How to generate equidistributed points on the surface of a sphere

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There are two qualitatively different ways in which we could try to achieve equidistribution of points on a surface. One is to randomly place them in such a way that the probability of ending up in some particular region is proportional to the area of that region (two-dimensional Poisson statistics). This gives equidistribution on average. The second is to regularly place points such that their distance in two orthogonal directions is locally always the same. This gives typically a better result (no fluctuations and no accidental overlap), but the emerging partial crystallinity could sometimes be undesirable. For the case of a sphere an example for both strategies is presented.

I. SPHERICAL COORDINATES

The most straightforward way to create points on the surface of a sphere are classical spherical coordinates, in which a point is addressed via its two angular coordinates, the polar angle \( \theta \in [0; \pi] \) and the azimuthal angle \( \varphi \in [0, 2\pi] \). If the sphere has radius \( r \), the Cartesian coordinates of that point are given by

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = r \begin{pmatrix}
\sin \theta \cos \varphi \\
\sin \theta \sin \varphi \\
\cos \theta
\end{pmatrix}.
\tag{1}
\]

II. RANDOM PLACEMENT

When one wants to evenly place points on the surface of a sphere, it is important to realize that it is not correct to simply choose the spherical angles \( \theta \) and \( \varphi \) equidistributed from their permissible intervals. In order to find out what has to be done instead, it is helpful to have a look back at the area element. Notice that it can be written as

\[
dA = r^2 \sin \theta \, d\theta \, d\varphi = r \, d(r \cos \theta) \, d\varphi = r \, dz \, d\varphi. \tag{2}
\]

We thus see that it is correct to choose \( \varphi \) equidistributed from \([0; 2\pi]\) and to also choose \( z \) equidistributed from \([-r; r]\). This then gives the following algorithm for placing \( N \) randomly equidistributed points on the surface of a sphere of radius \( r \):

```
repeat \( N \) times {
  Choose \( z \) equidistributed from \([-r; r]\).
  Choose \( \varphi \) equidistributed from \([0; 2\pi]\).
  Set \( x = \sqrt{r^2 - z^2} \cos \varphi \).
  Set \( y = \sqrt{r^2 - z^2} \sin \varphi \).
}
```

The result of such an algorithm is illustrated in Fig. 1a.

III. REGULAR PLACEMENT

Regular equidistribution can be achieved by choosing circles of latitude at constant intervals \( d\varphi \) and on these circles points with distance \( d\varphi \), such that \( d\varphi \approx d\varphi \) and that \( d\varphi d\varphi \) equals the average area per point. This then gives the following algorithm:

```
Set \( N_{\text{count}} = 0 \).
Set \( a = 4\pi r^2/N \) and \( d = \sqrt{a} \).
Set \( M_\varphi = \text{round}[\pi/d] \).
Set \( d_\theta = \pi/M_\varphi \) and \( d_\varphi = a/d_\varphi \).
For each \( m \) in 0...\( M_\varphi - 1 \) do {
  Set \( \varphi = \pi(m + 0.5)/M_\varphi \).
  Set \( M_\varphi = \text{round}[2\pi \sin \varphi / d_\varphi] \).
  For each \( n \) in 0...\( M_\varphi - 1 \) do {
    Set \( \varphi = 2\pi n/M_\varphi \).
    Create point using Eqn. (1).
    \( N_{\text{count}} += 1 \).
  }
}
```

At the end of this algorithm \( N_{\text{count}} \) points have been placed, with \( N_{\text{count}} \) very close to \( N \). The result of such an algorithm is illustrated in Fig. 1b.

Note that the regular placement is far more even: neither local clustering (points almost sitting on top of each other) nor global \( \sqrt{N} \) fluctuations occur. For setting up a liquid state the second algorithm thus appears more favorable.