(\sigma^2_{u_y})^3(d^2\sigma^2_{u_w}+\sigma^2_{u_x})^2\\
&(d^2\sigma^2_{u_w}(c^2k\sigma^2_{u_x}-k\sigma^2_{u_m}+\sigma^2_{u_m})+k\sigma^2_{u_x}(c^2\sigma^2_{u_x}-k\sigma^2_{u_m}+\sigma^2_{u_m}))),
\end{aligned}$$

where $k = \frac{P}{N}$.

F.2 Comparison with frontdoor and backdoor estimators

In this section, we show some examples of regimes where the combining partially observed datasets results in lower variance than applying either of the backdoor or frontdoor estimator even when the total number of samples are the same. In other words, there exist settings of model parameters such that, for some $k \in (0, 1)$, we have

$$V_e \leq \text{Var}(\sqrt{N}\widehat{a}\widehat{c})_{\text{backdoor}}, \text{ and } V_e \leq \text{Var}(\sqrt{N}\widehat{a}_f\widehat{c}).$$

Figure 3 shows three examples where the optimal value of k is between 0 and 1. We plot the variance as predicted by the expression for V_e versus the value of k. The plots show that in some cases, it is better to collect a mix of confounders and mediators rather than only mediators or only confounders. The expression for V_e in the previous section allows us to verify that. This happens when the variance of the frontdoor and backdoor estimators do not differ by too much.

In Figure 3a, the model parameters are $\{a = 10, b = 3.7, c = 5, d = 5, \sigma^2_{u_w} = 1, \sigma^2_{u_x} = 1, \sigma^2_{u_m} = 0.64, \sigma^2_{u_y} = 1\}$. In this case, the variance of the frontdoor estimator is lower than the backdoor estimator. Despite this, it is not optimal to only collect mediators. The optimal value of k is 0.303, that is, 30% of the collected samples should be confounders and the rest should be mediators to achieve lowest variance.

In Figure 3b, the model parameters are $\{a = 10, b = 3.955, c = 5, d = 5, \sigma^2_{u_w} = 1, \sigma^2_{u_x} = 1, \sigma^2_{u_m} = 0.64, \sigma^2_{u_y} = 1\}$. In this case, the variance of the frontdoor estimator is almost equal to that of the backdoor estimator. The optimal ratio k is 0.505, that is, we should collect the same of amount of confounders as mediators.

In Figure 3c, the model parameters are $\{a = 10, b = 4.3, c = 5, d = 5, \sigma^2_{u_w} = 1, \sigma^2_{u_x} = 1, \sigma^2_{u_m} = 0.64, \sigma^2_{u_y} = 1\}$. In this case, the variance of the frontdoor estimator is greater than the backdoor estimator. The optimal ratio k is 0.735, that is, we should collect the more confounders than mediators.

F.3 Parameter initialization for finding the MLE

The likelihood in Eq. 15 is non-convex. As a result, we cannot start with arbitrary initial values for model parameters because we might encounter a local minimum. To avoid this, we use the two datasets to initialize our parameter estimates. Each of the eight parameters can be identified using only data from one of the datasets. For example, d can be initialized using the revealed-confounder dataset (via OLS regression of X on W). The parameter e is can be identified using either dataset, so we pick the value with lower bootstrapped variance.

After initializing the eight model parameters, we run the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm [Fletcher, 2013] to find model parameters that minimize the negative log-likelihood.

G More details on experiments

G.1 Empirical vs theoretical variance on synthetic data

Here we provide more details for how results in Table 1 are generated. We initialize the model parameters by sampling 200 times from the following distributions:

$$\begin{aligned} a, b, c, d &\sim \text{Unif}[-10, 10] \\ \sigma^2_{u_w}, \sigma^2_{u_x}, \sigma^2_{u_m}, \sigma^2_{u_y} &\sim \text{Unif}[0.01, 2]. \end{aligned} \tag{55}$$

For each initialization, we compute the Mean Absolute Percentage Error (MAPE) of the theoretical variance as a predictor of empirical variance:

$$\text{MAPE} = \frac{|\text{Var}_{\text{theoretical}} - \text{Var}_{\text{empirical}}|}{\text{Var}_{\text{empirical}}} * 100\%$$

We report the mean and standard deviation of the MAPE across 1000 realizations of datasets sampled from Eq. 55. We find that the theoretical variance is close to the empirical variance even for small sample sizes (Table 1).