State-Dependent Utilities

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Several axiom systems for preference among acts lead to a unique probability and a state-independent utility such that acts are ranked according to their expected utilities. These axioms have been used as a foundation for Bayesian decision theory and subjective probability calculus. In this article we note that the uniqueness of the probability is relative to the choice of what counts as a constant outcome. Although it is sometimes clear what should be considered constant, in many cases there are several possible choices. Each choice can lead to a different "unique" probability and utility. By focusing attention on state-dependent utilities, we determine conditions under which a truly unique probability and utility can be determined from an agent's expressed preferences among acts. Suppose that an agent's preference can be represented in terms of a probability P and a utility U. That is, the agent prefers one act to another iff the expected utility of that act is higher than that of the other. There are many other equivalent representations in terms of probabilities Q, which are mutually absolutely continuous with P, and state-dependent utilities V, which differ from U by possibly different positive affine transformations in each state of nature. We describe an example in which there are two different but equivalent state-independent utility representations for the same preference structure. They differ in which acts count as constants. The acts involve receiving different amounts of one or the other of two currencies, and the states are different exchange rates between the currencies. It is easy to see how it would not be possible for constant amounts of both currencies to have simultaneously constant values across the different states. Savage (1954, sec. 5.5) discovered a situation in which two seemingly equivalent preference structures are represented by different pairs of probability and utility. He attributed the phenomenon to the construction of a "small world." We show that the small world problem is just another example of two different, but equivalent, representations treating different acts as constants. Finally, we prove a theorem (similar to one of Karni 1985) that shows how to elicit a unique state-dependent utility and does not assume that there are prizes with constant value. To do this, we define a new hypothetical kind of act in which both the prize to be awarded and the state of nature are determined by an auxiliary experiment.

KEY WORDS: Constant acts; Elicitation; Exchange rates; Preferences; Savage's axioms; Small worlds.

1. INTRODUCTION

Expected utility theory is founded on at least one of several axiomatic derivations of probabilities and utilities from expressed preferences over acts (Anscombe and Aumann 1963; deFinetti 1974; Ramsey 1926; Savage 1954). These derivations allow for the simultaneous existence of a unique personal probability over the states of nature and a unique (up to positive affine transformations) utility function over the prizes such that the acts are ranked by expected utility. For example, suppose that there are n states of nature that form the set $S = \{s_1, \ldots, s_n\}$ and m prizes in the set $Z = \{z_1, \ldots, z_m\}$. An example of an act is a function f mapping S to Z. That is, if $f(s_i) = z_j$, then we receive prize z_j if state s_i occurs. (We will consider more complicated acts later.) Now suppose that there is a probability over the states such that $p_i = \Pr(s_i)$ and that there is a utility U over prizes. By saying that acts are ranked by expected utility, we mean that we strictly prefer act g to act f iff

$$\sum_{i=1}^{n} p_i U[f(s_i)] < \sum_{i=1}^{n} p_i U[g(s_i)]. \quad (1)$$

If we allow the utilities of prizes to vary conditionally on which state of nature occurs, we can rewrite Equation (1) as

$$\sum_{i=1}^{n} p_i U_i[f(s_i)] < \sum_{i=1}^{n} p_i U_i[g(s_i)], \quad (2)$$

where $U_i(z_j)$ is the utility of prize z_j given that state s_i occurs. Without restrictions, however, on the degree to which U_i can differ from $U_{i'}$ for $i \neq i'$, the uniqueness of the personal probability no longer holds. For example, let $q_1, \ldots, q_n$ be another probability over the states such that $p_i > 0$ iff $q_i > 0$. Then for an arbitrary act f,

$$\sum_{i=1}^{n} q_i V_i[f(s_i)] = \sum_{i=1}^{n} p_i U_i[f(s_i)],$$

where $V_i(\cdot) = p_i U_i(\cdot)/q_i$ when $q_i > 0$ (V_i can be arbitrary when $q_i = 0$). In this case, it is impossible to determine an agent's personal probability by studying his or her preferences for acts. Rubin (1987) noted this and developed an axiom system that does not lead to a separation of probability and utility. Arrow (1974) considered the problem for insurance (a footnote credits Rubin with raising this same issue in an unpublished 1964 lecture).

DeGroot (1970) began his derivation of expected utility theory by assuming that the concept of "at least as likely as" is an undefined primitive. This allows the construction of probability without reference to preferences. DeGroot also needs to introduce preferences among acts, however, to derive a utility function. In Section 2, we will examine VonNeumann and Morgenstern's (1947) axiomatization, along with Anscombe and Aumann's (1963) extension, to

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see how it attempts to avoid the non-uniqueness problem just described. In Section 3, we look at Savage's system with the same goal in mind. Section 4 provides a critical examination of the theory of deFinetti (1974). In Section 5, we give an example illustrating the problem's persistence despite the best efforts of those who have derived the theories. While reviewing an example from Savage in Section 6, we see how close he was to discovering the non-uniqueness problem in connection with his own theory. In Section 7, we describe a method for obtaining a unique personal probability and state-dependent utility based on a proposal of Karni, Schmeidler, and Vind (1983).

2. STATE-INDEPENDENT UTILITY

Following VonNeumann and Morgenstern (1947), we generalize the concept of act introduced in Section 1 by allowing randomization. That is, suppose that the agent is comfortable declaring probabilities for an auxiliary experiment the results of which he or she believes would in no way alter his or her preferences among acts. Furthermore, assume that this auxiliary experiment has events with arbitrary probabilities (e.g., it may produce a random variable with continuous distribution). Define a *lottery* as follows: If $A_1, \ldots, A_m$ is a partition of the possible outcomes of the auxiliary experiment with $\alpha_j = \Pr(A_j)$ for each j, then the lottery $(\alpha_1, \ldots, \alpha_m)$ awards prize z_j if A_j occurs. We assume that the choice of the partition events $A_1, \ldots, A_m$ does not affect the lottery. That is, if $B_1, \ldots, B_m$ is another partition such that $\Pr(B_j) = \alpha_j$ for each j also, then the lottery that awards prize z_j when B_j occurs is, to the agent, the *same lottery* as the one described in terms of the A_j. In this way, a lottery is just a simple probability distribution over prizes that is independent of the state of nature. Any two lotteries that award the prizes with the same probabilities are considered the same lottery. If L_1 and L_2 are the two lotteries $(\alpha_1, \ldots, \alpha_m)$ and $(\beta_1, \ldots, \beta_m)$, respectively, then for $0 \leq \lambda \leq 1$, we denote by $\lambda L_1 + (1 - \lambda)L_2$ the lottery $[\lambda\alpha_1 + (1 - \lambda)\beta_1, \ldots, \lambda\alpha_m + (1 - \lambda)\beta_m]$.

If we consider only lotteries, we can introduce some axioms for preferences among lotteries. For convenience, we will henceforth assume that there are two lotteries such that the agent has a strict preference for one over the other. Otherwise, the preference relation is trivial and no interesting results are obtained.

Axiom 1 (Weak Order). There is a weak order, $\succcurlyeq$, among lotteries such that $L_1 \succcurlyeq L_2$ iff L_1 is not strictly preferred to L_2.

This axiom requires that weak preference among lotteries be transitive, reflexive, and connected. If we define *equivalence* to mean "no strict preference in either direction," then equivalence is transitive also.

Definition 1. Assuming Axiom 1, we say that L_1 *is equivalent to* L_2 (denoted by $L_1 \sim L_2$) if $L_1 \succcurlyeq L_2$ and $L_2 \succcurlyeq L_1$. We say L_2 *is strictly preferred to* L_1 (denoted by $L_1 \prec L_2$) if $L_1 \succcurlyeq L_2$ but not $L_2 \succcurlyeq L_1$.

The axiom that does most of the work is one that entails stochastic dominance.

Axiom 2 (Independence). For each L, L_1, L_2, and $0 < \alpha < 1$, $L_1 \succcurlyeq L_2$ iff $\alpha L_1 + (1 - \alpha)L \succcurlyeq \alpha L_2 + (1 - \alpha)L$.

A third axiom is often introduced to guarantee that utilities are real valued.

Axiom 3 (Archimedean). If $L_1 \prec L_2 \prec L_3$, then there exists $0 < \alpha < 1$ such that $L_2 \sim (1 - \alpha)L_1 + \alpha L_3$.

Axiom 3 prevents L_3 from being infinitely better than L_2 and L_1 from being infinitely worse than L_2.

With axioms equivalent to these three, VonNeumann and Morgenstern (1947) proved that there exists a utility over prizes U such that $(\alpha_1, \ldots, \alpha_m) \succcurlyeq (\beta_1, \ldots, \beta_m)$ iff $\sum_{i=1}^{m} \alpha_i U(z_i) \leq \sum_{i=1}^{m} \beta_i U(z_i)$. This utility is unique up to positive affine transformation. In fact, it is quite easy (and useful for the example in Sec. 5) to construct the utility function from the stated preferences. Pick an arbitrary pair of lotteries L_0 and L_1 such that $L_0 \prec L_1$. Assign these lotteries the utilities $U(L_0) = 0$ and $U(L_1) = 1$. For all other lotteries L, the utilities are assigned as follows: If $L_0 \succcurlyeq L \succcurlyeq L_1$, $U(L)$ is that α such that $(1 - \alpha)L_0 + \alpha L_1 \sim L$. If $L \prec L_0$, then $U(L) = -\alpha/(1 - \alpha)$, where $(1 - \alpha)L + \alpha L_1 \sim L_0$ (hence $\alpha \neq 1$). If $L_1 \prec L$, then $U(L) = 1/\alpha$, where $(1 - \alpha)L_0 + \alpha L \sim L_1$. The existence of these α values is guaranteed by Axiom 3 and their uniqueness follows from Axiom 2.

To handle acts in which the prizes vary with the state of nature, Anscombe and Aumann (1963) introduced a fourth axiom designed to say that the preferences among prizes did not vary with the state. Before stating this axiom, we introduce a more general act, known as a *horse lottery*.

Definition 2. A function mapping states of nature to lotteries is called a horse lottery.

That is, if H is a horse lottery such that $H(s_i) = L_i$ for each i, then if state s_i occurs, the prize awarded is the prize that lottery L_i awards. The L_i can be all different or some (or all) the same. If $H(s_i) = L$ for all i, then we say $H = L$. If H_1 and H_2 are two horse lotteries such that $H_j(s_i) = L_i^{(j)}$ for each i and J and if $0 \leq \alpha \leq 1$, then we denote by $\alpha H_1 + (1 - \alpha)H_2$ the horse lottery H such that $H(s_i) = \alpha L_1^{(i)} + (1 - \alpha)L_2^{(i)}$ for each i. Axioms 1 and 2, when applied to preferences among horse lotteries, imply that the choice of an act has no effect on the probabilities of the state of nature.

Definition 3. A state of nature s_i is called *null* if for each pair of horse lotteries H_1 and H_2 satisfying $H_1(s_j) = H_2(s_j)$ for all $j \neq i$, $H_1 \sim H_2$. A state is called *non-null* if it is not null.

Axiom 4 (State-Independence). For each non-null state s_i, each pair of lotteries (L_1, L_2), and each pair of horse lotteries H_1 and H_2 satisfying $H_1(s_j) = H_2(s_j)$ for $j \neq i$, $H_1(s_i) = L_1$, and $H_2(s_i) = L_2$, we have $L_1 \prec L_2$ iff $H_1 \prec H_2$.

Axiom 4 says that a strict preference between two lotteries is reproduced for every pair of horse lotteries that differ only in some non-null state, and their difference in that state is that each of them equals one of the two lotteries. With this setup, Anscombe and Aumann (1963) proved the following theorem.

Theorem 1 (Anscombe and Aumann). Under Axioms 1–4, there exist a unique probability P over the states and utility U over prizes (unique up to positive affine transformation) such that $H_1 \succcurlyeq H_2$ iff

$$\sum_{i=1}^{n} P(s_i)U[H_1(s_i)] \leq \sum_{i=1}^{n} P(s_i)U[H^2(s_i)],$$

where for each lottery $L = (\alpha_1, \ldots, \alpha_m)$, $U(L)$ stands for $\sum_{j=1}^{m} \alpha_j U(z_j)$.

Even when the four axioms hold, there is no requirement that the utility function U be the same, conditional on each state of nature. As we did when we constructed Equation (2), we could allow $U_i(z_j) = a_i U(z_j) + b_i$, where each $a_i > 0$. Then we could let $Q(s_i) = a_i P(s_i)/\sum_{k=1}^{n} a_k P(s_k)$. It would now be true that $H_1 \succcurlyeq H_2$ iff

$$\sum_{i=1}^{n} Q(s_i)U_i[H_1(s_i)] \leq \sum_{i=1}^{n} Q(s_i)U_i[H_2(s_i)].$$

The uniqueness of the probability in Theorem 1 depends on the use of a state-independent utility U. Hence one cannot determine an agent's probability from his or her stated preferences unless one assumes that the agent's utility is state-independent. This may not seem like a serious difficulty when Axiom 4 holds. We will see in Section 5, however, that the problem is more complicated.

3. SAVAGE'S POSTULATES

Savage (1954) gave a set of postulates that do not rely on an auxiliary randomization to extract probabilities and utilities from preferences. Rather, they rely on the use of prizes that can be considered "constant" across states. Savage's most general acts are functions from states to prizes. Because he did not introduce an auxiliary randomization, he required that there be infinitely many states. The important features of Savage's theory, for this discussion, are the first three postulates and a few definitions. Some of the axioms and definitions are stated in terms of *events,* which are sets of states. Savage's postulates are consistent with the axioms of Section 2 in that they provide models for preference by maximizing expected utility.

The first postulate is the same as Axiom 1. The second postulate requires a definition of conditional preference.

Definition 4. Let B be an event. We say that $f \succcurlyeq g$ *given B* iff

- $f' \succcurlyeq g'$ for each pair f' and g' such that $f'(s) = f(s)$ for all $s \in B$, $g'(s) = g(s)$ for all $s \in B$, and $f'(s) = g'(s)$ for all $s \notin B$
- and $f' \succcurlyeq g'$ for every such pair or for none.

The second postulate is an analog of Axiom 2 (see Fishburn 1970, p. 193).

Postulate 2. For each pair of acts f and g and each event B, either $f \succcurlyeq g$ given B or $g \preccurlyeq f$ given B.

Savage has a concept of *null event* that is similar to the concept of null state from Definition 3.

Definition 5. An event B is *null* if for every pair of acts f and g, $f \succcurlyeq g$ given B. An event B is *non-null* if it is not null.

Savage's third postulate concerns acts that are constant, such as $f(s) = z$ for all s, where z is a single prize. For convenience, we will call such an act f by the name z also.

Postulate 3. For each non-null event B and each pair of prizes z_1 and z_2 (considered as constant acts), $z_1 \succcurlyeq z_2$ iff $z_1 \succcurlyeq z_2$ given B.

Savage's definition of probability relies on Postulate 3.

Definition 6. Suppose that A and B are events. We say that *A is at least as likely as B* if for each pair of prizes z and w, with $z \prec w$, we have $f_B \succcurlyeq f_A$, where $f_A(s) = w$ if $s \in A$, $f_A(s) = z$ if $s \notin A$, $f_B(s) = w$ if $s \in B$, and $f_B(s) = z$ if $s \notin B$.

Postulate 2 guarantees that with f_A and f_B as defined in Definition 6, either $f_B \succcurlyeq f_A$ no matter which pair of prizes z and w one chooses (as long as $z \prec w$) or $f_A \succcurlyeq f_B$ no matter which pair of prizes one chooses.

Postulate 3 says that the *relative* values of prizes cannot change between states. Savage (1954, p. 25) suggested that problems in locating prizes that satisfy this postulate may be solved by a clever redescription. For example, rather than describing prizes as "receiving a bathing suit" and "receiving a tennis racket" (whose relative values change, depending on which of the two states "picnic at the beach" or "picnic in the park" occurs), Savage suggested that the prizes might be "a refreshing swim with friends," "sitting alone on the beach with a tennis racket," and so on. We do not see how to carry out such redescriptions, however, while satisfying Savage's structural assumption that each prize is available as an outcome under each state. (What does it mean to receive the prize "sitting alone on the beach with a tennis racket" when the state "picnic in the park" occurs?)

Our problem, however, is deeper than this. Definition 6 assumes that the *absolute* values of prizes do not change from state to state. For example, suppose that A and B are disjoint and the value of z is 1 for the states in A and 2 for the states in B. Similarly, suppose that the value of w is 2 for the states in A and 4 for the states in B. Then even if A is more likely than B, but is not twice as likely, we would get $f_A \prec f_B$ and we would conclude, by Definition 6, that B is more likely than A. The example in Section 5 (using just one of the currencies) and our interpretation of Savage's "small worlds" problem (in Sec. 6) suggest that it might be very difficult to find prizes with the property that their "absolute" values do not change from state to state even though their "relative" values remain the same from state to state.

4. DEFINETTI'S GAMBLING APPROACH

deFinetti (1974) assumed that there is a set of prizes with numerical values such that utility is linear in the numerical value. That is, a prize numbered 4 is worth twice as much as a prize numbered 2. More specifically, to say that utility is linear in the numerical values of prizes, we mean the following: For each pair of prizes, (z_1, z_2) with $z_1 < z_2$, and each $0 \leq \alpha \leq 1$, the lottery that pays z_1 with probability $1 - \alpha$ and pays z_2 with probability α (using the auxiliary randomization of Sec. 2) is equivalent to the lottery that pays $(1 - \alpha)z_1 + \alpha z_2$ for sure. Using such a set of prizes, deFinetti supposed that an agent will accept certain gambles that pay these prizes. If f is an act, to gamble on f means to accept a contract that pays the agent $c[f(s) - x]$ when state s occurs, where c and x are some values. A negative outcome means that the agent has to pay out, whereas a positive outcome means that the agent gains some amount.

Definition 7. The *prevision* of an act f is the number x that one would choose so that all gambles of the form $c[f - x]$ would be accepted for all small values of c, both positive and negative.

If an agent is willing to gamble on each of several acts, then it is assumed that he or she will also gamble on them simultaneously. (For a critical discussion of this point, see Kadane and Winkler 1988; Schick 1986.)

Definition 8. A collection of previsions for acts is *coherent* if for each finite set of the acts, say $f_1, \ldots, f_n$ with previsions $x_1, \ldots, x_n$, respectively, and each set of numbers $c_1, \ldots, c_n$, we have

$$\sup_{\text{all } s} \sum_{i=1}^{n} c_i[f_i(s) - x_i] \geq 0.$$

Otherwise, the previsions are *incoherent*.

deFinetti (1974) proved that a collection of previsions of bounded acts is coherent iff there exists a finitely additive probability such that the prevision of each act is its expected value. This provides a method of eliciting probabilities by asking an agent to specify previsions for acts, such as $f(s) = 1$ if $s \in A$ and $f(s) = 0$ if $s \notin A$. The prevision of such an act f would be its probability if the previsions are coherent. As plausible as this sounds, the following example casts doubt on the ability of deFinetti's program to elicit probabilities accurately.

5. AN EXAMPLE

Let the set of available prizes be various amounts of dollars. We suppose that there are three states of nature (which we will describe in more detail later) and that the agent expresses preferences that satisfy the axioms of Section 2 and Savage's postulates. Furthermore, suppose that the agent's utility for money is linear. That is, for each state i, $U_i(\$cx) = cU_i(\$x)$. In particular, $U_i(\$0) = 0$. We now offer the agent three horse lotteries, H_1, H_2, and H_3, whose outcomes are

State of Nature

	s_1	s_2	s_3
H_1	\$1	\$0	\$0
H_2	\$0	\$1	\$0
H_3	\$0	\$0	\$1

Suppose that the agent claims that these three horse lotteries are equivalent. If we assume that the agent has a state-independent utility, the expected utility of H_i is $U(\$1)P(s_i)$. It follows from the three horse lotteries' being equivalent that $P(s_i) = 1/3$ for each i.

Next we alter the set of prizes to be various Japanese yen amounts. Suppose that we offer the agent three yen horse lotteries, H_4, H_5, and H_6, whose outcomes are

State of Nature

	s_1	s_2	s_3
H_4	¥100	¥0	¥0
H_5	¥0	¥125	¥0
H_6	¥0	¥0	¥150

If the agent claims that these three horse lotteries are equivalent, and if we assume that he or she uses a state-independent utility for yen prizes, then $P(s_1)U(¥100) = P(s_2)U(¥125) = P(s_3)U(¥150)$. Supposing that the agent's utility is linear in yen, as it was in dollars, we conclude that $P(s_1) = 1.25P(s_2) = 1.5P(s_3)$. It follows that $P(s_1) = .4054$, $P(s_2) = .3243$, and $P(s_3) = .2703$. It would be incoherent for the agent to express both sets of equivalences, since he or she is apparently now committed to two different probability distributions over the three states. This is not correct, however, as we now see.

Suppose that the three states of nature represent three different exchange rates between dollars and yen. $s_1 = \{\$1$ is worth ¥100$\}$, $s_2 = \{\$1$ is worth ¥125$\}$, and $s_3 = \{\$1$ is worth ¥150$\}$. Suppose further that the agent can change monetary units at the prevailing rate of exchange without any penalty. As far as this agent is concerned, H_i and H_{3+i} are worth exactly the same for $i = 1, 2, 3$ because in each state the prizes awarded are worth the same amount. A problem arises in this example: The two probability distributions were constructed under incompatible assumptions. The discrete uniform probability was constructed under the assumption that $U(\$1)$ is the same in all three states, whereas the other probability was constructed under the assumption that $U(¥100)$ was the same in all three states. Clearly these cannot both be true given the nature of the states. Both Theorem 1 and Savage's theory are saved because preference can be represented by expected utility *no matter which of the two assumptions one makes.* Unfortunately, this same fact forces the uniqueness of the probability to be relative to the choice of which prizes count as constants in terms of utility. There are two different representations of the agent's preferences by probability and state-independent utility. What is state-independent in one representation, though, is state-dependent in the other.

If we allow both types of prizes at once, we can calculate the marginal exchange rate for the agent. That is, we can ask, "For what value x will the agent claim that \$1 and ¥x are equivalent?" This question can be answered by using either of the two probability–utility representations, and the answers will be the same. First, with dollars having constant value, the expected utility of a horse lottery paying \$1 in all three states is $U(\$1)$. The expected value of the horse lottery paying ¥x in all three states is

$$\frac{U_1(¥x) + U_2(¥x) + U_3(¥x)}{3}$$
$$= \frac{1}{3}\left[\frac{x}{100}U(\$1) + \frac{x}{125}U(\$1) + \frac{x}{150}U(\$1)\right]$$
$$= .008222xU(\$1),$$

using the linearity of utility and the state-specific exchange rates. By setting this expression equal to $U(\$1)$, we obtain $x = 121.62$. Equivalently, we can calculate the exchange rate assuming that yen have constant value over states. The act paying ¥x in all states has expected utility $U(¥x) = .01xU(¥100)$. The act paying \$1 in all states has expected utility.

$$.4054U_1(\$1) + .3243U_2(\$1) + .2703U_3(\$1)$$
$$= .4054U(¥100) + .3243U(¥125) + .2703U(¥150)$$
$$= U(¥100)[.4054 + .3243 \times 1.25 + .2703 \times 1.5]$$
$$= 1.2162U(¥100).$$

Setting this equal to $.01xU(¥100)$ yields $x = 121.62$, which is the same exchange rate as calculated earlier.

The implications of this example for elicitation are staggering. Suppose that we attempt to elicit the agent's probabilities over the three states by offering acts in dollar amounts and using deFinetti's gambling approach from Section 4. The agent has utility that is linear in both dollars and yen without reference to the states, hence deFinetti's program will apply. To see this, select two prizes, such as \$0 and \$1, to have utilities 0 and 1, respectively. Then for $0 < x < 1$, $U(\$x)$ must be the value c that makes the following two lotteries equivalent: $L_1 = \$x$ for certain, and $L_2 =$ \$1 with probability c and \$0 with probability $1 - c$. Assuming that dollars have constant utility, it is obvious that $c = x$. Assuming that yen have constant utility, the expected utility of L_1 is $1.2162xU(¥100)$ and the expected utility of L_2 is $cU(¥121.62)$. These two are the same iff $x = c$. Similar arguments work when x is not between 0 and 1 and when the two prizes with utilities 0 and 1 are yen prizes. Now suppose that the agent actually uses the state-independent utility for dollars and the discrete uniform distribution to rank acts, but the eliciter does not know this. The eliciter will try to elicit the agent's probabilities for the states by offering gambles in yen (linear in utility). For example, the agent claims that the gamble $c(f - 40.54)$ would be accepted for all small values of c, where $f(s) =$ ¥150 if $s = s_3$ and ¥0 otherwise. The reason for this is that since ¥150 equals \$1 when s_3 occurs, the winnings are \$1 when s_3 occurs, which has a probability of 1/3. The marginal exchange rate is ¥121.62 for \$1, so the appropriate amount to pay (no matter which state occurs), to win \$1 when s_3 occurs, is \$1/3, which equals ¥121.62/3 = ¥40.54. Realizing that utility is linear in yen, the eliciter now decides that $\Pr(s_3)$ must equal 40.54/150 = .2703. Hence the eliciter elicits the wrong probability, even though the agent is coherent!

The expressed preferences satisfy the four axioms of Section 2, all of Savage's postulates, and deFinetti's linearity condition, but we are still unable to determine the probabilities of the states based only on preferences. The problem becomes clearer if we allow both dollar and yen prizes at the same time. It is impossible, however, for a single utility to be state-independent for all prizes. That is, Axiom 4 and Postulate 3 would no longer hold. Things are more confusing in deFinetti's framework, because there is no room for state-dependent utilities. The agent appears to have two different probabilities for the same event, even though there would be no incoherency.

6. SAVAGE'S "SMALL WORLDS" EXAMPLE

In section 5.5 of Savage (1954), the topic of *small worlds* is discussed. An anomaly occurs in this discussion, and Savage implies that it is an effect of the construction of the small world. In this section, we briefly introduce small worlds and then explain why we believe that the anomaly discovered by Savage is actually another example of the non-uniqueness illustrated in Section 5. It is a mere coincidence that it arose in the discussion of small worlds. We show how precisely the same effect arises without any mention of small worlds.

A small world can be thought of as a description of the states of nature in which each state can actually be partitioned into several smaller states, but we do not actually do the partitioning when making comparisons between acts. For a mathematical example, Savage mentioned the following case. Consider the unit square $S = \{(x, y) : 0 \le x, y \le 1\}$ as the finest possible partition of the states of nature. Suppose, however, that we consider as states the subsets $\bar{x} = \{(x, y) : 0 \le y \le 1\}$ for each $x \in [0, 1]$. Savage discovered the following problem in this example: It is possible to define small world prizes in a natural way and for preferences among small world acts to satisfy all of his axioms and, at the same time, consistently define prizes in the "grand world" consisting of the whole square S. It is possible, however, for the preferences among small world acts to be consistent with the preferences among grand world acts in such a way that the probability measure determined from the small world preferences is not the marginal probability measure over the sets $\bar{x}$ induced from the grand world probability. As we will see, the problem that Savage discovered results from using different prizes as constants in the two problems. It is not due to the small world but will actually appear in the grand world as well.

Any grand world act can be considered a small world prize. In fact, the very reason for introducing small worlds is to deal with the case in which what we count as a prize is actually worth different amounts depending on which of the subdivisions of the small world state of nature occurs. Therefore, we let the grand world prizes be nonnegative numbers and the grand world acts all bounded measurable functions on S. The grand world probability is uniform over the square and the grand world utility is the numerical value of the prize. To guarantee that Savage's axioms hold in the small world, choose the small world prizes to be 0 and positive multiples of a single function h. Assuming that $U(h) = 1$, the small world probability of a set $\overline{B} = \{\overline{x} : x \in B\}$ is (from Savage 1954, p. 89) $Q(\overline{B}) = \int_{\overline{B}} q(x)\, dx$, where

$$q(x) = \frac{\int_0^1 h(x, y)\, dy}{\int_0^1 \int_0^1 h(x, y)\, dy\, dx}. \qquad (3)$$

Unless $\int_0^1 h(x, y)\, dy$ is constant as a function of x, Q will not be the marginal distribution induced from the uniform distribution over S. Even if $\int_0^1 h(x, y)\, dy$ is not constant, however, the ranking of small world acts is consistent with the ranking of grand world acts. Let $ch(\cdot, \cdot)$, considered as a small world prize, be denoted by $\overline{c}$. Let $U(\overline{c}) = c$ denote the small world utility of small world prize $\overline{c}$. If $\overline{f}$ is a small world act, then for each $\overline{x}$, $\overline{f}(\overline{x}) = \overline{c}$ for some c. The expected small world utility of $\overline{f}$ is $\int_0^1 U[\overline{f}(\overline{x})]q(x)\, dx$. Let the grand world act f corresponding to $\overline{f}$ be defined by $f(x, y) = \overline{f}(\overline{x})h(x, y)$. It follows from Equation (3) that

$$U[\overline{f}(\overline{x})]q(x) = \frac{\int_0^1 f(x, y)\, dy}{\int_0^1 \int_0^1 h(x, y)\, dy\, dx}.$$

Hence the expected small world utility of $\overline{f}$ is

$$\int_0^1 \frac{\int_0^1 f(x, y)\, dy}{\int_0^1 \int_0^1 h(x, y)\, dy\, dx}\, dx,$$

which is just a constant times the grand world expected utility of f. Hence small world acts are ranked in precisely the same order as their grand world counterparts, even though the small world probability is not consistent with the grand world probability.

We claimed that the inconsistency of the two probabilities is due to the choice of "constants" and not to the small worlds. To see this, let the grand world constants be 0 and the positive multiples of h. Then an act f in the original problem becomes an act f^* with $f^*(x, y) = f(x, y)/h(x, y)$. That is, the prize that f^* assigns to (x, y) is the number of multiples of $h(x, y)$ that $f(x, y)$ is. We define the new probability for B, a two-dimensional Borel set,

$$R(B) = \frac{\int_B h(x, y)\, dy\, dx}{\int_S h(x, y)\, dy\, dx}.$$

The expected utility of f^* is now

$$\frac{\int_S f^*(x, y)h(x, y)\, dy\, dx}{\int_S h(x, y)\, dy\, dx} = \frac{\int_S f(x, y)\, dy\, dx}{\int_S h(x, y)\, dy\, dx}.$$

This is just a constant times the original expected utility. Hence acts are ranked in the same order by both probability–utility representations. Both representations are state-independent but each one is relative to a different choice of constants. The constants in one representation have different utilities in different states in the other representation. Both representations satisfy Savage's axioms, however. (Note that the small world probability constructed earlier is the marginal probability associated with the grand world probability R, so Savage's small world problem evaporates when the definition of constant is allowed to change.) Remember that the uniqueness of the probability–utility representation for a collection of preferences is relative to what counts as a constant. To use Savage's notation in the example of Section 5, suppose that we use yen gambles to elicit probabilities. Instead, however, of treating multiples of ¥1 as constants, we treat multiples of gamble $f(s_1) =$ ¥100, $f(s_2) =$ ¥125, $f(s_3) =$ ¥150 as constants. Then we will elicit the discrete uniform probability rather than the nonuniform probability.

7. HOW TO ELICIT UNIQUE PROBABILITIES AND UTILITIES SIMULTANEOUSLY

There is one obvious way to avoid the confusion of the previous examples—elicit a unique probability without reference to preferences. This is DeGroot's (1970) approach. It requires that the agent have an understanding of the primitive concept "at least as likely as" in addition to the more widely understood primitive "is preferred to." Some decision theorists prefer to develop the theory solely from preference without reference to the more statistical primitive "at least as likely as"; they need an alternative to the existing theories in order to separate probability from utility.

Karni et al. (1983; see also Karni 1985) proposed a scheme for simultaneously eliciting probability and state-dependent utility. Essentially, in addition to stating preferences among horse lotteries, an agent is asked to state preferences among horse lotteries under the assumption that he or she holds a particular probability distribution over the states (explicitly, they say on p. 1024, "contingent upon a strictly positive probability distribution p' on S.") They also require the agent to compare acts with different "contingent" probabilities. Karni (1985) described these (in a slightly more general setting) as *prize–state lotteries* that are functions $\hat{f}$ from $Z \times S$ to $\Re^+$ such that $\sum_{\text{all}(z,s)} \hat{f}(z, s) = 1$ and the probability $\hat{f}(z, s)$ for each z and s is understood in the same sense as the probabilities involved in the lotteries of Section 2. That is, the results of a prize–state lottery are determined by an auxiliary randomization. The agent is asked to imagine that the state of nature could be chosen by the randomization scheme rather than by the forces of nature. This is intended to remove the uncertainty associated with how the state of nature is determined so that a pure utility can be extracted by using Axioms 1–3 applied to a preference relation among prize–state lotteries.

For example, suppose that the agent in Section 5 expresses a strict preference for the prize–state lottery that awards \$1 in state 2 with probability 1 [$\hat{f}(\$1, s_2) = 1$] over $\hat{g}(\$1, s_1) = 1$. This preference would not be consistent with a state-independent utility for dollar prizes; however, it would be consistent with a state-independent utility in yen prizes.

The pure utility elicited in this fashion is a function of both prizes and states, so it is actually a state-dependent utility. As long as the preferences among prize–state lotteries are consistent with the preferences among horse lotteries, the elicited state-dependent utility can then be assumed to be the agent's utility. There will then be a unique probability such that $H_1 \succcurlyeq H_2$ iff the expected utility of H_1 is at most as large as the expected utility of H_2. The type of consistency that Karni et al. (1983) require between the two sets of preferences is rather more complicated than necessary. The following simple consistency axiom will suffice.

Axiom 5 (Consistency). For each non-null state s and each pair $(\hat{f}_1, \hat{f}_2)$ of prize–state lotteries satisfying $\Sigma_{\text{all } z}\hat{f}_i(z, s) = 1$ and some pair of horse lotteries H_1 and H_2 satisfying $H_1(s_i) = H_2(s_i)$ for all $s_i \neq s$ and $H_1(s) = f_1$ and $H_2(s) = f_2$, we have $H_1 \succcurlyeq H_2$ iff $\hat{f}_1 \succcurlyeq \hat{f}_2$, where f_1 and f_2 are lotteries that correspond to $\hat{f}_1$ and $\hat{f}_2$ as follows: $f_i = [\hat{f}_i(z_1, s), \ldots, \hat{f}_i(z_m, s)]$, $i = 1, 2$, in the notation of Section 2.

This just says that preferences among prize–state lotteries with all of their probabilities on the same state must be reproduced as preferences between horse lotteries that differ only in that common state.

Theorem 2. Suppose that there are n states of nature and m prizes. Assume that preferences among horse lotteries satisfy Axioms 1–3. Also assume that preferences among prize–state lotteries satisfy Axioms 1–3. Finally, assume that Axiom 5 holds. Then there exists a unique probability P over the states and a utility $U : Z \times S \to \mathfrak{R}$, unique up to positive affine transformation, satisfying the following:

1. $H_1 \succcurlyeq H_2$ iff

$$\sum_{i=1}^{n} P(s_i)U[H_1(s_i), s_i] \leq \sum_{i=1}^{n} P(s_i)U[H_2(s_i), s_i],$$

where for each lottery $L = (\alpha_1, \ldots, \alpha_m)$, $U(L, s_i)$ stands for $\Sigma_{j=1}^{m} \alpha_j U(z_j, s_i)$.

2. $\hat{f} \succcurlyeq \hat{g}$ iff

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \hat{f}(z_j, s_i)U(z_j, s_i) \leq \sum_{i=1}^{n}\sum_{j=1}^{m} \hat{g}(z_j, s_i)U(z_j, s_i).$$

The proof of Theorem 2 makes use of the following theorem from Fishburn (1970, p. 176):

Theorem 3 (Fishburn). Under Axioms 1, 2, and 3, there exist real-valued functions $W_1, \ldots, W_n$ such that $H_1 \prec H_2$ iff

$$\sum_{i=1}^{n} W_i[H_1(s_i)] \leq \sum_{i=1}^{n} W_i[H_2(s_i)]. \tag{4}$$

The W_i that satisfy Equation (4) are unique up to a similar positive linear transformation, with W_i constant iff s_i is null.

We provide only a sketch of the proof of Theorem 2. Let $(W_1, \ldots, W_n)$ be the state-dependent utility for horse lotteries guaranteed by Theorem 3, and let $\hat{V}$ be the utility for prize–state lotteries guaranteed by the VonNeumann and Morgenstern's theorem (1947). All we need to show is that there are $c_1, \ldots, c_n$ and positive $a_1, \ldots, a_n$ such that for each $i = 1, \ldots, n$,

$$W_i(z) = a_i\hat{V}(z, s_i) + c_i, \qquad \text{for all } z. \tag{5}$$

If Equation (5) is true, then it follows directly from Equation (4) that $U = \hat{V}$ serves as the state-dependent utility and $P(s_i) = a_i/\Sigma_{k=1}^{n} a_k$ is the probability. The uniqueness follows from the uniqueness of the W_i and $\hat{V}$. To prove Equation (5), let $s = s_j$ for some j and suppose that H_1, H_2, $\hat{f}_1$, $\hat{f}_2$, f_1, and f_2 are as in the statement of Axiom 5. Now consider the set $\mathcal{H}_j$ of all horse lotteries H such that $H(s_i) = H_1(s_i)$ for all $i \neq j$. The stated preferences among this set of horse lotteries satisfy Axioms 1, 2, and 3. Hence there is a utility V_j for this set, and V_j is unique up to positive affine transformation. Clearly, W_j is such a utility, hence we assume that $V_j = W_j$. Next consider the set $\hat{\mathcal{H}}_j$ of all prize–state lotteries $\hat{f}$ that satisfy $\Sigma_{k=1}^{m} \hat{f}(z_k, s_j) = 1$. The stated preferences among elements of $\hat{\mathcal{H}}_j$ also satisfy Axioms 1, 2, and 3. Hence there is a utility $\hat{V}_j$ that is unique up to positive affine transformation. Clearly $\hat{V}$, with domain restricted to $\hat{\mathcal{H}}_j$, is such a utility, hence we will assume that $\hat{V}_j = \hat{V}$. The mapping $T_j : \mathcal{H}_j \to \hat{\mathcal{H}}_j$ defined by $T_j(H)(z, s) = 0$ for all (z, s) with $s \neq s_j$ and $T_j(H) = \alpha_i$ for $z = z_i$ and $s = s_j$, where $H(s_j) = (\alpha_1, \ldots, \alpha_m)$, is one to one and T_j preserves convex combination. It then follows from Axiom 5 that for $H_1, H_2 \in \mathcal{H}_j$, $W_j(H_1) \leq W_j(H_2)$ iff $\hat{V}[T_j(H_1)] \leq \hat{V}[T_j(H_2)]$. Since both $V_j = W_j$ and $\hat{V}_j = \hat{V}$ are unique up to positive affine transformation, we have $W_j = a_j\hat{V} + b_j$ for some positive a_j. This proves Equation (5).

8. DISCUSSION

The need for state-dependent utilities arises out of the possibility that what may appear to be a constant prize may not actually have the same value to an agent in all states of nature. Much of probability theory and statistical theory deals solely with probabilities and not with utilities. If probabilities are only unique relative to a specified utility, then the meaning of much of this theory is in doubt. Much of statistical decision theory makes use of utility functions of the form $U(\theta, d)$, where θ is a state of nature and d is a possible decision. The prize awarded when decision d is chosen and the state of nature is θ is not explicitly mentioned. Rather, the utility of the prize is specified without reference to the prize. Although it would appear that $U(\theta, d)$ is a state-dependent utility (as well it might be), one has swept comparisons between states "under the rug," For example, if $U(\theta, d) = -(\theta - d)^2$, one might ask how it was determined that an error of 1 when $\theta = a$ has the same utility as an error of 1 when $\theta = b$.

DeGroot (1970) avoided these problems by assuming that the concept of one event being at least as likely as another is understood without definition. He then proceeded to state axioms implying the existence of a unique subjective probability distribution over states of nature. (For a discussion of attempts to derive quantitative probability from qualitative probability, see Narens 1980.) Further axioms governing preference could then be introduced. These would then lead to a state-dependent utility function. Axioms such as those of Savage (1954), Von-Neumann and Morgenstern (1947) and Anscombe and Aumann (1963), and deFinetti (1974), which concern only preference among acts like horse lotteries, are not sufficient to guarantee a representation of preference by a unique state-dependent utility and probability. Direct comparisons must be made between lotteries in a specified state of nature and other lotteries in another specified state of nature. These are the prize–state lotteries introduced by Karni (1985). Assuming that preferences among prize–state lotteries are consistent with preferences among horse lotteries, a unique state-dependent utility and probability can be recovered from the preferences.

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REFERENCES

Anscombe, F. J., and Aumann, R. J. (1963), "A Definition of Subjective Probability," *Annals of Mathematical Statistics,* 34, 199–205.

Arrow, K. J. (1974), "Optimal Insurance and Generalized Deductibles," *Scandinavian Actuarial Journal,* 1, 1–42.

deFinetti, B. (1974), *Theory or Probability* (2 vols.), New York: John Wiley.

DeGroot, M. H. (1970), *Optimal Statistical Decisions,* New York: John Wiley.

Fishburn, P. (1970). *Utility Theory for Decision Making,* New York: John Wiley.

Kadane, J. B., and Winkler, R. L. (1988), "Separating Probability Elicitation From Utilities," *Journal of the American Statistical Association,* 83, 357–363.

Karni, E. (1985), *Decision Making Under Uncertainty,* Cambridge, MA: Harvard University Press.

Karni, E., Schmeidler, D., and Vind, K. (1983), "On State Dependent Preferences and Subjective Probabilities," *Econometrica,* 51, 1021–1031.

Narens, L. (1980), "On Qualitative Axiomatizations for Probability Theory," *Journal of Philosophical Logic,* 9, 143–151.

Ramsey, F. P. (1926), "Truth and Probability," in *Studies in Subjective Probability,* eds. H. E. Kyburg and H. E. Smokler, Huntington, NY: Krieger, pp. 23–52.

Rubin, H. (1987), "A Weak System of Axioms for "Rational" Behavior and the Nonseparability of Utility From Prior," *Statistics and Decisions,* 5, 47–58.

Savage, L. J. (1954), *Foundations of Statistics,* New York: John Wiley.

Schick, F. (1986), "Dutch Book and Money Pumps," *Journal of Philosophy,* 83, 112–119.

VonNeumann, J., and Morgenstern, O. (1947), *Theory of Games and Economic Behavior,* Princeton, NJ: Princeton University Press.