The probability density of the sum of two uncorrelated random variables is not necessarily the convolution of its two marginal densities

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If two random variables X and Y are *independent*, then the probability density of their sum is equal to the convolution of the probability densities of X and Y. With obvious notation, we have

$$p_{X+Y}(z) = \int \mathrm{d}x \, p_X(x) \, p_Y(z-x) \; . \tag{1}$$

The proof is simple: Independence of the two random variables implies that

$$p_{X,Y}(x,y) = p_X(x) \, p_Y(y) \; . \tag{2}$$

And by the transformation theorem for probability densities we immediately get

$$\begin{aligned} p_{X+Y}(z) &= \int \mathrm{d}x \int \mathrm{d}y \, p_{X,Y}(x,y) \, \delta(x+y-z) \\ &= \int \mathrm{d}x \, p_{X,Y}(x, z-x) \qquad (3) \\ &\stackrel{(2)}{=} \int \mathrm{d}x \, p_X(x) \, p_Y(z-x) \; . \qquad (4) \end{aligned}$$

We here want to convince ourselves by a counterexample that *uncorrelatedness* of the random variables does *not suffice* for the convolution formula to hold. To see this, let us look at the probability density

$$p_{X,Y}(x,y) = \frac{x^2+y^2}{4\pi} \mathrm{e}^{-\frac{1}{2}(x^2+y^2)} \; . \tag{5}$$

This probability density evidently does not factorize. Indeed, the marginal densities are given by

$$p_X(x) \;=\; \int \mathrm{d}y \, p_{X,Y}(x,y) \;=\; \frac{1+x^2}{\sqrt{8\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \; , \tag{6}$$

with the same functional form of course also holding for $p_Y(y)$. Evidently, Eqn. (2) does not hold for this choice of $p_{X,Y}(x,y)$ and its marginal densities $p_X(x)$ and $p_Y(y)$. However, since $p_{X,Y}(x,y)$ is rotationally symmetric about the origin, the covariance of X and Y vanishes, *hence X and Y are uncorrelated, and yet dependent.*

What is now the probability density of $X+Y$? From the transformation theorem we get

$$\begin{aligned} p_{X+Y}(z) &\stackrel{(3)}{=} \int \mathrm{d}x \, p_{X,Y}(x, z-x) \\ &\stackrel{(5)}{=} \int \mathrm{d}x \, \frac{x^2+(z-x)^2}{4\pi} \mathrm{e}^{-\frac{1}{2}[x^2+(z-x)^2]} \\ &= \frac{2+z^2}{8\sqrt{\pi}} \mathrm{e}^{-\frac{1}{4}z^2} \; . \qquad (7) \end{aligned}$$

On the other hand, the convolution of p_X and p_Y is

$$\begin{aligned} [p_X * p_Y](z) &= \int \mathrm{d}x \, p_X(x) \, p_Y(z-x) \\ &\stackrel{(6)}{=} \int \mathrm{d}x \, \frac{1+x^2}{\sqrt{8\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \, \frac{1+(z-x)^2}{\sqrt{8\pi}} \mathrm{e}^{-\frac{1}{2}(z-x)^2} \\ &= \frac{z^4+4z^2+44}{128\sqrt{\pi}} \mathrm{e}^{-\frac{1}{4}z^2} \; , \qquad (8) \end{aligned}$$

which differs from the correct answer. Fig. 1 illustrates the difference between these two functions.

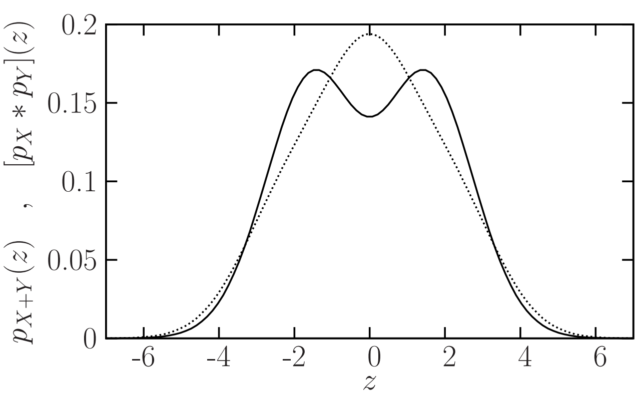

FIG. 1: True probability density of the sum random variable $p_{X+Y}(z)$ (solid line) and convolution of its marginal densities, $[p_X * p_Y](z)$ (dotted line).