Decomposing Congestion and Reliability

Seth Blumsack, Marija Ilić, and Lester B. Lave

***Abstract*—Policy surrounding the North American transmission grid, particularly in the wake of electric-industry restructuring and following the blackout of August, 2003, has treated network congestion and network reliability as if they were separable and independent system attributes. Except for a few special cases, congestion and reliability are not independent, and may not even be separable in any meaningful way. Using the DC power flow model with linear ATC, we provide a method for decomposing a change in network topology into a congestion effect and a reliability effect. We provide analytical expressions describing the topological conditions under which a given network addition or outage will affect congestion and reliability, and prove some sufficiency conditions and some necessary conditions for congestion and reliability to be independent. These include (i) the network is series-parallel; (ii) demand is completely price-inelastic; (iii) all customers value reliability identically; and (iv) the grid operator does not discriminate among customers when forced to physically ration consumption.**

***Index Terms*—Reliability, congestion, merchant transmission, transmission investment, available transfer capacity, Braess Paradox, Wheatstone network.**

I. Nomenclature

NL = Number of lines in the network
NB = Number of buses in the network
S_{ij} = Transmission line connecting buses i and j
B_{ij} = Susceptance of the link connecting buses i and j
X_{ij} = Reactance of the link connecting buses i and j
θ_i = Phase angle at the ith bus
P_i = Net real power injection at the ith bus; positive for net generation and negative for net withdrawal
P_{Li} = Real power demand at the ith bus
P_{Gi} = Real power demand at the ith bus
ρ_{li} = Power transfer distribution factor along line l with respect to a network resource at bus i.
δ_{ij} = Phase angle difference between buses i and j
F_{ij} = Real power flow between buses i and j
π_i = Nodal price at bus i
μ_{ij} = Shadow price of transmission between buses i and j
C_i = Total cost function at the ith bus.
ATC_i = Available transfer capacity into bus i.
$\mathbf{B}$ = ($NB \times NB$) system susceptance matrix
$\mathbf{B}^{\mathbf{diag}}$ = ($NL \times NL$) diagonal matrix of line susceptances
$\mathbf{N}$ = ($NB \times NB$) node-node adjacency matrix
$\mathbf{A}$ = ($NB \times NL$) system node-line adjacency matrix
$\mathbf{P}$ = ($NB \times 1$) vector of bus injections
$\mathbf{F}$ = ($NL \times 1$) vector of line flows
θ = ($NB \times 1$) vector of bus angles
δ = ($NL \times 1$) vector of bus angle differences
ρ = ($NL \times NB$) matrix of power transfer distribution factors

II. Introduction

Restructuring in the U.S. electric power sector has encouraged investment by the non-utility, or "merchant" sector. Initially, investment activity in the merchant generation sector was high, with a great deal of mostly gas-fired capacity added between 1995 and 2002 [1]. Market-based merchant transmission has been far less successful, existing for the most part only in theory. Language supporting merchant transmission exists both in regulatory documents (such as section 1221 of the 2005 Energy Policy Act) and in the tariffs of the regional transmission operators such as PJM, New York ISO, and ISO New England.

Merchant transmission and market-based transmission investment were originally synonymous. Persistent differences in locational marginal prices (LMP) would serve as signals for investors. Compensation for system upgrades would come in the form of contracts entitling the investor to some share of the congestion rent, related to the incremental capacity created by the upgrade. Proposed transmission congestion contracts initially took the form of point-to-point financial transmission rights (FTR, [2]) or line-by-line "flowgate" rights [3], [4]. More recent market-based compensation mechanisms include the admittance-rights formulation in [5], and the LMP/megawatt-mile formulation in [6].

The use of FTRs to encourage transmission investment has been supported by the analysis of Bushnell and Stoft [7], [8]. They demonstrate that if incremental FTRs are allocated according to Hogan's "feasibility allocation rule," [2], [9], and if other economic assumptions hold, then merchant transmission can be economically efficient – all socially beneficial network investments will also be privately profitable. Significant criticism has come from [4] and [10], who argue that congestion contracts based on LMPs will

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inadvertently enhance the market power of certain generators. More recently, [11] demonstrate that the economic efficiency of the market-based merchant transmission model falls apart when the underlying assumptions are relaxed. Blumsack [12, Ch. 4] discusses a network topology in which merchant investors can profit from modifying the network in ways that cause congestion, even without relaxing the assumptions of [7] and [8].

Globally, Argentina has claimed success in getting a small number of merchant lines built [13], [14]. The U.S. experience has involved a "weak" version of merchant transmission. Non-utility lines have been built, but compensation has come through long-term contracts and not through market prices.

Underlying the market-based merchant transmission model is an implicit assumption that transmission projects can be cleanly separated into those that relieve congestion and those that enhance reliability. Since merchant transmission investments would be compensated with contracts based on congestion rents, this model would presumably only apply to those investments justifiable on "economic" grounds; investments for reliability would need to be socialized since the benefits are more widespread. This distinction has been made even more explicit in the recent "participant funding" models of [15] and [16]

Paul Joskow [1] argues that the distinction between investments for economics and those for reliability amounts to a meaningless dichotomy, since most transmission investments in the U.S. have been made by regulated utilities on the basis of reliability criteria. This paper as well as [17] demonstrate that this distinction is not simply meaningless; in many cases, it is wrong. Lines that cause congestion in the network may be justified on reliability grounds, and vice versa.

III. A Stylized Example

The interaction between congestion and reliability can be illustrated using the simple four-bus test network shown in Figure 1. This test system is known as the Wheatstone network, and the link connecting buses 2 and 3 is known as the Wheatstone bridge. Bus 1 is assumed to have an inexpensive generator with $P_{G1}^{\max}$ = 100 MW, while bus 4 has an expensive generator and a load with a constant per-period real power demand of P_{L1} = 100 MW. Buses 2 and 3 are tie-points containing neither generation nor load. The susceptances of lines S_{12} and S_{34} are assumed to be equal, and the susceptances of lines S_{13} and S_{24} are assumed to be equal. The two "upstream" lines (S_{12} and S_{13}) have a rated limit of 55 MW each, and the two "downstream" lines (S_{24} and S_{34}) have a rated limit of 100 MW each.

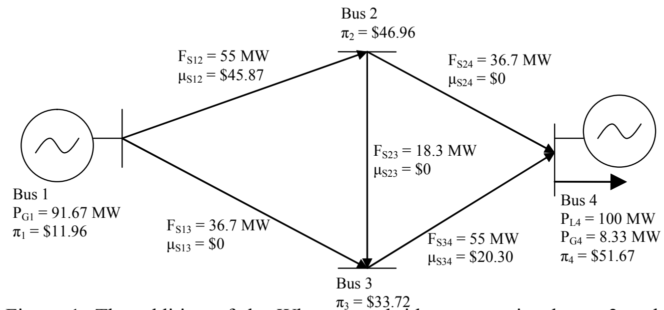

Figure 1: The addition of the Wheatstone bridge connecting buses 2 and 3 causes congestion along links S_{12} and S_{34}. The total system cost rises from $1,620 per hour without the Wheatstone bridge to $1,945 per hour with the bridge.

The cost curves of the two generators are given the following quadratic parameterization:

$$C(P_{G1}) = 200 + 10.3P_{G1} + 0.009P_{G1}^2 \quad (1)$$

$$C(P_{G4}) = 300 + 50P_{G4} + 0.1P_{G4}^2. \quad (2)$$

Consider a parallel version of the network in Figure 1, without the Wheatstone bridge. Abstracting from losses and reactive power demand, the inexpensive generator at bus 1 will produce 100 MW and serve the entire load. Due to the symmetry in the network, 50 MW will flow along each of the two paths from bus 1 to bus 4. A DC optimal power flow run on the parallel network yields identical LMPs of $12.11/MW-period at each of the four buses; the total system cost of serving the load is $1,620 per period.

Once the Wheatstone bridge is added to the network, the pattern of flows shifts, causing congestion on lines S_{12} and S_{34} in the network.[1] The congestion reduces the total transfer capacity from bus 1 to bus 4. The generator at bus 1 produces only 91.67 MW of real power, while the remainder must be made up by the expensive generator at bus 4 (or load-shedding must occur at bus 4). The presence of congestion alters the LMPs at all four buses in the system; in particular, the LMP at the load bus increases to $51.67/MW-period. The total system cost of serving the load rises to $1,945 per period. For a system with market-based nodal pricing for generators and loads, the price paid by the load would increase by more than a factor of four, resulting in both wealth transfers from consumers to generators (or the holders of congestion contracts) and a deadweight social loss ([11], [18], [12 Ch. 2 and 3]).

The phenomenon that congestion is caused or worsened by adding links to a network is known as the Braess Paradox, and was first observed in automotive highway networks [19], [20]. A detailed discussion of the conditions under which the Braess Paradox will arise for more general electric power networks can be found in [12 Ch. 3] and [18].

Thus, adding a Wheatstone bridge to a parallel system causes congestion and harms the network. However, the line

[1] These lines correspond to the high-susceptance lines in the network.

might be justified on reliability grounds. Without the Wheatstone bridge, the network would violate the *N – 1* criterion for transmission. Suppose an outage were to occur on either of the two "downstream" lines S_{24} or S_{34}. Without the Wheatstone bridge in the system, the loss of one of these downstream lines would effectively remove one of the electrical paths between buses 1 and 4. Thus, only 55 MW could be generated at bus 1. If the generator at bus 4 has $P_{G4}^{\max} < 45$ MW, then blackouts will result at bus 4.

IV. Decomposition of Single and Multiple Element Changes in Network Topology

In the four-bus Wheatstone test network discussed in Section 3, addition of the Wheatstone bridge causes congestion, but adds to the reliability of the system in the event of an outage in one of the network boundary links. Through the example in Section 3, we have implicitly defined the congestion effect of a given line as being related to a single-element change in the network topology (namely, the addition of the Wheatstone bridge to the network). Similarly, we have defined the reliability effect as being associated with a multiple-element change in the network topology (the addition of the Wheatstone bridge and the loss of one of the transmission lines on the boundary). We will explicitly define congestion and reliability metrics in Section 5, but the simple example of the Wheatstone network makes clear that any thorough mathematical exploration of congestion and reliability requires us to decompose the effects of multiple changes to the network topology. To arrive at this decomposition, we adopt the DC power flow approximation and generalize the method of [21], which models network outages as changes to the system admittance matrix.

The decomposition method proposed by [21] and expanded upon in [22], was developed for the purpose of analyzing and ranking single-element contingencies. Thus, their modifications to the system admittance matrix (or just the susceptance matrix in the case of the DC power flow) amounted to network outages, modeled as $\Delta B_k = -B_k$ for an outage on the *k*th line. We generalize their calculations to include network additions, modeled as $\Delta B_k = +B_k$, and multiple-element topology changes.

A. *Decomposition of Single-Element Topology Changes*

We first review the case of a single-element topology change; an equivalent calculation was performed in [12, Ch. 3] for the case of the Wheatstone network presented in Section 3. The goal is to decompose the effect of a change in network topology so as to be able to write:

$$\mathbf{F}^{new} = \mathbf{F}^{old} + \Delta\mathbf{F}$$

where $\Delta\mathbf{F}$ represents the adjustment factor due to a single change in the network topology. The decomposition will show that $\Delta\mathbf{F}$ depends only on the system node-line adjacency matrix, the system susceptance matrix prior to the topology change, and the bus voltage angles prior to the topology change.

We start with the DC model for net injections:

$$\mathbf{P} = \mathbf{B\theta}. \tag{4}$$

We model a change in the network topology as an adjustment to the susceptance matrix $\mathbf{B}$. The adjustment takes the form $\Delta\mathbf{B}_k = \mathbf{A}\Delta\mathbf{B}_k^{diag}\mathbf{A}'$, where $\Delta\mathbf{B}_k^{diag}$ is a diagonal matrix whose entries are all equal to zero except the *k,k*th entry, which is equal to ΔB_k (thus, ΔB_k represents a single-element change in the network topology affecting line *k*). In the case of a line outage, we will have $\Delta B_k = -B_k$, and the dimensionality of $\Delta\mathbf{B}_k^{diag}$ will be ($NL \times NL$). In the case of a network addition, we will have $\Delta B_k = +B_k$. The dimensionality of $\Delta\mathbf{B}_k^{diag}$ will be ($NL{+}1 \times NL{+}1$), and the dimensionality of $\mathbf{A}$ will be ($NB \times NL{+}1$), to account for the new line in the system.[2] Whether $\Delta\mathbf{B}_k^{diag}$ represents a line outage or a line addition in the network, we note that $\Delta\mathbf{B}_k^{diag}$ has rank one, and thus $\Delta\mathbf{B}_k$ also has rank one.

Following the change to the network topology, the DC equations can be written as:

$$\mathbf{P} = (\mathbf{B} + \Delta\mathbf{B}_k)\boldsymbol{\theta}. \tag{5}$$

Since $\Delta\mathbf{B}_k$ has rank one, we can write $\Delta\mathbf{B}_k = \Delta B_k \mathbf{A}_k \mathbf{A}'_k$, where $\mathbf{A}_k$ is the *k*th column of the node-line incidence matrix $\mathbf{A}$.

Solving for $\boldsymbol{\theta}$, and using the matrix inversion lemma [23, 24], we get:

$$\begin{aligned} \boldsymbol{\theta}^{new} &= (\mathbf{B} + \Delta\mathbf{B}_k)^{-1}\mathbf{P} \\ &= \left(\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{A}_k(1 + \Delta B_k \mathbf{A}'_k\mathbf{B}^{-1}\mathbf{A}_k)^{-1} B_k \mathbf{A}'_k\mathbf{B}^{-1}\right)\mathbf{P} \\ \Rightarrow \boldsymbol{\theta}^{new} &= \boldsymbol{\theta}^{old} - \frac{\Delta B_k \mathbf{B}^{-1}\mathbf{A}_k\mathbf{A}'_k\boldsymbol{\theta}^{old}}{1 + \Delta B_k \mathbf{A}'_k\mathbf{B}^{-1}\mathbf{A}_k}, \end{aligned} \tag{6}$$

where we note that $1 + B_k\mathbf{A}'_k\mathbf{B}^{-1}\mathbf{A}_k$ is a scalar. We can rewrite this as:

$$\boldsymbol{\theta}^{new} = \boldsymbol{\theta}^{old} - \gamma\mathbf{r}, \tag{7}$$

where $\mathbf{r} = \mathbf{B}^{-1}\mathbf{A}_k$ and $\gamma = \frac{\Delta B_k \mathbf{A}'_k\boldsymbol{\theta}^{old}}{1 + \Delta B_k\mathbf{A}'_k\mathbf{r}}$. In the discussion below, it will be easier to define $\Delta\boldsymbol{\theta} = \boldsymbol{\theta}^{new} - \boldsymbol{\theta}^{old}$. Note that γ is a scalar and $\mathbf{r}$ is a vector of dimension ($NB \times 1$). Plugging (7) into the DC flow equations yields:

[2] We could also model upgrades to the existing topology in a similar fashion. A line upgrade (not a line addition) would be represented by $\Delta B_k = +B_k$, with $+B_k$ being the magnitude of the upgrade. The dimensionality of $\Delta\mathbf{B}^{diag}$ would continue to be ($NL \times NL$).

$$\begin{aligned}\mathbf{F}^{new} &= \mathbf{B}^{diag}\mathbf{A}'(\boldsymbol{\theta}^{old} - \gamma \mathbf{r}) \\ &= \mathbf{F}^{old} - \gamma \mathbf{B}^{diag}\mathbf{A}'\mathbf{r}.\end{aligned} \tag{8}$$

The equivalent of (8) for flow on a single line F_l, is:

$$F_l^{new} = \begin{cases} F_l^{old} + b_k^{-1}\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{old}, & l \neq k \\ \left(F_l^{old} - \Delta B_l \delta_l^{old}\right)(1 - b_l^{-1}\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_l), & l = k, \end{cases}$$

(9)

where $b_k = (\Delta B_k^{-1} + \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k)$, $b_l = (\Delta B_l^{-1} + \mathbf{A}_l'\mathbf{B}^{-1}\mathbf{A}_l)$.

B. Decomposition of Multiple-Element Topology Changes

We now consider the case of a change in the network topology affecting *n* distinct network elements. The network adjustment is again modeled as an adjustment of the form $\Delta\mathbf{B} = \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}'$, except we allow $\Delta\mathbf{B}^{diag}$ to have multiple non-zero entries ΔB_{kk}, $k = \{1,\ldots,n\}$ on the main diagonal (the off-diagonal entries are still assumed to be zero). If we assume that *n1* of these network changes represent line outages, and *n2* of these network changes represent new lines (so we have *n1 + n2 = n*), then the dimensionality of $\Delta\mathbf{B}^{diag}$ will be (*NL + n2* × *NL + n2*). For example, we would model the addition of line *k* and an outage on line $m \neq k$ with the (*NL+1* × *NL+1*) matrix:

$$\Delta\mathbf{B}^{diag} = \Delta\mathbf{B}_k^{diag} + \Delta\mathbf{B}_m^{diag} = \begin{bmatrix} 0 & & \cdots & & 0 \\ & \ddots & & & \\ & & +B_k & & \\ \vdots & & & \ddots & & \vdots \\ & & & & -B_m & \\ & & & & & \ddots \\ 0 & & \cdots & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 \\ & \ddots & \\ \vdots & +B_k & \vdots \\ & & \ddots \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ & \ddots & \\ \vdots & -B_m & \vdots \\ & & \ddots \\ 0 & \cdots & 0 \end{bmatrix}.$$

More generally, we can write an *n*-element network topology change as:

$$\Delta\mathbf{B}^{diag} = \Delta\mathbf{B}_1^{diag} + \Delta\mathbf{B}_2^{diag} + \cdots + \Delta\mathbf{B}_n^{diag}. \tag{10}$$

We note that the rank of $\Delta\mathbf{B}^{diag}$ is equal to *n*, the number of distinct adjustments to the network topology. Thus, the rank of $\Delta\mathbf{B} = \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}'$ is also equal to *n*.

The modified DC model for a multi-element topology change can be written as:

$$\mathbf{P} = (\mathbf{B} + \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}')\boldsymbol{\theta}. \tag{11}$$

Solving for the vector of bus voltage angles, we get:

$$\boldsymbol{\theta} = (\mathbf{B} + \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}')^{-1}\mathbf{P}. \tag{12}$$

Invoking the matrix inversion lemma, this can be written as:

$$\boldsymbol{\theta} = \left(\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{A}(\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\right)\mathbf{P}, \tag{13}$$

where $\mathbf{I}$ is the (*NL* × *NL*) identity matrix. The adjustment factor for the vector of bus angles becomes:

$$\Delta\boldsymbol{\theta} = -\mathbf{B}^{-1}\mathbf{A}(\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\Delta\mathbf{B}^{diag}\mathbf{A}'\boldsymbol{\theta}^{old}. \tag{14}$$

If we let $\mathbf{R}=\mathbf{B}^{-1}\mathbf{A}$, and $\mathbf{C} = (\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})$, we get:

$$\Delta\boldsymbol{\theta} = -\mathbf{R}\mathbf{C}^{-1}\mathbf{A}'\boldsymbol{\theta}^{old}. \tag{15}$$

Based on [25] and [26], we note that $\mathbf{C}$ is invertible if and only if $\mathbf{B} + \Delta\mathbf{B}^{diag}\mathbf{A}\mathbf{A}'$ is invertible. If we let $\boldsymbol{\Gamma} = \mathbf{C}^{-1}\mathbf{A}'\boldsymbol{\theta}^{old}$, then we can write:

$$\Delta\boldsymbol{\theta} = -\mathbf{R}\boldsymbol{\Gamma} = -\gamma_1\mathbf{r}_1 - \gamma_2\mathbf{r}_2 - \cdots - \gamma_{NL}\mathbf{r}_{NL}, \tag{16}$$

where γ_i is the *i*th element of $\boldsymbol{\Gamma}$, and $\mathbf{r}_i$ is the *i*th column of $\mathbf{R}$.

Thus, the voltage-angle adjustment factor for an *n*-element topology change is a linear combination of *n* single-element topology changes. As with the single-element analysis in (8), we can use the voltage-angle adjustment factor $\Delta\boldsymbol{\theta}$ in the DC flow equations to calculate the effect of a multi-element topology change on the network flows, as follows:

$$\begin{aligned}\mathbf{F}^{new} &= \left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'(\boldsymbol{\theta}^{old} - \mathbf{R}\boldsymbol{\Gamma}) \\ &= \mathbf{F}^{old} - \left(\Delta\mathbf{B}^{diag}\boldsymbol{\delta}^{old} - \sum_{i=1}^{NL}\gamma_i\left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'\mathbf{r}_i\right).\end{aligned} \tag{17}$$

Similarly, the effect on the net nodal injections can be calculated as:

$$\begin{aligned}\mathbf{P}^{new} &= \mathbf{A}\left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'(\boldsymbol{\theta}^{old} - \mathbf{R}\boldsymbol{\Gamma}) \\ &= \mathbf{P}^{old} - \left(\mathbf{A}\Delta\mathbf{B}^{diag}\boldsymbol{\delta}^{old} - \sum_{i=1}^{NL}\gamma_i\mathbf{A}\left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'\mathbf{r}_i\right).\end{aligned} \tag{18}$$

Equations (17) and (18) give us a way to decompose the effects of multiple changes to the network topology. In particular, we see that the adjustment to the vector of network flows or injections due to a multi-element change in the network topology is a linear combination of the effects of a series of single-element changes to the network topology.

V. Variation of Distribution Factors with Topological Changes

We can use the same method developed in Section 4 to derive an expression showing how the matrix of power transfer distribution factors (PTDF) changes following alterations to the network topology. We begin with the definition of the PTDF matrix:

$$\boldsymbol{\rho} = \mathbf{H}\mathbf{B}^{-1} \ , \tag{19}$$

Where $\mathbf{H}$ is a ($NL \times NB$) matrix defined by $\mathbf{H} = \mathbf{B}^{diag}\mathbf{A}'$. Thus, we can rewrite (19) as:

$$\boldsymbol{\rho} = \mathbf{B}^{diag}\mathbf{A}'(\mathbf{A}\mathbf{B}^{diag}\mathbf{A}')^{-1} \, . \tag{19'}$$

Following a change to the network topology represented by $\Delta\mathbf{B}^{diag}$, we have:

$$\boldsymbol{\rho}^{new} = (\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\left[\mathbf{A}(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\right]^{-1} . \tag{20}$$

Distributing terms in the inverse matrix in (20), we get:

$$\boldsymbol{\rho}^{new} = (\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\left[\mathbf{B} + \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}'\right]^{-1} . \tag{20'}$$

If we define $\Delta\mathbf{B} = -\mathbf{B}^{-1}\mathbf{A}(\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}$, as in (13), then we can rewrite (20) as:

$$\begin{aligned}\boldsymbol{\rho}^{new} &= (\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\left[\mathbf{B}^{-1} + \Delta\mathbf{B}^{-1}\right] \\ &= \left(\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\right) + \mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1} \\ &\qquad + \Delta\mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1} \\ &= \boldsymbol{\rho}^{old} + \Delta\boldsymbol{\rho},\end{aligned} \tag{20''}$$

where $\Delta\boldsymbol{\rho} = \mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1} + \Delta\mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1}$.

Equation (20'') describes the change in the distribution factors for a general (multi-element) topological network change. The equivalent of (20'') for the effect of a single-element topological change ΔB_k on the lth line is given by:

$$\boldsymbol{\rho}_l^{new} = \boldsymbol{\rho}_l^{old} + B_l\mathbf{A}'_l\,\Delta\mathbf{B}^{-1} + \Delta B_k(\mathbf{A}'_l\,\Delta\mathbf{B}^{-1} + A'_l\,\mathbf{B}^{-1}) \, , \tag{21}$$

where $\boldsymbol{\rho}_l$ is the lth row of the distribution matrix, and $\mathbf{A}_l$ is the lth row of the node-line adjacency matrix.

We can use the DC flow model $\mathbf{F}=\boldsymbol{\rho}\mathbf{P}$ to link the distribution-matrix decomposition in (21) with the flow-vector decomposition (17). We note that:

$$\begin{aligned}\mathbf{F}^{new} &= (\boldsymbol{\rho}^{old} + \Delta\boldsymbol{\rho})\mathbf{P} \\ &= \boldsymbol{\rho}^{old}\mathbf{P} + \Delta\boldsymbol{\rho}\mathbf{P} \\ &= \mathbf{F}^{old} + \Delta\boldsymbol{\rho}\mathbf{P}.\end{aligned} \tag{22}$$

Since $\mathbf{F}^{new} = \mathbf{F}^{old} + \Delta\mathbf{F}$, it must be true that $\Delta\mathbf{F} = \Delta\boldsymbol{\rho}\mathbf{P}$, or equivalently:

$$\Delta\boldsymbol{\rho}\mathbf{P} = -\left[\Delta\mathbf{B}^{diag}\boldsymbol{\delta}^{old} - \left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'\mathbf{R}\boldsymbol{\Gamma}\right] . \tag{23}$$

Thus, the effects of a n-element topology change on the PTDF matrix can be broken down into a linear combination of n single-element topology changes. Equations (23) and (17) can essentially be used interchangeably.

VI. Congestion and Reliability Metrics

The example of the Wheatstone network in Section 3 suggests two conceptual distinctions between reliability and congestion. The first is that reliability refers to the state or robustness of the network under contingencies, while congestion is a system attribute associated with normal operations. In developed countries with robust power grids, contingencies should not occur very often, but particular paths may become congested on a regular basis, particularly during times of peak demand. Thus, congestion events may be common and perhaps even predictable. Reliability problems (which lead to demand not being fully served), on the other hand, are the result of contingencies that should be random and rare.

The second distinction is that the presence of congestion may increase the cost of filling customer demands, while a lack of reliability in the system results in the physical inability of the system to meet these demands. Here, we are explicitly defining both reliability and congestion from the point of view of the customer. If a particular piece of equipment in the network is prone to outages, but these outages do not restrict the amount of customer demand that the network can meet, then the network is sufficiently reliable for the customer.

Before discussing the extent to which these two system attributes can be decomposed, we define some explicit congestion and reliability metrics. Our focus in this paper is the effects of topological changes on both reliability and congestion in the network. For concreteness, we will define a network topology Ω as a triple $\Omega = \{\mathbf{B}, \mathbf{A}, \mathbf{F}^{\max}\}$, where $\mathbf{B}$ is the susceptance matrix, $\mathbf{A}$ is the node-line adjacency matrix, and $\mathbf{F}^{\max}$ is the vector of capacity limits for the transmission lines. Thus, there is a distinction between our definition of a topology and the usual definition of a graph or network as a collection of nodes and edges.

Unless stated otherwise, we will be comparing different network topologies under the following set of assumptions:

Assumption 1: The profile of desired nodal demands $\mathbf{P}_d = (P_{d1}, \ldots, P_{dNB})$ does not change with a change in network topology.[3]
Assumption 2: Demand is completely price-inelastic.
Assumption 3: The grid operator treats all customers as if they valued reliability equally. One important implication of this assumption is that unserved energy at a given node is inconsistent with excess generation capacity at that node.[4]

A. Congestion Metrics

Congestion in an electric network occurs whenever a transmission line is loaded up to its predetermined capacity limit. Actual transmission lines have a variety of constraints representing different operating states of the system. These include voltage stability loading limits, thermal limits, and short-term contingency limits. These are normally measured in MVA to accommodate both real and reactive power constraints, but to simplify the discussion we will treat transmission lines as if they had a single steady-state capacity limit, stated in MW for real power.

A single line *l* becomes congested when $F_l = F_l^{\max}$. From (9), we can see that a single-element topology change affecting line $k \neq l$ will cause congestion if $F_l^{new} \geq F_l^{\max}$, or:

$$\begin{aligned} &F_l^{old} + b_k^{-1}\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{old} \geq F_l^{\max} \\ &\Rightarrow \Delta B_k^{-1} \geq \frac{\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{old}}{F_l^{\max} - F_l^{old}} - \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k. \end{aligned} \tag{24}$$

We can also express (24) using the network distribution matrix, as in Section 5. An equivalent condition to (24) is:

$$\Delta B_k > \frac{F_l^{\max} - F_l^{old} - B_l(\mathbf{A}'_l\,\Delta\mathbf{B}^{-1}\mathbf{B}\boldsymbol{\theta})}{\delta_l^{old} + \mathbf{A}'_l\,\Delta\mathbf{B}^{-1}\mathbf{B}\boldsymbol{\theta}}. \tag{24'}$$

For a given demand profile in the network, (24) can serve as the definition of congestion caused by a specific network link. That is, for a given vector of nodal demands, we define line *k* as causing congestion on line *l* (relative to the network topology without line *k*) if (24) is satisfied.

Congestion may impose a cost on the system if it results in generating units being dispatched out of merit order. We measure the congestion cost *CC* of a single-element or multi-element change to the network topology by taking the difference in the total cost of serving the load before and after the topological change:

$$CC = \sum_i C_i\left(P_{Gi,opt}^{\Omega_a}\right) - \sum_i C_i\left(P_{Gi,opt}^{\Omega_b}\right), \tag{25}$$

where Ω_a and Ω_b represent two distinct network topologies, and $P_{Gi,opt}^{\Omega_a}, P_{Gi,opt}^{\Omega_b}$ represent the optimal output of the *i*th generating unit with respect to network topologies Ω_a and Ω_b.[5]

Equation (24) provides an explicit condition under which a topological change will result in network congestion (for a given network demand profile). Measuring the congestion cost requires optimization of the entire system under different network topologies. Thus, the magnitude of the network congestion cost is largely an empirical matter, rather than a theoretical one.

B. Reliability Metrics

Current industry practice assesses network reliability using a number of different metrics, such as:

1. The $N - k$ criterion; whether the system can continue to provide uninterrupted service to customers in the face of a contingency in which *k* out of *N* pieces of equipment are lost, damaged, or otherwise disconnected from the network;
2. the Loss of Load Probability (LOLP), defined as the probability over some period of time that the network will fail to provide uninterrupted service to customers;
3. the Loss of Energy Expectation (LOEE) and Loss of Energy Probability (LOEP), defined as the expected amount and proportion of customer demand not served over some time frame. These are also known as the Unserved Energy Expectation or Probability.

Note that in the discussion of the Wheatstone network in Section 3, we measured reliability using the unserved energy metric. Factoring in the probability of a contingency on one of the boundary links transforms this metric into the unserved energy expectation. Parameterizing customer demands using either a value of lost load (VOLL) or an explicit demand curve further transforms this metric into the expected cost of unserved energy [27], [17].

In this paper, we consider reliability from the point of view of the customer. That is, a reliable system (with respect to a particular topological configuration) minimizes the need of the grid operator to physically ration the consumption of electric power. We will define the system as reliable (with respect to a given topology Ω) if $P_{Li}^{\Omega} = P_{di}^{\Omega}$ for all *i*, *i* = {1,…,*NB*},

[3] There is some foreshadowing in this assumption. We will define reliability (from the point of view of the customer) as the situation where there is no difference between the actual demand profile and the profile of desired demand.

[4] Another implication is that without some cost-based or value-based decision criteria, it may not always be clear which customer or customers should be blacked out in the event of a physical shortage of network transmission resources. We abstract from this decision problem and focus on the end state of the network; i.e., the sufficient system conditions for a customer to be blacked out.

[5] By "optimal output," we mean the level of output resulting from an optimization problem which minimizes the total cost of generation.

where P_{di} is the amount of power the customer at bus i desires to consume, and P_{Li} is the amount of power actually consumed. We measure the effect of a single- or multi-element topological change on reliability by measuring the amount by which unserved energy increases following the topological change.

Mathematically, we define $\Delta P_{Li} = P_{Li}^{\Omega_a} - P_{Li}^{\Omega_b}$ to be the difference between consumption at the ith bus with two different network topologies represented by Ω_a and Ω_b. Thus, ΔP_{Li} represents the (relative) reliability benefit of topology Ω_a over topology Ω_b. If $\Delta P_{Li} = 0$, then topology Ω_a does not provide a reliability benefit over topology Ω_b. If $\Delta P_{Li} > 0$, then topology Ω_a provides a reliability benefit over topology Ω_b (and vice versa if $\Delta P_{Li} < 0$). We will also define a variable ΔP_{Gi} in a similar fashion to ΔP_{Li}.

Our focus in this paper is on situations where the initial network topology Ω_a represents a reliable system. For these systems, the assumption of a constant demand profile excludes the case $\Delta P_{Li} > 0$.

As an example, consider the Wheatstone network discussed in Section 3. Assume that there are no line outages in the network. Without the Wheatstone bridge, total consumption at bus 4 is 100 MW. When the Wheatstone bridge is added to the network, total consumption is still 100 MW. Thus, for these two topological configurations, $\Delta P_{Li} = 0$. Next, consider the same two topologies, but assume an outage on link S_{24}. Assume further that the generator at bus 4 has a capacity limit for real power of 10 MW. With the Wheatstone bridge, total consumption at bus 4 is 100 MW. Without the Wheatstone bridge, consumption at bus 4 is 65 MW. Thus, for these two topological configurations, ΔP_{Li} = 35 MW (that is, the Wheatstone bridge provides a reliability benefit of magnitude 35 MW for a load of P_{L4} = 100 MW).

VII. Separability and Decomposition of Reliability and Congestion

In Sections 4 and 5, we developed a method to decompose the effects of multiple topology changes on network flows and net injections, in the context of the DC power flow approximation. In Section 6, we used this decomposition method to derive metrics for determining whether a given topological change will affect congestion and reliability in the network. In this section, we use the results of Sections 4 through 6 to derive some explicit topological conditions under which congestion and reliability will be independent.

The first two results demonstrate that (in the steady state) under the assumption of price-inelastic demand, a decrease in net injections following a change in the network topology, as in (18), is equivalent to saying that the topological change has decreased system reliability.

Result 1: Let $\Delta P_{Li} = P_{Li}^{\Omega_a} - P_{Li}^{\Omega_b}$ denote the change in steady-state demand at the ith bus following a change in topology from Ω_a to Ω_b, and let $\Delta P_i = P_i^{\Omega_a} - P_i^{\Omega_b}$ be the change in steady-state net injection at the ith bus, as defined in (18), where $P_i = P_{Li} - P_{Gi}$. Assume that Ω_a represents a reliable system. Suppose that bus i represents a net demander of real power. A necessary and sufficient condition that the change in the network topology does not harm reliability at bus i is that $\Delta P_i \geq 0$.

Proof of Result 1: Proving the result involves demonstrating that $\Delta P_i \geq 0 \Leftrightarrow \Delta P_{Li} = 0$. The first step of the proof is to show that $\Delta P_i \geq 0 \Rightarrow \Delta P_{Li} = 0$. We first consider the case where $\Delta P_i > 0$. It must also be true that $\Delta P_{Gi} > 0$ and $\Delta P_{Li} = 0$; thus the change in network topology from Ω_a to Ω_b does not result in any unserved energy.

Next, we consider the case $\Delta P_i = 0$. We note that $\Delta P_i = 0$ could arise in one of two cases. The first is where both $\Delta P_{Gi} = 0$ and $\Delta P_{Li} = 0$. In this case there is clearly no unserved energy arising from the topology change Ω_a to Ω_b. The second case is where both $\Delta P_{Gi} > 0$ and $\Delta P_{Li} > 0$, with $\Delta P_{Gi} = \Delta P_{Li}$. But $\Delta P_{Li} > 0$ violates either Assumption 1 or the assumption that Ω_a represents a reliable system. Thus, we see that $\Delta P_i \geq 0$ can only occur if $\Delta P_{Gi} \geq 0$ and $\Delta P_{Li} = 0$.

The next step of the proof is to show that $\Delta P_i \geq 0 \Rightarrow \Delta P_{Li} = 0$. Assume that $\Delta P_i \geq 0$. By Assumption 1, and since bus i represents a net demander, it must be the case that $\Delta P_{Gi} \leq 0$.

Result 2: Let ΔP_i and ΔP_{Li} be the same as defined in Result 1, and suppose that P_{Gi} = 0. Assume also that Ω_a represents a reliable system. Then $\Delta P_i < 0 \Leftrightarrow \Delta P_{Li} < 0$.

Proof of Result 2: Since $P_{Gi} = 0 \Rightarrow P_{Li} = P_i$, and because of Assumption 1, the proof is trivial.

A more interesting situation arises when bus i is a net demander, but it may be true that $P_{Gi} \neq 0$. In this case, reliability at the ith bus decreases following a topological network change if the available transfer capacity (ATC) into bus i declines. In the DC model, we can measure the (linear) ATC from bus s to bus i using the formula derived in [28] and [29];

$$ATC_{s,i} = \min_m \{T_m^s\}, \tag{26}$$

where $T_m^s = (F_m^{\max} - F_m) / \rho_{m,i}$, and $\rho_{m,i}$ denotes the distribution factor of line m with respect to a unit injection/withdrawal at bus i. From the system Load Bus Transfer Capability derived in [30], we can write the system-wide ATC into bus i as:

$$ATC_i = \max_s \min_m \{T_m^s\} \quad (26')$$

As with the congestion and reliability metrics discussed in Section 6, we will define ATC with respect to a given network topology Ω.

Following a change in the network topology from Ω_a to Ω_b, the new ATC is expressed as:

$$ATC_i^{\Omega_b} = \max_s \min_m \left\{ \frac{F_m^{\max} - (F_m^{\Omega_a} + \Delta F_m)}{\rho_{m,i} + \Delta\rho_{m,i}} \right\}. \quad (27)$$

Suppose that both (26') and (27) are satisfied by line *l*. Then the change in ATC is given by:

$$\Delta ATC = ATC_i^{\Omega_b} - ATC_i^{\Omega_a} = \frac{F_l^{\max} - (F_l^{\Omega_a} + \Delta F_l)}{\rho_{l,i} + \Delta\rho_{l,i}} - \frac{F_l^{\max} - F_l^{\Omega_a}}{\rho_{l,i}}. \quad (28)$$

Note that $\Delta ATC < 0$ if $ATC^{\Omega_a} > ATC^{\Omega_b}$, or equivalently if $\frac{\Delta F_l}{F_l^{\max} - F_l^{\Omega_a}} < \frac{\Delta\rho_{l,i}}{\rho_{l,i}}$. This condition says that $ATC^{\Omega_a} > ATC^{\Omega_b}$ if a network topology change causes a proportionally larger change in the distribution factor of the limiting line connected to bus *i* than in the flow across the limiting line connected to bus *i*. This leads to the following results:

Result 3: Let $\Delta ATC = ATC_i^{\Omega_b} - ATC_i^{\Omega_a}$ denote the change in steady-state demand at the *i*th bus following a change in topology from Ω_a to Ω_b, and suppose that bus *i* represents a net demander. Assume that Ω_a represents a reliable system, and that the change in topology from Ω_a to Ω_b represents a single-element topology change. Then reliability at bus *i* decreases following the network topology change if $\Delta B_k^{-1} > \frac{\mathbf{A}'_l \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})} \frac{\rho_{l,i}}{\Delta\rho_{l,i}} - \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k$.

Proof of Result 3: If bus *i* represents a net demander, then a decrease in reliability is equivalent to the condition that $ATC^{\Omega_a} > ATC^{\Omega_b}$. Assuming that (28) is satisfied, this occurs when:

$$\frac{\Delta F_l}{F_l^{\max} - F_l^{\Omega_a}} < \frac{\Delta\rho_{l,i}}{\rho_{l,i}} \quad (29)$$

Multiplying through and inserting (9) yields the result:

$$\frac{\mathbf{A}'_l \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{\Delta B_k^{-1} + \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k}\rho_{l,i} < \Delta\rho_{l,i}(F_l^{\max} - F_l^{\Omega_a})$$

$$\Leftrightarrow \frac{\mathbf{A}'_l \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{\Delta\rho_{l,i}(F_l^{\max} - F_l^{\Omega_a})}\rho_{l,i} - < \Delta B_k^{-1} + \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k \quad (30)$$

$$\Leftrightarrow \Delta B_k^{-1} > \frac{\mathbf{A}'_l \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})}\frac{\rho_{l,i}}{\Delta\rho_{l,i}} - \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k.$$

Note that a reversal of the inequality in (30) implies that the topological change from Ω_a to Ω_b increases the available transfer capacity into bus *i* and thus provides a reliability benefit.

From (17), we see that a condition analogous to (30) for a multi-element topological change can be written as:

$$\sum_{k\neq 0}\Delta B_k \mathbf{A}_l{}'\mathbf{r}_k < (F_l^{\max} - F_l^{\Omega_a})\frac{\Delta\rho_{l,i}}{\rho_{l,i}} + \sum_{k\neq 0} B_k \mathbf{A}_l{}'\mathbf{r}_k - \Delta B_l \delta_l^{\Omega_a}. \quad (31)$$

Equations (30) and (31) can also be used to prove an exception to Result 3 for the case where bus *i* is a net generator, rather than a net demander.

Result 4: Let ΔATC and ΔP_i be as defined in Results 1 through 3, and assume that Ω_a represents a reliable system. Assume also that bus *i* represents a net generator. Then, a change in topology from Ω_a to Ω_b cannot harm reliability at bus *i*.

Proof of Result 4: Results 1 through 3 demonstrate that a necessary condition for reliability at the *i*th bus being degraded by a change in network topology is that $\Delta ATC_i < 0$. The proof of Result 4 demonstrates that this is not a sufficient condition. If bus *i* is a net generator, then it must be the case that $P_{Gi}^{\max} > P_{di}$ since $P_{Gi}^{\max} \geq P_{Gi} > P_{Li} = P_{di}$ (we have $P_{Li} = P_{di}$ by Assumption 2). Under Assumption 1, P_{di} does not change following a change in topology from Ω_a to Ω_b. Assumption 3 states that excess generation capacity at bus *i* is inconsistent with unserved energy at bus *i*. Thus, even if available transfer capacity into bus *i* declines following the change in network topology, the fact that there is excess generation at bus *i* prevents reliability at bus *i* from being degraded.

Equations (30) and (31) are particularly revealing. The topological condition under which reliability will be improved or degraded looks similar to the topological condition under which congestion will occur in the network. The topological condition for congestion takes the form $\Delta B_k^{-1} > \alpha + \beta$, where $\alpha = \frac{\mathbf{A}'_l\, \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})}$ and $\beta = -\mathbf{A}_k\mathbf{'B}^{-1}\mathbf{A}_k$, as in (24). Meanwhile, the topological condition for reliability, as in (30), takes the form $\Delta B_k^{-1} > \alpha \frac{\rho_{l,i}}{\Delta\rho_{l,i}} + \beta$ (the condition for reliability under a multi-element topological change, as in (31), is similar). Thus, decomposing reliability from congestion and assessing the degree to which the two are independent, complementary, or represent tradeoffs, is largely a function of how the distribution matrix changes following a shift in the network topology.

Result 5: Under the DC power flow assumptions with linear ATC, for a single-element topology change affecting line *k*, we have the following:

(*i*) $\Delta\rho_{l,i} = \rho_{l,i} \Leftrightarrow \Delta B_k^{-1} = \frac{\mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1})}{\rho_{l,i} - B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1}}$

(*ii*) $\Delta\rho_{l,i} > \rho_{l,i} \Leftrightarrow \Delta B_k^{-1} < \frac{\mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1})}{\rho_{l,i} - B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1}}$

(*iii*) $\Delta\rho_{l,i} < \rho_{l,i} \Leftrightarrow \Delta B_k^{-1} > \frac{\mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1})}{\rho_{l,i} - B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1}}.$

Proof of Result 5: We will prove the result for case (*i*); the proofs for cases (*ii*) and (*iii*) are identical. From (21), we see that $\Delta\rho_{l,i} = B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1} + \Delta B_k \mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1})$, where $\Delta\mathbf{B}_i^{-1}$ and $\mathbf{B}_i^{-1}$ represent the *i*th column of $\Delta\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}$. Thus, for case (*i*) to hold, we must have:

$$\begin{aligned} &\rho_{l,i} = B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1} + \Delta B_k \mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1}) \\ &\Leftrightarrow \rho_{l,i} - B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1} = \Delta B_k \mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1}) \\ &\Leftrightarrow \Delta B_k^{-1} = \frac{\mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1})}{\rho_{l,i} - B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1}}. \end{aligned} \tag{32}$$

Cases (*ii*) and (*iii*) can be verified using the same procedure as in (32).

By now, it should be clear that congestion and reliability are rarely independent. Case (*i*) in Result 5 demonstrates a necessary and sufficient topological condition under which congestion and reliability cannot be independent. Cases (*ii*) and (*iii*) demonstrate topological conditions under which congestion and reliability may be independent. Case (*ii*) shows a situation in which congestion is a sufficient condition for a degradation in reliability, but not a necessary condition. Case (*iii*) shows a situation in which a degradation in reliability is a sufficient condition for congestion, but not a necessary condition.

The next result strengthens Case (*iii*) to show that any topological change that weakens reliability also causes congestion.

Result 6: Suppose that bus *i* represents a steady-state net demander. Assume that Ω_a represents a reliable system, and that the change in topology from Ω_a to Ω_b represents a single-element topology change. If $\Delta ATC_i < 0$ (that is, reliability at bus *i* is degraded by the change in network topology), and if (28) is satisfied, then the change in network topology results in congestion on at least one line connected to bus *i*.

Proof of Result 6: We will actually prove a slightly stronger statement than Result 6; we will show that the limiting line in the formula for ATC_i is the one that becomes congested. Under the assumption that reliability at bus *i* is degraded by the change in the network topology, Result 3 tells us that $\Delta B_k^{-1} > \frac{\mathbf{A}'_l\, \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})} \frac{\rho_{l,i}}{\Delta\rho_{l,i}} - \mathbf{A}_k\mathbf{'B}^{-1}\mathbf{A}_k$.

Since $\Delta ATC_i < 0$, we also know that $\Delta\rho_{l,i} > 0$ and thus $\rho_{l,i} / \Delta\rho_{l,i} > 0$.[6] We must also have:

$$\begin{aligned} \Delta B_k^{-1} &> \frac{\mathbf{A}'_l\, \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})} - \mathbf{A}_k\mathbf{'B}^{-1}\mathbf{A}_k \\ &> \frac{\mathbf{A}'_l\, \mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})} \frac{\rho_{l,i}}{\Delta\rho_{l,i}} - \mathbf{A}_k\mathbf{'B}^{-1}\mathbf{A}_k. \end{aligned} \tag{33}$$

The first inequality represents the condition for a network topology change to cause congestion, thus establishing the result. Thus, a change in network topology that degrades reliability into bus *i* also causes congestion on the set of transmission lines connected to bus *i*.

Result 6 states that if a topological change decreases reliability at bus *i*, then it causes congestion on the set of transmission lines connected to bus *i*. The topological change may also cause (or relieve) congestion in other portions of the system.

The discussion of the Wheatstone network in Section 3 demonstrated that a congestion-causing topological change could also improve reliability. As a generalization of this phenomenon, suppose that bus *i* represents a steady-state net demander. Consider four topologies Ω_a, Ω_b, Ω_c, and Ω_d. Assume that Ω_a represents a reliable system and that Ω_c represents a system with degraded reliability relative to Ω_a. Consider two single-element topology changes; the first is from Ω_a to Ω_b and the second is from Ω_c to Ω_d. Assume that the first change in network topology causes congestion (with respect to topology shift Ω_a to Ω_b). Both sets of single-element topology changes involve the same line *k*. Through repeated application of (24) and (30), the change in network

[6] Since bus *i* is a net demander, the limiting line in the ATC calculation should have $\rho_{l,i} > 0$.

topology improves reliability (with respect to topology shift Ω_c to Ω_d) if:

$$\frac{\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_a}}{(F_l^{\max} - F_l^{\Omega_a})} \geq \frac{\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{\Omega_c}}{(F_l^{\max} - F_l^{\Omega_c})} \frac{\rho_{l,i}}{\Delta\rho_{l,i}}. \qquad (34)$$

In the specific case of the Wheatstone network in Section 3, we have $\delta_k^{\Omega_a} = \delta_k^{\Omega_c}$ and $F_l^{\Omega_a} = F_l^{\Omega_c}$, so (34) reduces to:

$$\frac{\rho_{l,i}}{\Delta\rho_{l,i}} \leq 1\,, \qquad (34')$$

or, equivalently, using Result 5,

$$\Delta B_k^{-1} \leq \frac{\mathbf{A}_l\,'(\Delta\mathbf{B}_i^{-1} + \mathbf{B}_i^{-1})}{\rho_{l,i} - B_l \mathbf{A}_l\,'\Delta\mathbf{B}_i^{-1}}\,. \qquad (34'')$$

Since Braess' Paradox in power systems is not uniquely associated with the Wheatstone network, not every Wheatstone network will present a tradeoff between congestion and reliability (Blumsack and Ilić 2006). However, as the last result demonstrates, the presence of an embedded Wheatstone network is a necessary condition for a tradeoff between congestion and reliability to exist in the DC model with linear ATC.

Result 7: Suppose that bus *i* represents a steady-state net demander. Assume that Ω_a represents a reliable system. Suppose that a single-element topology change from Ω_a to Ω_b causes congestion. A single-element topology change representing a network addition (so that $\Delta B_k > 0$), can also provide a reliability benefit only if the network (under topologies Ω_a and Ω_b) contains an embedded Wheatstone bridge. A single-element topology change representing a network outage (so that $\Delta B_k < 0$) can provide a reliability benefit in a network without an embedded Wheatstone bridge if $-\Delta B_k \geq \frac{1}{2}\sum_{j\in\Omega_a} B_j$.

Proof of Result 7: An equivalent statement to Result 7 is that (24) and (30) cannot simultaneously hold in a network that does not contain an embedded Wheatstone bridge. [31] and [32] show that any undirected graph is either series-parallel (or radial) or contains an embedded Wheatstone network. Thus, proving Result 7 amounts to showing the conditions under which (24) and (30) cannot simultaneously hold in a series-parallel network.

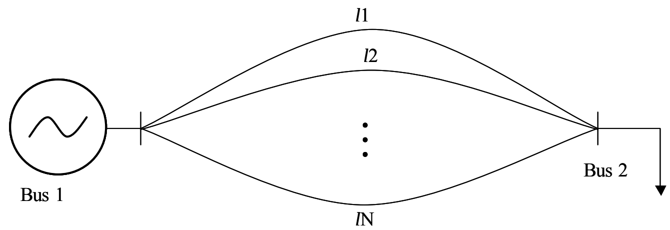

Figure 2: Parallel network with *N* lines and two buses.

We will first sketch out the proof for the lossless *N*-line parallel network shown in Figure 2. For systems without loop flows (but allowing for parallel flows), [33] show that there is at least one node in the system that is a pure "source," and at least one that is a pure "sink." In such a network, the distribution factors are given by:

$$\rho_{l,i} = \frac{B_l}{\sum_{j\in\Omega_a} B_j}\,. \qquad (35)$$

Thus, a single-element topology change ΔB_k will cause congestion if $\frac{\Delta B_k}{\sum_j B_j} > \frac{F_k^{\max}}{P_{Li}}$. Note that the series-parallel network admits the special form of (30) shown in (34''); thus a change in network topology will cause congestion but improve reliability if $\rho_{l,i}/\Delta\rho_{l,i} \leq 1$. The remainder of the proof will demonstrate that this results in a contradiction if the topology change represents a network addition, and results in a contradiction under certain conditions if the topology change represents a network outage.

If $\rho_{l,i}/\Delta\rho_{l,i} \leq 1$, then we must have:

$$\frac{B_l}{\sum_{j\in\Omega_a} B_j} \leq \frac{B_l}{\sum_{j\in\Omega_a} B_j + \Delta B_k} - \frac{B_l}{\sum_{j\in\Omega_a} B_j}$$

$$\Leftrightarrow \frac{1}{2}\sum_{j\in\Omega_a} B_j \geq \sum_{j\in\Omega_a} B_j + \Delta B_k \qquad (36)$$

$$\Leftrightarrow -\Delta B_k \geq \frac{1}{2}\sum_{j\in\Omega_a} B_j\,.$$

Since $B_j \geq 0$ for all *j* in a series-parallel network, (36) can only hold if $\Delta B_k < 0$; that is, if the topology change represents an outage on line *k* in the transmission system.

VIII. Discussion and Conclusions

In this paper, we have decomposed the congestion and reliability attributes of modifications to the transmission network. In particular, we focus on the question of whether a given change in the network topology causes or relieves congestion; and whether the topological change improves or degrades reliability. We find that the effects on congestion and reliability are rarely independent. Necessary assumptions and conditions for independence include (i) price-inelastic demand; (ii) all customers valuing reliability equally; (iii) a system operator that does not discriminate among consumers; and (iv) a network that does not contain any embedded Wheatstone bridges. Additions to series-parallel networks represent one set of circumstances where reliability and congestion may be truly independent; the effect of topology changes on net generator buses represents another.

Reliability and congestion are rarely independent system attributes. Network investments made with reliability in mind can have effects on congestion (positive or negative), and vice versa. Further, these externalities are highly variable with the network topology and the level of demand, and may not be captured in locational prices [13], [14]. One planning solution is thus to construct enough network capacity so that demand, even peak demand, looks small by comparison. Such a solution would be costly and also ignores the effects of future planned investments [34]. Transmission investment is a systems problem and should be treated as such.

The cost of congestion is easy to define. The interdependence of congestion and reliability, particularly when the two represent tradeoffs, suggests that reliability should be defined in terms of these costs and benefits. Particularly with the transition to markets in the new electric power industry, and with the increasing difficulty of siting and building new transmission lines, the value of reliability (either to the entire network or to individual customers) becomes a more important concept than physical reliability itself.

The four-bus test network in Section 3 provides a good framework for illustrating concepts, but there are important differences between the test network and actual highly interconnected networks. In real systems, to the degree that the matrix of distribution factors is nonsparse, resources throughout the network can affect the behavior of any particular sub-structure. An equally important issue is that while the distribution matrix may be approximately constant for a fixed network topology [35], network additions or outages change the distribution factors, and thus the total system cost and LMPs. Predicting the direction and magnitude of these changes can be difficult for complex networks.

The analysis in this paper provides a framework for assessing whether a new line will have congestion or reliability impacts (or both). Which effect dominates, and whether the impact is large or small, is ultimately an empirical matter.

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