Electric field of a plane with a periodic charge modulation

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An overall charge-neutral substrate can nevertheless have an electric field, which then stems from a modulation of a charge density with zero monopole moment. A plane with a $1d$- and $2d$-periodic charge density (the latter with square and hexagonal symmetry) is discussed in these notes.

I. $1d$ PERIODICITY

Consider the xy-plane equipped with a $1d$-periodic charge density of the form

$$\sigma_{k,\varphi}(x) \;=\; \sigma_0 \cos(kx+\varphi) \;, \tag{1}$$

where σ_0 is the amplitude and $k=\frac{2\pi}{\lambda}$ the wave vector of the periodic charge modulation (of wavelength λ) in x-direction, and φ is an offset phase (see Fig.1a for an illustration). The electric field $\boldsymbol{E}_{k,\varphi}$ at position $(0,0,z)^\top$ above the plane is obtained by integrating up all $\frac{1}{r^2}$-contributions over the entire plane, which readily leads to the expression

$$\begin{aligned}
\boldsymbol{E}_{k,\varphi}(z) \;&=\; \frac{1}{4\pi\varepsilon_0\varepsilon_\mathrm{r}} \int_{-\infty}^{\infty} \mathrm{d}\bar{x}\,\mathrm{d}\bar{y}\; \frac{\sigma(\bar{x})}{(\bar{x}^2+\bar{y}^2+z^2)^{3/2}} \begin{pmatrix} -\bar{x} \\ -\bar{y} \\ +z \end{pmatrix} \\
&=\; \frac{\sigma_0}{2\pi\varepsilon_0\varepsilon_\mathrm{r}} \int_{-\infty}^{\infty} \mathrm{d}\bar{x}\; \frac{\cos(k\bar{x}+\varphi)}{\bar{x}^2+z^2} \begin{pmatrix} -\bar{x} \\ 0 \\ z \end{pmatrix} \\
&=\; \frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}} \mathrm{e}^{-kz} \begin{pmatrix} \sin\varphi \\ 0 \\ \cos\varphi \end{pmatrix} \;.
\end{aligned} \tag{2}$$

Averaged over the offset-phase φ (equivalently: over all x-positions) the charge density is zero; but nevertheless, the electric field does not vanish identically. Still, just like the charge density, it also vanishes on average:

$$\langle \boldsymbol{E}_{k,\varphi}(z)\rangle_\varphi \;=\; \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\; \boldsymbol{E}_{k,\varphi}(z) \;=\; \mathbf{0} \;. \tag{3}$$

Therefore, on average there is no force on a (point-) charge a distance z above the plane.

However, what does *not* vanish is the *square* of the electric field; it does not even depend on φ:

$$E_k^2(z) \;=\; \left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz} \;. \tag{4}$$

Assume that there's a (point-sized) object of (scalar) polarizability α at position $(0,0,z)^\top$ above the plane. It will develop a polarization

$$\boldsymbol{P}_{k,\varphi}(z) = \alpha \boldsymbol{E}_{k,\varphi}(z) \tag{5}$$

and will thus have an electrostatic energy

$$\mathscr{E}_{k,\varphi}(z) \;=\; -\int \boldsymbol{P}_{k,\varphi}(z)\cdot \mathrm{d}\boldsymbol{E}_{k,\varphi}(z) \;=\; -\frac{1}{2}\alpha E_{k,\varphi}^2(z) \;. \tag{6}$$

The force in z direction on that object is thus given by

$$\boldsymbol{F}_{k,\varphi}(z) \;=\; -\frac{\partial \mathscr{E}_{k,\varphi}(z)}{\partial z}\boldsymbol{e}_z \;=\; -\alpha k \left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz}\boldsymbol{e}_z \;. \tag{7}$$

Note that this attractive force is independent of the phase shift φ! It thus depends only on the distance z from the plane and is laterally constant, even though the surface itself is laterally charge-modulated.

II. $2d$ PERIODICITY

Two dimensional periodic arrays can have different symmetries. We'll be looking at the two cases of square and hexagonal symmetry.

A. Square symmetry

The corresponding charge density is (see Fig. 1b)

$$\sigma_{k,\varphi_x,\varphi_y}(x,y) \;=\; \frac{\sigma_0}{2}\big[\cos(kx+\varphi_x)+\cos(ky+\varphi_y)\big] \;. \tag{8}$$

The electric field depends linearly on the charge density, so the field belonging to (8) follows readily from Eqn. (2):

$$\boldsymbol{E}_{k,\varphi_x,\varphi_y}(z) \;=\; \frac{\sigma_0}{4\varepsilon_0\varepsilon_\mathrm{r}}\mathrm{e}^{-kz} \begin{pmatrix} \sin\varphi_x \\ \sin\varphi_y \\ \cos\varphi_x+\cos\varphi_y \end{pmatrix} \;. \tag{9}$$

Hence, its square is given by

$$E_{k,\varphi_x,\varphi_y}^2(z) \;=\; \frac{1}{2}\left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz}\big[1+\cos\varphi_x\cos\varphi_y\big] \;. \tag{10}$$

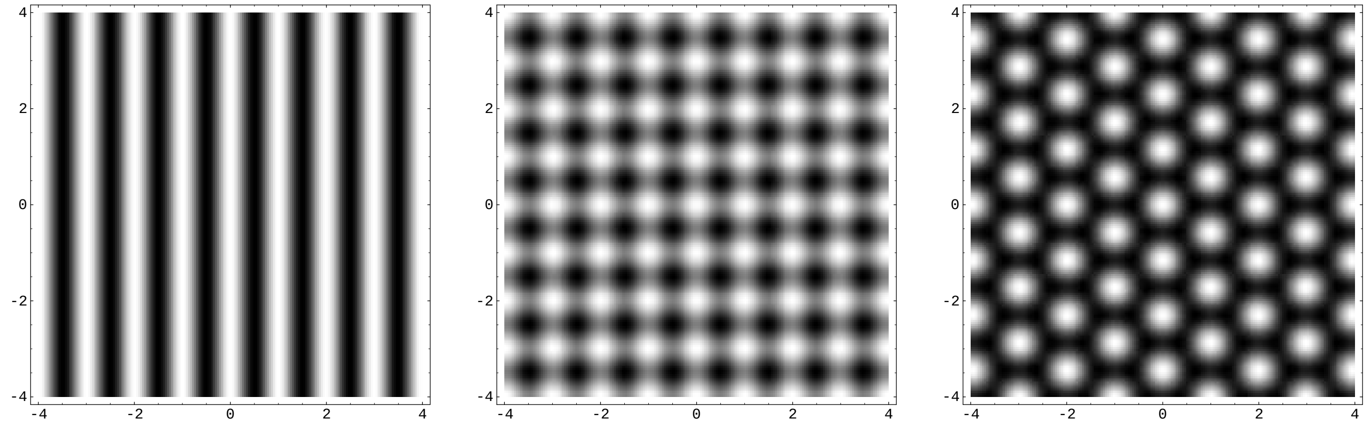
Therefore, the force on a point-object of polarizability α a distance z above the origin of the plane is

$$\boldsymbol{F}_{k,\varphi_x,\varphi_y}(z) = -\frac{1}{2}\alpha k \left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz}\big[1+\cos\varphi_x\cos\varphi_y\big]\boldsymbol{e}_z \;. \tag{11}$$

Upon phase-averaging, the cosine product vanishes:

$$\langle \boldsymbol{F}_{k,\varphi_x,\varphi_y}(z)\rangle \;=\; -\frac{1}{2}\alpha k \left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz}\boldsymbol{e}_z \;, \tag{12}$$

leaving a force *half* as big as in the $1d$-periodic case.



a) b) c)

FIG. 1: Illustration of the planar charge densities discussed in these notes. From left to right we have: a) the one-dimensional density, (1), b) the two-dimensional charge density with square symmetry, (8), and c) the two-dimensional charge density with hexagonal symmetry, (14). In all cases we have $k = 2\pi$ and all phases are zero.

B. Hexagonal symmetry

The charge density now corresponds to a superposition of three one-dimensional periodic charge densities, belonging to the wave vectors $\boldsymbol{k}_i = k\hat{\boldsymbol{n}}_i$, with the unit vectors $\hat{\boldsymbol{n}}_i$ pointing into the direction of the wave front. A hexagonal array results from giving them mutual angles of 120°, *i.e.*

$$\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{n}_2 = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad , \quad \boldsymbol{n}_3 = \frac{1}{2}\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} , \tag{13}$$

resulting in the charge density (see Fig. 1c)

$$\begin{aligned} \sigma_{k,\varphi_1,\varphi_2,\varphi_3}(x,y) \; = \; & \frac{\sigma_0}{3}\Big\{ \cos\left[kx + \varphi_1\right] \\ & + \cos\left[k(-x + \sqrt{3}\,y)/2 + \varphi_2\right] \\ & + \cos\left[k(-x - \sqrt{3}\,y)/2 + \varphi_3\right] \Big\} . \end{aligned} \tag{14}$$

As we see from Eqn. (9), the electric field has some exponential prefactor, a cosine term from each wave in the z component and a sine-term in the xy component into the direction of the wave. Hence, without doing the integral, we can immediately write down what the field of the hexagonal charge distribution (14) has to be:

$$\begin{aligned} \boldsymbol{E}_{k,\varphi_1,\varphi_2,\varphi_3}(z) \; = \; & \\ & \frac{\sigma_0}{6\varepsilon_0\varepsilon_\mathrm{r}} \mathrm{e}^{-kz} \begin{pmatrix} \sin\varphi_1 - \frac{1}{2}(\sin\varphi_2 + \sin\varphi_3) \\ \frac{1}{2}\sqrt{3}(\sin\varphi_2 - \sin\varphi_3) \\ \cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3 \end{pmatrix} . \end{aligned} \tag{15}$$

Hence, it's square is given by

$$\begin{aligned} E^2_{k,\varphi_x,\varphi_y}(z) \; = \; & \frac{1}{3}\left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz} \times \\ & \Big\{ 1 + \frac{1}{3}\Big[\cos(\varphi_1 + \varphi_2) + \cos\varphi_1\cos\varphi_2 \Big] \\ & \qquad \cos(\varphi_2 + \varphi_3) + \cos\varphi_2\cos\varphi_3 \Big] \\ & \qquad \cos(\varphi_3 + \varphi_1) + \cos\varphi_3\cos\varphi_1 \Big] \Big\} . \end{aligned} \tag{16}$$

This again depends in some awkward way on the three phases. However, upon averaging over them, all the cosine terms vanish. Hence, in analogy to Eqn. (12), we obtain an average force, given by

$$\langle \boldsymbol{F}_{k,\varphi_1,\varphi_2,\varphi_3}(z) \rangle \; = \; -\frac{1}{3}\alpha k \left(\frac{\sigma_0}{2\varepsilon_0\varepsilon_\mathrm{r}}\right)^2 \mathrm{e}^{-2kz}\boldsymbol{e}_z \; , \tag{17}$$

which is *one third* as strong as in the $1d$-periodic case.

III. MORAL

If one comes across an exponentially decaying force on a polarizable point source which displays a characteristic decay length $\ell = 1/2k$, this would correspond *within the present scenario* to a $1d$ or $2d$ surface charge modulation with a wavelength $\lambda = 4\pi\ell$. In order to check the *symmetry*, one would have to measure the phases.