Electric field of a plane with a one-dimensional rectangular-periodic charge density

Markus Deserno  
Max-Planck-Institut für Polymerforschung, Ackermannweg 10, 55128 Mainz, Germany

(Dated: January 12, 2005)

A charge density with one-dimensional periodicity can be expanded in a simple Fourier series. Since for each mode the electric field can be calculated, the complete field and its square follow readily. For the special case of “rectangular wave” (\(\nabla l\)) the resulting polarization force and its lateral average are calculated explicitly. To leading order (relevant for the far field) they decay exponentially with a characteristic length equal to the rectangular wavelength divided by 4\(\pi\).

Consider a periodic one-dimensional charge density in the \(xy\)-plane, given by \(\sigma(x) = \sigma_0 \cos(kx)\), where \(\sigma_0\) is the amplitude and \(k = \frac{2\pi}{\lambda}\) the wave vector of the periodic charge modulation (of wavelength \(\lambda\)). The electric field \(E_k\) at position \((x, y, z)\) above the plane is given by

\[
E_k(x, y, z) = \frac{1}{4\pi\varepsilon_0} \int_\infty \frac{d\tilde{z}}{[\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2]^{3/2}} \frac{x - \tilde{x}}{\tilde{y}} \frac{\sigma(\tilde{x})}{z} = \frac{\sigma_0}{2\pi\varepsilon_0} \int_\infty \frac{d\xi}{\xi^2 + z^2} \left( \frac{\sin(\xi)}{\xi} \right) .
\]

A charge density that is also periodic with wavelength \(\lambda\), but not a simple cosine function, can be expanded in a Fourier cosine series in the following way:

\[
\sigma(x) = \sum_{n=1}^{\infty} \sigma_n \cos(nkx) .
\]

The corresponding electric field is then given by

\[
E(x, y, z) = \frac{1}{2\varepsilon_0 \varepsilon_r} \sum_{n=1}^{\infty} \sigma_n e^{-nkz} \begin{pmatrix} \sin(nkz) \\ 0 \\ \cos(nkz) \end{pmatrix} .
\]

Let us look at the example of a rectangular-periodic charge density of amplitude \(\sigma_0\). Its Fourier expansion is

\[
\sigma_{\nabla l}(x) = \sigma_0 \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos[(2n+1)kx]}{2n+1} .
\]

Combining Eqns. (2,3), we obtain the electric field

\[
E(x, y, z) = \frac{2\sigma_0}{\pi\varepsilon_0\varepsilon_r} \sum_{n=0}^{\infty} (-1)^n \frac{\cos[(2n+1)kx]}{2n+1} \begin{pmatrix} \sin[(2n+1)kx] \\ 0 \\ \cos[(2n+1)kx] \end{pmatrix} .
\]

This expresses the result as a kind of “Laplace series”. For large distances \(z\) the first term in the series is the most important one, and up to the understandable prefactor \(4/\pi\) it coincides exactly with the field of a single cosine mode of wavelength \(\lambda\), see Eqn. (1). In other words: The far field of a rectangular-periodic charge density is \(4/\pi\) times the far field of a cosine-periodic charge density of the same wavelength.

With the help of MATHEMATICA™, the Laplace series can be written in a closed form:

\[
E(x, y, z) = \frac{4}{\pi} \times \frac{\sigma_0}{4\varepsilon_0\varepsilon_r} \times \left( \frac{\arctan \left( \frac{\sin(kz)}{\cos(kz)} \right)}{\arctan \left( \frac{\sin(kz)}{\cos(kz)} \right)} \right) .
\]

Assume that there’s a (point-sized) object of (scalar) polarizability \(\alpha\) at position \((x, y, z)\) above the plane. It will develop a polarization \(P = \alpha E\) and will thus have an electrostatic energy

\[
\mathcal{E} = -\int P \cdot dE = -\frac{1}{2} \alpha E^2 .
\]

The force in \(z\) direction on that object is thus given by

\[
F(x, y, z) = -\frac{\partial \mathcal{E}}{\partial z} = \frac{\sigma_0}{2\varepsilon_0\varepsilon_r} \left( \frac{4}{\pi} \times \frac{\sigma_0}{2\varepsilon_0\varepsilon_r} \right)^2 \times \left[ \cos(2kx) + \cosh(2kz) \right]^{-1} \times \left\{ \arctan \frac{\cos(kz)}{\sin(kz)} \cos(kz) \right. \\
+ \left. \arctanh \frac{\sin(kz)}{\cosh(kz)} \sin(kz) \right\} e_z .
\]

Unfortunately, it is hard to see what happens to this expression after averaging over all \(x\)-positions. However, we can first expand it for large \(kz\), by which we obtain

\[
F(x, y, z) \approx -k\alpha \left( \frac{4}{\pi} \times \frac{\sigma_0}{2\varepsilon_0\varepsilon_r} \right)^2 \times \left( e^{-2kz} - \frac{4}{3} \cos(2kx)e^{-4kz} + \left[ \frac{1}{3} + \frac{6}{5} \cos(4kx) \right] e^{-6kz} + \ldots \right) e_z .
\]

The leading order is \textit{independent} of \(x\) and decays like \(e^{-2kz}\), just as one would expect it from the leading order electric field. The expansion (4) can now be averaged over \(x\) quite easily. In fact, it turns out that the non-vanishing terms are very simple, hence the expansion can be re-summed to obtain the exact result in closed form:

\[
\langle F(x, y, z) \rangle = -k\alpha \left( \frac{4}{\pi} \times \frac{\sigma_0}{2\varepsilon_0\varepsilon_r} \right)^2 \times \sum_{n=0}^{\infty} e^{-2(2n+1)kz} \frac{\Gamma(2n+1)}{2n+1} e_z = -k\alpha \left( \frac{4}{\pi} \times \frac{\sigma_0}{2\varepsilon_0\varepsilon_r} \right)^2 \times \left( \frac{\arctanh e^{-2kz}}{4 \log \cosh e^{2kz}} \right) e_z .
\]