VECTOR FIELD MODELS FOR NEMATIC DISCLINATIONS

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Abstract. In this paper, a model for defects that was introduced in [ZANV21] is studied. In the literature, the setting of most models for defects is the function space SBV (special bounded variation functions) (see, e.g., [CGO15, GMPS21]). However, this model regularizes the director field to be in the Sobolev space by adding a second field to incorporate the defect. A relaxation result in the case of fixed parameters is proven along with some partial compactness results and conjectures in the limit.

1. Introduction

The purpose of this paper is to initiate the rigorous mathematical analysis of a model of the dynamics of disclination line defects in nematics proposed in [ZANV21]. Here, we focus on energetic aspects of the model. Combined with the ideas presented in [AD14] and the demonstrations provided in [PAD15, ZZA$^+$17, ZANV21], which include static fields of straight $\pm\frac{1}{2}$ disclinations, their annihilation, the dissociation of closely bound pairs of straight disclinations, as well as static fields of disclination loops, the model can be considered as a thermodynamically consistent generalization of the Ericksen-Leslie (EL) model to account for the dynamics of disclination lines, with total energy that remains bounded in finite bodies in the presence of these line defects.

The model introduces an extra second-order tensor field, B, beyond the EL director field, k. This new field is to be physically thought of as a locally integrable realization, at the mesoscale, of the 'singular' part of the director gradient field (Dk) in the presence of line defects, singular when viewed at the macroscale. Thus, at the mesoscale, both the director gradient field and the new field are integrable - with this clear, we nevertheless refer to B as the 'singular part of the director distortion.' Notably, the field B is not a gradient, and this allows it to encode information on the topological charge of line defects through its curl. The evolution of the director field k continues to be obtained from the balance of angular momentum, as shown by Leslie [Les92], and the evolution of B follows from a conceptually simple conservation law for the topological charge of the line defects, which is tautological before the introduction of constitutive assumption for the disclination velocity, the latter deduced from consistency with the second law of thermodynamics. The introduction of dynamics based on such a conservation law, rooted in the kinematics of defect lines, is a conceptual departure from what is done for dynamics with the Landau-DeGennes Q

tensor model (see, [SV12, Mac92]), or Ericksen's variable degree of orientation model [Eri91]. In doing so, the model also makes connections to the dynamics of dislocation line defects in elastic solids [ZAWB15, AZA20], as well as their statics [AA20]. At the length scales where individual line defects are resolved, partial differential equations-based dynamical models arising from continuum mechanical considerations involve Newtonian and thermodynamic driving forces that include nonlinear combinations of entities that represent director distortions and the disclination density fields. This requires a minimum amount of regularity in these fields and hence it is essential to have a formulation that utilizes at least locally integrable functions, and our model is designed to be consistent with this requirement (of course, this does not preclude the question of studying limiting situations of such models when such functions tend to singular limits, modeling fields that have discontinuities, and singularities in the macroscopic limit).

The heuristics behind the energy (1.1) below for the prediction $\pm\frac{1}{2}$ line defects is as follows: the nonconvex potential W assigns vanishing energy cost when $|B| = 0, \frac{2}{\varepsilon\xi}$. This along with the elastic energy term $|Dk - B|^2$ assigns approximately vanishing elastic cost for pointwise values of the director gradient of the type $Dk \approx \frac{n-(-n)}{\varepsilon\xi} \otimes l$, where $0 < \varepsilon \leq 1$ and n, l are unit vectors, the latter representing the direction along which the jump of n occurs. Thus, if the layer in Fig. 1 were not to terminate, the energy cost would be minimal for a jump in the orientation of k by π across the layer. However, with a termination, $\operatorname{curl} B$ is non-zero near the termination, and were $\xi = 0$, it would be singular. In such a case Dk cannot annihilate B (regardless of $\xi = 0$, or not). The Euler Lagrange equation of a functional with just the energy density $|Dk - B|^2$ for admissible variations in k with B a specified field is, with $Dk - B =: e$, $\operatorname{div} e = 0$, and $\operatorname{curl} e = -\operatorname{curl} B$. For $\operatorname{curl} B$ a (mollified) Dirac supported at the layer termination, this produces the approximate elastic energy density field, given here by $|e|^2$, of some canonical line defects in 2 dimensions (screw dislocation in solids, the wedge disclination in nematics with unit vector constraint imposed, either exactly or approximately). Since $e = Dk$ outside the layer, we have the right director distribution (using the penalized unit vector constraint represented by the first term in (2.5) and a specified value of k at one point of the domain). Within the layer, but outside core, the director field k flips orientation by π radians, with a somewhat more involved distribution in the core.

1.1. **Main Results.** Let $\Omega := (-1, 1)^2$ be the domain occupied by a nematic liquid crystal in the plane. We consider the following energy for two fields $k \in W^{1,2}(\Omega; \mathbb{R}^2)$ and $B \in H_{\mathrm{curl}}(\Omega; \mathbb{R}^{2\times 2})$,

$$E_{\varepsilon,\xi}[k, B] := \int_\Omega \left[\frac{(|k| - 1)^2}{\varepsilon\xi^2} + |\nabla k - B|^2 + \varepsilon\xi^2 |\operatorname{curl} B|^2 + \frac{1}{\varepsilon\xi^2} W(\varepsilon\xi|B|) \right] dx, \tag{1.1}$$

$W : [0, \infty) \to [0, \infty)$ is a nonconvex double-well continuous potential with wells at 0 and 2.

We first consider the case of when ε, ξ are fixed. We obtain an integral representation of the relaxation of the energy to be

$$\bar{E}_{\varepsilon,\xi}[k, B] := \int_\Omega \left[\frac{(|k|-1)^2}{\varepsilon\xi^2} + |\nabla k - B|^2 \right] dx + \int_\Omega \left[\varepsilon\xi^2 |\operatorname{curl} B|^2 + \frac{1}{\varepsilon\xi^2} Q(W(|\cdot|))(\varepsilon\xi B) \right] dx. \tag{1.2}$$

Here, Qf denotes the quasiconvex envelope of f. We also conjecture that $Q(W(|\cdot|))(p + \nabla z(x)) = W^{**}(p + \nabla z(x))$ due to the radial symmetry.

Afterwards, we consider the particular case of when B is supported in a layer as in Figure 1. We consider the limit $\varepsilon \to 0$ with $\xi > 0$ fixed. After a change of variables as in the dimension reduction problems, we prove a compactness theorem for the rescaled fields

Theorem 1.1. *Let $k_n, \tilde{B}_n$ with uniformly bounded energy $E_{\varepsilon_n,\xi}[k, \tilde{B}] = E^{bulk}_{\varepsilon_n,\xi}[k_n] + E^{layer}_{\varepsilon_n,\xi}[\tilde{k}_n, \tilde{B}_n]$. Then $k_n \chi_{L_{\varepsilon_n,\xi}} \to k$ strongly in $L^2(\Omega; \mathbb{R}^2)$ where $k \in SBV(\Omega; \mathbb{R}^2)$ and is S^1 valued. Furthermore, the jump set of k is precisely $\overline{L^0_\xi}$.*

In the layer, we can generate a rescaled k_n which are denoted $\tilde{k}_n$. We will have the convergences

$$\begin{aligned} &\tilde{k}_n \rightharpoonup \tilde{k} \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^2), \\ &\tilde{B}_n \rightharpoonup B \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^{2\times 2}), \\ &\operatorname{curl}_{\varepsilon_n} \tilde{B}_n \rightharpoonup \alpha \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^2), \end{aligned}$$

for some $\tilde{k} \in L^2(L_\xi; \mathbb{R}^2)$, $\alpha \in L^2(L_\xi; \mathbb{R}^2)$, and $B := \begin{bmatrix} 0 & \partial_2 \tilde{k}_1 \\ 0 & \partial_2 \tilde{k}_2 \end{bmatrix}$.

Furthermore, Define $[\tilde{k}] : (-\xi, 1) \to \mathbb{R}$ as $[\tilde{k}](x_1) := \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \partial_2 \tilde{k} \, dx_2$.. This allows us to generate compatibility conditions between $\tilde{k}$ and α to be

$$[\tilde{k}](s) = \int_{-\xi}^{s} \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \alpha \, dx.$$

$$|[\tilde{k}](1)| = 2$$

Furthermore, we show how a portion of the limiting energy can be considered to be similar to a Modica-Mortola functional for the vertical jump of the director with a transition layer on the order of the core length ξ.

There are many open questions stemming from this work. Foremost, is the integral representation of a precise limiting energy for the case $\varepsilon \to 0$ with $\xi > 0$ fixed. It is also possible to consider the case of $\xi \to 0$ at various rates compared to $\varepsilon \to 0$. However, any of the cases of $\xi \to 0$ will be complicated by the need to rescale the energy by $\log \xi$, which leads to a delicate Ginzbug-Landau type problem.

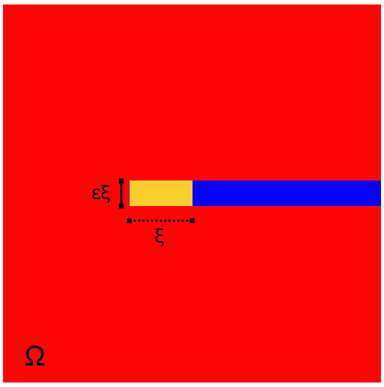

Figure 1. A representation of the domain and the layer where the discontinuity is supported

2. Model

Let $\Omega := (-1,1)^2$ be the domain of a liquid crystal in the plane. We assume the defect is at the origin, and the surface of discontinuity is within a layer $L_{\varepsilon,\xi} := (-\xi, 1) \times (-\frac{\varepsilon\xi}{2}, \frac{\varepsilon\xi}{2})$ with parameters $\varepsilon, \xi > 0$. In physical terms, ξ would be considered the core length of the crystalline defect and ε is a parameter which determines the thickness of the defect layer $L_{\varepsilon,\xi}$.

Let $W^{1,2}(\Omega;\mathbb{R}^2)$ denote the usual Sobolev space, and we designate by $H_{\text{curl}}(\Omega;\mathbb{R}^{2\times 2})$ the space of L^2 matrix valued tensors, whose row-wise distributional curl is also in L^2. Under this setting, we consider the following energy for two fields $k \in W^{1,2}(\Omega;\mathbb{R}^2)$ and $B \in H_{\text{curl}}(\Omega;\mathbb{R}^{2\times 2})$,

$$E_{\varepsilon,\xi}[k,B] := \int_\Omega \left[\frac{(|k|-1)^2}{\varepsilon\xi^2} + |\nabla k - B|^2 + \varepsilon\xi^2 |\operatorname{curl} B|^2 + \frac{1}{\varepsilon\xi^2} W(\varepsilon\xi|B|) \right] dx,$$

$W : [0,\infty) \to [0,\infty)$ is a nonconvex continuous potential satisfying the following coercivity and growth properties,

$$\frac{1}{C}|x|^2 - C \le W(x) \le C(1+|x|^2) \quad \text{for} \quad x \in [0,\infty) \tag{2.1}$$

for some $C > 0$, and

$$\{x \in [0,+\infty) : W(x) = 0\} = \{0, 2\}. \tag{2.2}$$

In order to impose the physical phenomena that a defect only affects the crystal in a short range, we impose the conditions

$$B = 0 \qquad \text{in} \quad \Omega \setminus L_{\varepsilon,\xi}, \tag{2.3}$$

$$Bt = 0 \qquad \text{on} \quad \partial L_{\varepsilon,\xi} \setminus \partial\Omega, \tag{2.4}$$

where t is the tangent vector to the boundary point. The condition (2.4) is necessary in order to ensure that $B \in H_{\text{curl}}(\Omega;\mathbb{R}^{2\times 2})$. This is because that functions in $H_{\text{curl}}(\Omega;\mathbb{R}^{2\times 2})$ have a well-defined tangential trace which must be matched to the condition (2.3) [BF13].

In this paper we are primarily concerned with $\pm\frac{1}{2}$ disclinations, and these satisfy the constraint

$$\left|\int_{\Omega} \operatorname{curl} B \, dx\right| = 2.$$

This is a model constraint which enforces the fact that a disclination must exist in the domain. By Stokes' theorem it is consistent with a layer field the form $B = \frac{n-(-n)}{\varepsilon\xi} \otimes l$ specified in a layer of width $\varepsilon\xi$ with normal in the direction l and n a unit vector, the layer running from the boundary of the domain and terminating in the interior. Since Dk must attempt to annihilate B to the extent possible to reduce elastic energy, this can be achieved by the field k flipping by π across such a layer, reflecting a strength $\pm\frac{1}{2}$ disclination represented by a vector field (also see heuristics in Sec. 1).

Thus, the model can be written as

$$E_{\varepsilon,\xi}[k,B] := \int_{\Omega \setminus L_{\varepsilon,\xi}} \left[\frac{(|k|-1)^2}{\varepsilon\xi^2} + |\nabla k|^2 \right] dx$$

$$+ \int_{L_{\varepsilon,\xi}} \left[\frac{(|k|-1)^2}{\varepsilon\xi^2} + |\nabla k - B|^2 + \varepsilon\xi^2 |\operatorname{curl} B|^2 + \frac{1}{\varepsilon\xi^2} W(\varepsilon\xi|B|) \right] dx, \quad (2.5)$$

$$\left|\int_{L_{\varepsilon,\xi}} \operatorname{curl} B \, dx\right| = 2, \quad (2.6)$$

$$Bt = 0 \qquad \text{on} \quad \partial L_{\varepsilon,\xi} \setminus \partial\Omega.$$

The ultimate goal is to investigate the limits as $\varepsilon, \xi \to 0$ at various rates. Here, we give an integral representation result for the relaxation of the energy with $\varepsilon, \xi > 0$ fixed, and, in particular, we provide partial compactness results and conjectures in the case when $\varepsilon \to 0$ with $\xi > 0$ fixed.

We seek to study the convergence of the functional (2.5) in the sense of $\Gamma-$convergence.

Definition 2.1. Given a metric space (X, d), let $F_n : X \to [0, \infty]$ be a sequence of functionals.We say that F_n Γ-converge to $F_0 : X \to [0, \infty]$ with respect to the metric d if the following two conditions hold:

(1) (Liminf Inequality) For every $u \in X$ and for every sequence $\{u_n\} \subset X$ such that $u_n \to u$ in the metric d, we have

$$F_0(u) \le \liminf_{n\to\infty} F_n(u_n).$$

(2) (Recovery Sequence) For every $u \in X$, there exists $\{u_n\} \subset X$ such that $u_n \to u$ in the sense of the metric d, and it recovers the energy, i.e.,

$$\limsup_{n\to\infty} F_\varepsilon(u_\varepsilon) = F_0(u).$$

3. Relaxation for fixed ε, ξ

Firstly, we study some properties of this energy with $\varepsilon, \xi > 0$ fixed. It is not clear that minimizers to the original problem exist nor is it clear what the value of the infimum is. Thus, in order to apply the direct method of calculus of variations, we will consider the lower semicontinuous envelope of the functional. Specifically, we will obtain an integral representation of this relaxation of energy in dimension 2 or 3. This restriction is to enable us to use the results for the Helmholtz decomposition and the corresponding Sobolev spaces in [BF13].

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$, with $N = 2$ or 3, be an open, bounded set with Lipschitz continuous boundary, and let E be defined in* (2.5). *The relaxation of E is given by*

$$\bar{E}_{\varepsilon,\xi}[k, B] := \inf \left\{ \liminf_{n\to\infty} E_{\varepsilon,\xi}[k_n, B_n] : (k_n, B_n) \to (k, B) \right\},$$

where the convergence is such that $k_n \to k$ and $B_n \rightharpoonup B$ in L^2. This relaxation has the integral representation, for every $k \in W^{1,2}(\Omega; \mathbb{R}^N)$ and $B \in H_{curl}(\Omega; \mathbb{R}^{N\times N})$,

$$\begin{aligned}\bar{E}_{\varepsilon,\xi}[k, B] := &\int_\Omega \left[\frac{(|k| - 1)^2}{\varepsilon\xi^2} + |\nabla k - B|^2 \right] dx \\ &+ \int_\Omega \left[\varepsilon\xi^2 |\operatorname{curl} B|^2 + \frac{1}{\varepsilon\xi^2} Q(W(|\cdot|))(\varepsilon\xi B) \right] dx. \end{aligned} \tag{3.1}$$

Here, Qf denotes the quasiconvex envelope of f..

To prove this result, we will introduce an intermediate functional, which will be related to (3.1) through the Helmholtz decomposition of B. We denote the space of the divergence-free fields given in the Helmholtz decomposition as

$$\mathcal{C} := \{u \in H_{\text{curl}}(\Omega; \mathbb{R}^{N\times N}) : \quad \operatorname{div} u = 0, \quad u \cdot \nu = 0 \quad \text{on } \partial\Omega\}. \tag{3.2}$$

Define the functional $I : (L^2(\Omega; \mathbb{R}^N))^2 \times L^2(\Omega; \mathbb{R}^{N\times N}) \to [0, +\infty]$ by

$$\begin{aligned} I_{\varepsilon,\xi}[\tilde{k}, z, p] := &\int_\Omega \left[\frac{(|\tilde{k} + z| - 1)^2}{\varepsilon\xi^2} + |\nabla \tilde{k}|^2 \right] dx \\ &+ \int_\Omega \left[|p|^2 + \varepsilon\xi^2 |\operatorname{curl} p|^2 + \frac{1}{\varepsilon\xi^2} W(\varepsilon\xi |\nabla z + p|) \right] dx, \end{aligned} \tag{3.3}$$

if $(\tilde{k}, z, p) \in \mathcal{X}$, and $+\infty$ otherwise. Here,

$$\mathcal{X} := W^{1,2}(\Omega; \mathbb{R}^N) \times \left(W^{1,2}(\Omega; \mathbb{R}^N) \cap \left\{ \int_\Omega z \, dx = 0 \right\} \right) \times \mathcal{C}. \tag{3.4}$$

First, we investigate compactness for this new functional $I_{\varepsilon,\xi}$. We write it in a more general setting than in the model though we still need $N = 2, 3$ in order to directly use the results in [BF13].

Lemma 3.2 (Compactness of $I_{\varepsilon,\xi}$)**.** *Let $\Omega \subset \mathbb{R}^N$, with $N = 2$ or 3, be an open, bounded set with Lipschitz continuous boundary. Consider a sequence*

$\{(\tilde{k}_n, z_n, p_n)\}$ such that $\sup_{n\in\mathbb{N}} I_{\varepsilon,\xi}[\tilde{k}_n, z_n, p_n] \le C$. Then there is $(\tilde{k}, z, p) \in (W^{1,2}(\Omega;\mathbb{R}^N))^2 \times \mathcal{C}$ such that up to subsequence (not relabeled)

$$\begin{aligned} \tilde{k}_n &\rightharpoonup \tilde{k} \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^N), \\ z_n &\rightharpoonup z \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^N), \\ p_n &\rightharpoonup p \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^{N\times N}). \end{aligned}$$

In particular, we can assume

$$(\tilde{k}_n, z_n, p_n) \to (\tilde{k}, z, p) \quad \text{strongly in } (L^2(\Omega;\mathbb{R}^N))^2 \times L^2(\Omega;\mathbb{R}^{N\times N}).$$

Proof. The key is the inequality (see [BF13])

$$\|p_n\|_{W^{1,2}} \le C(\Omega)\left(\|p_n\|_{L^2} + \|\operatorname{div} p_n\|_{L^2} + \|\operatorname{curl} p_n\|_{L^2}\right). \tag{3.5}$$

From the definition of $\mathcal{C}$ in (3.2) and $I_{\varepsilon,\xi}$ in (3.3), we can conclude that, in fact, we have $\|p_n\|_{W^{1,2}(\Omega,\mathbb{R}^{N\times N})} \le C < \infty$, with convergence following from weak compactness. By (2.1) and (3.3), we have $\|\nabla z_n\|_{L^2(\Omega)} \le C < \infty$, and the desired convergence follows from Poincaré's inequality because $\int_\Omega z\,dx = 0$. Finally, the uniform bound of the energy (3.3) implies a uniform bound on $\|\nabla\tilde{k}_n\|_{L^2(\Omega)}$. Combining this with control of $\|z_n\|_{L^2(\Omega;\mathbb{R}^N)}$ and $\|\tilde{k}_n\|_{L^2(\Omega;\mathbb{R}^N)}$, shows $\|\tilde{k}_n\|_{W^{1,2}(\Omega;\mathbb{R}^N)} \le C < \infty$.

To conclude strong convergence in L^2, we apply the Rellich-Kondrachov compactness theorem. □

We will prove the relaxation of Theorem 3.1 using the technique of $\Gamma-$convergence.

Now we define our candidate limiting functional to be

$$\begin{aligned} \bar{I}_{\varepsilon,\xi}[\tilde{k}, z, p] := &\int_\Omega \left[\frac{(|\tilde{k}+z|-1)^2}{\varepsilon\xi^2} + |\nabla\tilde{k}|^2\right] dx \\ &+ \int_\Omega \left[|p|^2 + \varepsilon\xi^2|\operatorname{curl} p|^2 + \frac{1}{\varepsilon\xi^2} Q(W(|\cdot|))(\varepsilon\xi(\nabla z + p))\right] dx, \end{aligned} \tag{3.6}$$

if $(\tilde{k}, z, p) \in \mathcal{X}$, and $+\infty$ otherwise.

We note that this functional is similar to the original functional with W replaced by $Q(W(|\cdot|))$. In order to prove that this is indeed the correct limiting functional, we will first show that the liminf inequality in the $\Gamma-$convergence conditions (see Def 2.1 (1)) is satisfied.

Lemma 3.3 (Liminf of I)**.** *Let $\Omega \subset \mathbb{R}^N$, with $N = 2$ or 3, be an open, bounded set with Lipschitz continuous boundary, and assume that W satsfies (2.1) and (2.2). For all sequences such that*

$$(\tilde{k}_n, z_n, p_n) \to (\tilde{k}, z, p) \quad \text{strongly in } (L^2(\Omega;\mathbb{R}^N))^2 \times L^2(\Omega;\mathbb{R}^{N\times N}),$$

we have

$$\bar{I}_{\varepsilon,\xi}[\tilde{k}, z, p] \le \liminf_{n\to\infty} I_{\varepsilon,\xi}[\tilde{k}_n, z_n, p_n].$$

Proof. We define the function $h(p,\eta) := W(|p+\eta|)$ and $\mathcal{Q}h(p,\cdot)$ to be the greatest quasiconvex function below $h(p,\cdot)$. We **claim**

$$\begin{aligned}\int_\Omega Q(W(|\cdot|))(p+\nabla z)\,dx &= \int_\Omega \mathcal{Q}h(p,\nabla z)\,dx\\ &\le \liminf_{n\to\infty}\int_\Omega h(p_n,\nabla z_n) = \liminf_{n\to\infty}\int_\Omega W(|\nabla z_n + p_n|). \quad (3.7)\end{aligned}$$

The first equality is easy to see because we note that there is a translational symmetry to h which gives the equality $\mathcal{Q}h(p,\nabla z) = \mathcal{Q}h(0,\nabla(px+z(x))) = Q(W(|\cdot|))(p+\nabla z)$.

So it suffices to prove the lower semicontinuity portion of the claim. The proof consists of a blow up argument. By taking an appropriate subsequence, we may assume that the liminf is actually a limit and is finite. Define the measures μ_n and ν_n by

$$\mu_n := h(p_n,\nabla z_n)\mathcal{L}^N|_\Omega, \qquad \nu_n := \|\nabla z_n\|^2\mathcal{L}^N|_\Omega. \quad (3.8)$$

As the energy is bounded, $\mu_n \overset{*}{\rightharpoonup} \lambda$. By Lemma 3.2, z_n are bounded in $W^{1,2}$ and $\nu_n \overset{*}{\rightharpoonup} \nu$. To prove the claim, it suffices to show for $x_0\in\Omega$ a.e., we have the bound

$$\lim_{\delta\to 0}\frac{\lambda(\overline{Q(x_0,\delta)})}{\delta^N} \ge \mathcal{Q}h(p(x_0),\nabla z(x_0)), \quad (3.9)$$

where $Q(x_0,\delta)$ is a cube centered at x_0 with sides of length $\delta>0$.

Let $x_0\in\Omega$ be a Lebesgue point of p_n, z_n, λ, ν and satisfy

$$\lim_{\delta\to 0}\frac{1}{\delta^{N+2}}\int_{Q(x_0,\delta)} |z(x)-(z(x_0)+\nabla z(x_0)(x-x_0))|^2\,dx = 0. \quad (3.10)$$

We have

$$\begin{aligned}\frac{d\lambda}{d\mathcal{L}^N}(x_0) = \lim_{\delta\to 0}\frac{\lambda(\overline{Q(x_0,\delta)})}{\delta^N} &\ge \limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{\delta^N}\int_{Q(x_0,\delta)} h(p_n,\nabla z_n)\,dx\\ &= \limsup_{\delta\to 0}\limsup_{n\to\infty}\int_{Q(0,1)} h(p_{n,\delta},\nabla z_{n,\delta})\,dy,\end{aligned} \quad (3.11)$$

where we have used the change of variables $y=\frac{(x-x_0)}{\delta}$ and defined

$$\begin{aligned}p_{n,\delta}(y) &:= p_n(x_0+\delta y),\\ z_{n,\delta}(y) &:= \frac{z_n(x_0+\delta y)-z(x_0)}{\delta}.\end{aligned}$$

As x_0 is a Lebesgue point of ν, by arguing as in (3.11), we have

$$\frac{d\nu}{d\mathcal{L}^N}(x_0) \ge \limsup_{\delta\to 0}\limsup_{n\to\infty}\int_\Omega \|\nabla z_{n,\delta}\|^2\,dy.$$

Defining $z_0(y) := \nabla z(x_0)y$, we use (3.10) and properties of x_0 to conclude

$$\lim_{\delta\to 0}\lim_{n\to\infty}\Big(\|z_{\epsilon,\delta}-z_0\|_{L^2(\Omega;\mathbb{R}^N)} + \|p_{\epsilon,\delta}-p(x_0)\|_{L^2(\Omega;\mathbb{R}^{N\times N})}\Big) = 0.$$

We now diagonalize to find a sequences $p_n := p_{n,\delta_n} \to p(x_0)$ in L^2 and $z_n := z_{n,\delta_n} \rightharpoonup z_0$ in $W^{1,2}$ such that

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \lim_{n\to\infty} \int_{Q(0,1)} h(p_n, \nabla z_n)\, dy.$$

We go down to the quasi-convex envelope of h to find

$$\frac{d\lambda}{d\mathcal{L}^N}(x_0) \geq \lim_{n\to\infty} \int_{Q(0,1)} \mathcal{Q}h(p_n, \nabla z_n)\, dy.$$

As W satisfies (2.1), $\mathcal{Q}h$ is 2-Lipschitz, and specifically satisfies the bound

$$\begin{aligned}\mathcal{Q}h(p(x_0), \nabla z_n) \leq &\mathcal{Q}h(p_n, \nabla z_n) \\ &\qquad + C(1 + \|\nabla z_n\| + \|p_n\| + \|p(x_0)\|)\|p_n - p(x_0)\| \\ \leq &\mathcal{Q}h(p_n, \nabla z_n) + \eta(1 + \|\nabla z_n\|^2 + \|p_n\|^2 + \|p(x_0)\|^2) \\ &\qquad + C(\eta)\|p_n - p(x_0)\|^2.\end{aligned}$$

As $p_n \to p(x_0)$ in L^2, we have

$$\begin{aligned}&\lim_{n\to\infty} \int_{Q(0,1)} \mathcal{Q}h(p(x_0), \nabla z_n)\, dy \\ &\leq \lim_{n\to\infty} \int_{Q(0,1)} \mathcal{Q}h(p_n, \nabla z_n)\, dy \\ &\qquad + \eta \left(C(p(x_0)) + \sup \left\{ \|\nabla z_n\|^2_{L^2(Q(0,1);\mathbb{R}^N)} + \|p_n\|^2_{L^2(Q(0,1);\mathbb{R}^{N\times N})} \right\} \right).\end{aligned}$$

As $\eta > 0$ is arbitrary, we conclude

$$\lim_{n\to\infty} \int_{Q(0,1)} \mathcal{Q}h(p(x_0), \nabla z_n)\, dy \leq \lim_{n\to\infty} \int_{Q(0,1)} \mathcal{Q}h(p_n, \nabla z_n)\, dy.$$

From here, a standard relaxation result gives the bound

$$\lim_{n\to\infty} \int_{Q(0,1)} \mathcal{Q}h(p(x_0), \nabla z_n)\, dy \geq \mathcal{Q}h(p(x_0), \nabla z(x_0)),$$

proving the claim.

The lower semi-continuity of the L^2 norm shows that

$$\liminf_{n\to\infty} \int_\Omega \left(|\nabla \tilde{k}_n|^2 + |\mathrm{curl}\, p_n|^2 \right) dx \geq \int_\Omega \left(|\nabla \tilde{k}|^2 + |\mathrm{curl}\, p|^2 \right) dx, \tag{3.12}$$

with strong convergence of z_n and k_n in L^2 giving that

$$\lim_{n\to\infty} \int_\Omega (|\tilde{k}_n + z_n| - 1)^2\, dx = \int_\Omega (|\tilde{k} + z| - 1)^2\, dx. \tag{3.13}$$

Combining (3.7), (3.12), and (3.13), the lemma is proven. □

In order to complete the integral representation, we show the existence of a recovery sequence (see Def 2.1(2)).

Lemma 3.4 (Recovery Sequence of $\bar{I}_{\varepsilon,\xi}$)**.** *If $(\tilde{k}, z, p) \in \mathcal{X}$, then there exists a sequence $\{(\tilde{k}_n, z_n, p_n)\}$ such that*

$$(\tilde{k}_n, z_n, p_n) \to (\tilde{k}, z, p) \quad \textit{strongly in } (L^2(\Omega;\mathbb{R}^N))^2 \times L^2(\Omega;\mathbb{R}^{N\times N}) \tag{3.14}$$

$$\limsup_{n\to\infty} I_{\varepsilon,\xi}[\tilde{k}_n, z_n, p_n] \leq \bar{I}_{\varepsilon,\xi}[\tilde{k}, z, p]. \tag{3.15}$$

Proof. First, note that $\tilde{k}, p$ are also admissible in the original energy. Thus, we can simplify further by just taking $p_n \equiv p$ and $\tilde{k}_n = \tilde{k}$. First note that for any sequence converging as in (3.14), we must have:

$$\int_\Omega \left[(|\tilde{k} + z_n| - 1)^2 + |p|^2\right] \, dx \to \int_\Omega \left[(|\tilde{k} + z| - 1)^2 + |p|^2\right] \, dx \tag{3.16}$$

Now define the function $g : \Omega \times \mathbb{R}^{N\times N} \to [0, \infty)$ by $g(x, \eta) = W(|p(x) + \eta|)$. Note that as W is continuous and p is measurable, we have the property that g is Caratheodory and has polynomial growth in ψ. Then again by standard relaxation results, we can find a sequence, $\{z_n\} \subset H^1(\Omega;\mathbb{R}^N)$ such that

$$z_n \to z \quad \text{strongly in } L^2(\Omega;\mathbb{R}^N), \tag{3.17}$$

$$\limsup_{n\to\infty} \int_\Omega g(x, \nabla z(x)) \, dx \leq \int_\Omega \mathcal{Q}g(x, \nabla z(x)) \, dx \tag{3.18}$$

where $\mathcal{Q}g(x, \eta)$ denotes the quasiconvex envelope of $g(x, \cdot)$ (in the sense of greatest quasiconvex function below $g(x, \cdot)$). Again, by a similar argument of translational symmetry as in the liminf, we have that $\mathcal{Q}g(x, \nabla z(x)) = Q(W(|\cdot|))(p + \nabla z(x))$. Combining (3.16) and (3.18), we obtain the desired result. □

Combining the last two lemmas, we have proved Theorem 3.1 that the relaxation of the energy is given by (3.1).

4. Effective Energies

In the following, we will be concerned with the case where $N = 2$, $\varepsilon \to 0$, and $\xi > 0$ is fixed. $L_{\varepsilon,\xi}$ is becoming thin in the limit. As it is typical in dimension reduction problems, we perform the change of variables

$$\tilde{k}(x_1, x_2) := k\left(x_1, \varepsilon x_2\right), \tag{4.1}$$

$$\tilde{B}(x_1, x_2) := \varepsilon B\left(x_1, \varepsilon x_2\right). \tag{4.2}$$

We note that we have rescaled the B as well because the quadratic coercivity of W only gives compactness on εB. With $L_\xi := (-\xi, 1) \times (-\frac{\xi}{2}, \frac{\xi}{2})$, this leads to the energy

$$\begin{aligned}
E_{\varepsilon,\xi}[k,\tilde{B}] &:= \int_{\Omega\setminus L_{\varepsilon,\xi}} \left[\frac{(|k|-1)^2}{\varepsilon\xi^2} + |\nabla k|^2\right] dx \\
&+ \varepsilon \int_{L_\xi} \left[\frac{(|\tilde{k}|-1)^2}{\varepsilon\xi^2} + \left|\nabla_\varepsilon \tilde{k} - \frac{\tilde{B}}{\varepsilon}\right|^2 + \frac{\xi^2}{\varepsilon}|\mathrm{curl}_\varepsilon \tilde{B}|^2 + \frac{1}{\varepsilon\xi^2}W(\xi|\tilde{B}|)\right] dx, \\
&= \int_{\Omega\setminus L_{\varepsilon,\xi}} \left[\frac{(|k|-1)^2}{\varepsilon\xi^2} + |\nabla k|^2\right] dx \\
&+ \int_{L_\xi} \left[\frac{(|\tilde{k}|-1)^2}{\xi^2} + \varepsilon\left|\nabla_\varepsilon \tilde{k} - \frac{\tilde{B}}{\varepsilon}\right|^2 + \xi^2|\mathrm{curl}_\varepsilon \tilde{B}|^2 + \frac{1}{\xi^2}W(\xi|\tilde{B}|)\right] dx,
\end{aligned}$$

where $\nabla_\varepsilon := [\partial_1, \frac{1}{\varepsilon}\partial_2]$ and the scaled curl operator is $\mathrm{curl}_\varepsilon\, g := \partial_1 g_2 - \frac{1}{\varepsilon}\partial_2 g_1$. Furthermore, the curl constraint (2.6) becomes

$$\left|\int_{L_\xi} \mathrm{curl}_\varepsilon \tilde{B}\, dx\right| = 2. \tag{4.3}$$

Denote $L^0_\xi := (-\xi, 1) \times \{0\}$. We take an arbitrary subsequence $\varepsilon_n \to 0$, and we would like to investigate the effective energy and keep track of the dependence on ξ for the following envelope

$$E_\xi[k] := \inf\{\liminf_{n\to\infty} E_{\varepsilon_n,\xi}[k_n, \tilde{B}_n] : k_n\chi_{L_{\varepsilon_n,\xi}} \to k \quad \text{strongly in} \quad L^2(\Omega;\mathbb{R}^2)\}. \tag{4.4}$$

5. Compactness

We consider any sequence with uniformly bounded energy and write it as the sum of the non-negative energies:

$$E^{bulk}_{\varepsilon_n,\xi}[k_n] + E^{layer}_{\varepsilon_n,\xi}[\tilde{k}_n, \tilde{B}_n].$$

5.1. **Bulk Energy.** In this portion of the energy, we have

$$\sup_n \int_{\Omega\setminus L_{\varepsilon_n,\xi}} \left[\frac{(|k_n|-1)^2}{\varepsilon_n\xi^2} + |\nabla k_n|^2\right] dx \le C. \tag{5.1}$$

In particular, for any U smooth open set which is compactly contained in the set $\Omega \setminus \overline{L^0_\xi}$, we have that

$$\sup_n \|k_n\|_{W^{1,2}(U;\mathbb{R}^2)} \le C$$

Thus, up to a subsequence, we have that we can assume $k_n \to k$ strongly in $L^2(\Omega \setminus \overline{L^0_\xi};\mathbb{R}^2)$. Because of the unit norm regularization, we have that $|k| = 1$ almost everywhere. Furthermore, since $k \in W^{1,2}(\Omega \setminus \overline{L^0_\xi};\mathbb{R}^2)$, it is a standard integration by parts argument [AFP00] to show $k \in SBV(\Omega;\mathbb{R}^2)$ where the jump set of k is precisely $\overline{L^0_\xi}$.

5.2. **Layer Energy.** In this portion of the energy, we have

$$\int_{L_\xi} \left[\frac{(|\tilde{k}_n| - 1)^2}{\xi^2} + \varepsilon_n \left| \nabla_{\varepsilon_n} \tilde{k}_n - \frac{\tilde{B}_n}{\varepsilon_n} \right|^2 \right] dx \\ + \int_{L_\xi} \left[\xi^2 |\mathrm{curl}_{\varepsilon_n} \tilde{B}_n|^2 + \frac{1}{\xi^2} W(\xi |\tilde{B}_n|) \right] dx \leq C. \quad (5.2)$$

Using the quadratic coercivity of W in (2.1), we have

$$\|\tilde{k}_n\|_{L^2} \leq C, \quad (5.3)$$

$$\|\varepsilon_n \nabla_{\varepsilon_n} \tilde{k}_n - \tilde{B}_n\|_{L^2} \leq C \varepsilon_n^{\frac{1}{2}}, \quad (5.4)$$

$$\|\mathrm{curl}_{\varepsilon_n} \tilde{B}_n\|_{L^2} + \|\tilde{B}_n\|_{L^2} \leq C, \quad (5.5)$$

This implies that up to a subsequence, not relabeled, we obtain

$$\tilde{B}_n \rightharpoonup B \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^{2\times 2}), \quad (5.6)$$

$$\mathrm{curl}_{\varepsilon_n} \tilde{B}_n \rightharpoonup \alpha \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^2), \quad (5.7)$$

for some $B \in L^2(L_\xi; \mathbb{R}^{2\times 2})$ and $\alpha \in L^2(L_\xi; \mathbb{R}^2)$.

Furthermore, using the quadratic bounds on $\tilde{k}$ and (5.4), we deduce that

$$\tilde{k}_n \rightharpoonup \tilde{k} \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^2), \quad (5.8)$$

$$\varepsilon_n \nabla_{\varepsilon_n} \tilde{k}_n \rightharpoonup B \quad \text{weakly in } L^2(L_\xi; \mathbb{R}^{2\times 2}), \quad (5.9)$$

for some $\tilde{k} \in L^2(L_\xi; \mathbb{R}^2)$.

In order to further characterize B, we can analyze componentwise for $i = 1, 2$ for $\phi \in C_c^\infty(L_\xi)$ using (5.9)

$$\int_{L_\xi} B^{i1} \phi \, dx = \lim_{n\to\infty} \int_{L_\xi} \varepsilon_n \partial_1 \tilde{k}_n^i \phi \, dx = - \lim_{n\to\infty} \varepsilon_n \int_{L_\xi} \tilde{k}_n^i \partial_1 \phi \, dx = 0,$$

$$\int_{L_\xi} B^{i2} \phi \, dx = \lim_{n\to\infty} \int_{L_\xi} \partial_2 \tilde{k}_n^i \phi \, dx = - \lim_{n\to\infty} \int_{L_\xi} \tilde{k}_n^i \partial_2 \phi \, dx = \int_{L_\xi} \partial_2 \tilde{k}^i \phi \, dx$$

where we have applied (5.8) after integrating by parts.

Thus, we conclude that

$$B = \begin{bmatrix} 0 & \partial_2 \tilde{k}_1 \\ 0 & \partial_2 \tilde{k}_2 \end{bmatrix}.$$

Now we can get information on $\mathrm{curl}_{\varepsilon_n} \tilde{B}_n$ by integrating by parts. We will do it componentwise for $i = 1, 2$. For any $\phi \in C^\infty(L_\xi)$, we have

$$\int_{L_\xi} \alpha^i \phi \, dx = \lim_{n\to\infty} \int_{L_\xi} [\mathrm{curl}_{\varepsilon_n} \tilde{B}_n]^i \, \phi \, dx,$$

$$= \lim_{n\to\infty} - \int_{L_\xi} \tilde{B}_n^{i2} \partial_1 \phi - \frac{1}{\varepsilon_n} \tilde{B}_n^{i1} \partial_2 \phi \, dx + \int_{\partial L} \phi [\tilde{B}_n^{i2} \nu_1 - \frac{1}{\varepsilon_n} \tilde{B}_n^{i1} \nu_2] \, d\mathcal{H}^1,$$

where $\mathcal{H}^1$ is the one dimensional Hausdorff measure in $\mathbb{R}^2$.

Using the tangential relations in (2.4) and the weak convergence of $\tilde{B}_n$, we simplify

$$\int_{L_\xi} \alpha^i \phi \, dx + \int_{L_\xi} \partial_2 \tilde{k}^i \partial_1 \phi \, dx$$
$$= \lim_{n\to\infty} \left[\int_{L_\xi} \frac{1}{\varepsilon_n} \tilde{B}^n_{i1} \partial_2 \phi \, dx + \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \phi(1, x_2) \tilde{B}^{i2}_n(1, x_2) dx_2 \right]. \quad (5.10)$$

Taking $\phi \equiv 1$ leads to the relation

$$\int_{L_\xi} \alpha^i \, dx = \lim_{n\to\infty} \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \tilde{B}^{i2}_n(1, x_2) dx_2. \quad (5.11)$$

Allowing $\phi \in C^\infty((-\xi, 1))$ which means that $\partial_2 \phi = 0$ leads to the equation

$$\int_{L_\xi} \alpha^i \phi \, dx + \int_{L_\xi} \partial_2 \tilde{k}^i \partial_1 \phi \, dx = \lim_{n\to\infty} \phi(1) \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \tilde{B}^{i2}_n(1, x_2) dx_2. \quad (5.12)$$

Define $[\tilde{k}^i] : (-\xi, 1) \to \mathbb{R}$ as

$$[\tilde{k}^i](x_1) := \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \partial_2 \tilde{k}^i \, dx_2.$$

This is well-defined as an L^2 function since $\partial_2 \tilde{k}^i \in L^2(L_\xi)$. In the sense of traces it encodes the vertical jump across the layer of $\tilde{k}^i$. Using (5.11) and the fact that ϕ only depends on x_1, we can simplify the relation (5.12) further as

$$\int_{L_\xi} \alpha^i \phi \, dx + \int_{-\xi}^{1} [\tilde{k}^i] \partial_1 \phi \, dx_1 = \phi(1) \int_{L_\xi} \alpha^i \, dx. \quad (5.13)$$

In particular, taking $\phi \in C_c^\infty((-\xi, 1))$, we deduce that $[\tilde{k}^i] \in W^{1,2}((-\xi, 1))$ with

$$\frac{d}{dx_1} [\tilde{k}^i](x_1) = \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \alpha^i \, dx_2.$$

Now, for generic $\phi \in C^\infty((-\xi, 1))$ we can integrate by parts in (5.13) to get

$$[\tilde{k}^i](1)\phi(1) - [\tilde{k}^i](-\xi)\phi(-\xi) = \phi(1)([\tilde{k}^i](1) - [\tilde{k}^i](-\xi)), \quad (5.14)$$
$$\text{and so} \qquad [\tilde{k}^i](-\xi)(\phi(1) - \phi(-\xi)) = 0. \quad (5.15)$$

Since the equation has to hold for every such ϕ, we have that $[\tilde{k}^i](-\xi) = 0$. This gives us a complete characterization of the vertical jump as

$$[\tilde{k}^i](s) = \int_{-\xi}^{s} \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \alpha^i \, dx. \quad (5.16)$$

By our convergences, we also have that (4.3) passes to the limit. To be precise,

$$2 = \left| \int_{L_\xi} \operatorname{curl}_\varepsilon \tilde{B} \, dx \right| \to \left| \int_{L_\xi} \alpha \, dx \right| = \left| [\tilde{k}](1) \right|. \quad (5.17)$$

We can summarize the previous results in the following theorem

Theorem 5.1. *Let $k_n, \tilde{B}_n$ with uniformly bounded energy $E_{\varepsilon_n,\xi}[k,\tilde{B}] = E^{bulk}_{\varepsilon_n,\xi}[k_n] + E^{layer}_{\varepsilon_n,\xi}[\tilde{k}_n,\tilde{B}_n]$. Then $k_n\chi_{L_{\varepsilon_n,\xi}} \to k$ strongly in $L^2(\Omega;\mathbb{R}^2)$ where $k \in SBV(\Omega;\mathbb{R}^2)$ and is S^1 valued. Furthermore, the jump set of k is precisely $\overline{L^0_\xi}$.*

In the layer, we can generate a rescaled k_n which are denoted $\tilde{k}_n$. We will have the convergences

$$\begin{aligned}
&\tilde{k}_n \rightharpoonup \tilde{k} \quad \text{weakly in } L^2(L_\xi;\mathbb{R}^2),\\
&\tilde{B}_n \rightharpoonup B \quad \text{weakly in } L^2(L_\xi;\mathbb{R}^{2\times 2}),\\
&\operatorname{curl}_{\varepsilon_n}\tilde{B}_n \rightharpoonup \alpha \quad \text{weakly in } L^2(L_\xi;\mathbb{R}^2),
\end{aligned}$$

for some $\tilde{k} \in L^2(L_\xi;\mathbb{R}^2)$, $\alpha \in L^2(L_\xi;\mathbb{R}^2)$, and $B := \begin{bmatrix} 0 & \partial_2\tilde{k}_1 \\ 0 & \partial_2\tilde{k}_2\end{bmatrix}$.

Define $[\tilde{k}] : (-\xi,1) \to \mathbb{R}$ as $[\tilde{k}](x_1) := \int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \partial_2\tilde{k}\, dx_2$. This allows us to define a compatibility condition between $\tilde{k}$ and α to be

$$[\tilde{k}](s) = \int_{-\xi}^{s}\int_{-\frac{\xi}{2}}^{\frac{\xi}{2}} \alpha\, dx, \qquad |[\tilde{k}](1)| = 2.$$

We note that even though $k, \tilde{B}$ were independent in the beginning, these fields become coupled in the limit.

6. Conjectures about the Limiting Energy

There are two main challenges to computing an integral representation for the energy directly. Firstly, we would need to understand better the relationship between $\tilde{k}$ and k. Secondly, it is difficult to characterize the limiting energy due to the coupled term

$$\int_{L_\xi} \varepsilon_n \left|\nabla_{\varepsilon_n}\tilde{k}_n - \frac{\tilde{B}_n}{\varepsilon_n}\right|^2 dx.$$

However, consider the terms which depend purely on $\tilde{B}_n$ and using the convergences given in Theorem 5.1, we can find a recovery sequence for relaxation of weak convergence such that

$$\int_{L_\xi}\left[\xi^2|\operatorname{curl}_{\varepsilon_n}\tilde{B}_n|^2 + \frac{1}{\xi^2}W(\xi|\tilde{B}_n|)\right] dx \tag{6.1}$$

$$\to \int_{L_\xi}\left[\xi^2|\alpha|^2 + \frac{1}{\xi^2}Q(W(|\cdot|))(\xi\partial_2\tilde{k})\right] dx \tag{6.2}$$

Since the envelope (4.4) we are considering does not depend on $\alpha, \tilde{k}$, we can take an infimum over these variables. An easy lower bound is achieved through Jensen's inequality and the definition of quasiconvexity. In particular defining $[\tilde{k}]$ as in Theorem 5.1, we can see that a lower bound is

$$\int_{L_\xi} \left[\xi^2 |\alpha|^2 + \frac{1}{\xi^2} Q(W(|\cdot|))(\xi \partial_2 \tilde{k}) \right] \, dx \geq \int_{-\xi}^{1} \left[\xi \left| \frac{d}{dx_1} [\tilde{k}](x_1) \right|^2 + \frac{1}{\xi} Q(W(|\cdot|))([\tilde{k}]) \right] dx. \quad (6.3)$$

We also make the conjecture that $[\tilde{k}] = [k]$. In this setting, this equation is quite similar to a Modica-Mortola functional for the vertical jump of the director with a transition layer on the order of the core length ξ. This is the type of picture predicted by the numerical experiments in [ZANV21]. Ideally, we would like to further modify the original energy so that we can obtain strong convergence (in at least L^1) in $\tilde{B}_n$, so that the quasiconvexification of $W(|\cdot|)$ is unnecessary. This would make the analogy to the Modica-Mortola functional stronger.

Furthermore, we can also make a connection to the recent preprint [GMPS21]. In this preprint, they propose a SBV model for $\pm\frac{1}{2}$ disclinations. However, they already strictly impose the constraint that $[k] = 2$ along the jump set. We should view the envelope in (4.4) as an attempt to also relax this condition by allowing for various jumps but in the limit $\xi \to 0$ they do have to be 2 on most of the jump set. However, this limit will be further complicated by the fact that the jump set is changing $(-\xi, 1) \times \{0\} \to [0, 1) \times \{0\}$ as $\xi \to 0$.

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