A variant of the double-negation translation*

Jeremy Avigad

August 21, 2006

Abstract

An efficient variant of the double-negation translation explains the relationship between Shoenfield's and Gödel's versions of the Dialectica interpretation.

Fix a classical first-order language, based on the connectives $\vee$, $\wedge$, $\neg$, and $\forall$. We will define a translation to intuitionistic (even minimal) logic, based on the usual connectives. The translation maps each formula φ to the formula $\varphi^* = \neg\varphi_*$, so φ_* is supposed to represent an intuitionistic version of the *negation* of φ. The map from φ to φ_* is defined recursively, as follows:

$$\begin{array}{rcl}
\varphi_* & = & \neg\varphi, \quad \text{when } \varphi \text{ is atomic} \\
(\neg\varphi)_* & = & \neg\varphi_* \\
(\varphi \vee \psi)_* & = & \varphi_* \wedge \psi_* \\
(\varphi \wedge \psi)_* & = & \varphi_* \vee \psi_* \\
(\forall x\ \varphi)_* & = & \exists x\ \varphi_*
\end{array}$$

Note that we can eliminate either $\vee$ or $\wedge$ and retain a complete set of connectives. If Γ is the set of classical formulas $\{\varphi_1, \ldots, \varphi_k\}$, let Γ^* denote the set of formulas $\{\varphi_1^*, \ldots, \varphi_k^*\}$. The main theorem of this note is the following:

Theorem 0.1 *1. Classical logic proves $\varphi \leftrightarrow \varphi^*$.*

2. If φ is provable from Γ in classical logic, then φ^* is provable from Γ^* in minimal logic.

Note that both these claims hold for the usual Gödel-Gentzen translation $\varphi \mapsto \varphi^N$. Thus the theorem is a consequence of the following lemma:

Lemma 0.2 *For every φ, minimal logic proves $\varphi^* \leftrightarrow \varphi^N$.*

*This note was written in response to a query from Grigori Mints. After circulating a draft, I learned that Ulrich Kohlenbach and Thomas Streicher had hit upon the same solution, and that the version of the double-negation translation described below is due to Jean-Louis Krivine (see [3]). These results now appear as exercises in [1]. Since Krivine's translation and its application to the Dialectica translation are not well known, however, posting this note seemed worthwhile.

Proof. By induction on φ. The cases where φ is atomic or a negation are immediate. For $\vee$, we have

$$(\varphi \vee \psi)^* = \neg(\varphi_* \wedge \psi_*) \equiv \neg(\neg\neg\varphi_* \wedge \neg\neg\psi_*) \equiv \neg(\neg\varphi^N \wedge \neg\psi^N) = (\varphi \vee \psi)^N.$$

For $\wedge$, we have

$$(\varphi \wedge \psi)^* = \neg(\varphi_* \vee \psi_*) \equiv \neg\varphi_* \wedge \neg\psi_* \equiv \varphi^N \wedge \psi^N = (\varphi \wedge \psi)^N.$$

For $\forall$, we have

$$(\forall x\ \varphi)^* = \neg\exists x\ \varphi_* \equiv \forall x\ \neg\varphi_* \equiv \forall x\ \varphi^N N = (\forall x\ \varphi)^N.$$

This concludes the proof. $\square$

In his textbook [2], Shoenfield defines a version of the Dialectica translation for the language of arithmetic based on the connectives $\vee$, $\neg$, and $\forall$. Each formula φ is mapped to a formula φ^S of the form $\forall a\ \exists b\ \varphi_S(a, b)$, where a and b are sequences of variables. Assuming φ^S is as above and ψ^S is $\forall c\ \exists d\ \psi_S(c, d)$, the translation is defined recursively, as follows:

$$\begin{aligned} \theta^S &= \theta, \quad \text{when } \theta \text{ is atomic} \\ (\neg\varphi)^S &= \forall B\ \exists a\ \varphi_S(a, B(a)) \\ (\varphi \vee \psi)^S &= \forall a, c\ \exists b, d\ (\varphi_S(a, b) \vee \psi_s(c, d)) \\ (\forall x\ \varphi)^S &= \forall x, a\ \exists b\ \varphi_S(a, b) \end{aligned}$$

Thus Shoenfield's main result is this:

Theorem 0.3 *If φ is provable in classical arithmetic, there are terms B such that $\varphi_S(a, B(a))$ is provable in Gödel's theory T.*

If η is a formula in the language of intuitionistic logic, let η^D denote the usual Dialectica translation. It is straightforward to verify the following by recursion on formulas:

Proposition 0.4 *Suppose φ^S is $\forall a\ \exists b\ \varphi_S(a, b)$. Then $(\varphi^*)^D$ is obtained from $\exists B\ \forall a\ \varphi_S(a, B(a))$ by adding double-negations before each atomic formula.*

Note that the proposition holds even if we extend the Shoenfield translation to $\wedge$ with the clause

$$(\varphi \wedge \psi)^S = \forall a, c\ \exists b, d\ (\varphi_S(a, b) \wedge \psi_S(c, d)).$$

Thus Shoenfield's result is just a corollary of Gödel's, together with the $*$ mapping of classical to intuitionistic arithmetic.

As I have presented it, the $*$ translation is remarkably parsimonious in adding negations to a formula. It fares slightly worse on the connectives $\rightarrow$ and $\forall$:

$$\begin{aligned} (\varphi \rightarrow \psi)_* &= \neg\varphi_* \wedge \psi_* \\ (\exists x\ \varphi)_* &= \forall x\ \neg\neg\varphi_*. \end{aligned}$$

Thus it adds a negation for each $\to$, and *two* negations for each $\exists$. This is reminiscent of the Kuroda translation, which adds two negations after each universal quantifier, and two at the beginning of the formula. (Note, however, that verifying the Kuroda translation of a classical theorem requires *intuitionistic* logic, not just minimal logic.)

The nice thing is that when translating formulas from classical to intuitionstic logic, one can use the Kuroda and the $*$ translations interchangeably, since the resulting formulas are equivalent. When carrying out the Dialectica interpretation of a classical theorem, the $*$-based Shoenfield translation is often more convenient.

References

[1] Ulrich Kohlenbach. *Proof interpretations and the computational content of proofs*. Draft available online, http://www.mathematik.tu-darmstadt.de/~kohlenbach/

[2] Joseph Shoenfield. *Mathematical Logic*. Addison Wesley, Reading, MA, 1967.

[3] T. Streicher and B. Reus. Classical logic: continuation semantics and abstract machines. Journal of Functional Programming, 8:543–572, 1998.