Generalized Do-Calculus Without Graphs

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Abstract

Inferring the potential consequences of an unobserved event is a fundamental scientific question. To this end, Pearl's celebrated do-calculus provides a set of inference rules to derive an interventional probability from an observational probability. In this framework, the primitive causal relations are encoded on a directed acyclic graph (DAG), which can be limitative for some applications. In this paper, we capture causality without reference to a graph and we extend the rules of do-calculus to systems that do not possess a fixed causal ordering. For this purpose, we introduce a new framework which relies on the theory developed by Witsenhausen for multi-agent stochastic control. The mapping from graphs to so-called Witsenhausen's intrinsic models is natural: the primitives of the problem are the agents' information fields; the random variables are synthesized by the agents whose strategies encode the informational constraints. All in all, our framework offers a richer language than DAGs and provides a generalized do-calculus.

Contents

1 Introduction

1.1 Causality and do-calculus

As the world shifts toward more and more data-driven decision making, causal inference is taking more space in applied sciences, statistics and machine learning. This is because it allows for better, more robust statistical interpretation, and provides a way to undertsand the data that goes beyond correlation [13]. For instance, causal inference provides a language and tools to describe and solve Simpson's paradox, which embodies the "correlation is not causation" principle as can be found in any "Statistics 101" basic course. The main concern in causal inference is to compute post-intervention probabilities distribution from pre-intervention data. For this purpose, graphical models are practical because they allow to represent assumptions easily and benefit from an extensive scientific literature.

In his seminal work, Pearl builds on graphical models [3] to introduce the so-called do-calculus. Several extensions to this do-calculus have been proposed recently [21, 9, 18, 2]. As asserted by Pearl, language is an important element in this research program [11]:

> Another ramification of the sharp distinction between associational and causal concepts is that any mathematical approach to causal analysis must acquire *new notation* for expressing causal relations – probability calculus is insufficient.

We completely concur with this idea. Moreover, our contribution is based on the work of another scholar, Witsenhausen, who developed a mathematical framework to capture the notion of causality and who wrote in [22]

> The main difficulty is one of *notations*.

Witsenhausen introduced the intrinsic model that turns the focus from dependencies represented by functions to dependencies represented by σ-fields and measurability constraints (this will be clarified thereafter) in multi-agent stochastic control problems.

Causal graphical models move the focus from joint probability distributions to functional dependencies thanks to the Structural Causal Model (SCM). It is our hope that the introduction of Witsenhausen intrinsic model will move the focus from functional dependencies to informational relations, hence bringing a new, complementary view to the causal reasoning toolbox.

So, we introduce a general, unifying framework for causal inference that may be used for both recursive and non-recursive systems [4]. The cost for this generalization is a bit of abstraction because the model cannot be captured with a DAG, and rely on a more generic product structure. In particular, while DAG modeling does not rely directly on the notion of random variable but on the joint, pushforward probability distribution ([12, footnote 3] or [14, Appendix A]), our approach requires to go back to the primitives of the probabilistic model: sample set, σ-fields, measurable maps. This is why we depart from the usual presentation of causal inference papers.

1.2 Our contributions

We generalise causal modeling thanks to the notion of information fields. In this new abstraction, we redefine the notion of d-separation and provide a rigorous analysis of the model structure thank to the equivalence result provided by the companion paper [5]. Our main results are Theorem 8 which expresses that independence induces a factorization property of the solution map, and Theorem 11, which is a generalization of do-calculus, and which subsumes several recent results.

We start with a brief presentation of Witsenhausen's intrinsic model (§2), which we have adapted to our needs. We then (§3) build the path from Pearl's causal model to Witsenhaysen intrinsic model. A key new notion in our investigation is the one of *topological separation* which, as shown in the companion paper [5], is equivalent to a generalization of Pearl's d-separation. We cast a new light on the notion of conditional independence (§4), which allows us in particular to generalize Pearl's Do-calculus.

2 Witsenhausen's intrinsic model and formalism for causality

We present the so-called *Witsenhausen's intrinsic model*, followed by its formalism for causality.

2.1 The Witsenhausen's intrinsic model

Because Witsenhausen introduced the intrinsic model to the control community some five decades ago [22, 23], we expect that most readers will not be familiar with it. We provide tentative correspondences between Pearl's DAG and Witsenhausen's intrinsic model in Table 1.

	Pearl	*Witsenhausen*
Structure	DAG	Nature and agents decision sets, with their respective fields
Parent relation	$\rightarrow$	precedence relation
	node	agent
	edge	agents related by the precedence relation
Dependence	SCM	agents information fields
	functional relation	policy profiles measurable w.r.t. information fields
Resolution	induction	solution map
	random variable	policy profile composed with solution map
Intervention	do operator	change of information fields
Causal ordering	fixed	existence depends on agents information fields

Table 1: Correspondences between Pearl's DAG and Witsenhausen's intrinsic model

Witsenhausen used the language of σ-fields to handle the concept of information in control theory. We present it below using the more general notion of field. A *field* (resp. *σ-field*) over the set $\mathbb{D}$ is a subset $\mathcal{D} \subset 2^{\mathbb{D}}$, containing $\mathbb{D}$, and which is stable under complementation and union (resp. countable union). The trivial field over the set $\mathbb{D}$ is $\{\emptyset, \mathbb{D}\}$. The complete field over the set $\mathbb{D}$ is $2^{\mathbb{D}}$. When $\mathcal{D}' \subset \mathcal{D}$ are two fields over the set $\mathbb{D}$, we say that $\mathcal{D}'$ is a *subfield* of $\mathcal{D}$. All the Witsenhausen theory can be developed with fields, but σ-fields are needed when probability distributions are involved.

Definition 1. *(Adapted from [22, 23]) A* W-model *is a collection* $\big(\mathbb{A}, (\mathbb{U}_a, \mathcal{U}_a)_{a\in\mathbb{A}}, (\Omega, \mathcal{F}), (\mathcal{I}_a)_{a\in\mathbb{A}}\big)$, *where*

- $\mathbb{A}$ *is a finite set, whose elements are called* agents*;*
- *for any* $a \in \mathbb{A}$, $\mathbb{U}_a$ *is a set, the* set of decisions *for agent* a*;* $\mathcal{U}_a$ *is a field over* $\mathbb{U}_a$*;*
- Ω *is a set made of* states of Nature*;* $\mathcal{F}$ *is a field over* Ω*;*
- *for any* $a \in \mathbb{A}$, $\mathcal{I}_a$ *is a subfield of the following product field*

$$\mathcal{I}_a \subset \mathcal{F} \otimes \bigotimes_{b\in\mathbb{A}} \mathcal{U}_b \;, \quad \forall a \in \mathbb{A} \tag{1}$$

and is called the information field *of the agent* a*.*

The *configuration space* is the product space (also called *hybrid space*, hence the $\mathbb{H}$ notation)

$$\mathbb{H} = \Omega \times \prod_{a\in\mathbb{A}} \mathbb{U}_a \;. \tag{2a}$$

The following product *configuration field* is a field over $\mathbb{H}$

$$\mathcal{H} = \mathcal{F} \otimes \bigotimes_{a\in\mathbb{A}} \mathcal{U}_a \;. \tag{2b}$$

A *policy* of agent $a \in \mathbb{A}$ is a mapping

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_a, \mathcal{U}_a) \tag{3a}$$

from configurations to decisions, which satisfies the following measurability property

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{I}_a \;. \tag{3b}$$

Condition (3b) expresses the property that any policy of agent a may only depend upon the information $\mathcal{I}_a$ available to the agent. We denote by Λ_a the set of all policies of agent $a \in \mathbb{A}$. A *policy profile* λ is a collection of policies, one per agent $a \in \mathbb{A}$:

$$\lambda = (\lambda_a)_{a\in\mathbb{A}} \in \prod_{a\in\mathbb{A}} \Lambda_a \;. \tag{3c}$$

In what follows, we will need some notations. For any nonempty subset $B \subset \mathbb{A}$ of agents, we define

$$\mathcal{H}_B = \mathcal{F} \otimes \bigotimes_{b \in B} \mathcal{U}_b \otimes \bigotimes_{a \not\in B} \{\emptyset, \mathbb{U}_a\} \subset \mathcal{H} \; , \tag{4a}$$

$$\lambda_B = (\lambda_b)_{b \in B} \in \prod_{b \in B} \Lambda_b \; , \;\; \forall \lambda = (\lambda_a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a \; . \tag{4b}$$

2.2 Solvability and solution map

With any given policy profile $\lambda = (\lambda_a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a$ we associate the set-valued mapping

$$\mathcal{M}_\lambda : \Omega \rightrightarrows \prod_{b \in \mathbb{A}} \mathbb{U}_b \; , \;\; \omega \mapsto \Big\{ (u_b)_{b \in \mathbb{A}} \in \prod_{b \in \mathbb{A}} \mathbb{U}_b \,\Big|\, u_a = \lambda_a\big(\omega, (u_b)_{b \in \mathbb{A}}\big) \; , \;\; \forall a \in \mathbb{A} \Big\} \; . \tag{5}$$

With this definition, we slightly reformulate below how Witsenhausen introduced solvability.

Definition 2. *([22, 23]) The* (measurable) solvability property *holds true for the W-model of Definition 1 — or the W-model is said to be* (measurable) solvable *— when, for any policy profile $\lambda = (\lambda_a)_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a$, the set-valued mapping $\mathcal{M}_\lambda$ in* (5) *is a (measurable) mapping whose domain is Ω, that is, the cardinal of $\mathcal{M}_\lambda(\omega)$ is equal to one, for any state of nature $\omega \in \Omega$.*

Thus, under solvability property, for any state of nature $\omega \in \Omega$, there exists one, and only one, decision profile $(u_b)_{b \in \mathbb{A}} \in \prod_{b \in \mathbb{A}} \mathbb{U}_b$ which is a solution of the closed-loop equations

$$u_a = \lambda_a\big(\omega, (u_b)_{b \in \mathbb{A}}\big) \; , \;\; \forall a \in \mathbb{A} \; . \tag{6a}$$

In this case, we define the solution map

$$S_\lambda : \Omega \to \mathbb{H} \; , \;\; S_\lambda(\omega) = \big(\omega, M_\lambda(\omega)\big) \tag{6b}$$

where $M_\lambda(\omega)$ is the unique element contained in the image set $\mathcal{M}_\lambda(\omega)$ that is, for all $(u_b)_{b \in \mathbb{A}} \in \prod_{b \in \mathbb{A}} \mathbb{U}_b$, $M_\lambda(\omega) = (u_b)_{b \in \mathbb{A}} \iff \mathcal{M}_\lambda(\omega) = \{(u_b)_{b \in \mathbb{A}}\}$.

Thus, when the solvability property holds true, for each state of Nature, there is a single family of decisions compatible with a given policy profile. This family is the unique solution of the closed-loop equations (6a). In some cases, these equations can be solved sequentially (where the order may depend on the state of Nature, and on the given policy profile). This is the case when causality holds true.

2.3 Causality

In his articles [22, 23], Witsenhausen introduces a notion of causality that relies on suitable configuration-orderings. Here, we introduce our own notations, as they make possible a compact formulation of the causality property.

For any finite set $\mathbb{D}$, let $|\mathbb{D}|$ denote the cardinal of $\mathbb{D}$. Thus, $|\mathbb{A}|$ denotes the cardinal of the set $\mathbb{A}$, that is, $|\mathbb{A}|$ is the number of agents. For $k \in \{1, \dots, |\mathbb{A}|\}$, let $\Sigma^k = \big\{\kappa : \{1, \dots, k\} \to \mathbb{A} \,\big|\, \kappa \text{ is an injection} \big\}$ denote the set of k-orderings, that is, injective mappings from $\{1, \dots, k\}$ to $\mathbb{A}$. The set $\Sigma^{|\mathbb{A}|}$ is the set of *total orderings* of agents in $\mathbb{A}$, that is, bijective mappings from $\{1, \dots, |\mathbb{A}|\}$ to $\mathbb{A}$ (in contrast with *partial orderings* in Σ^k for $k < |\mathbb{A}|$). We define the *set of orderings* by $\Sigma = \bigcup_{k \in \{0, \dots, |\mathbb{A}|\}} \Sigma^k$, where $\Sigma^0 = \{\emptyset\}$. For any $k \in \{1, \dots, |\mathbb{A}|\}$, any ordering $\kappa \in \Sigma^k$, and any integer $\ell \leq k$, $\kappa_{|\{1,\dots,\ell\}}$ is the restriction of the ordering κ to the first ℓ integers, and we introduce the mapping $\psi_k : \Sigma^{|\mathbb{A}|} \to \Sigma^k \ , \ \ \rho \mapsto \rho_{|\{1,\dots,k\}}$ which performs the restriction of any total ordering of $\mathbb{A}$ to $\{1, \dots, k\}$. For any $k \in \{1, \dots, |\mathbb{A}|\}$, and any k-ordering $\kappa \in \Sigma^k$, we define the *range* $\|\kappa\| = \big\{\kappa(1), \dots, \kappa(k)\big\} \subset \mathbb{A}$, the *cardinal* $|\kappa| = k \in \{1, \dots, |\mathbb{A}|\}$, the *last element* $\kappa^\star = \kappa(k) \in \mathbb{A}$, and the *restriction* $\kappa^- = \kappa_{|\{1,\dots,k-1\}} \in \Sigma^{k-1}$.

The following definition of causality originates from [22]. In a causal W-model, there exists a configuration-ordering with the property that when an agent is called to play — as he is the last one in an ordering — what he knows cannot depend on decisions made by agents that are not his predecessors (in the range of the ordering under consideration).

Definition 3. *([22, 23]) A W-model (as in Definition 1) is* causal *if there exists (at least) one causal configuration-ordering $\varphi : \mathbb{H} \to \Sigma^{|\mathbb{A}|}$, that is, with the property that*

$$\mathbb{H}^{\varphi}_{\kappa} \cap H \in \mathcal{H}_{\|\kappa^-\|} \ , \ \ \forall H \in \mathcal{I}_{\kappa^\star} \ , \ \ \forall \kappa \in \Sigma \ , \tag{7}$$

where the subset $\mathbb{H}^{\varphi}_{\kappa} \subset \mathbb{H}$ of configurations is defined by (by convention, we put $\mathbb{H}^{\varphi}_{\emptyset} = \mathbb{H}$)

$$\mathbb{H}^{\varphi}_{\kappa} = \big\{ h \in \mathbb{H} \,\big|\, \psi_{|\kappa|}\big(\varphi(h)\big) = \kappa \big\} \ , \ \ \forall \kappa \in \Sigma \ . \tag{8}$$

The set $\mathbb{H}^{\varphi}_{\kappa}$ contains all the configurations for which the agent $\kappa(1)$ is acting first, the agent $\kappa(2)$ is acting second,..., till the last agent $\kappa^\star = \kappa(|\kappa|)$ acting at stage $|\kappa|$. Hence, otherwise said, causality means that, once we know the first $|\kappa|$ agents, the information of the agent $\kappa^\star$ depends at most on the decisions of the agents in the range $\|\kappa^-\|$, as represented by the subfield (see Equation (4a))

$$\mathcal{H}_{\|\kappa^-\|} = \mathcal{F} \otimes \bigotimes_{a \in \|\kappa^-\|} \mathcal{U}_a \otimes \bigotimes_{b \not\in \|\kappa^-\|} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{H} \ . \tag{9}$$

In [22], Witsenhausen proves that causal W-models are measurable solvable. The reverse is false: in [22, Theorem 2], Witsenhausen exhibits an example of noncausal W-model that is solvable.

Toy example. The following W-model summarizes the DAG in Figure 1. The set of agents is $\mathbb{A} = \{Z, T, Y\}$. Suppose that the values of each of the three random variables represented on the DAG belong to $\{0, 1\}$. Then, the sets of decisions for the agents are $\mathbb{U}_Z = \mathbb{U}_T = \mathbb{U}_Y = \{0, 1\}$, each equipped with the complete field $\mathcal{U}_Z = \mathcal{U}_T = \mathcal{U}_Y = \big\{\emptyset, \{0\}, \{1\}, \{0, 1\}\big\}$. We take $\Omega = \{0, 1\}^3$ as Nature set, equipped with the complete field $\mathcal{F} = 2^{\Omega}$ made of all subsets

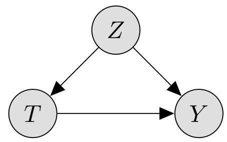

Figure 1: Toy example

of Ω. We write $\Omega = \Omega_Z \times \Omega_T \times \Omega_Y$, where $\Omega_Z = \Omega_T = \Omega_Y = \{0,1\}$, and $\mathcal{F} = \mathcal{F}_Z \otimes \mathcal{F}_T \otimes \mathcal{F}_Y$, where $\mathcal{F}_Z = \mathcal{F}_T = \mathcal{F}_Y = \big\{\emptyset, \{0\}, \{1\}, \{0,1\}\big\}$. To represent the arrows in the DAG in Figure 1 (as well as implicit assumptions about information on Nature), we require that the information fields $\mathcal{I}_Z$, $\mathcal{I}_T$ and $\mathcal{I}_Y$ satisfy:

$$\mathcal{I}_Z \subset \mathcal{F}_Z \otimes \{\emptyset, \Omega_T\} \otimes \{\emptyset, \Omega_Y\} \otimes \{\emptyset, \mathbb{U}_Z\} \otimes \{\emptyset, \mathbb{U}_T\} \otimes \{\emptyset, \mathbb{U}_Y\} \; , \tag{10a}$$
$$\mathcal{I}_T \subset \{\emptyset, \Omega_Z\} \otimes \mathcal{F}_T \otimes \{\emptyset, \Omega_Y\} \otimes \mathcal{U}_Z \otimes \{\emptyset, \mathbb{U}_T\} \otimes \{\emptyset, \mathbb{U}_Y\} \; , \tag{10b}$$
$$\mathcal{I}_Y \subset \{\emptyset, \Omega_Z\} \otimes \{\emptyset, \Omega_T\} \otimes \mathcal{F}_Y \otimes \mathcal{U}_Z \otimes \mathcal{U}_T \otimes \{\emptyset, \mathbb{U}_Y\} \; . \tag{10c}$$

These relations express that the information of agent Z depends at most on its own "hazard generator" (the term $\mathcal{F}_Z$), that the information of agent T depends at most on its own "hazard generator" (the term $\mathcal{F}_T$) and on the decisions of agent Z (the term $\mathcal{U}_Z$), and that the information of agent Y depends at most on its own "hazard generator" (the term $\mathcal{F}_Y$) and on the decisions of both agent Z and agent T (the term $\mathcal{U}_Z \otimes \mathcal{U}_T$). This W-model is clearly causal, as it has the constant causal ordering (Z, T, Y) (see Definition 3), hence is measurable solvable.

3 Conditional parentality and topological separation in the intrinsic model

In the companion paper [5] we show an equivalence between three abstract binary relations on finite sets, namely blocking, directional separability and topological separability. We are now going to instentiate a topological separability relation for the instrinsic model.

We recall that a *(binary) relation* $\mathfrak{R}$ on $\mathbb{A}$ is a subset $\mathfrak{R} \subset \mathbb{A}^2$ and that $a \,\mathfrak{R}\, b$ means $(a,b) \in \mathfrak{R}$. For any subset $B \subset \mathbb{A}$, the *(sub)diagonal relation* is $\Delta_B = \big\{(a,b) \in \mathbb{A}^2 \,\big|\, a = b \in B\big\}$ and the *diagonal relation* is $\Delta = \Delta_{\mathbb{A}}$. A *foreset* of a relation $\mathfrak{R}$ is any set of the form $\mathfrak{R}\, b = \big\{a \in \mathbb{A} \,\big|\, a \,\mathfrak{R}\, b\big\}$ or, by extension, of the form $\mathfrak{R}\, B = \big\{a \in \mathbb{A} \,\big|\, \exists b \in B \;,\; a \,\mathfrak{R}\, b\big\}$, where $B \subset \mathbb{A}$. The *opposite* or *complementary* $\mathfrak{R}^{\mathsf{c}}$ of a binary relation $\mathfrak{R}$ is the relation $\mathfrak{R}^{\mathsf{c}} = \mathbb{A}^2 \setminus \mathfrak{R}$, that is, defined by $a \,\mathfrak{R}^{\mathsf{c}}\, b \iff \neg(a \,\mathfrak{R}\, b)$. The *converse* $\mathfrak{R}^{-1}$ of a binary relation $\mathfrak{R}$ is defined by $a \,\mathfrak{R}^{-1}\, b \iff b \,\mathfrak{R}\, a$ (and $\mathfrak{R}$ is *symmetric* if $\mathfrak{R}^{-1} = \mathfrak{R}$). The *composition* $\mathfrak{R}\mathfrak{R}'$ of two binary relations $\mathfrak{R}, \mathfrak{R}'$ is defined by $a(\mathfrak{R}\mathfrak{R}')b \iff \exists \delta \in \mathbb{A}$, $a \,\mathfrak{R}\, \delta$ and $\delta \,\mathfrak{R}'\, b$. The *transitive closure* of a binary relation $\mathfrak{R}$ is $\mathfrak{R}^+ = \cup_{k=1}^{\infty} \mathfrak{R}^k$ (and $\mathfrak{R}$ is *transitive* if $\mathfrak{R}^+ = \mathfrak{R}$) and the *reflexive and transitive closure* is $\mathfrak{R}^* = \mathfrak{R}^+ \cup \Delta$.

3.1 Conditional parentality in the intrinsic model

We suppose to be given a W-model as in Definition 1. Witsenhausen defines the *precedence relation* $\mathcal{P}$ on the set $\mathbb{A}$ of agents by

$$\mathcal{P}a = \bigcap_{B \subset \mathbb{A}; \mathcal{I}_a \subset \mathcal{H}_B} B \;, \; \forall a \in \mathbb{A} \quad \text{and} \quad b \, \mathcal{P} \, a \iff b \in \mathcal{P}a \; . \tag{11}$$

When the precedence relation $\mathcal{P}$ is acyclic, we recover a DAG. However, a W-model is much richer than the DAG it can induce. Instead of the limited precedence relation $\mathcal{P}$ in (11), we introduce a new and more flexible definition of parentality.

Definition 4. *Let $H \subset \mathbb{H}$ be a subset of configurations, and $W \subset \mathbb{A}$ be a subset of agents. We set*

$$\mathcal{P}_{W,H}a = \bigcap_{B \subset \mathbb{A}; \mathcal{I}_a \cap H \subset \mathcal{H}_{B \cup W}} B \;, \; \forall a \in \mathbb{A} \;, \tag{12a}$$

and we define the (conditional) parental relation $\mathcal{P}_{W,H}$ *on $\mathbb{A}$ (w.r.t. (W,H)) by*

$$b \, \mathcal{P}_{W,H} \, a \iff b \in \mathcal{P}_{W,H}a \;, \; \forall (a,b) \in \mathbb{A}^2 \; . \tag{12b}$$

We define the (conditional) ancestral relation $\mathcal{A}_{W,H}$ *(w.r.t. (W,H)) as the transitive and reflexive closure of the conditional parental relation $\mathcal{P}_{W,H}$, that is,*

$$\mathcal{A}_{W,H} = \mathcal{P}_{W,H}^{+} \cup \Delta = \mathcal{P}_{W,H}^{*} \; . \tag{13}$$

Thus, when $b \, \mathcal{P}_{W,H} \, a$, it means that the information available to agent a, on the subset $H \subset \mathbb{H}$ of configurations, necessarily involves the decisions of the agent b and, possibly of the agents in W. More precisely, we have the relation

$$\mathcal{P}_{W,H}C = W^{\mathsf{c}} \cap \mathcal{P}_{\emptyset,H}C \;, \; \forall C \subset \mathbb{A} \; . \tag{14}$$

Witsenhausen's precedence relation $\mathcal{P}$ is the special case $\mathcal{P}_{\emptyset,\mathbb{H}}$.

Proposition 5. *We have that*

$$\mathcal{P}_{W,H}\mathbb{A} \subset W^{\mathsf{c}} \;, \; \mathcal{P}_{W,H} = \Delta_{W^{\mathsf{c}}}\mathcal{P}_{\emptyset,H} \; . \tag{15}$$

Proof. By (14), we have that $\mathcal{P}_{W,H}\mathbb{A} \subset W^{\mathsf{c}}$.

Proving that $\mathcal{P}_{W,H} = \Delta_{W^{\mathsf{c}}}\mathcal{P}_{\emptyset,H}$ is equivalent to proving that $\mathcal{P}_{W,H}a = W^{\mathsf{c}} \cap \big(\mathcal{P}_{\emptyset,H}a\big)$ for all $a \in \mathbb{A}$. Let $a \in \mathbb{A}$ be a given agent. We introduce the two subsets of agents defined by $\Gamma_a = \big\{B \in \mathbb{A} \,\big|\, \mathcal{I}_a \cap H \subset \mathcal{H}_B\big\}$ and $\Gamma_{a,W} = \big\{B \in \mathbb{A} \,\big|\, \mathcal{I}_a \cap H \subset \mathcal{H}_{B \cup W}\big\}$. Then, the two subsets $\mathcal{P}_{\emptyset,H}a$ and $\mathcal{P}_{W,H}a$ read as $\mathcal{P}_{\emptyset,H}a = \bigcap_{B \in \Gamma_a} B$ and $\mathcal{P}_{W,H}a = \bigcap_{B \in \Gamma_{a,W}} B$ and we have that

$$
\begin{aligned}
W^{\mathsf{c}} \cap \big(\mathcal{P}_{\emptyset,H}a\big) &= W^{\mathsf{c}} \cap \Big(\bigcap_{B\in\Gamma_a} B\Big) && (\text{as } \mathcal{P}_{\emptyset,H}a = \textstyle\bigcap_{B\in\Gamma_a} B) \\
&\subset W^{\mathsf{c}} \cap \Big(\bigcap_{C\in\Gamma_{a,W}} (C\cup W)\Big) && (\text{because } C\in\Gamma_{a,W} \implies C\cup W\in\Gamma_a) \\
&= W^{\mathsf{c}} \cap \Big(\big(\bigcap_{C\in\Gamma_{a,W}} C\big)\cup W\Big) \\
&= \Big(W^{\mathsf{c}} \cap \big(\bigcap_{C\in\Gamma_{a,W}} C\big)\Big) \cup \Big(W^{\mathsf{c}}\cap W\Big) \\
&= W^{\mathsf{c}} \cap \big(\mathcal{P}_{W,H}a\big) && (\text{as } \mathcal{P}_{W,H}a = \textstyle\bigcap_{C\in\Gamma_{a,W}} C) \\
&= \mathcal{P}_{W,H}a\ . && (\text{as } \big(\mathcal{P}_{W,H}a\big)\subset W^{\mathsf{c}})
\end{aligned}
$$

Therefore, we have obtained that $W^{\mathsf{c}} \cap \big(\mathcal{P}_{\emptyset,H}a\big) \subset \mathcal{P}_{W,H}a$, that is, $\Delta_{W^{\mathsf{c}}}\mathcal{P}_{\emptyset,H} \subset \mathcal{P}_{W,H}$.

Now, we prove the reverse inclusion. For this purpose, we consider $b \in \mathcal{P}_{W,H}a$ and we suppose that $b \notin \mathcal{P}_{\emptyset,H}a$. Then, as $\mathcal{P}_{\emptyset,H}a = \bigcap_{B\in\Gamma_a} B$, there exists $B \in \Gamma_a$ such that $b \notin B$. Now, we have that $B\cup W \in \Gamma_a$ using the fact that $\mathcal{I}_a\cap H \subset \mathcal{H}_B$ implies that $\mathcal{I}_a\cap H \subset \mathcal{H}_{B\cup W}$. We therefore obtain that $B \in \Gamma_{a,W}$ with $b \notin B$. This contradicts the fact that $b \in \mathcal{P}_{W,H}a$. Therefore, we have obtained that $\mathcal{P}_{W,H}a \subset \mathcal{P}_{\emptyset,H}a$. As $\mathcal{P}_{W,H}a \subset W^{\mathsf{c}}$, we conclude that $\mathcal{P}_{W,H}a \subset W^{\mathsf{c}} \cap \mathcal{P}_{\emptyset,H}a$, hence that $\mathcal{P}_{W,H} \subset \Delta_{W^{\mathsf{c}}}\mathcal{P}_{\emptyset,H}$.

Finally, we get that $W^{\mathsf{c}} \cap \big(\mathcal{P}_{\emptyset,H}a\big) = \mathcal{P}_{W,H}a$ and this ends the proof. □

3.2 Topological separation in the intrinsic model

We introduce a suitable topology on the set of agents, extending the topology introduced in [23], and we define a new notion of *conditional topological separation.*

Proposition 6. *The following subset* $\mathcal{T}_{W,H}$ *of* $2^{\mathbb{A}}$*, defined by*

$$\mathcal{T}_{W,H} = \big\{ B \subset \mathbb{A} \,\big|\, \mathcal{A}_{W,H}(\mathbb{A}\setminus B) \subset \mathbb{A}\setminus B \big\} \tag{16}$$

where the ancestral relation $\mathcal{A}_{W,H}$ *has been defined in* (13)*, is a topology on the set* $\mathbb{A}$ *of agents. In this topology, a subset* $C \subset \mathbb{A}$ *is closed iff* $\mathcal{A}_{W,H}C \subset C$ *iff* $\mathcal{A}_{W,H}C = C$*, and the topological closure* $\overline{B}^{W,H}$ *of a subset* $B \subset \mathbb{A}$ *is the* $\mathcal{A}_{W,H}$*-foreset*

$$\overline{B}^{W,H} = \mathcal{A}_{W,H}B\ . \tag{17}$$

In this topology, the subset W *is open.*

Proof. We prove that the set $\mathcal{T}_{W,H}$ in (16) is a topology. As $\mathbb{A}$ is finite, it is equivalent to prove that the set

$$\mathcal{F}_{W,H} = \big\{ C \subset \mathbb{A} \,\big|\, \mathcal{A}_{W,H}C \subset C \big\}$$

contains both $\emptyset, \mathbb{A}$ and is stable under the union and intersection operations. Now, both $\emptyset, \mathbb{A} \in \mathcal{F}_{W,H}$ as $\mathcal{A}_{W,H}\emptyset = \emptyset$ and $\mathcal{A}_{W,H}\mathbb{A} \subset \mathbb{A}$. Let $B, C \in \mathcal{F}_{W,H}$, that is, $\mathcal{A}_{W,H}B \subset B$ and $\mathcal{A}_{W,H}C \subset C$. We have $\mathcal{A}_{W,H}(B \cup C) = \mathcal{A}_{W,H}B \cup \mathcal{A}_{W,H}C \subset B \cup C$, hence stability by union. We have $\mathcal{A}_{W,H}(B \cap C) \subset \mathcal{A}_{W,H}B \cap \mathcal{A}_{W,H}C \subset B \cap C$, hence stability by intersection.

By definition, a subset $C \subset \mathbb{A}$ is closed iff $\mathcal{A}_{W,H}C \subset C$. This is also equivalent to $\mathcal{A}_{W,H}C = C$ because $C \subset \mathcal{A}_{W,H}C$ since the relation $\mathcal{A}_{W,H} = \mathcal{P}^{+}_{W,H} \cup \Delta$ in (13) is reflexive.

Now, we consider a subset $B \subset \mathbb{A}$ and we characterize its topological closure $\overline{B}^{W,H}$, the smallest closed subset that contains B. On the one hand, we have that $B \subset \mathcal{A}_{W,H}B$ because the relation $\mathcal{A}_{W,H} = \mathcal{P}^{+}_{W,H} \cup \Delta$ in (13) is reflexive. On the other hand, the set $\mathcal{A}_{W,H}B$ is closed since $\mathcal{A}_{W,H}(\mathcal{A}_{W,H}B) = \mathcal{A}^{2}_{W,H}B \subset \mathcal{A}_{W,H}B$, because the relation $\mathcal{A}_{W,H} = \mathcal{P}^{+}_{W,H} \cup \Delta$ in (13) is transitive. By definition of the topological closure $\overline{B}^{W,H}$, we deduce that $\overline{B}^{W,H} \subset \mathcal{A}_{W,H}B$. Now, let $C \subset \mathbb{A}$ be a closed subset such that $B \subset C$. We necessarily have that $\mathcal{A}_{W,H}B \subset \mathcal{A}_{W,H}C \subset C$. As we have just proven that $B \subset \mathcal{A}_{W,H}B$ and that $\mathcal{A}_{W,H}B$ is closed, we conclude that $\overline{B}^{W,H} = \mathcal{A}_{W,H}B$, by definition of the topological closure $\overline{B}^{W,H}$.

The subset W is open because its complementary set W^{c} satisfies $\mathcal{A}_{W,H}W^{\mathsf{c}} = \mathcal{P}^{+}_{W,H}W^{\mathsf{c}} \cup W^{\mathsf{c}} \subset W^{\mathsf{c}}$, as $\mathcal{P}_{W,H}\,\mathbb{A} \subset W^{\mathsf{c}}$.

This ends the proof. □

For any subsets $B \subset \mathbb{A}$ and $B_j \subset \mathbb{A}$, $j = 1, \ldots, n$, we write $B_1 \sqcup \cdots \sqcup B_n = B$ when we have both $B_j \cap B_k = \emptyset$ for all $j \neq k$ and $B_1 \cup \cdots \cup B_n = B$. This notion differs from the one of partition, as we allow some of the subsets $B_j \subset \mathbb{A}$ to be possibly empty.

Definition 7. *Let $W \subset \mathbb{A}$ be a subset of agents.*

1. *Let $B, C \subset \mathbb{A}$ be two subsets of agents. We say that B and C are (conditionally)* topologically separated *(w.r.t. W), denoted by $B \perp\!\!\!\perp C \mid W$, when there exists $W_B, W_C \subset W$ such that*
$$W_B \sqcup W_C = W \ \textit{ and } \ \overline{B \cup W_B}^{W,H} \cap \overline{C \cup W_C}^{W,H} = \emptyset \,, \tag{18a}$$
or, equivalently,
$$W_B \sqcup W_C = W \ \textit{ and } \ \mathcal{A}_{W,H}(B \cup W_B) \cap \mathcal{A}_{W,H}(C \cup W_C) = \emptyset \,. \tag{18b}$$

2. *Let $b, c \in \mathbb{A}$ be two agents. We say that b and c are (conditionally)* topologically separated *(w.r.t. W), denoted by $b \perp\!\!\!\perp c \mid W$, when the subsets $\{b\}$ and $\{c\}$ are topologically separated.*

When $B, C \subset \mathbb{A}$ are topologically separated, we necessarily have that $B \cap C = \emptyset$ as $B \cap C \subset \overline{B \cup W_B}^{W,H} \cap \overline{C \cup W_C}^{W,H} = \emptyset$ by (18a).

In the light of the main result from the companion paper [5], we will equivalently say that such B and C are *(conditionally) separated.*

4 Topological separation, factorization and do-calculus in the intrinsic model

In this section, where we deal with probability, we consider a finite W-model (as in Definition 1), that is, where all sets are finite, to avoid technical measurability issues. Moreover, we suppose that the set Ω of states of Nature, and its field $\mathcal{F}$ have the following product form:

$$\Omega = \prod_{a\in\mathbb{A}} \Omega_a \ , \ \ \mathcal{F} = \bigotimes_{a\in\mathbb{A}} \mathcal{F}_a \ . \tag{19a}$$

For any nonempty subset $B \subset \mathbb{A}$ of agents, we denote

$$\Omega_B = \prod_{b\in B} \Omega_b \ , \ \ \mathcal{F}_B = \bigotimes_{b\in B} \mathcal{F}_b \ , \ \ \mathbb{U}_B = \prod_{b\in B} \mathbb{U}_b \ , \ \ \mathcal{U}_B = \bigotimes_{b\in B} \mathcal{U}_b \ , \tag{19b}$$

and we denote by π_B the projection from $\mathbb{H}$ to its (decision) factors in B as follows:

$$\pi_B : \mathbb{H} = \Omega_B \times \Omega_{B^{\mathsf{c}}} \times \mathbb{U}_B \times \mathbb{U}_{B^{\mathsf{c}}} \to \mathbb{U}_B \ . \tag{19c}$$

4.1 Conditional separation implies factorization

We are now going to show how conditional topological separation induces a factorization of the solution map (see Definition 2).

Theorem 8. *We consider a solvable finite W-model, where the set Ω of states of Nature has the product form* (19a)*, where all fields contain the singletons, where each information field $\mathcal{I}_a$ in* (1) *is such that*

$$\mathcal{I}_a \subset \mathcal{F}_a \otimes \bigotimes_{b\neq a} \{\emptyset, \Omega_b\} \otimes \bigotimes_{c\in\mathbb{A}} \mathcal{U}_c \ , \ \ \forall a \in \mathbb{A} \ . \tag{20}$$

We also consider a policy profile $\lambda = (\lambda_a)_{a\in\mathbb{A}} \in \prod_{a\in\mathbb{A}} \Lambda_a$, a subset $H \subset \mathbb{H}$ of configurations, and Y, W and Z three subsets of $\mathbb{A}$, two by two disjoint and such that (see the comment following Definition7)

$$Y \perp\!\!\!\perp Z \mid (W, H) \ . \tag{21}$$

Then, there exist five subsets $Y', Z', W_Y, W_Z, \langle\emptyset\rangle \subset \mathbb{A}$ such that

$$\mathbb{A} = \widetilde{Y} \sqcup \widetilde{Z} \sqcup \langle\emptyset\rangle \quad \textit{where} \quad \widetilde{Y} = Y \sqcup Y' \sqcup W_Y \ , \ \ \widetilde{Z} = Z \sqcup Z' \sqcup W_Z \ , \ \ W = W_Y \sqcup W_Z \ , \tag{22a}$$

and there exist three measurable mappings (reduced solution maps)

$$\widetilde{M}_{\lambda_{\widetilde{Y}}} : \Omega_{\widetilde{Y}} \times \mathbb{U}_{W_Z} \to \mathbb{U}_{\widetilde{Y}} \ , \ \ \widetilde{M}_{\lambda_{\widetilde{Z}}} : \Omega_{\widetilde{Z}} \times \mathbb{U}_{W_Y} \to \mathbb{U}_{\widetilde{Z}} \ , \ \ \widetilde{M}_{\lambda_{\langle\emptyset\rangle}} : \Omega_{\langle\emptyset\rangle} \times \mathbb{U}_{\widetilde{Y}\cup\widetilde{Z}} \to \mathbb{U}_{\langle\emptyset\rangle} \tag{22b}$$

such that the solution map $S_\lambda(\omega) = \big(\omega, M_\lambda(\omega)\big)$ in (6b) *splits in three factors as follows: $\forall \omega \in S_\lambda^{-1}(H)$, we have that*

$$M_\lambda(\omega) = \bigg(\widetilde{M}_{\lambda_{\widetilde{Y}}}\Big(\omega_{\widetilde{Y}}, \lambda_{W_Z}\big(S_\lambda(\omega)\big)\Big), \widetilde{M}_{\lambda_{\widetilde{Z}}}\Big(\omega_{\widetilde{Z}}, \lambda_{W_Y}\big(S_\lambda(\omega)\big)\Big), \widetilde{M}_{\lambda_{\langle\emptyset\rangle}}\Big(\omega_{\langle\emptyset\rangle}, \lambda_{\widetilde{Y}\cup\widetilde{Z}}\big(S_\lambda(\omega)\big)\Big)\bigg) \ , \tag{22c}$$

or, equivalently,

$$M_\lambda(\omega) = \left(\widetilde{M}_{\lambda_{\widetilde{Y}}}\Big(\omega_{\widetilde{Y}}, \pi_{W_Z}\big(S_\lambda(\omega)\big)\Big), \widetilde{M}_{\lambda_{\widetilde{Z}}}\Big(\omega_{\widetilde{Z}}, \pi_{W_Y}\big(S_\lambda(\omega)\big)\Big), \widetilde{M}_{\lambda_{\langle\emptyset\rangle}}\Big(\omega_{\langle\emptyset\rangle}, \pi_{\widetilde{Y}\cup\widetilde{Z}}\big(S_\lambda(\omega)\big)\Big)\right) . \tag{22d}$$

More precisely, with the notation (19c), Equation (22c) has to be understood as $\pi_{\widetilde{Y}}\big(M_\lambda(\omega)\big) = \widetilde{M}_{\lambda_{\widetilde{Y}}}\Big(\omega_{\widetilde{Y}}, \lambda_{W_Z}\big(S_\lambda(\omega)\big)\Big)$, $\pi_{\widetilde{Z}}\big(M_\lambda(\omega)\big) = \widetilde{M}_{\lambda_{\widetilde{Z}}}\Big(\omega_{\widetilde{Z}}, \lambda_{W_Y}\big(S_\lambda(\omega)\big)\Big)$, and $\pi_{\langle\emptyset\rangle}\big(M_\lambda(\omega)\big) = \widetilde{M}_{\lambda_{\langle\emptyset\rangle}}\Big(\omega_{\langle\emptyset\rangle}, \lambda_{\widetilde{Y}\cup\widetilde{Z}}\big(S_\lambda(\omega)\big)\Big)$.

Proof of Theorem 8. The proof is in five steps.

• First, we identify five subsets $Y', Z', W_Y, W_Z, \langle\emptyset\rangle \subset \mathbb{A}$ such that (22a) holds true.

By assumption, we have that $Y \cap Z = Y \cap W = Z \cap W = \emptyset$ and $Y \perp\!\!\!\perp Z \mid (W, H)$. By the comment following Definition7, this means that the two subsets Y, $Z \subset \mathbb{A}$ are (conditionally) topologically separated (w.r.t. (W, H)). As a consequence, by Definition 7, there exists $W_Y, W_Z \subset W$ such that $W_Y \sqcup W_Z = W$ and $\overline{Y \cup W_Y}^{W,H} \cap \overline{Z \cup W_Z}^{W,H} = \emptyset$, that is,

$$W_Y \cap W_Z = \emptyset \ , \ \ W_Y \cup W_Z = W \ , \ \ \mathcal{A}_{W,H}(Y \cup W_Y) \cap \mathcal{A}_{W,H}(Z \cup W_Z) = \emptyset \ . \tag{23a}$$

We set

$$\widetilde{Y} = \mathcal{A}_{W,H}(Y \cup W_Y) \ \text{ and } \ \widetilde{Z} = \mathcal{A}_{W,H}(Z \cup W_Z) \ . \tag{23b}$$

By definition of the ancestral relation $\mathcal{A}_{W,H} = \mathcal{P}^+_{W,H} \cup \Delta$ in (13), we have that $Y \cup W_Y \subset \widetilde{Y}$ and $Z \cup W_Z \subset \widetilde{Z}$, where we can write $Y \cup W_Y = Y \sqcup W_Y$ and $Z \cup W_Z = Z \sqcup W_Z$ because $Y \cap W_Y \subset Y \cap W = \emptyset$ and $Z \cap W_Z \subset Z \cap W = \emptyset$ by assumption. Then, we set

$$Y' = \widetilde{Y} \setminus \big(Y \sqcup W_Y\big) \ , \ \ Z' = \widetilde{Z} \setminus \big(Z \sqcup W_Z\big) \ .$$

• Second, we show that

$$\mathfrak{I}_{\widetilde{Y}} \subset \mathcal{H}_{\widetilde{Y}\cup W_Z} \ , \ \ \mathfrak{I}_{\widetilde{Z}} \subset \mathcal{H}_{\widetilde{Z}\cup W_Y} \ . \tag{23c}$$

By definition (11) of the precedence relation $\mathcal{P}$ on the set $\mathbb{A}$ of agents, it is equivalent to show that

$$\mathcal{P}\widetilde{Y} \subset W_Z \cup \widetilde{Y} \ \text{ and } \ \mathcal{P}\widetilde{Z} \subset W_Y \cup \widetilde{Z} \ . \tag{23d}$$

To begin with, we establish that $\mathcal{P}_{W,H}\widetilde{Y} \subset \widetilde{Y}$. Indeed, we have

$$\begin{aligned} \mathcal{P}_{W,H}\widetilde{Y} &= \mathcal{P}_{W,H}\mathcal{A}_{W,H}(Y \cup W_Y) && \text{(as } \widetilde{Y} = \mathcal{A}_{W,H}(Y \cup W_Y) \text{ by definition (23b))} \\ &\subset \mathcal{A}_{W,H}(Y \cup W_Y) \end{aligned}$$

because $\mathcal{P}_{W,H}\mathcal{A}_{W,H} \subset \mathcal{A}_{W,H}$ as $\mathcal{A}_{W,H} = \mathcal{P}^+_{W,H} \cup \Delta$ in (13) implies that $\mathcal{P}_{W,H}\mathcal{A}_{W,H} = \mathcal{P}_{W,H}(\mathcal{P}^+_{W,H} \cup \Delta) = \mathcal{P}_{W,H}\mathcal{P}^+_{W,H} \cup \mathcal{P}_{W,H} = \mathcal{P}^+_{W,H} \cup \mathcal{P}_{W,H} = \mathcal{P}^+_{W,H} \subset \mathcal{P}^+_{W,H} \cup \Delta = \mathcal{A}_{W,H}$

$$= \widetilde{Y} \ . \qquad \text{(by definition (23b))}$$

Then, we deduce that

$$\begin{aligned}
\mathcal{P}\widetilde{Y} &= \big(W^{\mathsf{c}} \cap \mathcal{P}\widetilde{Y}\big) \cup \big(W \cap \mathcal{P}\widetilde{Y}\big) \\
&\subset \mathcal{P}_{W,H}\widetilde{Y} \cup W && \text{(as } \mathcal{P}_{W,H}\widetilde{Y} = W^{\mathsf{c}} \cap \mathcal{P}\widetilde{Y} \text{ by (14))} \\
&\subset \widetilde{Y} \cup W && \text{(as we have just shown that } \mathcal{P}_{W,H}\widetilde{Y} \subset \widetilde{Y}) \\
&= \widetilde{Y} \cup W_Y \cup W_Z && \text{(as } W_Y \cup W_Z = W \text{ by (23a))} \\
&= \widetilde{Y} \cup W_Z \; .
\end{aligned}$$

(as $W_Y \subset \mathcal{A}_{W,H}(Y \cup W_Y) = \widetilde{Y}$ by (23b) and since the relation $\mathcal{A}_{W,H}$ is reflexive)

In the same way, we obtain that $\mathcal{P}\widetilde{Z} \subset W_Y \cup \widetilde{Z}$.

• Third, we prepare the existence of a factorization as in (22c). By the equality $\mathbb{A} = \widetilde{Y} \sqcup \widetilde{Z} \sqcup \langle\emptyset\rangle$, the solution map (6b) splits in three factors

$$M_\lambda(\omega) = \Big(\pi_{\widetilde{Y}}\big(S_\lambda(\omega)\big), \pi_{\widetilde{Z}}\big(S_\lambda(\omega)\big), \pi_{\langle\emptyset\rangle}\big(S_\lambda(\omega)\big)\Big) \; .$$

Let us examine the term $\pi_{\widetilde{Y}}\big(M_\lambda(\omega)\big)$, as the other two terms can be treated in the same way. On the one hand, by (20), we have that $\mathfrak{I}_{\widetilde{Y}} \subset \mathcal{F}_{\widetilde{Y}} \otimes \bigotimes_{b \not\in \widetilde{Y}} \{\emptyset, \Omega_b\} \otimes \bigotimes_{c \in \mathbb{A}} \mathcal{U}_c$. On the other hand, by (23c), we have that $\mathfrak{I}_{\widetilde{Y}} \subset \mathcal{H}_{\widetilde{Y} \cup W_Z} = \mathcal{F} \otimes \bigotimes_{b \in \widetilde{Y} \cup W_Z} \mathcal{U}_b \otimes \bigotimes_{c \not\in \widetilde{Y} \cup W_Z} \{\emptyset, \mathbb{U}_c\}$. Therefore, we deduce that

$$\begin{aligned}
\mathfrak{I}_{\widetilde{Y}} &\subset \Big(\mathcal{F}_{\widetilde{Y}} \otimes \bigotimes_{b \not\in \widetilde{Y}} \{\emptyset, \Omega_b\} \otimes \bigotimes_{c \in \mathbb{A}} \mathcal{U}_c\Big) \cap \Big(\mathcal{F} \otimes \bigotimes_{b \in \widetilde{Y} \cup W_Z} \mathcal{U}_b \otimes \bigotimes_{c \not\in \widetilde{Y} \cup W_Z} \{\emptyset, \mathbb{U}_c\}\Big) \\
&= \mathcal{F}_{\widetilde{Y}} \otimes \bigotimes_{b \not\in \widetilde{Y}} \{\emptyset, \Omega_b\} \otimes \bigotimes_{b \in \widetilde{Y} \cup W_Z} \mathcal{U}_b \otimes \bigotimes_{c \not\in \widetilde{Y} \cup W_Z} \{\emptyset, \mathbb{U}_c\} \; .
\end{aligned}$$

As all fields contain the singletons by assumption, and by the measurability property (3b) of the policies in the policy profile λ, there exists a "reduced" measurable mapping $\overline{\lambda}_{\widetilde{Y}} : \Omega_{\widetilde{Y}} \times \mathbb{U}_{\widetilde{Y}} \times \mathbb{U}_{W_Z} \to \mathbb{U}_{\widetilde{Y}}$ such that

$$\lambda_{\widetilde{Y}}\Big(\omega_{\widetilde{Y}}, \omega_{\mathbb{A}\setminus\widetilde{Y}}, u_{\widetilde{Y}}, u_{W_Z}, u_{\mathbb{A}\setminus(\widetilde{Y}\cup W_Z)}\Big) = \overline{\lambda}_{\widetilde{Y}}\Big(\omega_{\widetilde{Y}}, u_{\widetilde{Y}}, u_{W_Z}\Big) \; , \; \forall \Big(\omega_{\widetilde{Y}}, \omega_{\mathbb{A}\setminus\widetilde{Y}}, u_{\widetilde{Y}}, u_{W_Z}, u_{\mathbb{A}\setminus(\widetilde{Y}\cup W_Z)}\Big) \in \mathbb{H} \; .$$

In the same way, there exists a "reduced" measurable mapping $\overline{\lambda}_{\widetilde{Z}} : \Omega_{\widetilde{Z}} \times \mathbb{U}_{\widetilde{Z}} \times \mathbb{U}_{W_Y} \to \mathbb{U}_{\widetilde{Z}}$ such that

$$\lambda_{\widetilde{Z}}\Big(\omega_{\widetilde{Z}}, \omega_{\mathbb{A}\setminus\widetilde{Z}}, u_{\widetilde{Z}}, u_{W_Y}, u_{\mathbb{A}\setminus(\widetilde{Y}\cup W_Z)}\Big) = \overline{\lambda}_{\widetilde{Z}}\Big(\omega_{\widetilde{Z}}, u_{\widetilde{Z}}, u_{W_Y}\Big) \; , \; \forall \Big(\omega_{\widetilde{Z}}, \omega_{\mathbb{A}\setminus\widetilde{Z}}, u_{\widetilde{Z}}, u_{W_Y}, u_{\mathbb{A}\setminus(\widetilde{Z}\cup W_Y)}\Big) \in \mathbb{H} \; ,$$

and a measurable mapping $\overline{\lambda}_{\langle\emptyset\rangle} : \Omega_{\langle\emptyset\rangle} \times \mathbb{U}_{\langle\emptyset\rangle} \times \mathbb{U}_{\langle\emptyset\rangle^{\mathsf{c}}} \to \mathbb{U}_{\langle\emptyset\rangle}$ such that

$$\lambda_{\langle\emptyset\rangle}\Big(\omega_{\langle\emptyset\rangle}, \omega_{\langle\emptyset\rangle^{\mathsf{c}}}, u_{\langle\emptyset\rangle}, u_{\langle\emptyset\rangle^{\mathsf{c}}}\Big) = \overline{\lambda}_{\langle\emptyset\rangle}\Big(\omega_{\langle\emptyset\rangle}, u_{\langle\emptyset\rangle}, u_{\langle\emptyset\rangle^{\mathsf{c}}}\Big) \; , \; \forall \Big(\omega_{\langle\emptyset\rangle}, \omega_{\langle\emptyset\rangle^{\mathsf{c}}}, u_{\langle\emptyset\rangle}, u_{\langle\emptyset\rangle^{\mathsf{c}}}\Big) \in \mathbb{H} \; .$$

By Definition 2 of $S_\lambda(\omega)$, we can regroup the closed-loop equations (6a) in three parts as

$$\begin{aligned}
\pi_{\widetilde{Y}}\big(S_\lambda(\omega)\big) &= \overline{\lambda}_{\widetilde{Y}}\Big(\omega_{\widetilde{Y}}, \pi_{\widetilde{Y}}\big(S_\lambda(\omega)\big), \pi_{W_Z}\big(S_\lambda(\omega)\big)\Big)\\
\pi_{\widetilde{Z}}\big(S_\lambda(\omega)\big) &= \overline{\lambda}_{\widetilde{Z}}\Big(\omega_{\widetilde{Z}}, \pi_{\widetilde{Z}}\big(S_\lambda(\omega)\big), \pi_{W_Y}\big(S_\lambda(\omega)\big)\Big)\\
\pi_{\langle\emptyset\rangle}\big(S_\lambda(\omega)\big) &= \overline{\lambda}_{\langle\emptyset\rangle}\Big(\omega_{\langle\emptyset\rangle}, \pi_{\langle\emptyset\rangle}\big(S_\lambda(\omega)\big), \pi_{\widetilde{Y}}\big(S_\lambda(\omega)\big), \pi_{\widetilde{Z}}\big(S_\lambda(\omega)\big)\Big)
\end{aligned}$$

so that the reduced closed-loop equations

$$u_{\widetilde{Y}} = \overline{\lambda}_{\widetilde{Y}}\Big(\omega_{\widetilde{Y}}, u_{\widetilde{Y}}, u_{W_Z}\Big) \tag{24a}$$

$$u_{\widetilde{Z}} = \overline{\lambda}_{\widetilde{Z}}\Big(\omega_{\widetilde{Z}}, u_{\widetilde{Z}}, u_{W_Y}\Big) \tag{24b}$$

$$u_{\langle\emptyset\rangle} = \overline{\lambda}_{\langle\emptyset\rangle}\Big(\omega_{\langle\emptyset\rangle}, u_{\langle\emptyset\rangle}, u_{\widetilde{Y}}, u_{\widetilde{Z}}\Big) \tag{24c}$$

have (at least) the solution $(u_{\widetilde{Y}}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle}) = \big(\pi_{\widetilde{Y}}\big(S_\lambda(\omega)\big), \pi_{\widetilde{Z}}\big(S_\lambda(\omega)\big), \pi_{\langle\emptyset\rangle}\big(S_\lambda(\omega)\big)\big)$ when $u_W = \pi_W\big(S_\lambda(\omega)\big)$.

• Fourth, we show the existence of three mappings as in (22b).

Let $\omega \in \Omega$. We denote by $\mathcal{U}_{\widetilde{Y}}(\omega)$ the set of elements $u_{\widetilde{Y}} \in \mathbb{U}_{\widetilde{Y}}$ such that there exists at least one $(\omega'_{\widetilde{Z}}, \omega'_{\langle\emptyset\rangle}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle}) \in \Omega_{\widetilde{Z}} \times \Omega_{\langle\emptyset\rangle} \times \mathbb{U}_{\widetilde{Z}} \times \mathbb{U}_{\langle\emptyset\rangle}$ that satisfies $(u_{\widetilde{Y}}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle}) = \lambda\Big((\omega_{\widetilde{Y}}, \omega'_{\widetilde{Z}}, \omega'_{\langle\emptyset\rangle}), (u_{\widetilde{Y}}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle})\Big)$ and $\pi_{W_Z}\big(S_\lambda(\omega)\big) = \pi_{W_Z}\big(S_\lambda(\omega_{\widetilde{Y}}, \omega'_{\widetilde{Z}}, \omega'_{\langle\emptyset\rangle})\big)$. We are going to show that $\mathcal{U}_{\widetilde{Y}}(\omega)$ is a singleton. For this purpose, we consider $(u_{\widetilde{Y}}, \omega_{\widetilde{Z}}, \omega_{\langle\emptyset\rangle}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle})$ and $(u'_{\widetilde{Y}}, \omega'_{\widetilde{Z}}, \omega'_{\langle\emptyset\rangle}, u'_{\widetilde{Z}}, u'_{\langle\emptyset\rangle})$ satisfying the two conditions that define the set $\mathcal{U}_{\widetilde{Y}}(\omega)$.

As we have, on the one hand, that

$$(u_{\widetilde{Y}}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle}) = \lambda\Big((\omega_{\widetilde{Y}}, \omega_{\widetilde{Z}}, \omega_{\langle\emptyset\rangle}), (u_{\widetilde{Y}}, u_{\widetilde{Z}}, u_{\langle\emptyset\rangle})\Big) \tag{25a}$$

and, on the other hand, that

$$(u'_{\widetilde{Y}}, u'_{\widetilde{Z}}, u'_{\langle\emptyset\rangle}) = \lambda\Big((\omega_{\widetilde{Y}}, \omega'_{\widetilde{Z}}, \omega'_{\langle\emptyset\rangle}), (u'_{\widetilde{Y}}, u'_{\widetilde{Z}}, u'_{\langle\emptyset\rangle})\Big) , \tag{25b}$$

we deduce that, by using (24),

$$(u_{\widetilde{Y}}, u'_{\widetilde{Z}}, \hat{u}_{\langle\emptyset\rangle}) = \lambda\Big((\omega_{\widetilde{Y}}, \omega'_{\widetilde{Z}}, \omega'_{\langle\emptyset\rangle}), (u_{\widetilde{Y}}, u'_{\widetilde{Z}}, \hat{u}_{\langle\emptyset\rangle})\Big) , \tag{25c}$$

where $\hat{u}_{\langle\emptyset\rangle} = \overline{\lambda}_{\langle\emptyset\rangle}\Big(\omega'_{\langle\emptyset\rangle}, \hat{u}_{\langle\emptyset\rangle}, \lambda_{\widetilde{Y}\cup\widetilde{Z}}\big(S_\lambda(\omega)\big)\Big)$. The Equations (25b) and (25c) imply, by the solvability assumption (see Definition 2), that $u_{\widetilde{Y}} = u'_{\widetilde{Y}}$.

Thus, we have defined, for any $u_{W_{\widetilde{Z}}} = \pi_{W_{\widetilde{Z}}}\big(S_\lambda(\omega)\big)$ a unique element $u_{\widetilde{Y}} = \widetilde{M}_{\lambda_{\widetilde{Y}}}(\omega_{\widetilde{Y}}, u_{W_{\widetilde{Z}}})$. We do the same for $\widetilde{Z}$ and for $\langle\emptyset\rangle$. Thus, we have defined reduced solution maps as follows

$$\widetilde{M}_{\lambda_{\widetilde{Y}}} : \Omega_{\widetilde{Y}} \times \mathbb{U}_{W_Z} \to \mathbb{U}_{\widetilde{Y}} , \tag{26a}$$

$$\widetilde{M}_{\lambda_{\widetilde{Z}}} : \Omega_{\widetilde{Z}} \times \mathbb{U}_{W_Y} \to \mathbb{U}_{\widetilde{Z}} , \tag{26b}$$

$$\widetilde{M}_{\lambda_{\langle\emptyset\rangle}} : \Omega_{\langle\emptyset\rangle} \times \mathbb{U}_{\widetilde{Y}\cup\widetilde{Z}} \to \mathbb{U}_{\langle\emptyset\rangle} . \tag{26c}$$

This ends the proof. □

4.2 Conditional independence in a closed-loop system

This subsection provides tools to study conditional independence in the presence of nonrecursive systems. We also discuss an instance where such independence is not captured by Pearl's [12] d-separation criterion.

4.2.1 Key technical lemma for closed-loop systems

We state and prove a lemma that will serve as a main argument for the proof of the coming Theorem 10. As far as we know, this result is novel. It cannot be deduced from Pearl's rules.

Lemma 9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let Ξ_1, Ξ_2, Υ_1, Υ_2, Θ_1, Θ_2 be six measurables spaces and*

$$\Psi_1 : \Xi_1 \times \Upsilon_2 \to \Theta_1 \; , \;\; \Psi_2 : \Xi_2 \times \Upsilon_1 \to \Theta_2 \; , \;\; \Phi_1 : \Xi_1 \times \Upsilon_2 \to \Upsilon_1 \; , \;\; \Phi_2 : \Xi_2 \times \Upsilon_1 \to \Upsilon_2 \tag{27}$$

be four measurable mappings. Let

$$\xi_1 : \Omega \to \Xi_1 \; , \;\; \xi_2 : \Omega \to \Xi_2 \; , \;\; \theta_1 : \Omega \to \Theta_1 \; , \;\; \theta_2 : \Omega \to \Theta_2 \; , \;\; \upsilon_1 : \Omega \to \Upsilon_1 \; , \;\; \upsilon_2 : \Omega \to \Upsilon_2 \tag{28}$$

be six random variables satisfying

$$\theta_1 = \Psi_1(\xi_1, \upsilon_2) \; , \tag{29a}$$

$$\theta_2 = \Psi_2(\xi_2, \upsilon_1) \; , \tag{29b}$$

$$\upsilon_1 = \Phi_1(\xi_1, \upsilon_2) \; , \tag{29c}$$

$$\upsilon_2 = \Phi_2(\xi_2, \upsilon_1) \; . \tag{29d}$$

Suppose that the couple (υ_1, υ_2) of random variables takes values in the countable[1] *product subset $\Upsilon'_1 \times \Upsilon'_2 \subset \Upsilon_1 \times \Upsilon_2$, and that the system of equations*

$$w_1 = \Phi_1(x_1, w_2) \tag{30a}$$

$$w_2 = \Phi_2(x_2, w_1) \tag{30b}$$

has a unique solution (w_1, w_2) in $\Upsilon'_1 \times \Upsilon'_2$, for any $(x_1, x_2) \in \Xi_1 \times \Xi_2$.

Then, if the random variables ξ_1 and ξ_2 are independent, the random variables θ_1 and θ_2 are independent when conditioned on (υ_1, υ_2).

Proof. By assumption, there exists a unique solution (w_1, w_2) to the implicit system (30) of equations. Thus, there exists mappings

$$\widetilde{\Phi}_1 : \Xi_1 \times \Xi_2 \to \Upsilon'_1 \; , \;\; \widetilde{\Phi}_2 : \Xi_1 \times \Xi_2 \to \Upsilon'_2 \; , \tag{31a}$$

[1]The proof of Lemma 9 in case of general random variables is postponed to future work.

such that, for any (w_1, w_2) in $\Upsilon'_1 \times \Upsilon'_2$ and $(x_1, x_2) \in \Xi_1 \times \Xi_2$, we have

$$\Big(w_1 = \Phi_1(x_1, w_2)\ ,\ \ w_2 = \Phi_2(x_2, w_1)\Big) \iff \Big(w_1 = \widetilde{\Phi}_1(x_1, x_2)\ ,\ \ w_2 = \widetilde{\Phi}_2(x_1, x_2)\Big)\ . \tag{31b}$$

We suppose that the random variables ξ_1 and ξ_2 are independent.

• First, we establish that, for any couple $(w_1, w_2) \in \Upsilon'_1 \times \Upsilon'_2$:

$$\Big\{\Phi_1(\xi_1, w_2) = w_1\ ,\ \ \Phi_2(\xi_2, w_1) = w_2\Big\} = \Big\{\upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\Big\}\ . \tag{32}$$

Indeed, on the one hand, we have

$$\begin{aligned}
&\Big\{\Phi_1(\xi_1, w_2) = w_1\ ,\ \ \Phi_2(\xi_2, w_1) = w_2\Big\} \\
&\quad = \Big\{w_1 = \widetilde{\Phi}_1(\xi_1, \xi_2)\ ,\ \ w_2 = \widetilde{\Phi}_2(\xi_1, \xi_2)\Big\} && \text{(by (31))} \\
&\quad = \Big\{w_1 = \widetilde{\Phi}_1(\xi_1, \xi_2)\ ,\ \ w_2 = \widetilde{\Phi}_2(\xi_1, \xi_2)\Big\} \cap \Big\{\upsilon_1 = \Phi_1(\xi_1, \upsilon_2)\ ,\ \ \upsilon_2 = \Phi_2(\xi_2, \upsilon_1)\Big\} \\
&\qquad\qquad \text{(because } \Big\{\upsilon_1 = \Phi_1(\xi_1, \upsilon_2)\ ,\ \ \upsilon_2 = \Phi_2(\xi_2, \upsilon_1)\Big\} = \Omega \text{ by (29c) and (29d))} \\
&\quad = \Big\{w_1 = \widetilde{\Phi}_1(\xi_1, \xi_2)\ ,\ \ w_2 = \widetilde{\Phi}_2(\xi_1, \xi_2)\Big\} \cap \Big\{\upsilon_1 = \widetilde{\Phi}_1(\xi_1, \xi_2)\ ,\ \ \upsilon_2 = \widetilde{\Phi}_2(\xi_1, \xi_2)\Big\} \\
&\qquad\qquad && \text{(by (31))} \\
&\quad \subset \Big\{\upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\Big\}\ .
\end{aligned}$$

On the other hand, the reverse inclusion can be proved in the same way. Thus, we have obtained the equality (32).

• Second, we show that, for any subsets $\Theta'_1 \subset \Theta_1$ and $\Theta'_2 \subset \Theta_2$, and for any couple $(w_1, w_2) \in \Upsilon'_1 \times \Upsilon'_2$:

$$\begin{aligned}
&\Big\{\theta_1 \in \Theta'_1\ ,\ \ \theta_2 \in \Theta'_2\ ,\ \ \upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\Big\} \\
&\quad = \Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1\ ,\ \ \Phi_1(\xi_1, w_2) = w_1\Big\} \cap \Big\{\Psi_2(\xi_2, w_1) \in \Theta'_2\ ,\ \ \Phi_2(\xi_2, w_1) = w_2\Big\}\ .
\end{aligned} \tag{33}$$

Indeed, we have

$$\begin{aligned}
&\Big\{\theta_1 \in \Theta'_1\ ,\ \ \theta_2 \in \Theta'_2\ ,\ \ \upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\Big\} \\
&\quad = \Big\{\Psi_1(\xi_1, \upsilon_2) \in \Theta'_1\ ,\ \ \Psi_2(\xi_2, \upsilon_1) \in \Theta'_2\ ,\ \ \upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\Big\} \quad \text{(by (29a) and (29b))} \\
&\quad = \Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1\ ,\ \ \Psi_2(\xi_2, w_1) \in \Theta'_2\ ,\ \ \upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\Big\}
\end{aligned}$$

by substitution of the last two terms $\upsilon_1 = w_1$ and $\upsilon_2 = w_2$ in the first two terms

$$\begin{aligned}
&= \{\ \Psi_1(\xi_1, w_2) \in \Theta'_1\ ,\ \ \Psi_2(\xi_2, w_1) \in \Theta'_2\ ,\ \ \Phi_1(\xi_1, \upsilon_2) = w_1\ ,\ \ \Phi_2(\xi_2, \upsilon_1) = w_2\ , \\
&\qquad \upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\} \qquad \text{(by (29c) and (29d))} \\
&= \{\ \Psi_1(\xi_1, w_2) \in \Theta'_1\ ,\ \ \Psi_2(\xi_2, w_1) \in \Theta'_2\ ,\ \ \Phi_1(\xi_1, w_2) = w_1\ ,\ \ \Phi_2(\xi_2, w_1) = w_2\ , \\
&\qquad \upsilon_1 = w_1\ ,\ \ \upsilon_2 = w_2\}
\end{aligned}$$

by substitution of the last two terms $\upsilon_1 = w_1$ and $\upsilon_2 = w_2$ in the two middle terms

$$= \Big\{ \Psi_1(\xi_1, w_2) \in \Theta_1' \ , \ \ \Psi_2(\xi_2, w_1) \in \Theta_2' \ , \ \ \Phi_1(\xi_1, w_2) = w_1 \ , \ \ \Phi_2(\xi_2, w_1) = w_2 \Big\} \ .$$

because the system (30) of equations has a unique solution on $\Upsilon_1' \times \Upsilon_2'$, so that $\Phi_1(\xi_1, w_2) = w_1$ and $\Phi_2(\xi_2, w_1) = w_2$ imply that $\upsilon_1 = w_1$ and $\upsilon_2 = w_2$ hold true by (29c) and (29d). Thus, we have obtained (33).

• Third, we show that the random variables θ_1 and θ_2 are independent when conditioned on (υ_1, υ_2). For this purpose, we calculate, for any subsets $\Theta_1' \subset \Theta_1$ and $\Theta_2' \subset \Theta_2$, and for any couple $(w_1, w_2) \in \Upsilon_1' \times \Upsilon_2'$:

$$\begin{aligned} &\mathbb{P}\Big\{ \theta_1 \in \Theta_1' \ , \ \ \theta_2 \in \Theta_2' \ \Big| \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2 \Big\} \\ &\quad = \frac{\mathbb{P}\Big\{ \theta_1 \in \Theta_1' \ , \ \ \theta_2 \in \Theta_2' \ , \ \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2 \Big\}}{\mathbb{P}\Big\{ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2 \Big\}} \end{aligned}$$

by definition of the conditional probability, and where all quantities are zero if the denominator is zero

$$= \frac{\mathbb{P}\Big\{ \theta_1 \in \Theta_1' \ , \ \ \theta_2 \in \Theta_2' \ , \ \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2 \Big\}}{\mathbb{P}\Big\{ \theta_1 \in \Theta_1 \ , \ \ \theta_2 \in \Theta_2 \ , \ \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2 \Big\}} \ \ \text{(because } \Big\{ \theta_1 \in \Theta_1 \ , \ \ \theta_2 \in \Theta_2 \Big\} = \Omega\text{)}$$

$$= \frac{\mathbb{P}\Big\{ \Psi_1(\xi_1, w_2) \in \Theta_1' \ , \ \ \Phi_1(\xi_1, w_2) = w_1 \Big\} \times \mathbb{P}\Big\{ \Psi_2(\xi_2, w_1) \in \Theta_2' \ , \ \ \Phi_2(\xi_2, w_1) = w_2 \Big\}}{\mathbb{P}\Big\{ \Psi_1(\xi_1, w_2) \in \Theta_1 \ , \ \ \Phi_1(\xi_1, w_2) = w_1 \Big\} \times \mathbb{P}\Big\{ \Psi_2(\xi_2, w_1) \in \Theta_2 \ , \ \ \Phi_2(\xi_2, w_1) = w_2 \Big\}}$$

by (33) using the assumption that the random variables ξ_1 and ξ_2 are independent

$$= \frac{\mathbb{P}\Big\{ \Psi_1(\xi_1, w_2) \in \Theta_1' \ , \ \ \Phi_1(\xi_1, w_2) = w_1 \Big\}}{\mathbb{P}\Big\{ \Phi_1(\xi_1, w_2) = w_1 \Big\}} \times \frac{\mathbb{P}\Big\{ \Psi_2(\xi_2, w_1) \in \Theta_2' \ , \ \ \Phi_2(\xi_2, w_1) = w_2 \Big\}}{\mathbb{P}\Big\{ \Phi_2(\xi_2, \upsilon_1) = w_2 \Big\}} \ .$$

Then, we focus on the first term of the product and we write

$$\begin{aligned}
&\frac{\mathbb{P}\Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1 \ , \ \ \Phi_1(\xi_1, w_2) = w_1\Big\}}{\mathbb{P}\Big\{\Phi_1(\xi_1, w_2) = w_1\Big\}} \\
&= \frac{\mathbb{P}\Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1 \ , \ \ \Phi_1(\xi_1, w_2) = w_1\Big\} \times \mathbb{P}\Big\{\Phi_2(\xi_2, \upsilon_1) = w_2\Big\}}{\mathbb{P}\Big\{\Phi_1(\xi_1, w_2) = w_1\Big\} \times \mathbb{P}\Big\{\Phi_2(\xi_2, \upsilon_1) = w_2\Big\}} \\
&= \frac{\mathbb{P}\Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1 \ , \ \ \Phi_1(\xi_1, w_2) = w_1 \ , \ \ \Phi_2(\xi_2, \upsilon_1) = w_2\Big\}}{\mathbb{P}\Big\{\Phi_1(\xi_1, w_2) = w_1 \ , \ \ \Phi_2(\xi_2, \upsilon_1) = w_2\Big\}} \\
&\hspace{12em} \text{(because the random variables } \xi_1 \text{ and } \xi_2 \text{ are independent)} \\
&= \frac{\mathbb{P}\Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1 \ , \ \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\}}{\mathbb{P}\Big\{\upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\}} \qquad \text{(by the equality (32))} \\
&= \mathbb{P}\Big\{\Psi_1(\xi_1, w_2) \in \Theta'_1 \ \Big| \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\} \\
&\hspace{12em} \text{(by definition of the conditional probability)} \\
&= \mathbb{P}\Big\{\theta_1 \in \Theta'_1 \ \Big| \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\} \ . \qquad \text{(by (29a))}
\end{aligned}$$

Doing the same with the second term of the product, we get that

$$\begin{aligned}
&\mathbb{P}\Big\{\theta_1 \in \Theta'_1 \ , \ \ \theta_2 \in \Theta'_2 \ \Big| \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\} \\
&\quad = \mathbb{P}\Big\{\theta_1 \in \Theta'_1 \ \Big| \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\} \times \mathbb{P}\Big\{\theta_2 \in \Theta'_2 \ \Big| \ \upsilon_1 = w_1 \ , \ \ \upsilon_2 = w_2\Big\} \ .
\end{aligned}$$

This ends the proof.

□

4.2.2 Graphical discussion on Lemma 9

We consider a system of equations between random variables as in Lemma 9, that is, we have six random variables satisfying (29). We suppose that the random variables ξ_1 and ξ_2 are independent. We are going to show that the graphical analysis of such system fails to provide a result on conditional independence, whereas Lemma 9 allows to conclude.

We can draw a causality diagram corresponding to (29). For this purpose, we use the mappings (31) so that the system (29) of equations is equivalent to the system

$$\theta_1 = \Psi_1(\xi_1, \upsilon_2) \ , \tag{36a}$$

$$\theta_2 = \Psi_2(\xi_2, \upsilon_1) \ , \tag{36b}$$

$$\upsilon_1 = \widetilde{\Phi}_1(\xi_1, \xi_2) \ , \tag{36c}$$

$$\upsilon_2 = \widetilde{\Phi}_2(\xi_1, \xi_2) \ . \tag{36d}$$

We depict the above relations in the DAG in Figure 2. We observe that there exists an unblocked path $\theta_1 \leftarrow \xi_1 \rightarrow \nu_1 \leftarrow \xi_2 \rightarrow \theta_2$ from θ_1 to θ_2. As a consequence, we cannot use the Pearl's rules to conclude about conditional independence of θ_1 and θ_2 with respect to (υ_1, υ_2). Otherwise said, we lost some information about the system structure when we represent it with a DAG. The DAG representation does not permit the full causal analysis of this system. By contrast, by Lemma 9 we reach the conclusion that the random variables θ_1 and θ_2 are independent when conditioned on (υ_1, υ_2).

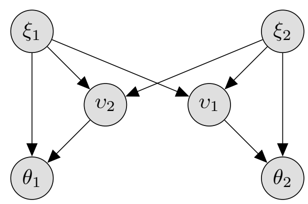

Figure 2: DAG representation of the system (36) of equations

4.3 Probabilistic implications and do-calculus in intrinsic models

Theorem 10. *We suppose that the assumptions of Theorem 8 are satisfied. Moreover, we suppose that the set* Ω *in* (19a) *is equipped with a probability* $\mathbb{P} = \bigotimes_{a \in \mathbb{A}} \mathbb{P}_a$ *where each* $\mathbb{P}_a$ *is a probability on* $(\Omega_a, \mathcal{F}_a)$.

We define the following pushforward probability $\mathbb{Q}_\lambda$ *on* $(\mathbb{H}, \mathcal{H})$ *by*

$$\mathbb{Q}_\lambda = \mathbb{P} \circ S_\lambda^{-1} \; . \tag{37}$$

Then, $\big(\mathbb{H}, \mathcal{H}, \mathbb{Q}_\lambda\big)$ *is a probability space, and the two projections* $\pi_{\overline{Y}^{W,H}} : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_{\overline{Y}^{W,H}}, \mathcal{U}_{\overline{Y}^{W,H}})$ *and* $\pi_{\overline{Z}^{W,H}} : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_{\overline{Z}^{W,H}}, \mathcal{U}_{\overline{Z}^{W,H}})$ *as in* (19c) *are independent under* $\mathbb{Q}_\lambda$*, conditionally on the subset* $H \subset \mathbb{H}$ *and on the projection* $\pi_W : (\mathbb{H}, \mathcal{H}) \to (\mathbb{U}_W, \mathcal{U}_W)$.

The Theorem claims that, for any decisions $u_Y \in \mathbb{U}_{\overline{Y}^{W,H}}$ $u_Z \in \mathbb{U}_{\overline{Z}^{W,H}}$ and $u_W \in \mathbb{U}_W$, we have that

$$\begin{aligned} \mathbb{Q}_\lambda\big(\pi_{\overline{Y}^{W,H}}(h) = u_Y \; , \; \pi_{\overline{Z}^{W,H}}(h) = u_Z \mid h \in H \; , \; \pi_W(h) = u_W\big) \\ = \mathbb{Q}_\lambda\big(\pi_{\overline{Y}^{W,H}}(h) = u_Y \mid h \in H \; , \; \pi_W(h) = u_W\big) \\ \times \; \mathbb{Q}_\lambda\big(\pi_{\overline{Z}^{W,H}}(h) = u_Z \mid h \in H \; , \; \pi_W(h) = u_W\big) \; . \end{aligned} \tag{38}$$

Proof. If $\mathbb{P}\big(S_\lambda^{-1}(H)\big) = 0$, conditional independence is trivial (and meaningless!). We suppose that $\mathbb{P}\big(S_\lambda^{-1}(H)\big) > 0$ and we instantiate Lemma 9 with

- probability space $\tilde{\Omega} = S_\lambda^{-1}(H)$ with renormalized probability $\tilde{\mathbb{P}} = \mathbb{P} / \mathbb{P}\big(S_\lambda^{-1}(H)\big)$,
- six measurable spaces $\Xi_1 = \Omega_{\widetilde{Y}}$, $\Xi_2 = \Omega_{\widetilde{Z}}$, $\Upsilon_1 = \mathbb{U}_{W_Y}$, $\Upsilon_2 = \mathbb{U}_{W_Z}$, $\Theta_1 = \mathbb{U}_{Y \cup Y'}$, $\Theta_2 = \mathbb{U}_{Z \cup Z'}$

- four measurable mappings $\Psi_1 = \pi_{Y \cup Y'} \circ \widetilde{M}_{\lambda_Y}$, $\Psi_2 = \pi_{Z \cup Z'} \circ \widetilde{M}_{\lambda_Z}$, $\Phi_1 = \pi_{W_Y} \circ \widetilde{M}_{\lambda_Y}$, $\Phi_2 = \pi_{W_Z} \circ \widetilde{M}_{\lambda_Z}$,
- six random variables $\xi_1(\omega) = \omega_{\widetilde{Y}}$, $\xi_2(\omega) = \omega_{\widetilde{Z}}$, for all $\omega \in \tilde{\Omega}$, and $\theta_1 = \pi_{Y \cup Y'} \circ S_\lambda$, $\theta_2 = \pi_{Z \cup Z'} \circ S_\lambda$, $\nu_1 = \pi_{W_Y} \circ S_\lambda$, $\nu_2 = \pi_{W_Z} \circ S_\lambda$ on $\tilde{\Omega}$.

By assumption, the set Ω in (19a) is equipped with a probability $\mathbb{P} = \bigotimes_{a \in \mathbb{A}} \mathbb{P}_a$ where each $\mathbb{P}_a$ is a probability on $(\Omega_a, \mathcal{F}_a)$. Because of the product structure, the random variables ξ_1 and ξ_2 are independent with respect to $\tilde{\mathbb{P}}$.

As the assumptions of Theorem 8 are satisfied, Equation (22d) holds true, that is, we have that

$$M_\lambda(\omega) = \Big(\widetilde{M}_{\lambda_{\widetilde{Y}}} \big(\omega_{\widetilde{Y}}, \pi_{W_Z}\big(S_\lambda(\omega)\big)\big), \widetilde{M}_{\lambda_{\widetilde{Z}}} \big(\omega_{\widetilde{Z}}, \pi_{W_Y}\big(S_\lambda(\omega)\big)\big), \widetilde{M}_{\lambda_{\langle\emptyset\rangle}} \big(\omega_{\langle\emptyset\rangle}, \pi_{\widetilde{Y} \cup \widetilde{Z}}\big(S_\lambda(\omega)\big)\big) \Big) , \quad \forall \omega \in S_\lambda^{-1}(H) .$$

Thus, the assumptions of Lemma 9 are satisfied, and we conclude that the random variables θ_1 and θ_2 are independent under the probability $\tilde{\mathbb{P}}$, when conditioned on (ν_1, ν_2).

In other words, we have obtained that $\pi_{Y \cup Y'} \circ S_\lambda = \pi_{\bar{Y}} \circ S_\lambda$ and $\pi_{Z \cup Z'} \circ S_\lambda = \pi_{\bar{Z}} \circ S_\lambda$ are independent random variables, when conditioned on $\pi_{W_Y} \circ S_\lambda$ and $\pi_{W_Z} \circ S_\lambda$ under the probability $\tilde{\mathbb{P}}$. We deduce that $\pi_{\bar{Y}}$ and $\pi_{\bar{Z}}$ are independent when conditioned on π_{W_Y} and π_{W_Z} under the probability $\mathbb{Q}_\lambda = \mathbb{P} \circ S_\lambda^{-1}$.

This ends the proof. □

4.4 Intervention variables, do-calculus

In this section, we introduce the notions of intervention, and we derive a generalization of the do-calculus from Theorem 10. We show that this generalization subsumes two recent results.

4.4.1 Intervention variables model

We now introduce the possibility to intervene on a variable. We can encode this possibility in the model using a simple procedure. Suppose we are interested in an intervention policy profile $\hat{\lambda}_Z$ for a subset $Z \subset \mathbb{A}$ of agents. For this purpose, we consider a new family of fields $\hat{\mathcal{I}}_z \subset \mathcal{H}$, for $z \in Z$, as in (1), and we suppose that $\hat{\lambda}_Z$ is $\hat{\mathcal{I}}_z$-measurable, for any $z \in Z$, as in (3b). Then, we enrich the W-model as follows: (i) we introduce a new *intervention agent* I, equipped with $\Omega_I = \{0, 1\}$ and $\mathbb{U}_I = \{0, 1\}$, and who only has access to her/his private information in Ω_I; (ii) we straightforwardly adapt the information fields for $\mathbb{A} \backslash (Z \cup I)$ and the probability $\mathbb{P}$; (iii) we replace the information field $\mathcal{I}_z$ by $\big(\{0\} \otimes \mathcal{I}_z\big) \cup \big(\{1\} \otimes \hat{\mathcal{I}}_z\big)$, for $z \in Z$.

More formally, we introduce the new W-model $\big(\tilde{\mathbb{A}}, (\tilde{\Omega}, \tilde{\mathcal{F}}), (\tilde{\mathbb{U}}_a, \tilde{\mathcal{U}}_a)_{a\in\tilde{\mathbb{A}}}, (\tilde{\mathcal{I}}_a)_{a\in\tilde{\mathbb{A}}}\big)$, where $\tilde{\mathbb{A}} = \mathbb{A} \cup \{I\}$, $\tilde{\Omega} = \Omega \times \{0,1\}$, $\tilde{\mathbb{U}}_I = \{0,1\}$, $\tilde{\mathbb{U}}_a = \mathbb{U}_a$ for any $a \in \mathbb{A}$, and

$$\tilde{\mathcal{I}}_a = \mathcal{I}_a \otimes \{\emptyset, \mathbb{U}_I\} \; , \;\; \forall a \in \mathbb{A} \setminus Z \; , \tag{39a}$$

$$\tilde{\mathcal{I}}_z = \hat{\mathcal{I}}_z \otimes \mathcal{U}_I \; , \;\; \forall z \in Z \; , \tag{39b}$$

$$\mathcal{I}_I = \bigotimes_{a\in\mathbb{A}} \{\emptyset, \Omega_a\} \otimes \big\{\emptyset, \{0\}, \{1\}, \{0,1\}\big\} \otimes \bigotimes_{a\in\mathbb{A}} \{\emptyset, \mathbb{U}_a\} \; . \tag{39c}$$

We also extend the probability $\mathbb{P}$ as a product probability $\tilde{\mathbb{P}} = \mathbb{P} \otimes \mu$ on $\tilde{\Omega}$, where μ is a full support probability on $\{0,1\}$.

4.4.2 Do-calculus

In the usual expositions of do-calculus, there is a set of variables — often denoted by X — over which the do operator is applied on both sides of the do-calculus identities. In our framework, this simply translates by the fact that X does not have any ancestors, and since it loses its specificity, there is not reason to refer to it other than by the policy profile.

Theorem 11 (Do-calculus in W-models)**.** *Under the assumptions of Theorem 10, we have that the projection π_Y has the same conditional distribution under $\mathbb{Q}_\lambda$, whether the conditioning is w.r.t. the subset $H \subset \mathbb{H}$, the projection π_W and the projection $\pi_{\overline{Z}^{W,H}}$, or is only w.r.t. the subset $H \subset \mathbb{H}$ and the projection π_W.*

Proof. We combine Theorem 10 and Proposition 2.4 (c) from [19]. □

We have proved, loosely speaking, that

$$Y \perp\!\!\!\perp Z \mid (W, H) \implies \mathbb{Q}_\lambda(h_Y | h_W, h_{\overline{Z}^{W,H}}, H) = \mathbb{Q}_\lambda(h_Y | h_W, H) \; . \tag{40}$$

We stress the conciseness of Theorem 11 — permitted by the notions introduced in this paper — as we now show that it implies the three rules of Pearl, as well as the following two recent results.

Tikka et al. [18] manage to summarize the three rules of do-calculus thanks to the notion of context specific independence. They rely on so-called *labeled DAG* that can be turned into a context specific DAG by removing the arcs that are desactivated (spurious) in the context of interest. In the formalism that we propose, such context is represented by a subset of $\mathbb{H}$. Indeed, if we denote by $H \in \mathbb{H}$ the context for which an arc (a, b) is deactivated (in the language of [18]), we represent this by the following two properties

$$a \notin \mathcal{P}_{\emptyset,H} b \tag{41a}$$

$$a \in \mathcal{P}_{\emptyset,H^c} b \; . \tag{41b}$$

Such a property can be also be encoded in the information set of agent b. As a consequence, there is a mapping from the model introduced in [18] to W-models.

To introduce the next result, we will allow some abuse of notations to make our notations as close as possible to the literature we are comparing with. We will use, for $B \subset \mathbb{A}$ and $u_B \in \mathbb{U}_B$, the notation

$$[h_B = u_B] = \{h \in \mathbb{H}; h_B = u_B\} \tag{42}$$

Then Rule 1 in [18] rewrites in our setting

$$Y \perp\!\!\!\perp_{(X, h_{\tilde{X}} = u_{\tilde{X}})} Z \implies \mathbb{Q}\left(h_Y | h_Z, h_X, h_{\tilde{X}} = u_{\tilde{X}}\right) = \mathbb{Q}\left(h_Y | h_X, h_{\tilde{X}} = u_{\tilde{X}}\right) \tag{43}$$

where $X, \tilde{X} \subset \mathbb{A}$ and for a given $u_{\tilde{x}}$.

Proposition 12. *Rule 1 from [18] can be deduced from Theorem 11. In particular, Theorem 11 subsumes Pearl's do-calculus from [10].*

Proof. First, we show that Theorem 11 extends a result from [18]. Then we use the fact that [18] extends Pearl's do-calculus [10]. Last we show that Theorem 1 from [2] can be deduced from Theorem 11.

• We show that Theorem 11 implies (43).

If we set $W = X$, $H = \{h \in H; h_{\tilde{X}} = u_{\tilde{X}}\}$ in

$$Y \perp\!\!\!\perp_{W,H} Z \implies \mathbb{Q}(h_Y | h_Z, h_W, H) = \mathbb{Q}(h_Y | h_W, H) \ , \tag{44}$$

we obtain Rule 1 in [18].

• To show that Theorem 11 implies Pearl do-calculus, we re-employ an argument in [18], stating that the previous rule implies the rules of Pearl's do-calculus. □

Remark 13. *In the same manner we could derive the 3 rules of Theorem 1 from [2] from Theorem 11. Next we explain informaly how this could be done. For this purpose, we extend the W-model to account for the potential interventions $\hat{\lambda}_Z$ and $\tilde{\lambda}_Z$ using an additional agent I such that $U_I = \{0, 1, 2\}$.*

1. *Rule 1 in [18] writes, in our setting, as*

$$Y \perp\!\!\!\perp_{W,[h_i=0]} T \implies \mathbb{Q}(h_Y | h_W, h_T, [h_i = 0]) = \mathbb{Q}(h_Y | h_W, [h_i = 0]) \ . \tag{45}$$

2. *Rule 2 in [18] writes, in our setting, as*

$$Y \perp\!\!\!\perp_{W \cup Z, [h_i=1] \cup [h_i=2]} I \implies \mathbb{Q}(h_Y | h_Z, h_W, [h_i = 1]) = \mathbb{Q}(h_Y | h_Z, h_W, [h_i = 2]) \ . \tag{46}$$

3. *By reproducing the argument from [18] (proof of Theorem 2 in their companion paper). We can show that Rule 3 in [18] writes, in our setting, as*

$$Y \perp\!\!\!\perp_{W,[h_i=1] \cup [h_i=2]} I \implies \mathbb{Q}(h_Y | h_W, [h_i = 1]) = \mathbb{Q}(h_Y | h_W, [h_i = 2]) \ . \tag{47}$$

Conclusion: those assertions are direct applications of Theorem 11.

5 Discussion

In this paper, we simplify and generalize the do-calculus by leveraging the concepts of information field and solution map. The do-calculus is reduced to one rule. Causality is presented as a property of the W-model, and is not encoded by design. The intrinsic model is richer than DAGs, and allows for cycles, conditioning, and even noncausality as we only suppose the more general assumption of (measurable) solvability, while DAGs warrant the modeling of situations that do not possess a fixed causal ordering [7]. Statistical independence follows from the factorization property of the solution map implied by separation (related to a form of informational independence). We underline that the results come from the information structure, not the probability. Also, because our approach is not based on graphical models, our work provides a new proof of Pearl's original result.

Let us add that the Witsenhausen intrinsic model offers promising developements in game theory, too. Indeed, in [22], Witsenhausen places his own model in the context of game theory by referring notably to Kuhn's extensive games [8, 20], where information plays a central role. The question of information modeling is still debated in game theory [1], and [6] presents a contribution based on the Witsenhausen model.

Apart from technical measurability issues that have to be addressed when dealing with infinite sets, further work includes drawing connections with other research programs, such as Proposition 12 or questions related to identification [15, 16, 17], using the framework developed in this paper.

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