(1) Gouy-Chapman Treatment of a Charged Membrane in Buffer  

A planar interface that separates a dielectric volume from an electrolytic solution (just like the surface of a membrane) carries an area charge density \( \sigma = Ze/a \), where \( Z \) and \( a \) are the valency of an ion and the average area it occupies (or the charge of a lipid and its area within the membrane), and \( e \) is the elementary charge. The direction perpendicular to the interface is \( \hat{x} \).

(a) Derive the concentrations of dissolved positive and negative ions near the surface, \( c_+(x) \) and \( c_-(x) \), respectively, and the electrostatic potential in the solution, \( V(x) \), using the linearized Poisson-Boltzmann approach. Determine the numerical value of the Debye length \( \lambda_D \) at \( T = 300 \text{ K} \) for \( Z = 1 \) and a salt concentration in the bulk solution of \( c_\infty = 100 \text{ mM} \). Using the Bjerrum length \( l_B = 7 \text{ Å} \) in \( \text{H}_2\text{O} \), determine the limiting area charge density for which the linearized approach is valid. For these conditions, plot the potential \( V(x) \) and charge distributions \( c_+(x) \) and \( c_-(x) \) in the solution near the interface.

(b) Show explicitly that the charge accumulated in the ion cloud above the interface compensates the surface charge.

(c) A typical charge density of a biomembrane surface can be estimated by assuming that 10% of all lipids are anionic. The average area of a dual-chain phospholipid is 0.7 nm\(^2\) per molecule. Determine the Debye length \( \lambda_D \) and the surface potential \( V(x = 0) \) of such a membrane at 300 K in 100 mM NaCl and calculate the binding energy of the trivalent polyamine spermidine to the membrane surface in units of thermal energy.