The probability density of the sum of two uncorrelated random variables is not necessarily the convolution of its two marginal densities

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If two random variables \(X\) and \(Y\) are independent, then the probability density of their sum is equal to the convolution of the probability densities of \(X\) and \(Y\). With obvious notation, we have

\[
p_{X+Y}(z) = \int dx \ p_X(x) \ p_Y(z-x) .
\] (1)

The proof is simple: Independence of the two random variables implies that

\[
p_{X,Y}(x,y) = p_X(x) \ p_Y(y) .
\] (2)

And by the transformation theorem for probability densities we immediately get

\[
p_{X+Y}(z) = \int dx \ \int dy \ p_{X,Y}(x,y) \ \delta(x+y-z) = \int dx \ p_{X,Y}(x,z-x) \equiv \int dx \ p_X(x) \ p_Y(z-x) .
\] (3)

We here want to convince ourselves by a counterexample that uncorrelatedness of the random variables does not suffice for the convolution formula to hold. To see this, let us look at the probability density

\[
p_{X,Y}(x,y) = \frac{x^2 + y^2}{4\pi} e^{-\frac{1}{2}(x^2+y^2)} .
\] (5)

This probability density evidently does not factorize. Indeed, the marginal densities are given by

\[
p_X(x) = \int dy \ p_{X,Y}(x,y) = \frac{1+x^2}{\sqrt{8\pi}} e^{-\frac{1}{2}x^2} ,
\] (6)

with the same functional form of course also holding for \(p_Y(y)\). Evidently, Eqn. (2) does not hold for this choice of \(p_{X,Y}(x,y)\) and its marginal densities \(p_X(x)\) and \(p_Y(y)\). However, since \(p_{X,Y}(x,y)\) is rotationally symmetric about the origin, the covariance of \(X\) and \(Y\) vanishes, hence \(X\) and \(Y\) are uncorrelated, and yet dependent.

What is now the probability density of \(X+Y\)? From the transformation theorem we get

\[
p_{X+Y}(z) = \int dx \ p_X(x) \ p_Y(z-x) = \int dx \ \frac{x^2 + (z-x)^2}{4\pi} e^{-\frac{1}{2}(x^2+(z-x)^2)} = \frac{2+z^2}{8\sqrt{\pi}} e^{-\frac{1}{2}z^2} .
\] (7)

On the other hand, the convolution of \(p_X\) and \(p_Y\) is

\[
[p_X \ast p_Y](z) = \int dx \ p_X(x) \ p_Y(z-x) = \int dx \ \frac{1+x^2}{\sqrt{8\pi}} e^{-\frac{1}{2}x^2} \ \frac{1+(z-x)^2}{\sqrt{8\pi}} e^{-\frac{1}{2}(z-x)^2} = \frac{z^4 + 4z^2 + 44}{128\sqrt{\pi}} e^{-\frac{1}{2}z^2} ,
\] (8)

which differs from the correct answer. Fig. 1 illustrates the difference between these two functions.

FIG. 1: True probability density of the sum random variable \(p_{X+Y}(z)\) (solid line) and convolution of its marginal densities, \([p_X \ast p_Y](z)\) (dotted line).