Effective field theory of Dislocation dynamics in Elastic solids

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We use effective field theories to study the dynamics of dislocations within a solid. We describe the dynamics of the phonons in terms of two-form gauge fields which allows us to couple the phonons non-derivatively to the dislocations. Using this formalism we are able calculate the forces between dislocations and also stresses induced by dislocations. We present new results for these quantities due to contribution from the dynamical phonon.

I. INTRODUCTION

In this paper, we explore the dynamics of dislocations within a solid using an Effective field theory approach. The physics of dislocations within solids has been explored for a long time and still is a very active research area. Traditionally, such a system has been studied by explicitly solving a system of partial differential equations corresponding to the equations of motion. One is able to formulate approximate solutions that describe the dynamics in a given setting. Here, we instead choose to use symmetries of the system to formulate an effective action for dislocations within a solid. This has numerous advantages and allows one to study phenomena which are beyond the scope of traditional methods.

Recently, there has been a renewed interest in the behavior of defects in solids using symmetric tensor gauge theories[1–4]. Here we instead focus on the gauge theory of phonons, where they are represented by anti-symmetric tensor gauge fields [5–7]. These mediate the interactions between the dislocations which are represented as one-dimensional strings. This construction completely relies on symmetries and allows one to calculate quantities in a systematic expansion based on simple power counting rules.

The low energy effective theory of solids can be easily understood just from the symmetry breaking pattern of the solid [8, 9]. This allows one to describe the dynamics in terms of scalar fields which correspond to the comoving coordinates of the solid. However the couplings to the defects are complicated by multi-valuedness of these fields which leads to a non-local action. As we will see, this can be circumvented by describing the dynamics in terms of two-form gauge fields which allows one to write down an effective action with local couplings to the dislocations [5–7]. These gauge fields describe a non-dynamical "stress-photon" and the dynamical phonon. The effective action is written as a derivative expansion in these fields with the validity upto a certain cutoff scale of the order of the dislocation radius.

We use the formalism to calculate forces between dislocations and also stresses mediated by dislocations and make connections with standard results. Furthermore, we are able to calculate the contribution due to phonons to the above quantities which have not been presented in the literature so far.

We will begin by reviewing the scalar field theory of a relativistic solid in Sec.II. In Sec.III, we explore the connections between the scalar field theory and the anti-symmetric dual gauge theory of solids. We then construct a dual effective action for solids in Sec.IV. We then derive the non-relativistic limit of this action and also write the action for a non-relativistic dislocation using the method of coset construction. In Sec.V, we detour to a variational approach to constructing the action for a non-linear dislocation mechanics. In Sec.VI, we explain the power counting rules that we use to perform systematic calculations within the EFT. Finally, we present some results concerning the potentials and stresses generated due to dislocations in Sec.VII.

II. EFFECTIVE FIELD THEORY OF SOLIDS

To describe the effective theory of relativistic solids in $D+1$ dimensions, we begin with a Lagrangian description where one introduces D scalar fields $\phi^I(\vec{x},t)$ for the comoving coordinates of the solids. These comoving coordinates can be thought of as the labels of the atoms within the solid. To write down the action for a solid, one needs to be consistent with the symmetries of a solid which in this case are poincare invariance, internal shift symmetry and internal rotational invariance. Hence all the dynamics of a solid is captured by a $SO(3)$ invariant function of

$B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J$ [8, 9].

$$S = \int d^{D+1}x\, \mathcal{F}(B^{IJ}) \tag{1}$$

The low energy properties of a system are dominated by the dynamics of the ground state. We are thus interested in studying the fluctuations of the system about its ground state. The fields ϕ^I obtain an expectation value in the ground state, spontaneously breaking a host of symmetries [8]. The expectation value can be chosen such that the internal coordinates align with the spatial coordinates i.e

$$\langle \phi^I(\vec{x}, t) \rangle = x^I \tag{2}$$

Since the ground state breaks a host of symmetries, one would expect multiple gapless particles in the spectrum from Goldstones theorem. But this need not be the case for space-time symmetries[10], which in the case of solids implies that only the goldstone bosons associated with translations (phonons) are needed to realize all the symmetries of the system. Hence the fluctuations can be parameterized as

$$\phi^I(\vec{x}, t) = x^I + \pi^I(\vec{x}, t) \tag{3}$$

where $\pi^I(\vec{x}, t)$ are the phonons. The function $\mathcal{F}(B^{IJ})$ can now be expanded around the background $B^{IJ} = \delta^{IJ}$.

$$\mathcal{F}(B^{IJ}) = c_0 + c_1(2\partial^I \pi^I + \partial_\mu \pi^I \partial^\mu \pi^I) + \frac{1}{2}\mathcal{C}_{IJKL}(\partial^I \pi^J + \partial^J \pi^I)(\partial^K \pi^L + \partial^L \pi^K) + ... \tag{4}$$

where the ... represent higher order terms in the phonon fields. For an isotropic solid, the elastic moduli tensor C_{IJKL} takes the form

$$C_{IJKL} = c_2 \delta_{IJ} \delta_{KL} + c_3(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK}) \tag{5}$$

Hence at the quadratic level, the Lagrangian takes the form

$$\mathcal{L}^{(2)} = c_1 \partial_\mu \pi^I \partial^\mu \pi^I + (2c_2 + c_3)(\partial_I \pi^I)^2 + c_3 \partial_I \pi^J \partial_I \pi^J \tag{6}$$

From the above equation, one can see that the phonons have a linear dispersion relation $\omega \sim c_s k$. The phonons represent D degrees of freedom, with $D-1$ transverse components $\vec{\pi}_T(\vec{x}, t)$ and 1 longitudinal component $\vec{\pi}_L(\vec{x}, t)$. The longitudinal phonons are responsible for compression whereas the transverse phonons cause shear forces in the solid. The stress-energy tensor of the solid is given by

$$T_{\mu\nu} = -\frac{\partial \mathcal{F}}{\partial(\partial_\mu \phi^I)} \partial_\nu \phi^I + \eta_{\mu\nu}\mathcal{F} \tag{7}$$

In the ground state, the energy density and pressure of the solid are given by

$$\rho = -\mathcal{F} \qquad p = \mathcal{F} - \frac{\partial \mathcal{F}}{\partial(\partial_i \phi^I)} \partial_i \bar{\phi}^I \tag{8}$$

where $\bar{\phi}^I$ represents the ground state value.

III. DUALITY

Until now we have discussed the effective field theory for solids in terms of the co-moving coordinates $\phi^I(\vec{x}, t)$ of the individual particles. Whenever topological defects are present within the material, the coupling of the dynamical fields to these defects is not straightforward. Consider the case of a superfluid, where there is U(1) symmetry associated with the superfluid phase ϕ. The defects in this case are the superfluid vortex lines and the phase ϕ becomes the winding angle around the vortex line which is multi-valued. For a vortex line with a winding number N, we get

$$\oint_C dx^i \partial_i \phi = 2\pi N \tag{9}$$

where C is a closed contour around the superfluid vortex. This multi-valuedness of the fields ϕ necessitates the use of dualities in physics. To write a consistent effective theory for superfluids, one needs to introduce dual gauge fields[11–13]. The dual fields are necessary to make the effective theory local. A scalar field is dual to an anti-symmetric $D-1$

form in $D+1$ space-time dimensions[14, 15]. As we will see for solids, the scalar fields ϕ^I are similarly singular in the presence of a dislocation. Hence in 3+1 dimensions, one introduces anti-symmetric two-form fields $b^I_{\mu\nu}$ with $I = 1, 2, 3$ to describe the low energy dynamics of a solid.

Let us understand how the duality works in the case of solids. Dislocations are topological defects in a solid which carry vectorial charges $\vec{n}$, called the burgers vector. Dislocations can be visualized as the end of a half-plane within a 3D solid or a half-line in a 2D solid. Hence they are line/point defects in a 3D/2D material. In the presence of a dislocation line, the net displacement traversed by a closed contour enclosing the dislocation is non-zero and equal to the burgers vector n^I.

$$\oint_C dx^i \partial_i \pi^I = n^I \tag{10}$$

This implies that the phonon fields are singular near the dislocation and one needs to use a dual effective theory to describe them. The dislocation line traces a worldsheet in space-time. We denote the dislocation line by the four vector $X^\mu(\tau,\rho)$, where τ and ρ are arbitrary world-sheet coordinates. The current associated with a p-dimensional defect is a $p+1$ form and hence a dislocation current in 3+1 dimensions is a two-form given by

$$J^{\mu\nu}_I = n_I \int d\rho \; \partial_\tau X^\mu \partial_\rho X^\nu \; \delta^{(3)}(\vec{x} - \vec{X}(\tau,\rho)) \tag{11}$$

where $v^\mu = \partial_\tau X^\mu$ is the velocity of the dislocation line. The dislocation density is given by the temporal component J^{0j}_I and the spatial components J^{ij}_I correspond to a dislocation line along the jth-direction moving in the ith-direction. The dislocations can be edge or screw type, depending on whether the burgers vector is perpendicular or parallel to the dislocation line respectively. Edge dislocations can only move in the direction of their burgers vectors since the motion perpendicular to their burgers vector involves the addition or removal of atoms.

In the presence of dislocations, the scalar fields are multi-valued; alternatively they can be viewed as single-valued, but with discontinuities across 2-d spatial surfaces, at least, with singular derivatives on such surfaces. The dislocation lines are the 1-d curves of termination of such surfaces. The inverse elastic distortion, W^I_σ, is defined by removing these singular parts from the derivative of the scalar fields, but since the 'curl' of the distributional derivative of the scalar fields have to vanish while there are terminating jumps in the latter, the skew-symmetric derivative of W^I_σ must be non-vanishing at these jump-terminations, equaling the dislocation density, $J^{\mu\nu}_I$, there: $\epsilon^{\mu\nu\rho\sigma}\partial_\rho W_{\sigma I} = J^{\mu\nu}_I$. This constraint can be imposed using an anti-symmetric two-form $b^I_{\mu\nu}$, which acts like a Lagrange multiplier:

$$S = \int d^D x \;\; \mathcal{F} + \frac{1}{2} b^I_{\mu\nu}(\epsilon^{\mu\nu\rho\sigma}\partial_\rho W_{\sigma I} - J^{\mu\nu}_I) \tag{12}$$

The equation of motion for the scalar field is given by

$$\frac{1}{2}\epsilon^{\sigma\rho\mu\nu}\partial_\rho b^I_{\mu\nu} = -\frac{\partial \mathcal{F}}{\partial W_{\sigma I}} \tag{13}$$

From the above relation one can see that the two-form fields are a non-local and non-linear functional of the phonon fields. We introduce the stress fields $\hat{T}^{\sigma I} = \frac{1}{2}\epsilon^{\sigma\rho\mu\nu}\partial_\rho b^I_{\mu\nu}$ which are identically conserved $\partial_\mu \hat{T}^{\mu I} = 0$.

From (13), one can see that this conservation law is equivalent to the equations of motion for the scalar fields, when W^I_σ is a gradient. In this sense, the two-form description of the elastic solid with dislocations may be considered dual to the scalar description.

We can go further and construct a Lagrangian $\mathcal{G}(T^{\sigma I})$ invariant under the Poincare and internal symmetries.

This Lagrangian describes the dynamics of the solid in terms of the dual gauge fields. In the ground state, the energy density and pressure are given by

$$\rho = -\mathcal{G} + \frac{\partial \mathcal{G}}{\partial T^{iI}}\bar{T}^{iI} \qquad p = \mathcal{G} \tag{14}$$

where $\bar{T}^{iI}$ represents the ground state value. Comparing (8) and (14), while it is tempting to conclude that $\mathcal{G}$ is a Legendre transform of $\mathcal{F}$, it must be realized that rotational invariance of $\mathcal{F}$, i.e. its dependence on $\partial_\mu\phi^I$ only through B^{IJ}, prevents it from being convex in $\partial_\mu\phi^I$ and this prevents a Legendre transform to be performed on it w.r.t. $\partial_\mu\phi^I$.

It is this lack of a direct link between the the scalar field description and the its dual description for a nonlinear elastic solid (whether with or without dislocations [16]) that motivates our study in Sec. V. There it is shown that while $\mathcal{F}$ may not admit a strict dual functional through a Legendre transform, the Euler-Lagrange equations of its action can be transformed to produce a family of dual functionals whose Euler-Lagrange equations are the same as the ones related to $\mathcal{F}$, interpreted through a well-defined dual-to-primal mapping.

IV. DUAL EFFECTIVE ACTION

Now we have all the necessary ingredients to construct an effective action for solids with dislocations. One can introduce the three-form field strength for the two-form field which is written as

$$G^I_{\rho\mu\nu} = 3\partial_{[\rho}b^I_{\mu\nu]} = \epsilon_{\rho\mu\nu\sigma}T^{\sigma I} \tag{15}$$

Symmetries force the dual Lagrangian to be a Lorentz invariant and $SO(3)$(internal) invariant function of the Field strength. This implies that the dual Lagrangian is a $SO(3)$ invariant function of $Y^{IJ} = T_\mu{}^I T^{\mu J}$. Hence the dual action is given by

$$S_{bulk} = \int d^4x \; \mathcal{G}(Y^{IJ}) \tag{16}$$

Since we are interested in the dynamics of the ground state of the system we need to expand this about its background. Evaluating (13) on the background, the most general solution of the dual fields turns out to be

$$\begin{aligned} \bar{b}^I_{jk} &= f^I_{[jk]}(t) \\ \bar{b}^I_{0k} &= \frac{1}{2}(\partial_t f^I_{jk}(t) + c_1\epsilon^I{}_{jk})x^j + g^I_k(t) \end{aligned} \tag{17}$$

But the action in (16) has a gauge invariance under the transformation $b^I_{\mu\nu} \to b^I_{\mu\nu} + \partial_\mu\lambda^I_\nu - \partial_\nu\lambda^I_\mu$. This allows one to perform a gauge transformation and set the functions f and g to zero.

One can now expand around the background and introduce fields A^a_i and B^a_i to describe the perturbations of $b^I_{\mu\nu}$.

$$b^I_{jk} = \epsilon_{ijk}B^I_i \qquad b^I_{0k} = \frac{c_1}{2}\epsilon^I{}_{jk}x^j + A^I_k \tag{18}$$

As we will see later, A describes a non-dynamical field which is responsible for forces between dislocations in static elasticity. On the other hand, B describes the phonons in the solid. The stress fields T^I_μ and the function Y^{IJ} expanded around the background give

$$\begin{aligned} T^I_\mu =& \sigma_0\Big(\delta^I_\mu + \delta^0_\mu\partial^i B^I_i + \delta^i_\mu(\epsilon_i{}^{jk}\partial_j A^I_k - \dot{B}^I_i)\Big) \\ Y^{IJ} =& (\sigma_0)^2\Big(\delta^{IJ} - (\partial^i B^I_i)(\partial^i B^J_i) + (\epsilon^{ijk}\partial_j A^I_k - \dot{B}^I_i)(\epsilon_i{}^{jk}\partial_j A^J_k - \dot{B}^J_i) \\ &+ \delta^{Ji}(\epsilon_i{}^{jk}\partial_j A^I_k - \dot{B}^I_i) + \delta^{Ii}(\epsilon_i{}^{jk}\partial_j A^J_k - \dot{B}^J_i)\Big) \end{aligned} \tag{19}$$

where we have replaced c_1 with σ_0 to denote the expectation value of the stress fields. After expanding around the background value $\bar{Y}^{IJ} = \sigma_0^2\delta^{IJ}$ with the perturbations in (19) denoted by $\tilde{Y}^{IJ}$, one obtains

$$S_{bulk} = \int d^dx \; (g + g_{IJ}\tilde{Y}^{IJ} + g_{IJKL}\tilde{Y}^{IJ}\tilde{Y}^{KL} + .. \tag{20}$$

where the ... represent higher order corrections to Lagrangian which will become important when we look at the non-linearities. We define the couplings as

$$g_{a_1a_2..a_n} = \left.\frac{d^n\mathcal{G}}{dY^{a_1a_2}dY^{a_3a_4}..}\right|_{\bar{Y}} \tag{21}$$

The fluctuations of the stress fields can be separated into temporal and spatial components.

$$\tilde{T}^I_m = (\vec{\nabla}\times\vec{A}^I)_m - \dot{\vec{B}}^I_m \qquad \tilde{T}^I_0 = \vec{\nabla}.\vec{B}^I \tag{22}$$

Note here that "I" is the Burgers vector index carried by the vectors $\vec{A}$ and $\vec{B}$. From now on, we will keep all indices lower case with the upper index denoting the burgers vector index. Hence one can write down the quadratic action for the stress fields.

$$\begin{aligned} S_{bulk} = \sigma_0^2\int d^dx \; & d_1(\tilde{T}^i_j\tilde{T}^i_j - \tilde{T}^i_0\tilde{T}^i_0) + 4d_2\tilde{T}^i_i\tilde{T}^j_j \\ &+ 4d_3(\tilde{T}^i_j\tilde{T}^i_j + \tilde{T}^i_j\tilde{T}^j_i) \end{aligned} \tag{23}$$

where we have used isotropy to define the couplings

$$g^{ij} = d_1\delta^{ij} \qquad g^{ijkl} = d_2\delta^{ij}\delta^{kl} + d_3(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{24}$$

One can see that the field A has no time-derivative and hence it is not dynamical. The only dynamical degree of freedom is the B field, which plays the role of phonons in the dual formalism.

The action defined in (23) is gauge invariant under the transformations $b^I_{\mu\nu} \to b^I_{\mu\nu} + \partial_\mu\lambda^I_\nu - \partial_\nu\lambda^I_\mu$. The gauge transformation parameter itself has a redundancy $\lambda_\mu \to \lambda_\mu + \partial_\mu\eta$. We can gauge fix by choosing either the lorentz $\partial^\mu b^I_{\mu\nu} = 0$ or the coloumb gauge $\partial^i b^I_{i\nu} = 0$. Coulomb gauge is not covariant but since we will be interested in the non-relativistic case, we will use the coulomb gauge. To do this we include a term in the action

$$S_{GF} = \frac{1}{2\zeta}\int d^4x(\partial^i b^I_{i\nu})^2 \tag{25}$$

Choosing $\zeta = 0$ amounts to working in the coulomb gauge. In this gauge, the fields A and B obey $\vec{\nabla}.\vec{A}^i = 0$ and $\nabla \times \vec{B}^i = 0$ respectively. Hence A is transverse whereas B is longitudinal in this gauge choice.

Until now, we have looked at the bulk action which describes the dynamics of the two-form fields. Since we are interested in the forces between the dislocation, we need to understand the dynamics of the dislocation and their interactions with the bulk fields. This is encoded in the Kalb-Ramond action and the dislocation action. The two-form dislocation current sources the gauge field $b^i_{\mu\nu}$ and hence the leading order interaction can be written as

$$\begin{aligned} S_{KR} &= \int d^4x\ J^{\mu\nu}_i b^i_{\mu\nu} \\ &= \int d^4x\ n_i C \int d\rho\ \partial_\rho X^\mu\ \delta^{(3)}(\vec{x} - \vec{X}(t,\rho))v^\nu b^i_{\mu\nu} \\ &= n_i C\int d\tau d\rho\ \partial_\rho X^\mu \partial_\tau X^\nu b^i_{\mu\nu} \end{aligned} \tag{26}$$

In the second line we have used (11) and in the third line we have substituted for the velocity and C is a constant. One can notice that this term has no-derivatives acting on the two-form field $b^i_{\mu\nu}$. This should be contrasted with the scalar field theory for phonons where ϕ^I is always comes with a derivative acting on it. One can write this action in terms of the stress photon and phonon fields.

$$\begin{aligned} S_{KR} &= \int d^4x\ J^{0k}_i \tilde{b}^i_{0k} + J^{ik}_i \tilde{b}^i_{ik} \\ &= \int d^4x\ J^{0k}_i A^i_k + J^{ik}_i \epsilon_{ikj} B^I_j \end{aligned} \tag{27}$$

Note that phonons are only sourced by moving dislocations. On the other hand static dislocations can still source stress photons since they couples to the dislocation density.

The dislocations within a solid are described by dynamical fields $X^\mu(\tau,\rho)$ and hence one can write an action which describes their dynamics. Since dislocation lines are similar to string-like defects their dynamics can be described by generalizing the Nambu-Goto action for string-like defects to dislocation lines. Just like the action for a relativistic point particle is given by the length(proper time) of its world-line, the action for the dislocation line is given by the area of its worldsheet. The induced metric on the worldsheet is

$$\gamma_{ab} = f_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{28}$$

where a, b run over τ, ρ. $f_{\mu\nu}$ is the space-time metric as seen by the dislocation line. One can think of $f_{\mu\nu}$ as the pull-back of the material metric to co-ordinate space. The action is therefore the Nambu-Goto action for strings generalized to the case of the dislocation.

$$S_{dis} = \int d\tau d\rho\ \sqrt{-det\ \gamma}\ T \tag{29}$$

where T represents the Tension in the dislocation. The tension can depend on the stresses in the material and hence can be generalized to be a function of the bulk fields $T(Y^{ij})$.

One can now write the action for a solid with dislocations as the sum of the terms described above.

$$S = S_{bulk} + S_{KR} + S_{dis} \tag{30}$$

This action has all the ingredients to calculate the forces between the dislocations and also the stresses induced by dislocations. One needs to expand the action using certain power counting rules which we will describe later. Using this we are able to calculate the required quantities in a systematic manner.

A. Non-Relativistic limit

Ultimately we are interested in the non-relativistic limit of the action described in (30). To derive the NR limit, one needs to take the limit $c \to \infty$. We will first look at the bulk action. Hence we explicitly reintroduce the factors of c in the stress fields.

$$\tilde{T}^i_m = (\nabla \times \vec{A}^i - \frac{1}{c}\dot{\vec{B}}^i)_m \qquad \tilde{T}^i_0 = \vec{\nabla}.\vec{B}^i \tag{31}$$

The Taylor coefficients in the bulk action can depend on the velocity ratio $\frac{c}{c_{T/L}}$. Here $c_{T/L}$ is the velocity of transverse/longitudinal phonons respectively. We will take them to be similar $c_T \sim c_L$ and just work with c_T. Also from dimensional analysis, we see that it depends on powers of σ_0. Let us assume that the scaling is given by

$$g_{a_1 a_2 .. a_n} \sim \frac{1}{\sigma_0^{n-1}}\Big(\frac{c^2}{c_T^2}\Big)^n \tag{32}$$

Using the above scaling, one can write the quadratic action as

$$\begin{aligned} S^{(2)}_{bulk} = \sigma_0 \int d^d x \; d_1 \frac{c^2}{c_T^2}(\tilde{T}^j_i \tilde{T}^j_i - \tilde{T}^j_0 \tilde{T}^j_0) + 4 d_2 \frac{c^4}{c_T^4}(\tilde{T}^i_i \tilde{T}^j_j) \\ + 4 d_3 \frac{c^4}{c_T^4}(\tilde{T}^j_i \tilde{T}^j_i + \tilde{T}^j_i \tilde{T}^i_j) \end{aligned} \tag{33}$$

where d_1, d_2, d_3 are dimensionless O(1) coefficients. To obtain the normalized kinetic term, one has to rescale the fields A and B.

$$A \to \frac{1}{\sigma_0}\frac{c_T^2}{c^2} A \qquad B \to \frac{1}{\sigma_0}\frac{c_T^2}{c} B \tag{34}$$

Using the canonical fields obtained above, one can rewrite the action as

$$\begin{aligned} S^{(2)}_{bulk} =& \frac{1}{\sigma_0}\int d^d x \; d_1 \Big(\frac{c_T^2}{c^2}(\vec{\nabla} \times \vec{A}^i - \dot{\vec{B}}^i)^2 - c_T^2 (\vec{\nabla}.\vec{B}^i)^2\Big) \\ &+ 4 d_2 ((\vec{\nabla} \times \vec{A}^i - \dot{\vec{B}}^i)_i)^2 + 4 d_3 \Big((\vec{\nabla} \times \vec{A}^i - \dot{\vec{B}}^i)^2 \\ &+ (\vec{\nabla} \times \vec{A}^j - \dot{\vec{B}}^j)_i (\vec{\nabla} \times \vec{A}^i - \dot{\vec{B}}^i)_j\Big) \end{aligned} \tag{35}$$

By sending $c \to \infty$, one sees that the first term vanishes in the above Lagrangian and we obtain the non-relativistic Lagrangian. One can indeed check that this is the correct dual Lagrangian for a non-relativistic solid by deriving the equations of motion for the dual fields. The variation of the action in (35) w.r.t to the dual fields A^I_i and B^I_i yields

$$\begin{aligned} \delta A^I_i : \quad & \frac{1}{\sigma_0}(8 d_3 \; \epsilon_{ijm}\partial_j(\tilde{T}^I_m + \tilde{T}^m_I) + 8 d_2 \; \epsilon_{ijI}\partial_j \tilde{T}^k_k) = J^I_{0i} \\ \delta B^I_i : \quad & \frac{1}{\sigma_0}(2 d_1 c_T^2 (\partial_i \tilde{T}^I_0) + (8 d_3 \; \partial_t(\tilde{T}^I_i + \tilde{T}^i_I) \\ & \qquad + 8 d_2 \; \delta^I_i \partial_t \tilde{T}^k_k)) = -\epsilon_{ijk} J^I_{jk} \end{aligned} \tag{36}$$

In Linear elasticity, the quadratic Lagrangian for a isotropic elastic solid is given by

$$\mathcal{L}_e = \frac{\rho}{2}(\partial_t \pi^i)^2 - \frac{1}{2} C_{ijkl}\partial_j \pi^i \partial_l \pi^k \tag{37}$$

Here ρ is the mass density of the object. C_{ijkl} is called the elastic moduli tensor whose form is given by

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{38}$$

where μ and λ are the Lame' constants. μ is called the shear-modulus and the compression modulus κ can be written as $\kappa = \lambda + \frac{2}{D}\mu$ where D is the number of spatial dimensions. Let us define $x_0 = c_T t$. The stress fields in this case are given

$$\tilde{T}^a_0 = \frac{\partial \mathcal{L}_e}{\partial(\partial_0 \pi^a)} = \rho c_T \partial_t \pi^a \qquad \tilde{T}^a_i = \frac{\partial \mathcal{L}_e}{\partial(\partial_i \pi^a)} = -C_{aibl}\partial_l \pi^b \tag{39}$$

Substituting the above obtained relations in the (36), one obtains the following compatibility equations after the identification $4\mu d_1 = \sigma_0, 16\mu(1+\nu)d_2 = -\sigma_0\nu$ and $32\mu d_3 = \sigma_0$.

$$\begin{aligned} \epsilon_{ijk}\partial_j\varepsilon_{kI} &= \alpha_i^I \\ \partial_t\varepsilon_{iI} + \partial_i v^I &= \epsilon_{ijk}\alpha_j^I v_k \end{aligned} \tag{40}$$

Note that this differs from the usual compatibility conditions (61) because only the symmetric strain contributes to the above equations.

The non-linear eom's can be derived similarly which are given by

$$\begin{aligned} &\epsilon^{ijk}\partial_j\Big(\frac{\partial G}{\partial Y^{kI}} + \frac{\partial G}{\partial Y^{Ik}}\Big) = \alpha_I^i \\ &\partial_t\Big(\frac{\partial G}{\partial Y^{iI}} + \frac{\partial G}{\partial Y^{Ii}}\Big) + \partial_i\Big(\Big(\frac{\partial G}{\partial Y^{kI}} + \frac{\partial G}{\partial Y^{Ik}}\Big)\vec{\nabla}.\vec{B}^k\Big) = \epsilon_{ijk}V^j\alpha_I^k \end{aligned} \tag{41}$$

Now lets look at the interactions which arise in the bulk action. This allows to calculate the sub-leading corrections to the forces between dislocations. One can show that there are only three invariants one can construct out of Y^{ij} which can be taken to be the traces

$$[\tilde{Y}], \quad [\tilde{Y}^2], \quad [\tilde{Y}^3] \tag{42}$$

The functional $G(Y)$ can be constructed using these invariants. The cubic interactions arise from the quantities

$$\begin{aligned} \mathcal{L}^{(3)} = \frac{1}{\sigma_0}\Big(d_2[Y']^2 + 2d_3[Y'^2]\Big) + \frac{1}{\sigma_0^2}\Big(&d_4[Y']^3 \\ &+ d_5[Y'^2][Y'] + d_6[Y'^3]\Big) \end{aligned} \tag{43}$$

where we have replaced $\tilde{Y}^{ij}$ with its non-relativistic limit given by

$$\tilde{Y}^{ij} \longrightarrow Y'^{ij} = \Big(\tilde{T}^i_j + \tilde{T}^j_i - \frac{1}{\sigma_0}\tilde{T}^i_0\tilde{T}^j_0\Big) \tag{44}$$

Inserting this into (43), we have the following interactions

$$\begin{aligned} \mathcal{L}^3_{bulk} = \frac{1}{\sigma_0^2}\Bigg(&-4d_2(\tilde{T}^I_0)^2\tilde{T}^J_J + 8d_4(\tilde{T}^J_J)^3 - 4d_3\tilde{T}^I_0\tilde{T}^J_0(\tilde{T}^I_J + \tilde{T}^J_I) \\ &+ d_5(\tilde{T}^I_J + \tilde{T}^J_I)^2\tilde{T}^K_K + d_6(\tilde{T}^J_I + \tilde{T}^I_J)(\tilde{T}^K_I + \tilde{T}^I_K)(\tilde{T}^J_K + \tilde{T}^K_J)\Bigg) \end{aligned} \tag{45}$$

One can expand the $\tilde{T}$ in A and B to obtain the interactions for these fields. We will not write them explicitly here since it complicates the above expressions.

Next we look at the non-relativistic limit of the Kalb-Ramond term. As we will see the Kalb-Ramond term remains the same in the NR limit. From dimensional analysis, one can see that the coupling C is just ρc. We can choose $\tau = ct$ and use the fact that $X^\mu = (ct, \vec{X})$. Rescaling as in (35), one gets

$$\begin{aligned} S_{KR} &= \rho c\frac{1}{\sigma_0}\frac{c_T}{c}\int d\tau\, d\rho\, \frac{c_T}{c}\partial_\rho X^k\partial_\tau X^0 A^I_k + c_T\partial_\rho X^k\partial_\tau X^i\epsilon_{ikj}B^I_j \\ &= \frac{n_I}{c_T}\int cdt\, d\rho\ \ \frac{c_T}{c}\partial_\rho X^k\partial_\tau X^0 A^I_k + \frac{c_T}{c}\partial_\rho X^k\partial_t X^i\epsilon_{ikj}B^I_j \\ &= n_I\int dt\, d\rho\ \ \partial_\rho X^k A^I_k + \partial_\rho X^k\partial_t X^i\epsilon_{ikj}B^I_j \end{aligned} \tag{46}$$

Hence we see that the form of the Kalb-Ramond term is unchanged upon taking the NR limit.

Next we are interested in the non-relativistic action for the dislocation. One can do so by taking the NR limit of the Nambu-goto action described in (29). But instead we will construct such an action using the method of coset construction which we describe in the next section.

B. Coset construction for dislocations in a solid

The coset construction is a tool to construct the action for non-linearly symmetries just from the symmetry breaking pattern of the system [17–20]. We will not go into details but just mention the main results. For an introduction, we refer the reader to [8, 21]. A dislocation within a solid breaks a host of symmetries in addition to the symmetry breaking by the ground state of the solid. The symmetry breaking pattern is given by

$$\text{unbroken} = \begin{cases} H \\ P_3 + T_3 \equiv \bar{P}_3 \ , \\ M \end{cases} \quad \text{broken} = \begin{cases} K_i \\ P_a + T_a \equiv \bar{P}_a \\ L_i + R_i \equiv \bar{L}_i \ , \\ T_i \\ R_i \end{cases} \tag{47}$$

where $a = 1, 2$. Note that the rotations along the Hence one can parametrize the coset as

$$\Omega = e^{-iHt} e^{i\bar{P}_z z} e^{iX^a \bar{P}_a} e^{i\eta^i K_i} e^{i\pi^i T_i} e^{i\theta^i R_i} e^{i\chi^i \bar{L}_i}. \tag{48}$$

where we have chosen the 3-direction to be in the z-direction. We will generalize to $X^i = (X^a, z)$ while performing our calculations. Using Ω, one can calculate the Maurer-Cartan form defined as $\Omega^{-1}d\Omega$ and obtain:

$$\begin{aligned} \Omega^{-1}d\Omega = i\Big\{ & -Hdt + \bar{P}_j Q_i{}^j(\chi)(\eta^i dt + dX^i) - M(\eta_i dX^i + \tfrac{1}{2}\vec{\eta}\cdot\vec{\eta}\,dt) + R_i \tfrac{1}{2}\epsilon^{ijk}\left[R^{-1}(\theta)dR(\theta)\right]_{jk} + K_j Q_i{}^j(\chi)d\eta^i \\ & + T_k Q_j{}^k(\chi)\left[(dX^i + d\pi^i)R_i{}^j(\theta) - \eta^j dt - dX^j\right] + \bar{L}_i \tfrac{1}{2}\epsilon^{ijk}\left[Q^{-1}(\chi)dQ(\chi)\right]_{jk}\Big\}, \end{aligned} \tag{49}$$

where we have introduced the matrices $R_{ij} \equiv \left(e^{i\theta^i R_i}\right)_{ij}$ and $Q_{ij} \equiv \left(e^{i\chi^i \bar{L}_i}\right)_{ij}$. Using the coordinates $x^\mu = (t, z)$ and defining $P_\mu = (-H, \bar{P}_z)$, we can rewrite the MC form as

$$\begin{aligned} \Omega^{-1}d\Omega \equiv i dx^\nu e_\nu{}^\mu \big(&\bar{P}_\mu + \nabla_\mu X^a \bar{P}_a + \nabla_\mu \pi^i T_i + \nabla_\mu \theta^i\, R_i \\ &+ \nabla_\mu \eta^i K_i + \nabla_\mu \chi^i \bar{L}_i + A_\mu M\big). \end{aligned}$$

This equation gives us the covariant derivatives of the goldstone fields. Remember here that X^a act as goldstones since $\bar{P}_a$ are broken generators. We can read off the vierbien

$$e_0^0 = 1 \quad e_0^z = Q^{iz}(\chi)(v_i + \eta_i) \quad e_z^0 = 0 \quad e_z^z = \partial_z X^i Q^{iz}(\chi) \tag{50}$$

where $v^i = \frac{dX^i}{dt}$ is the velocity of the dislocation. Using the inverse Higgs constraints in the bulk [10], we can eliminate the goldstones η^i and θ^i in favor of the π^i fields. For the boost goldstone, we have $\eta^i = \partial_t \pi^j (D^{-1})^i_j$. We also have an additional inverse higgs constraint which arises from the commutator

$$[\bar{L}^a, \bar{P}^z] \sim \bar{P}^b \tag{51}$$

Hence we can eliminate the goldstones χ^a in favor of X^a. The constraint that arises as a result of this is $\nabla_z X^a = Q^{ia}\partial_z X^i = 0$. Note that we still have an rotational goldstone χ^3 which is present in the spectrum. Using this we are finally able to write the covariant derivatives for the goldstones X^a,π^i and χ^3.

$$\begin{aligned} \nabla_t X^a &= Q^{ka}(\chi)(v_k + n_k) \\ \nabla_t \pi^i &= \frac{1}{det\ e}(v_k + n_k)\mathcal{D}_l \pi^j Q_j{}^i(\chi)(\delta^{kl} det\ e - Q^{zk}(\chi)\partial_z X^l) \\ \nabla_z \pi^i &= \frac{1}{det\ e}\partial_z X^l \mathcal{D}_l \pi^j Q_j{}^i(\chi) \\ \nabla_z \chi^3 &= \frac{1}{2}\frac{1}{det\ e}\epsilon^{3jk}\left[Q^{-1}(\chi)\partial_z Q(\chi)\right]_{jk} \\ \nabla_t \chi^3 &= \frac{1}{2}\epsilon^{3jk}\left[Q^{-1}(\chi)(\partial_t - \frac{Q^{iz}(\chi)(v_i + n_i)}{det\ e}\partial_z)Q(\chi)\right]_{jk} \end{aligned} \tag{52}$$

where we have introduced the bulk covariant derivatives $\mathcal{D}_l\pi^j$. Using these quantities, one can write down the action for the dislocations

$$S_{disl} = \int dzdt\ det\ e\ \mathcal{L}(\nabla_t X^a, \nabla_t \pi^i, \nabla_z \pi^i, \nabla_t \chi^3, \nabla_z \chi^3) \tag{53}$$

For a straight dislocation $\vec{X}(t,z) = (\vec{x}_\perp, z)$, the leading order action is given by

$$\begin{aligned} S_{disc} = \int dzdt\ |\partial_z \vec{X}| \Big[& a_0 + a_1(v^a - u^a)^2 + a_2((\vec{v} - \vec{u})^a \mathcal{D}_a \pi^i)^2 \\ & + a_3(\partial_z X^l \mathcal{D}_l \pi^i)^2 + a_1(\chi^3)^2(v^a - u^a)^2 \\ & + a_4(\partial_z \chi^3)^2 + a_5(\partial_t \chi^3 - (v^z - u^z)\partial_z \chi^3)^2 + ... \Big] \end{aligned} \tag{54}$$

where we have used the material velocity $u_i = -\eta_i$. Since a runs over 1,2 one can see that only the perpendicular component of dislocation velocity plays a role. The linearized bulk covariant derivatives give the linear strain ε_{ij} within the material which in the dual theory can be written as $\varepsilon_{ij} = C^{-1}_{ijkl}\tilde{T}^{kl}$. This generates the coupling of the dislocation to the stress photons and phonons in the Nambu-Goto action.

V. INTERLUDE: VARIATIONAL PRINCIPLE(S) FOR NONLINEAR DISLOCATION MECHANICS - A CONTINUUM MECHANICS-CUM-MATERIALS SCIENCE POINT OF VIEW

Unlike the physics of the microscopic structure of sub-atomic particles (e.g. ‘core’ of an electron), much is physically known, through direct experimental observation and lattice statics/molecular dynamics/density functional theory calculations about the microscopic structure of dislocations and their mutual interactions, as well as with applied loads through boundary conditions, within a (nonlinear) elastic crystal. Due to this knowledge, physically well-justified and transparent mathematical models can be posited for the phenomena, with the possibility of systematic refinement to include more detail when deemed necessary after mathematical study and comparison with experiment. There is a long and distinguished history of the study of dislocations in elasticity in the classical setting, see, e.g., [22–26], the continuously distributed setting, e.g., [27–29], [30, including second-order effects] and [31], and the connections of some of the kinematic aspects of dislocations to non-Riemannian Geometry [28, 32, 33]. As well, techniques for developing well-set, classical thermomechanical theories of the mechanics of continuous media comprising different types of materials exhibiting strongly nonlinear behavior and satisfying the relevant invariances and material symmetries are available [34–37] and [31]. These ideas and techniques have been synthesized and extended to produce the theory/model of dislocation mechanics stated in [38], as reviewed in [39]. The theory admits the minimal specification of an energy density function $\psi(W)$ and that of a dislocation velocity field, the function $V_s(\alpha, W, \rho)$ in (56), which, when guided by the requirements of being proportional to its derived thermodynamic driving force, is a specified function of the thermodynamically derived Cauchy stress tensor $T_{ij} = -\rho W_{ki}\psi'_{kj}$ and the dislocation density tensor α_{ij}, admitting a scalar or matrix of material constants representing dislocation mobility. Here, $\psi'_{ij} = \partial_{W_{ij}}\psi$, and it suffices to use a rectangular Cartesian coordinate system and tensor components w.r.t its basis in this Section. The time variable is represented by the symbol t and not used as an index.

For prescribed static dislocation fields the framework is shown to be able to compute the stress and energy fields of such distributions in bodies of arbitrary geometry and general elastic symmetries [40, 41]. Similarly for prescribed dislocation velocity field, the setup is shown to be able to compute the evolution of the dislocation field [41]. And the evolution in the fully coupled case also has been shown to work well to predict nonsingular dislocation cores, dislocation annihilation, dissociation and stress-mediated interaction when restricted to dislocation motion within a planar layer in a 3–d body [42] within a ‘small deformation’ ansatz.

The phenomenon of macroscopic plasticity of crystalline materials corresponds to the collective dynamical behavior of a very large number of dislocation curves in an elastic body under generally time-dependent loads. While experimental observations and real practical applications of plasticity abound, it is fair to say that there does not exist a fundamental theory that arises as a coarse-graining of nonlinear dislocation dynamics as described above (or by any other model). The phenomenon of plasticity shows fascinating dynamical changes as a function of initial conditions and tamely evolving driving loads - e.g., yielding, Stage I, II, III, IV behaviors as a function of applied load temperature and initial crystal orientation, intricate patterned dislocation microstructure formation such as cells and sub-grain boundaries to name only a few - with no established fundamental theory for understanding them (the phenomenon is even richer, with rapidly driven situations also being of theoretical and practical interest). It is in this context that we would like to use a path integral implementation of the dynamics represented by (56) to evaluate how much of the reality of macroscopic plasticity can be understood by the combination of the model and the technique.

The rough expectation is to be able to interpret drastic changes of overall behavior observed in reality as statistical phase transitions as understood in Effective/Statistical Field Theory.

A first step in this program is to define an action functional for the system (56) which, in the first instance, does not emanate from a variational principle; it is this objective that is tackled in this Section, refining the works [39, 43].

A. The essential idea: A variational principle for a finite-dimensional system of equations

Consider a generally nonlinear system of algebraic equations in the variables $x \in \mathbb{R}^n$ given by

$$A_\alpha(x) = 0,$$

where $A : \mathbb{R}^n \to \mathbb{R}^N$ is a given function (a simple example would be $A_\alpha(x) = \bar{A}_{\alpha i}\, x^i - b_\alpha$, $\alpha = 1 \ldots N, i = 1 \ldots n$, where $\bar{A}$ is a constant matrix, *not necessarily symmetric* (when $n = N$), and b is a constant vector). We allow for all possibilities $0 < n \lesseqqgtr N$ and the goal is to construct an objective function whose critical points solve the system (when a solution exists) for $x \in \mathbb{R}^n$.

For this, consider first the auxiliary function

$$\widehat{S}_H(x, z) = z^\alpha A_\alpha(x) + H(x)$$

(where H belongs to a class of scalar-valued function to be defined shortly) and define

$$S_H(z) = z^\alpha A_\alpha(x_H(z)) + H(x_H(z))$$

with the requirement that the system of equations

$$z^\alpha \frac{\partial A_\alpha}{\partial x^i}(x) + \frac{\partial H}{\partial x^i}(x) = 0 \tag{55}$$

be solvable for the function $x = x_H(z)$ through the choice of H, and *any* function H that facilitates such a solution qualifies for the proposed scheme. In other words, given a specific H, it should be possible to define a function $x_H(z)$ that satisfies $z^\alpha \partial_{x^i} A_\alpha(x_H(z)) + \partial_{x^i} H(x_H(z)) = 0$ for all $z \in \mathbb{R}^N$ (the domain of the function x_H may accommodate more intricacies, but for now we stick to the simplest possibility). Note that (55) is a set of n equations in n unknowns regardless of N (z for this argument is a parameter).

Assuming this is possible, we have

$$\frac{\partial S_H}{\partial z^\beta}(z) = A_\beta(x_H(z)) + \left(z^\alpha \frac{\partial A_\alpha}{\partial x^i}(x_H(z)) + \frac{\partial H}{\partial x^i}(x_H(z)) \right) \frac{\partial x_H^i}{\partial z^\beta}(z) = A_\beta(x_H(z)),$$

using (55). Thus,

- if z_0 is a critical point of the objective function S_H satisfying $\partial_{z^\beta} S_H(z_0) = 0$, then the system $A_\alpha(x) = 0$ has a solution defined by $x_H(z_0)$;
- if the system $A_\alpha(x) = 0$ has a unique solution, say y, and if z_0^H is any critical point of S_H, then $x_H\left(z_0^H\right) = y$, for all admissible H.
- If $A_\alpha(x) = 0$ has non-unique solutions, but $\partial_{z^\beta} S(z) = 0$ (N equations in N unknowns) has a unique solution for a specific choice of the function $z \mapsto x_H(z)$ related to a choice of H, then such a choice of H may be considered a selection criterion for imparting uniqueness to the problem $A_\alpha(x) = 0$.

B. A class of variational principles for nonlinear dislocation mechanics

We implement the idea of Sec. V A to define an action(s) for the nonlinear partial differential equations of dislocation mechanics given by

$$\begin{aligned}
0 &= e_{jrs} \partial_r W_{is} + \alpha_{ij} \\
0 &= \partial_t W_{ij} + \partial_j (W_{ik} v_k) - v_k e_{rkj} \alpha_{ir} - e_{jrs} \alpha_{ir} V_s(\alpha, W, \rho) = \partial_t W_{ij} + v_k \partial_k W_{ij} + W_{ik} \partial_j v_k - e_{jrs} \alpha_{ir} V_s(\alpha, W, \rho) \\
0 &= \partial_t \rho + \partial_k (\rho v_k) \\
0 &= \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) + \partial_j (\rho W_{ki} \psi'_{kj}).
\end{aligned} \tag{56}$$

First define the functional

$$\widehat{S}_H[A,W,\theta,\rho,\lambda,v,B,\alpha] = \int_{[0,T]\times\Omega} dtd^3x \; - W_{ij}\partial_t A_{ij} - W_{ik}v_k\partial_j A_{ij} - A_{ij}v_k e_{rkj}\alpha_{ir} - A_{ij}e_{jrs}\alpha_{ir}V_s(\alpha,W,\rho)$$
$$- \rho\partial_t\theta - \rho v_k\partial_k\theta$$
$$- \rho v_i\partial_t\lambda_i - \rho v_i v_j\partial_j\lambda_i - \rho W_{ki}\psi'_{kj}\partial_j\lambda_i$$
$$- e_{jrs}W_{is}\partial_r B_{ij} + B_{ij}\alpha_{ij}$$
$$+ H(W,\rho,v,\alpha),$$

which is obtained by converting (56) to scalar form by taking inner products with the 'dual' fields

$$D = (A,\theta,\lambda,B),$$

integrating by parts on the space-time domain assuming the dual fields vanish on the boundary of the domain, and adding the potential H. Now define

$$U := (W,\rho,v,\alpha) \qquad \text{and} \qquad \mathcal{D} := (\partial_t A, \nabla A, A, \partial_t\theta, \nabla\theta, \partial_t\lambda, \nabla\lambda, \nabla B, B)$$

(note '$\mathcal{D} \neq D$') and require that there exists a function

$$U_H(\mathcal{D}) = (W_H(\mathcal{D}), \rho_H(\mathcal{D}), v_H(\mathcal{D}), \alpha_H(\mathcal{D})) \tag{57}$$

such that *for the functional $S_H[A,\theta,\lambda,B]$ of the dual fields defined as*

$$\int_{[0,T]\times\Omega} dtd^3x \; \mathcal{L}_H(\mathcal{D}, U_H(\mathcal{D})) = S_H[A,\theta,\lambda,B] := \widehat{S}_H[A, W_H(\mathcal{D}), \theta, \rho_H(\mathcal{D}), \lambda, v_H(\mathcal{D}), B, \alpha_H(\mathcal{D})], \tag{58}$$

the first variation is given by (we suppress the subscript $_H$ on the elements of U_H for notational simplicity)

$$\begin{aligned}\delta S_H = \int_{[0,T]\times\Omega} dtd^3x \; & - W_{ij}(\mathcal{D})\partial_t\delta A_{ij} - W_{ik}(\mathcal{D})v_k(\mathcal{D})\partial_j\delta A_{ij} - \delta A_{ij}v_k(\mathcal{D})r_{rkj}\alpha_{ir}(\mathcal{D}) \\ & - \delta A_{ij}e_{jrs}\alpha_{ir}(\mathcal{D})V_s(\alpha(\mathcal{D}), W(\mathcal{D}), \rho(\mathcal{D})) \\ & - \rho(\mathcal{D})\partial_t\delta\theta - \rho(\mathcal{D})v_k(\mathcal{D})\partial_k\delta\theta \\ & - \rho(\mathcal{D})v_i(\mathcal{D})\partial_t\delta\lambda_i - \rho(\mathcal{D})v_i(\mathcal{D})v_j(\mathcal{D})\partial_j\delta\lambda_i - \rho(\mathcal{D})W_{ki}(\mathcal{D})\psi'_{kj}(W(\mathcal{D}))\partial_j\delta\lambda_i \\ & - e_{jrs}W_{is}(\mathcal{D})\partial_r\delta B_{ij} + \delta B_{ij}\alpha_{ij}(\mathcal{D}),\end{aligned} \tag{59}$$

a condition that is satisfied if the system

$$\begin{aligned}\frac{\partial\mathcal{L}_H}{\partial W_{lp}} &= -\partial_t A_{lp} - v_p\partial_j A_{lj} - A_{ij}e_{jrs}\alpha_{ir}\frac{\partial V_s}{\partial W_{lp}}(\alpha,W,\rho) - e_{jrp}\partial_r B_{lj} - \rho\left(\psi'_{lj}(W)\partial_j\lambda_p + W_{ki}\psi''_{kjlp}\partial_j\lambda_i\right) \\ &\qquad + \frac{\partial H}{\partial W_{lp}}(W,\rho,v,\alpha) = 0 \\ \frac{\partial\mathcal{L}_H}{\partial\rho} &= -A_{ij}e_{jrs}\alpha_{ir}\frac{\partial V_s}{\partial\rho}(\alpha,W,\rho) - \partial_t\theta - v_k\partial_k\theta - v_i\partial_t\lambda_i - v_i v_j\partial_j\lambda_i - W_{ki}\psi'_{kj}\partial_j\lambda_i \\ &\qquad + \frac{\partial H}{\partial\rho}(W,\rho,v,\alpha) = 0 \\ \frac{\partial\mathcal{L}_H}{\partial v_p} &= -W_{ip}\partial_j A_{ij} - A_{ij}e_{rpj}\alpha_{ir} - \rho\partial_p\theta - \rho\partial_t\lambda_p - \rho v_j\partial_j\lambda_p - \rho v_i\partial_p\lambda_i \\ &\qquad + \frac{\partial H}{\partial v_p}(W,\rho,v,\alpha) = 0 \\ \frac{\partial\mathcal{L}_H}{\partial\alpha_{lp}} &= -A_{lj}v_k e_{pkj} - A_{lj}e_{jps}V_s(\alpha,W,\rho) - A_{ij}e_{jrs}\alpha_{ir}\frac{\partial V_s}{\partial\alpha_{lp}}(\alpha,W,\rho) + B_{lp} \\ &\qquad + \frac{\partial H}{\partial\alpha_{lp}}(W,\rho,v,\alpha) = 0,\end{aligned} \tag{60}$$

can be solved in the form of

$$(W,\rho,v,\alpha) = U_H(\mathcal{D}).$$

This is so, since solving (60) defines $U_H(\mathcal{D})$ that ensures $\frac{\partial \mathcal{L}_H}{\partial U}(U_H(\mathcal{D})) = 0$ which then implies

$$\frac{\partial \mathcal{L}_H}{\partial U}(U_H(\mathcal{D})) \cdot \frac{\partial U_H}{\partial \mathcal{D}}(\mathcal{D}) \cdot \delta\mathcal{D} = 0 \qquad \text{for all } \mathcal{D}.$$

Note that (59) then is simply

$$\delta S_H = \int_{[0,T]\times\Omega} dt d^3x \; \frac{\partial \mathcal{L}_H}{\partial \mathcal{D}} \cdot \delta\mathcal{D},$$

and requiring

$$\delta S_H = 0 \text{ for all variations } \delta D \text{ that vanish on the boundary of } \Omega \times [0,T]$$

shows that the Euler-Lagrange equations of the functional S_H defined in (58) are the equations of (56) with the substitution

$$(W, \rho, v, \alpha) = U_H(\mathcal{D}).$$

To summarize, the primal equations (56) of dislocation mechanics are the Euler-Lagrange equations of any of the dual functionals, written in terms of particular specific combinations (mappings) of the dual fields for each choice of the function H, each specific mapping defining the primal fields. Thus one may think of the primal fields as "gauge invariant" observable combinations of the dual fields ("gauge fields") satisfying one specific set of equations (the primal system). While this is not how gauge fields appear in traditional gauge theories of physics, it is interesting that a completely different starting point and approach raise somewhat similar invariance structures that may be interpreted as symmetries.

As for the plausibility of being able to solve the *algebraic* system (60) given a specific $\mathcal{D}$, consider H to be separately quadratic in each of its arguments, say U_A, with large in magnitude coefficient, so, e.g., $H = \frac{1}{2}\alpha_W W_{ij} W_{ij} + \cdots$, with $1 \ll |\alpha_W|$. Then assuming, the solution of the Euler-Lagrange equations are bounded in some appropriate sense, (60) can indeed be solved to defined $U_H(\mathcal{D})$, and it has to be made sure that the solutions of the Euler-Lagrange equations (using this function) indeed satisfy the assumed bounds. To ensure that this latter condition is satisfied one has a large class of H functions to operate with and, if all else fails, it appears reasonable to demand admitting only those extremals at which the second variation of the functional S_H is (semi)-definite, i.e., in the space of all paths, we admit only those extremal paths as admissible that are local minima/maxima of the action functional S_H (which, of course, has direct conceptual similarity with analyzing the dual problem by a path integral approach based on the action S_H). Another alternative is to regularize the dual E-L system with appropriate terms in the gradients of the dual fields involving small parameters, say ε, and the limit of the solutions to the Euler-Lagrange equations as $\varepsilon \to 0$ may be studied. This second approach need not lend itself to a direct variational characterization of the regularized Euler-Lagrange equations, but can be dualized again, following exactly the procedure outlined above, to produce a variational characterization, if so desired.

We end this interlude with the following remarks:

- Our system (56) does not involve multi-valued fields or non-simply connected domains for defining dislocation dynamics, but is fully capable of representing the topological charge of dislocation lines with its ingredients.
- Based on the explorations of stress-coupled dislocation motion presented in [42, 44], the 'primal' system requires a 'core-energy' in the form of the dependence of the energy function ψ on the dislocation density α as well. This results in the dislocation velocity depending on the curl α. Such a dependence is accommodated within our 'action-generating' scheme by adding an extra variable and equation to the system (56) of the form $e_{jrs}\partial_r \alpha_{is} = \beta_{ij}$ and writing the dislocation velocity as $V_s = V_s(\alpha, W, \rho, \beta)$. This would have the effect of increasing the number of fields in the dual problem as well.

  It is an interesting question whether the precise definition of a formally 'small' core energy contribution with a small parameter representing microscopic physics can make a difference in the development of an accurate model for the prediction of macroscopic behavior, and whether such a device should be allowed in the class of models admitted. Physically, in the context of the physics of dislocation dynamics, there appears to be no reason to exclude the possibility of the importance of such effects and, in fact, allows more precise physics to be incorporated in the description of gross macroscopic behavior (which is, admittedly, a double-edged sword in the context of coarse-graining). Some evidence to support such an expectation is also provided by the mathematically rigorous study of the inviscid Burgers equation, 'regularized' by a small viscous effect in one case and by dispersion in another [45–47].

  Based on the above observation, one advantage of the 'dual' formalism proposed herein may be that when the microscopic physics to be added is not even qualitatively understood with certainty, working with a regularization on the dual side, may be guided solely by the aim of producing a 'good' dual extremal, i.e., with guaranteed existence in an appropriate function space, since this appears to require no modification to the physics of the primal problem.

- In the context of an action functional that simply has as its Euler-Lagrange equation the given system of pde, the proposed scheme delivers, at least formally and under the stated requirements, what is needed. However, if the action functional is to be used in a path integral, dual fields D other than extremals matter as well. In this sense it is reasonable to demand that the added potential H in $\mathcal{L}_H$ be subject to further requirements of invariance that may obstruct the inversion process required to define the function $U_H(\mathcal{D})$. In case such a restriction is so severe as not to allow the definition of even a single 'change of variables' ($U_H(\mathcal{D})$), through the choice of some H), one can retain both the fields W, A and still obtain a relevant action functional, as shown in [39].

C. Linear dislocation mechanics

We illustrate the proposed technique with a very closely related one (using a Legendre transform, cf. [48]) in the simplified setting of linear dislocation mechanics with a prescribed dislocation velocity field V in space-time along with the ansatz

$$
\begin{aligned}
U_{ij} &:= \delta_{ij} - W_{ij} \\
T_{ij} &:= C_{ijkl} U_{kl},
\end{aligned}
$$

ignoring all nonlinearities in (56) and assuming the mass density field ρ to a be specified field. The ansatz is justified for small elastic distortions (U) about the ground state (cf., [39]). We note that C_{ijkl} is necessarily symmetric in (k,l) and (i,j) so that it is not invertible on the space of all second order tensors (and hence the stress only depends on the elastic strain, the symmmetric part of U). With these assumptions, the system (56) may be expressed as

$$
\begin{aligned}
0 &= \partial_j v_i - \partial_t U_{ij} - e_{jrs}\alpha_{ir} V_s \\
0 &= e_j rs \partial_r U_{is} - \alpha_{ij} \\
0 &= \partial_t(\rho v_i) - \partial_j (C_{ijkl} U_{kl}).
\end{aligned}
\tag{61}
$$

Taking inner products of these equations with the dual fields $D = (A, B, \lambda)$ that vanish on the boundary and utilizing an arbitrary function M convex in the list of arguments $M(U, \alpha, v)$ we define the functional

$$
\begin{aligned}
\widehat{S}[A, U, B, \alpha, \lambda, v] = \int_{[0,T]\times\Omega} dt d^3x \; & v_i(-\partial_j A_{ij} - \rho \partial_t \lambda_i) \\
& + U_{ij}(\partial_t A_{ij} - e_{sjr}\partial_r B_{is} + C_{ijkl}\partial_l \lambda_k) \\
& + \alpha_{ir}(-A_{ij} e_{jrs} V_s - B_{ir}) - M(U, \alpha, v)
\end{aligned}
$$

Defining

$$
p := (-\partial_j A_{ij} - \rho \partial_t \lambda_i, \partial_t A_{ij} - e_{sjr}\partial_r B_{is} + C_{ijkl}\partial_l \lambda_k, -A_{ij} e_{jrs} V_s - B_{ir}); \qquad Q := (v.U, \alpha)
$$

and $M^*(p)$ the Legendre transform of $M(Q)$ given by

$$
\begin{aligned}
M^*(p) &= Q_M(p) \cdot p - M(Q_M(p)) \\
(v_M(p), U_M(p), \alpha_M(p)) =: Q_M(p) &= (\partial_Q M)^{-1}(p) \\
\partial_p M^*(p) &= Q_M(p)
\end{aligned}
\tag{62}
$$

(well-defined because of the convexity of $M(Q)$), we define the dual action, $S_M[D]$,

$$
\widehat{S}[A, U_M(p), B, \alpha_M(p), \lambda, v_M(p)] =: S_M[D] = \int_{[0,T]\times\Omega} dt d^3x \; M^*(p)
$$

whose first variation is given by (after an integration by parts)

$$
\begin{aligned}
\delta S_M &= \int_{[0,T]\times\Omega} dt d^3x \; Q_M(p)\, \delta p \\
&= \int_{[0,T]\times\Omega} dt d^3x \; \delta\lambda_i \left(\partial_t(\rho v_i(p)) - \partial_j(C_{ijkl} U_{kl}(p)\right) \\
&\qquad + \; \delta A_{ij} \left(\partial_j (v_i(p)) - \partial_t(U_{ij}(p)) - e_{jrs}\alpha_{ir}(p) V_s\right) \\
&\qquad + \; \delta B_{is}(e_{srj}\partial_r(U_{ij}(p)) - \alpha_{is}(p)),
\end{aligned}
$$

(where we have dropped the subscript $_M$ on the dual-to-primal mapping fields for notational convenience). Thus, the dual Euler-Lagrange equations are the system (61) expressed in terms of the dual fields through the mapping codified in $(62)_2$, *regardless of the convex potential M chosen to define the dual functional S_M.*

This exercise exposes an interesting fact in a simple setting. Clearly, for M to be convex in U it cannot be invariant as it has to depend on the skew-symmetric part of the latter - and rotational invariance/invariance under superposed rigid deformations in the linear setting precludes such a dependence. However, the use of such a potential in the dual theory does not in any way obstruct the definition of correct physics as embodied in the Euler-Lagrange equations solved.

It is a curiosity as to what connection this idea has to proceeding with a notional dual model as discussed in Sec. III when a direct connection to the primal model through a Legendre transform cannot be made due to obstructions arising essentially from invariance requirements.

VI. POWER COUNTING

In any EFT, one needs to develop certain power counting rules for it to be useful. This is because the action that we wrote down contains an infinite number of operators and hence one cannot make any predictions because one would need to determine infinite Wilson coefficients. The observables calculated in the EFT are arranged in a systematic expansion in powers of the expansion parameter(s). This parameter is a small quantity which is usually the ratio of some IR scale/energies to some UV scale. Hence to make a prediction at any given accuracy we need to include only a finite number of operators.

In our theory, we will power count in ratios of lengths and velocities. First we need to identify the scales in our EFT. Since a solid has both longitudinal and transverse phonons, there are two velocities c_T and c_L associated with it. In general, these velocities differ but for purposes we will take them to be similar and use only a single velocity scale which we will take to be c_T. Hence we can use v/c_T as one expansion parameter, v being the velocity of the dislocations.

The lattice spacing a and the dislocation core radius R act as UV scales in the theory. The dislocation radius is of the order of the lattice spacing and hence we can take it to be the cut-off scale. Since we are interested in calculating the forces and stresses due to dislocations, we will only be working with potential modes. For these modes the distance r over which the stress-photons/phonons are exchanged determine the energy and momenta of the modes.

$$\omega \sim \frac{v}{r} \qquad k \sim \frac{1}{r} \tag{63}$$

This is a potential mode because the modes are off-shell. Hence we can also power counting in R/r. Also for these modes $\omega \ll c_T k$ and hence one has expand the propagator for these modes.

Using the scaling of the frequency and momenta, one can read off the scaling of the fields from the action in (35). One can see that the fields scale as

$$\vec{A}(\vec{x},t) \sim \frac{\sqrt{\sigma_0 v}}{r} = \qquad \vec{B}_{pot}(\vec{x},t) \sim \frac{\sqrt{\sigma_0 v}}{c_T r} \tag{64}$$

where $\vec{B}_{pot}$ refers to the potential mode of the phonon. Also the dislocation fields scale as

$$\vec{X} \sim l \qquad \partial_t \vec{X} \sim v \qquad \partial_\rho \vec{X} \sim 1 \tag{65}$$

Using the above equations one can power count the interaction terms originating from the action. The leading contribution to the observables we consider should come from the Kalb-Ramond term since the bulk fields are non-derivatively coupled to the dislocation. The scaling of the KR terms is given by

$$\begin{aligned} n_I \int dt\, d\rho\, \vec{A}^I.\partial_\rho \vec{X} &\sim a\frac{r}{v} l \frac{\sqrt{\sigma_0 v}}{r} \sim \sqrt{m a c_T}\frac{l}{a}\sqrt{\frac{c_T}{v}} \\ n_I \int dt\, d\rho\, \vec{B}^I.(\partial_\rho \vec{X} \times \partial_t \vec{X}) &\sim a\frac{r}{v} l \frac{\sqrt{\sigma_0 v}}{r c_T} v \sim \sqrt{m a c_T}\frac{l}{a}\sqrt{\frac{v}{c_T}} \end{aligned} \tag{66}$$

We can see that coupling of the phonon $\vec{B}$ is suppressed relative to the coupling of the static mode $\vec{A}$. Hence the effective potential due to stress-photon exchange should be dominant as compared to a phonon exchange. Power counting tells us that the effective potentials should scale as

$$V^A_{eff} \sim \sigma_0 a^2 l \qquad V^B_{eff} \sim \sigma_0 a^2 l \frac{v^2}{c_T^2} \tag{67}$$

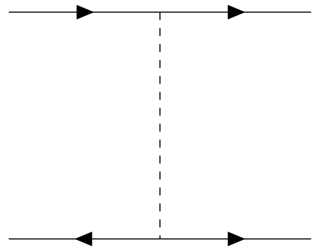

FIG. 1: This diagram shows the leading order contribution to the potential between dislocations

Now lets look at the interactions in the bulk which give corrections to the effective potential. This should be sub-leading to that obtained from the Kalb-Ramond term. The cubic interactions are of the form

$$S_{int} \sim \int d^3x\ dt\ \frac{1}{\sigma_0^2}\Big[(\vec{\nabla}\times\vec{A}^I - \dot{\vec{B}}^I)^3 + c_T^2(\vec{\nabla}.\vec{B}^I)^2(\vec{\nabla}\times\vec{A}^I - \dot{\vec{B}}^I)\Big] \tag{68}$$

One can see from the scaling in (65) that the leading contribution come from the interactions of the form

$$\begin{aligned}\int d^3x\ dt\ \frac{1}{\sigma_0^2}(\vec{\nabla}\times\vec{A}^I)^3 &\sim \frac{1}{\sqrt{mac_T}}\frac{a^2}{r^2}\sqrt{\frac{v}{c_T}}\\ \int d^3x\ dt\ \frac{c_T^2}{\sigma_0^2}(\vec{\nabla}.\vec{B}^I_{pot})^2(\vec{\nabla}\times\vec{A}^I) &\sim \frac{1}{\sqrt{mac_T}}\frac{a^2}{r^2}\sqrt{\frac{v}{c_T}}\end{aligned} \tag{69}$$

We can check that the other cubic bulk interactions are suppressed by powers of v/c_T. For examples we can consider

$$\int d^3x\ dt\ \frac{1}{\sigma_0^2}(\dot{\vec{B}}^I_{pot})^2(\vec{\nabla}\times\vec{A}^I) \sim \frac{1}{\sqrt{mac_T}}\frac{a^2}{r^2}\Big(\frac{v}{c_T}\Big)^{5/2} \tag{70}$$

which is suppressed by (v^2/c_T^2) relative to the contribution in (68). Similarly one can show that the other interaction vertices are also suppressed.

Notice that the KR action includes just a linear coupling of the bulk modes to the dislocation line. For non-linear couplings, we need to invoke the NG action. Also, we need to calculate the size of the coefficients in (54) which can be easily done from dimensional analysis.

$$a_0, a_3 \sim \sigma_0(n_I)^2 \quad a_1, a_2 \sim \rho(n_I)^2 \quad a_4 \sim \sigma_0(n_I)^4 \quad a_5 \sim \rho(n_I)^4 \tag{71}$$

The linear coupling between the bulk modes and the dislocation arising from the NG action is derivatively suppressed and hence is sub-leading in the power counting to the interaction arising from the Kalb-Ramond term. The non-linear interactions can be power counted in a similar way as was done for the Bulk action and Kalb-Ramond term.

VII. APPLICATIONS

Let us apply our effective theory we to calculate the effective potential between two dislocations and also stresses due to individual dislocations. As we have seen from the power counting, the leading contributions are derived from the Kalb-Ramond term. For simplicity we will consider straight infinite dislocation lines parametrized by $\vec{X}(t,z) = (x(t), y(t), z)$. We will first look at the effective potential generated due to stress-photon and phonon exchange.

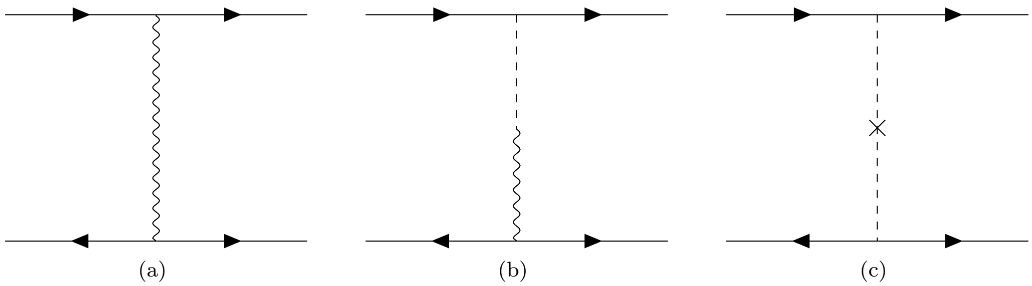

FIG. 2: Diagrams contributing to the potential between moving dislocations. The diagram in (a) is the phonon exchange and (b) is the mixed propagator. (c) shows the contribution from expanding the stress-photon propagator.

A. Effective potentials

The leading contribution to the effective potential is generated due to exchange of a stress-photon as shown in the Feynman diagram**??**. This potential is generated even for a static dislocation. This diagram can be calculated as

$$\begin{aligned} V_{1a} &= \int d^4x\; d^4y\; J_i^a(x) G_{ij}^{ab}(x-y) J_j^b(y) \\ &= \int dt_1 dt_2 \int dz_1 dz_2\; n_1^a n_2^b\; G_{zz}^{ab}(x-y) \\ &= \int dt \int dz_1 dz_2 \int \frac{d^3q}{(2\pi)^3}\; n_1^a n_2^b\; G_{zz}^{ab}(q) e^{i\vec{q}.\vec{r}} \end{aligned} \tag{72}$$

where $G_{ij}^{ab}(q)$ is the stress-photon propagator and $\vec{n}_1$ and $\vec{n}_2$ are the burgers vectors for the dislocations . The propagator matrix in the last line of (72) is given by

$$G_{zz}^{ab}(q) = \begin{pmatrix} -\frac{2q_y^2\mu}{q^4(1-\nu)} & \frac{2q_xq_y\mu}{q^4(1-\nu)} & 0 \\ \frac{2q_xq_y\mu}{q^4(1-\nu)} & -\frac{2q_x^2\mu}{q^4(1-\nu)} & 0 \\ 0 & 0 & -\frac{\mu}{q^2} \end{pmatrix} \tag{73}$$

Expanding the integrand in (72), one finds

$$\begin{aligned} V_{1a} = \int dz \int \frac{d^2q_\perp}{(2\pi)^2} \Big[(\vec{n}_1.\hat{z})(\vec{n}_2.\hat{z}) \frac{-\mu}{q_\perp^2} + (\vec{n}_1 \times \hat{z})_i (\vec{n}_2 \times \hat{z})_j \\ \left(\frac{2\mu}{1-\nu}\right) \frac{\partial_i \partial_j}{q_\perp^4} \Big] e^{i\vec{q}_\perp.\vec{r}_\perp} \end{aligned} \tag{74}$$

where $\vec{q}_\perp = (q_x, q_y)$. We use the general formula to evaluate this integral

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(\vec{k}^2)^\alpha} e^{i\vec{k}.\vec{r}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \Big(\frac{\vec{x}^2}{4}\Big)^{\alpha-d/2} \tag{75}$$

Evaulating this in $d = 2-\epsilon$ in the $\bar{MS}$ scheme, one has

$$\frac{V_{1a}}{L} = \frac{\mu}{2\pi} (\vec{n}_1.\hat{z})(\vec{n}_2.\hat{z}) ln(\frac{r}{2\Lambda}) + (\vec{n}_1 \times \hat{z})_i (\vec{n}_2 \times \hat{z})_j \left(\frac{\mu}{2\pi(1-\nu)}\right) (\delta_{ij} ln(\frac{r}{2\Lambda}) + \frac{x_ix_j}{r^2}) \tag{76}$$

where we have introduced a regularization scale Λ. We see that this agrees with the standard result of classical elasticity.

Next we look at the potential generated due to a phonon exchange. This diagram is shown in Fig. 2a. This potential is only generated by moving dislocations as can be seen from (46). The interaction potential generated by the phonons can be calculated just like we did for the stress-photons. The potential phonon propagators are given by $G_{ij}^{ab}(q) = \mu \delta^{ab} \frac{q_iq_j}{q^4}$. Hence the potential evaluates to

$$\frac{V_{2a}}{L} = \frac{\mu}{4\pi c_T^2}\vec{n}_1.\vec{n}_2\ v_1^i(t)v_2^j(t)\Big(\delta_{ij}ln(\frac{r(t)}{2\Lambda}) + \frac{x_i(t)x_j(t)}{r(t)^2}\Big) \tag{77}$$

There is also a contribution from the mixed propagators which is at the same order in power-counting as the phonon contribution. This is shown in Fig. 2b and evaluates to

$$\frac{V_{2b}}{L} = \frac{2\mu}{(1-\nu)}G_{ijkl}(\vec{r})\Big(\epsilon_{3mj}v_{1i}v_{2m}(\epsilon_{3nl}n_{1k}n_{2n} + \nu\delta_{kl}(\vec{n}_1\times\vec{n}_2).\hat{z}) + v_1 \leftrightarrow v_2, n_1 \leftrightarrow n_2\Big) \tag{78}$$

where

$$\begin{aligned} G_{ijkl}(\vec{x}) = \partial_i\partial_j\partial_k\partial_l\int\frac{d^2q}{(2\pi)^2}\frac{e^{i\vec{q}.\vec{x}}}{\vec{q}^6} =& \frac{1}{256\pi}\Big(16\frac{(\delta_{ij}x_lx_k + \delta_{ik}x_jx_l + \delta_{il}x_kx_j + \delta_{jl}x_ix_k + \delta_{jk}x_ix_l + \delta_{lk}x_ix_j)}{\vec{x}^2} \\ &+(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})(12 + 16ln(\frac{r}{2\Lambda})) - 32\frac{x_ix_jx_kx_l}{x^4}\Big) \end{aligned} \tag{79}$$

To the best of our knowledge, the results (77) and (78) have not been obtained in the literature.

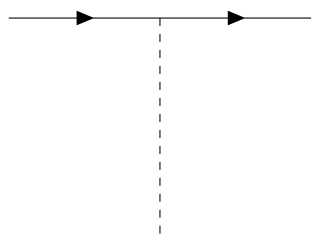

FIG. 3: Diagram contributing to the one point function of the stress fields for a stationary dislocation.

B. Stress due to dislocations

We can use our formalism to also calculate the stresses due to dislocations. In the dual theory the linear stress is described by $\tilde{T}^i_j$ as can be seen from (39). Hence the one-point function of $\tilde{T}^i_j$ gives us the stress.

$$\sigma_i^{\ j} = \langle T_i^j\rangle = \epsilon_{imn}\langle\partial_m A_n^j\rangle - \langle\partial_t B_i^j\rangle \tag{80}$$

We will first look at the stress due to a straight static dislocation. The only contribution comes from the stress-photon in this case. This contribution is shown in Fig. 3a. The stress is given by

$$\begin{aligned} \sigma_i^j(\vec{r},t) &= \int_p\int_k\ \epsilon_{imn}p_m\langle A_n^j(\vec{p},t)A_k^l(\vec{k},t)\rangle J_k^l(\vec{k},t)e^{i\vec{p}.\vec{r}} \\ &= n^l\int_p\ \epsilon_{imn}p_m(G_A)^{jl}_{nz}(\vec{p},t)\delta(p_z)e^{i\vec{p}.\vec{r}} \end{aligned} \tag{81}$$

where in the second line we have used the fourier transform of the current due to a straight dislocation in the z-direction $J_k^l(\vec{p},t) = \delta(p_z)n^l\delta_{kz}$. We also denote the momentum integral $\int\frac{d^3p}{(2\pi)^3}$ by $\int_p$. Let us calculate the stress fields for a screw dislocation, where the burgers vector is along the dislocation line i.e. in the z-direction. Using the propagators defined in (73) one finds,

$$\begin{aligned} \sigma_i^j(\vec{x},t) &= \mu n^z\int\frac{d^2p_\perp}{(2\pi)^2}\frac{(\delta_{ix}\delta_{jz}p_y - \delta_{iy}\delta_{jz}p_x)}{p_\perp^2}e^{i\vec{p}_\perp.\vec{r}} + i\leftrightarrow j \\ &= \frac{\mu n_z}{(2\pi)}\Big((\delta_{ix}\delta_{jz} + \delta_{iz}\delta_{jx})\frac{y}{r^2} - (\delta_{iy}\delta_{jz} + \delta_{iz}\delta_{jy})\frac{x}{r^2}\Big) \end{aligned} \tag{82}$$

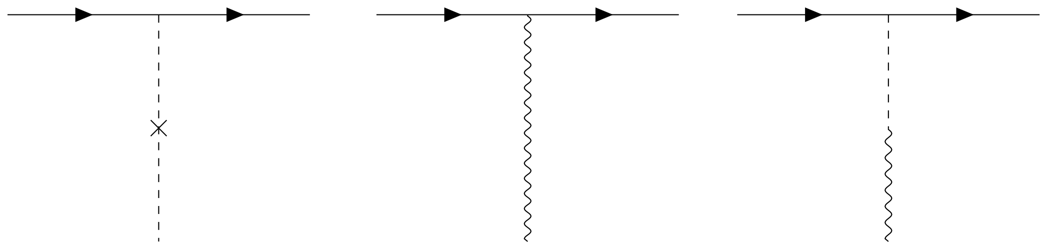

FIG. 4: Diagrams contributing to the one point function of the stress fields for a moving dislocation.

which agrees with the standard results. Similarly for a edge dislocation where the burgers vector is perpendicular to the dislocation line, one obtains

$$\sigma_i^j(\vec{x},t) = \frac{\mu}{\pi(1-\nu)r^2}\Big(\epsilon_{zkl}n_l x_k g_{ij} - \frac{1}{2}\epsilon_{zkl}n_l(\delta_{ij}x_k + \delta_{jk}x_i + \delta_{ik}x_j - \frac{2x_i x_j x_k}{r^2})\Big) \tag{83}$$

where g_{ij} =diag$(1,1,\nu)$. This also agrees with the standard results.

Let us now calculate the stress due to a moving dislocation. In this case, the leading contribution still comes from (82) and(83) but there are also subleading corrections which arise from Fig. 4a, 4b and 4c. Fig. 4a is the result of expanding the stress-photon propagator to next-to-leading order whereas 4b and 4c arise from the phonon and the mixed propagator respectively. The sum of the subleading contributions is given by

$$\begin{aligned}\sigma_{ij} = \text{Fig. 4a + 4b + 4c} = n^l\delta_{kz}\int\frac{d^2p_\perp}{(2\pi)^2}\ &\Big[\epsilon_{imn}p_m(G_A)^{jl}_{nk}(\vec{p},\omega_p) + \epsilon_{imn}p_m\epsilon_{skr}v^s_\perp(G_{AB})^{jl}_{nr}(\vec{p},\omega_p) + \omega_p\epsilon_{skr}v^s_\perp(G_B)^{jl}_{ir}(\vec{p},\omega_p)\\ &+ \omega_p(G_{BA})^{jl}_{ik}(\vec{p},\omega_p)\Big]e^{i\omega_p t - i\vec{p}.\vec{r}}|_{\omega_p=\vec{p}_\perp.\vec{v}_\perp}\\ =\ &\frac{(\delta_{lm}x_n + \delta_{ln}x_m + \delta_{mn}x_l - 2x_m x_n x_l)}{r^2}\Big(\frac{\mu v_m v_n}{2\pi(1-\nu)^2}(\epsilon_{lk}n_k(\delta_{ij}(1-2\nu) - \delta_{iz}\delta_{jz}(1-\nu)^2)\\ &+ (n_i\epsilon_{zjl} + \delta_{jl}\epsilon_{zik}n_k)(1-\nu)^2) - \frac{\mu v_m v_k\epsilon_{nk}}{2\pi(-1+\nu)}(n_l\delta_{ij}\nu + (1-\nu)n_i\delta_{jl} + n_j\delta_{il})\Big)\\ &+ \partial_m G_{ijkl}(\vec{r})\Big(\frac{2\mu v_k v_l}{(1-\nu)^2}(\epsilon_{zml}n_l\nu) - \frac{2\mu v_m}{(-1+\nu)}\epsilon_{zkn}v_n n_l\Big)\end{aligned} \tag{84}$$

with $J^l_k(\vec{k},\omega_k) = \delta(k_z)\delta(\omega_k - \vec{k}.\vec{v}_\perp)n^l\delta_{kz}$ and G_{ijkl} was defined in equation (79). Hence the stress due to a moving dislocation is given by the sum of diagrams in Fig. 3 and Fig 4. This is a new analytical result that has not be presented in the literature before.

VIII. CONCLUSION

In this paper, we have presented an effective field theory of phonons and their interactions with dislocations using a dual gauge theory. We use symmetries as our guiding principles to construct the most general theory of phonons and dislocations. We systematically derive the non-linear Lagrangian for the phonons and also use the coset construction to derive the dislocation dynamics. Using this we are able to calculate the forces between dislocations and the stresses induced by individual dislocation, including the contribution of dynamical phonons. We leave the study of non-linearities for future work.

IX. ACKNOWLEDGEMENTS

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Appendix A: Couple-stress theory

The e.o.m's we derived in (36) received contributions only from the symmetric incompatibility tensor $W_{(ij)}$. In general this is not true and includes anti-symmetric contributions as well. To derive the exact equations of motion, one needs to consider contributions of anti-symmetric phonon gradients in the action. This is known as couple-stress theory. These contributions enter the action at higher order in derivatives per field and are hence subleading. From a symmetry breaking point of view, these are precisely the rotational goldstones θ^i which arise due to spontaneous breaking of rotations by the ground state of the solid. The action for the rotational goldstones can be written as

$$\mathcal{L}_\theta = c_4(\nabla_t \theta^i)^2 - c_5(\nabla_j \theta^i)^2 \tag{A1}$$

But these goldstones are redundant and hence using certain Inverse Higgs constraints, one can rewrite them in terms of phonons **??**. At leading order, the rotational goldstones can be written as $\theta^i = \frac{1}{2}\epsilon^{ijk}W_{jk}$. As one can see, these are anti-symmetric in the incompatibilty tensor. Including these contributions the quadratic action for the phonons is

$$\begin{aligned}\mathcal{L}^{(2)} = \frac{\rho}{2}(W^{0i})^2 - \frac{1}{2}C_{ijkl}W_{(ij)}W_{(kl)} + c_4(\partial_t W_{[mn]})^2 \\ - c_5(\partial_j W_{[mn]})^2\end{aligned} \tag{A2}$$

The coefficients scale as $c_4 \sim \frac{\rho}{\Lambda}$ and $c_5 \sim \frac{\mu}{\Lambda}$ where the cutoff scale Λ is determined by the inverse lattice spacing. We can now define the tensor $\tilde{C}_{ijkl} = C_{ijkl} - (c_4\, \partial_t^2 - c_5\partial^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. Using the Hubbard-stratonovich transformation one obtains the Lagrangian

$$\mathcal{L}^{(2)} = \frac{1}{2\rho}(\tilde{T}^{0i})^2 - \frac{1}{2}\tilde{T}^{ij}\tilde{C}^{-1}_{ijkl}\tilde{T}^{kl} \tag{A3}$$

where now the stress fields are now given by $\tilde{T}^{0i} = \rho W^{0i}$ and $\tilde{T}^{ij} = \tilde{C}^{ijkl}W_{kl}$. Using the dual definition of the stress fields $T^{0i} = \vec{\nabla}.\vec{B}^i$ and $T^{ij} = (\vec{\nabla} \times \vec{A}^i)_j - \dot{B}^i_j$ and varying the action, one obtains

$$\begin{aligned}&\epsilon_{ijk}\partial_j W_{kI} = \alpha^I_i \\ &\partial_t W_{iI} + \partial_i v^I = \epsilon_{ijk}\alpha^I_j v_k\end{aligned} \tag{A4}$$

These agree with the standard incompatibility equations for elasticity.

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