Electric field of a plane with a one-dimensional rectangular-periodic charge density

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A charge density with one-dimensional periodicity can be expanded in a simple Fourier series. Since for each mode the electric field can be calculated, the complete field and it's square follow readily. For the special case of "rectangular wave" ($\sqcap\!\sqcup$) the resulting polarization force and its lateral average are calculated explicitly. To leading order (relevant for the far field) they decay exponentially with a characteristic length equal to the rectangular wavelength divided by 4π.

Consider a periodic one-dimensional charge density in the xy-plane, given by $\sigma_k(x) = \sigma_0 \cos(kx)$, where σ_0 is the amplitude and $k = \frac{2\pi}{\lambda}$ the wave vector of the periodic charge modulation (of wavelength λ). The electric field $\boldsymbol{E}_k$ at position $(x,y,z)^\top$ above the plane is given by

$$\begin{aligned} \boldsymbol{E}_k(x,y,z) &= \frac{1}{4\pi\varepsilon_0\varepsilon_\mathrm{r}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\;\sigma(\bar{x})}{[(x-\bar{x})^2+\bar{y}^2+z^2]^{3/2}} \begin{pmatrix} x-\bar{x} \\ \bar{y} \\ z \end{pmatrix} \\ &= \frac{\sigma_0}{2\pi\varepsilon_0\varepsilon_\mathrm{r}} \int_{-\infty}^{\infty} \mathrm{d}\xi\; \frac{\cos[k(\xi+x)]}{\xi^2+z^2} \begin{pmatrix} -\xi \\ 0 \\ z \end{pmatrix} \\ &= \frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}} \mathrm{e}^{-kz} \begin{pmatrix} \sin(kx) \\ 0 \\ \cos(kx) \end{pmatrix} . \end{aligned} \tag{1}$$

A charge density that is also periodic with wavelength λ, but *not* a simple cosine function, can be expanded in a Fourier cosine series in the following way:

$$\sigma(x) = \sum_{n=1}^{\infty} \sigma_n \cos(nkx) \; .$$

The corresponding electric field is then given by

$$\boldsymbol{E}(x,y,z) = \frac{1}{2\varepsilon_0\varepsilon_\mathrm{r}} \sum_{n=1}^{\infty} \sigma_n\,\mathrm{e}^{-nkz} \begin{pmatrix} \sin(nkx) \\ 0 \\ \cos(nkx) \end{pmatrix} . \tag{2}$$

Let us look at the example of a rectangular-periodic charge density of amplitude σ_0. It's Fourier expansion is

$$\sigma_{\sqcap\sqcup}(x) = \sigma_0\, \frac{4}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos[(2n+1)kx]}{2n+1} \; . \tag{3}$$

Combining Eqns. (2,3), we obtain the electric field

$$\boldsymbol{E}(x,y,z) = \frac{2\sigma_0}{\pi\varepsilon_0\varepsilon_\mathrm{r}} \sum_{n=0}^{\infty} (-1)^n \frac{\mathrm{e}^{-(2n+1)kz}}{2n+1} \begin{pmatrix} \sin[(2n+1)kx] \\ 0 \\ \cos[(2n+1)kx] \end{pmatrix} .$$

This expresses the result as a kind of "Laplace series". For large distances z the first term in the series is the most important one, and up to the understandable prefactor $4/\pi$ it coincides exactly with the field of a single cosine mode of wavelength λ, see Eqn. (1). In other words: *The far field of a rectangular-periodic charge density is $4/\pi$ times the far field of a cosine-periodic charge density of the same wavelength.*

With the help of MATHEMATICA™, the Laplace series can be written in a closed form:

$$\boldsymbol{E}(x,y,z) = \frac{4}{\pi} \times \frac{\sigma_0}{4\varepsilon_0\varepsilon_\mathrm{r}} \times \begin{pmatrix} \operatorname{artanh} \frac{\sin(kx)}{\cosh(kz)} \\ 0 \\ \arctan \frac{\cos(kx)}{\sinh(kz)} \end{pmatrix} .$$

Assume that there's a (point-sized) object of (scalar) polarizability α at position $(x,y,z)^\top$ above the plane. It will develop a polarization $\boldsymbol{P} = \alpha\boldsymbol{E}$ and will thus have an electrostatic energy

$$\mathcal{E} = -\int \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{E} = -\frac{1}{2}\alpha E^2 \; .$$

The force in z direction on that object is thus given by

$$\begin{aligned} \boldsymbol{F}(x,y,z) &= -\frac{\partial\mathcal{E}(x,y,z)}{\partial z}\boldsymbol{e}_z \\ &= -\frac{1}{2}k\alpha\left(\frac{4}{\pi}\times\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \times \big[\cos(2kx)+\cosh(2kz)\big]^{-1} \times \\ &\quad \Big\{ \arctan\frac{\cos(kx)}{\sinh(kz)}\cos(kx)\cosh(kz) \\ &\quad + \operatorname{artanh}\frac{\sin(kx)}{\cosh(kz)}\sin(kx)\sinh(kz) \Big\}\,\boldsymbol{e}_z \; . \end{aligned}$$

Unfortunately, it is hard to see what happens to this expression after averaging over all x-positions. However, we can first expand it for large kz, by which we obtain

$$\begin{aligned} \boldsymbol{F}(x,y,z) &\overset{kz\gg 1}{=} -k\alpha\left(\frac{4}{\pi}\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \times \Big\{\mathrm{e}^{-2kz} - \frac{4}{3}\cos(2kx)\mathrm{e}^{-4kz} \\ &\quad + \Big[\frac{1}{3}+\frac{6}{5}\cos(4kx)\Big]\mathrm{e}^{-6kz} \mp \cdots\Big\}\,\boldsymbol{e}_z \; . \end{aligned} \tag{4}$$

The leading order is *independent* of x and decays like e^{-2kz}, just as one would expect it from the leading order electric field. The expansion (4) can now be averaged over x quite easily. In fact, it turns out that the non-vanishing terms are very simple, hence the expansion can be re-summed to obtain the exact result in closed form:

$$\begin{aligned} \langle\boldsymbol{F}(x,y,z)\rangle &= -k\alpha\left(\frac{4}{\pi}\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \times \sum_{n=0}^{\infty}\frac{\mathrm{e}^{-2(2n+1)kz}}{2n+1}\,\boldsymbol{e}_z \\ &= -k\alpha\left(\frac{4}{\pi}\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \times \underbrace{\operatorname{artanh}\mathrm{e}^{-2kz}}_{=\frac{1}{2}\log\coth\mathrm{e}^{kz}}\,\boldsymbol{e}_z \; . \end{aligned} \tag{5}$$