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The fundamental theorems of prevision and asset pricing

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Abstract

We explore two connections between the concepts of coherence, as defined by de Finetti, and arbitrage-free asset pricing in financial markets. We contrast these concepts when random quantities may be unbounded. And we discuss some of the consequences for arbitrage theory when coherent previsions are merely finitely (but not countably) additive.

1. Introduction

de Finetti's theory of coherent previsions provides necessary and sufficient conditions that a set of gambles avoids a sure-loss, called a "Book". Parallel to this idea, there is the concept in finance theory of an arbitrage-free set of prices for a set of risky assets. Here, we contrast de Finetti's theory of coherent previsions, notably what is called his Fundamental Theorem of Previsions (Proposition 6) with the Fundamental Theorem of Asset Pricing (Proposition 5). Respectively, each of these fundamental theorems describes how to extend a coherent or arbitrage-free scheme to another one with the same desirable feature over the larger set of gambles or larger set of risky assets, while preserving those previsions or prices already settled. Each theorem provides interval-valued constraints on how the extension may be achieved. In the case of coherent previsions, it is well known that these interval valued constraints may be interpreted also as fixing lower and upper one-sided previsions, where one-sided previsions allow different buy and sell prices, as explained below.

We focus our comparison between these two theorems on two aspects of the resulting theories:

1. What if the extensions are to include unbounded random quantities?
2. Coherence, in de Finetti's sense, permits the use of merely finitely additive (and not countably additive) previsions. What is the counterpart issue for asset pricing? Specifically, by changing the numeraire in which gambles are determined, a coherent countably additive prevision may be changed into an equivalent one that is only merely finitely additive. How does this affect the theory of arbitrage-free pricing?

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de Finetti took the concept of random variables as gambles very seriously, and used that to motivate the familiar concepts of probability and expectation. For each gamble X, he assumed that "You" would assign a value $P(X)$, called the *prevision* of X so that you would be willing to accept the gamble $\beta[X - P(X)]$ as fair for all positive and negative values β. The only constraint that de Finetti envisioned for you and your previsions is that you insisted that there be no positive amount that you had to lose for sure. For example, you would not be allowed to call a gamble fair if its supremum were negative. On the other hand, the criterion is weak enough to allow you call a gamble fair if its supremum is 0, even if all of its possible values are negative.

Let Ω be a set of states with a σ-field of subsets $\mathscr{A}$. Let $\mathscr{X}$ stand for a set of measurable real-valued functions defined on Ω. Whether $\mathscr{X}$ contains unbounded functions will be made clear in each context. The elements of $\mathscr{X}$ will be called gambles, risky assets, or random variables. Functions of elements of $\mathscr{X}$ will also be called by those same names.

Definition 1. Let $\mathscr{X}$ be an arbitrary collection of gambles. Suppose that each gamble $X \in \mathscr{X}$ has a prevision $P(X)$. The collection of previsions is called *coherent* if, for every finite n (no larger than the cardinality of $\mathscr{X}$) and every $X_1, \ldots, X_n \in \mathscr{X}$ and every $\beta_1, \ldots, \beta_n \in \mathbb{R}$,

$$\sup_{\omega \in \Omega} \sum_{i=1}^{n} \beta_i [X_i(\omega) - P(X_i)] \geqslant 0.$$

If the previsions are not coherent, they are called *incoherent*.

Notice that previsions are incoherent if and only if there exist finite n, $\epsilon > 0$, and real $\beta_1, \ldots, \beta_n$ such that for all ω

$$\sum_{i=1}^{n} \beta_i [X_i(\omega) - P(X_i)] < -\epsilon. \tag{1}$$

In other words, your previsions are incoherent if and only if there is some positive amount that you can be forced to lose by combining finitely many of your fair gambles. A combination of gambles that produces the inequality in (1) is called a *book*, and previsions are coherent if and only if no book can be constructed.

The motivation for the definition of coherent previsions is that, if a collection of gambles are individually fair, then a finite sum of them should also be fair. Infinite sums were not of interest to deFinetti. One reason might have been the fact that infinite sums of real numbers are not necessarily defined when both positive and negative values are included. Even in cases in which limits of partial sums exist, the limits can depend on the order in which the sums are arranged.

The concept of arbitrage is similar to, but slightly stronger than, that of incoherence. The formation of a fair gamble as a multiple of $X - P(X)$ makes it natural to think of $P(X)$ as a price to pay for a risky asset X. To avoid arbitrage, it is necessary that your prices do not allow you to lose almost for certain with no chance of winning. The sticky part is defining "almost for certain". To do this, we introduce a subcollection $\mathscr{N} \subset \mathscr{A}$ called the *null events*. These events must satisfy:

- if $A \in \mathscr{N}$ and $B \subseteq A$, then $B \in \mathscr{N}$;
- if $A, B \in \mathscr{N}$, then $A \cup B \in \mathscr{N}$;
- $\Omega \notin \mathscr{N}$.

The three conditions above can be recognized as the conditions defining an *ideal* of subsets of Ω. A set is called *non-null* if it is not null.

Definition 2. Let $\mathscr{X}$ be a collection of risky assets. Suppose that each $X \in \mathscr{X}$ has a price $P(X)$. An *arbitrage opportunity* (or simply an *arbitrage*) exists if there exist a finite n, $X_1, \ldots, X_n \in \mathscr{X}$, and $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \beta_i P(X_i) \leqslant 0$ and $\sum_{i=1}^{n} \beta_i X_i(\omega) \geqslant 0$ for all ω with strict inequality for ω in a "non-null" set.

There are some connections between arbitrage and incoherence. Suppose, for example, that all constant gambles (assets) are in $\mathscr{X}$. That is, for each real c, the gamble X_c with $X_c(\omega) = c$ for all ω is in $\mathscr{X}$. Then, previsions will be incoherent and an arbitrage will exist unless $P(X_c) = c$ for all c. For the remainder of this paper, we will assume that all $X_c \in \mathscr{X}$ and that the price and/or prevision of X_c is c for all real c. These assumptions cannot affect whether or not the previsions are coherent nor can they affect whether or not arbitrages exist. The reader will note that arbitrage, as provided for in Definition 2, is equivalent to the following reformulation, which parallels the formulation in Definition 1.

Definition 2′. An *arbitrage opportunity* (or simply an *arbitrage*) exists if there exist a finite n, $X_1, \dots, X_n \in \mathscr{X}$, and $\beta_1, \dots, \beta_n \in \mathbb{R}$ such that $\sum_{i=1}^n \beta_i [X_i(\omega) - P(X_i)] \geqslant 0$ for all ω with strict inequality for ω in a "non-null" set.

Incoherence implies the existence of arbitrage but not vice versa, as Proposition 1 and Example 1 show.

Proposition 1. *If previsions are incoherent, there is an arbitrage opportunity, no matter which sets count as null.*

Proof. If previsions are incoherent, there exist n, $X_1, \dots, X_n$, $\epsilon > 0$, and $\gamma_1, \dots, \gamma_n$ such that $\sum_{i=1}^n \gamma_i [X_i - P(X_i)] < -\epsilon$. Let $\beta_i = -\gamma_i$ for $i = 1, \dots, n$ and $c = \epsilon + \sum_{i=1}^n \beta_i P(X_i)$ and let $X_0(\omega) = c$ with $\beta_0 = -1$. Then $\sum_{i=0}^n \beta_i X_i(\omega) > 0$ for all ω and $\sum_{i=0}^n \beta_i P(X_i) = -\epsilon$. So, there is an arbitrage no matter which sets count as null. □

Example 1. Consider a simple state space with $\Omega = \{0, 1\}$. Let $X(\omega) = \omega$ and $P(X) = 0$. Suppose that $\mathscr{N} = \{\emptyset\}$ is the collection of null events. Then this single prevision is coherent, but it leads to the obvious arbitrage opportunity.

Example 1 could be "fixed" by declaring $\{1\}$ to be another null event. The next example, however, cannot be fixed.

Example 2. Let $\Omega = \mathbb{Z}^+$. Let $X(\omega) = 1/\omega$ and $P(X) = 0$. Since $\sup_\omega \beta[X(\omega) - 0] \geqslant 0$ for all real β, this prevision is coherent. On the other hand, $P(X) \leqslant 0$ while $X(\omega) > 0$ for all ω, hence there is an arbitrage no matter which events we declare to be null.

2. Unbounded random variables

When $\mathscr{X}$ includes unbounded quantities, it may be impossible to assign finite previsions to all of them.

Example 3. Let $\Omega = \mathbb{Z}^+$, and let $Y(\omega) = 2^\omega$. Also, define $X_i(\omega) = I_{\{\omega\}}(i)$ for $i \in \mathbb{Z}^+$. Suppose that $P(X_i) = 1/2^i$ for all i, corresponding to a geometric distribution over Ω. Finally, let

$$Y_i(\omega) = Y(\omega) I_{[1,i]}(\omega) = \sum_{j=1}^{i} 2^j X_j(\omega),$$

for all $i > 0$, so that Y_i is Y truncated to the interval $[1, i]$. It is easy to see that $Y \geqslant Y_i$ for all i and that $P(Y_i) = i$ for all $i > 0$. If $P(Y)$ could take a value, it would have to be ∞, but such a prevision is not consistent with idea that $\beta[Y - P(Y)]$ is a fair gamble for some non-zero β.

In cases like Example 3, we will use the notation $P(Y) = \infty$ to mean that $\beta[Y - p]$ is acceptable for all finite p and all $\beta \geqslant 0$. Similarly, $P(X) = -\infty$ means that $\beta[X - p]$ is acceptable for all finite p and all $\beta \leqslant 0$. In this way, infinite previsions mean that only one-sided bets are acceptable for the corresponding unbounded random variables.

The reader can find a treatment of previsions for unbounded random variables in Crisma et al. [1], based on a two-point compactification of the real line. Also, an account of one-sided previsions for both bounded and unbounded random variables is given in Troffaes and de Cooman [9], which extends many of the ideas on one-sided previsions found in Walley [10]. Seidenfeld et al. [8] show that once (one-sided) previsions for unbounded random variables are infinite, or once finite previsions for unbounded random variables are not continuous from below, it is not possible to preserve indifference between equivalent random variables. We illustrate the latter problem with the following example, taken from Seidenfeld et al. [8].

Example 4. Let $\Omega = \{1,2,3,\ldots\} \times \{0,1\}$. Let the probability on Ω be $\Pr[\omega = (n,i)] = 2^{-(n+1)}$ for $n = 1,2,\ldots$ and $i = 0,1$. Define three random variables X, W_1, and W_2 on Ω as follows:

$$X(n,i) = n,$$
$$W_1(n,i) = \begin{cases} n+1 & \text{if } i = 1, \\ 1 & \text{if } i = 0, \end{cases}$$
$$W_2(n,i) = \begin{cases} n+1 & \text{if } i = 0, \\ 1 & \text{if } i = 1. \end{cases}$$

In this way, all three of X, W_1, and W_2 have the same Geometric(1/2) distribution, that is each random variable equals the integer m with probability 2^{-m} for $m = 1,2,\ldots$ The expected values are all equal to 2. It is also easy to verify that $W_1 + W_2 - X = 2$. All three of X, W_1, and W_2 can have the same prevision if and only if that prevision is 2. But de Finetti's theory of coherence requires only that $P(X) \geqslant 2$. (See Proposition 6 below for a proof of this fact.) Adding that equivalent random variables carry the same prevision compels continuity (from below) for non-negative, unbounded random variables. We develop this theme in Section 4, below, where we discuss some of the effects of using finitely additive previsions in the theory of arbitrage.

3. Extending previsions and prices

Both coherence and arbitrage have equivalent formulations in terms of linear inequalities. In what follows, X stands for an arbitrary linear combination of gambles or assets. When X is a linear combination of gambles, we use $P(X)$ to mean $\sum_{i=1}^{n} \beta_i P(X_i)$ if $X = \sum_{i=1}^{n} \beta_i X_i$ where the $X_i \in \mathscr{X}$.

Say that $X \leqslant c$ if $X(\omega) \leqslant c$ for all ω. Such an inequality will be called a *weak linear inequality*. Say that $X \prec c$ if $X(\omega) \leqslant c$ for all ω and $X(\omega) < c$ for all ω in some non-null set. Such an inequality will be called a *non-null linear inequality*.

Proposition 2. *The previsions for gambles in a set $\mathscr{X}$ are coherent if and only if every weak linear inequality satisfied by the gambles is also satisfied by the previsions.*

To prove Proposition 2, notice that $\sum_{i=1}^{n} \beta_i [X_i - P(X_i)] < -\epsilon$ if and only if $\sum_{i=1}^{n} \beta_i X_i + \epsilon < \sum_{i=1}^{n} \beta_i P(X_i)$. A similar idea establishes the following.

Proposition 3. *The prices for assets in a set $\mathscr{X}$ lead to no arbitrage opportunities if and only if every non-null linear inequality satisfied by the assets is satisfied as a strict inequality by the prices.*

Definition 3. A *linear functional* on a linear space $\mathscr{X}$ is a real-valued linear function. A *positive* linear functional is a linear functional L such that $L(X) \geqslant 0$ whenever $X \geqslant 0$. A *strictly positive* linear functional is a positive linear functional L such that $L(X) > 0$ if $X \succ 0$. A positive linear functional L is *countably additive* if, for every non-negative increasing sequence $\{X_n\}_{n=1}^{\infty}$ that has a pointwise limit X, $\lim_n L(X_n) = L(X)$. A positive linear functional is *merely finitely additive* if it is not countably additive.

Both coherence and arbitrage have equivalent formulations in terms of linear functionals. The following two results have straightforward proofs. See [2] for Proposition 4 and Delbaen and Schachermayer [3] for Proposition 5.

Proposition 4. *The previsions for gambles in a set $\mathscr{X}$ are coherent if and only there is a positive linear functional L defined on the linear span of $\mathscr{X}$ such that $L(X) = P(X)$ for all $X \in \mathscr{X}$ and $L(1) = 1$.*

Proposition 5 (Fundamental Theorem of Asset Pricing). *The prices for assets in a set $\mathscr{X}$ admit no arbitrage opportunities if and only there is a strictly positive linear functional L defined on the linear span of $\mathscr{X}$ such that $L(X) = P(X)$ for all $X \in \mathscr{X}$ and $L(1) = 1$.*

Extending a coherent set of previsions to include another gamble not in the linear span of $\mathscr{X}$ is similar to extending an arbitrage-free set of prices to include another asset not in the linear span of $\mathscr{X}$.

Proposition 6 (Fundamental Theorem of Prevision). *Suppose that coherent previsions are given for all gambles in a set $\mathcal{X}$. Let Y be a real-valued function not in $\mathcal{X}$. Let,*

$$\underline{A} = \{X : X \leqslant Y \text{ and } X \text{ is in the linear span of } \mathcal{X}\},$$
$$\overline{A} = \{X : X \geqslant Y \text{ and } X \text{ is in the linear span of } \mathcal{X}\}.$$

Define

$$\underline{P}(Y) = \sup_{X \in \underline{A}} P(X),$$
$$\overline{P}(Y) = \inf_{X \in \overline{A}} P(X).$$

Then $P(Y)$ can be taken to be any number in the closed interval $[\underline{P}(Y), \overline{P}(Y)]$ and the resulting previsions are still coherent. Furthermore, no value outside of that closed interval would be a coherent value of $P(Y)$.

Proposition 6 is a version of the theorem in Section 3.10 of de Finetti [2]. The proof of this version is similar to the proof of Proposition 7 below. One example of Proposition 6 is contained in Example 3, assuming that $\mathcal{X}$ contains the bounded gambles in the example but not Y. In that example $\underline{P}(Y) = \overline{P}(Y) = \infty$. The interpretation of infinite prevision in Proposition 6 is precisely the one given immediately after Example 3.

A more intriguing example of Proposition 6 is embedded in Example 4.

Example 5. Consider the random variable X in Example 4. Here X has a Geometric(1/2) distribution with expected value equal to 2. In Proposition 6, $\underline{P}(X) = 2$ and $\overline{P}(X) = \infty$. As we stated in Example 4, we now have many choices for $P(X)$, namely anything in the closed interval $[2, \infty]$.

We will return to Example 5 later to illustrate some other interesting features of prevision for unbounded gambles. In particular, the prevision of a random variable is not merely a function of its distribution as is the mathematical expectation.

For arbitrage-free asset prices, we have the following similar result.

Proposition 7. *Suppose that prices are given for all assets in a set $\mathcal{X}$ such that there are no arbitrage opportunities. Let Y be a real-valued function not in $\mathcal{X}$. Let,*

$$\underline{B} = \{X \prec Y : X \text{ is in the linear span of } \mathcal{X}\},$$
$$\overline{B} = \{X \succ Y : X \text{ is in the linear span of } \mathcal{X}\}.$$

Define

$$\underline{P}(Y) = \sup_{X \in \underline{B}} P(X), \quad \overline{P}(Y) = \inf_{X \in \overline{B}} P(X).$$

Then $P(Y)$ can be taken to be any number in the open interval $(\underline{P}(Y), \overline{P}(Y))$ and there will be no arbitrage opportunities. Furthermore, choosing a price for Y outside of the closed interval $[\underline{P}(Y), \overline{P}(Y)]$ would lead to arbitrage.

Proof. First, we show that prices outside of the closed interval lead to arbitrage. Suppose that $P(Y) < \underline{P}(Y)$ (the case of $P(Y) > \overline{P}(Y)$ is similar). Let $X \in \underline{B}$ be such that $P(X) \geqslant [P(Y) + \underline{P}(Y)]/2$, so that $P(Y) - P(X) < 0$. Since $X \prec Y$, we have $Y - X \geqslant 0$ for all ω with strict inequality on a non-null set and $P(Y) - P(X) < 0$, which constitutes an arbitrage.

For the main assertion, assume, to the contrary, that $P(Y)$ is chosen inside the open interval, but that there is an arbitrage. The coefficient of Y in the arbitrage must be non-zero or there would have been an arbitrage even without Y. So, suppose that there are $X_1, \ldots, X_n \in \mathcal{X}$ and $\beta_1, \ldots, \beta_n$ and β such that

$$\beta P(Y) + \sum_{i=1}^{n} \beta_i P(X_i) \leqslant 0, \tag{2}$$

$$\beta Y(\omega) + \sum_{i=1}^{n} \beta_i X_i(\omega) \geqslant 0, \tag{3}$$

for all ω with strict inequality for $\omega \in A$, a non-null set. Assume that $\beta < 0$ (the other case is similar). It follows from (3) that

$$\sum_{i=1}^{n} -\frac{\beta_i}{\beta} X_i(\omega) \geqslant Y(\omega), \tag{4}$$

for all ω with strict inequality for $\omega \in A$. Hence, the random variable on the left side of (4), call it X, must be an element of $\overline{B}$ in the statement of the proposition. The fact that $P(Y) < P(X)$ is a contradiction to (2). □

The following example illustrates why the interval of possible prices is open in the main assertion of Proposition 7.

Example 6. Let $\Omega = \mathbb{Z}^+$. Let $\mathscr{N}$ be the collection of all finite subsets. Let $\mathscr{X}$ consist of the linear span of all constant functions and all indicators of singletons, i.e., $I_{\{n\}}$ for all n. Let $P(I_{\{n\}}) = 0$ for all n and $P(c) = c$ for each constant c. If $X = \sum_{i=1}^{k} \beta_i X_i \succ 0$, then $X(n)$ is a positive constant for all but finitely many n and $P(X)$ equals that constant. There are no arbitrage opportunities. Now, suppose that we want to add the random variable $Y(n) = 1/n$ for all n. Then $\underline{P}(Y) = 0 = \overline{P}(Y)$, and Proposition 7 gives us no leeway to choose an arbitrage-free price for Y. Indeed, $P(Y) = 0$ leads to arbitrage all by itself as in Example 2.

Sometimes it is possible to avoid arbitrage by choosing $P(Y)$ equal to an endpoint of the open interval in Proposition 7. For example, if Y is itself a linear combination of elements of $\mathscr{X}$, then $\underline{P}(Y) = \overline{P}(Y)$ and the common value avoids arbitrage.

The difference between coherence and lack of arbitrage hinges on considerations of continuity. The following definition introduces a stronger continuity condition than is required for lack of arbitrage.

Definition 4. A *free lunch* is a net $\{(X_\alpha, Y_\alpha) : \alpha \in \aleph\}$ where each X_α is in the linear span of $\mathscr{X}$ and each Y_α is arbitrary and such that $X_\alpha \succ Y_\alpha$ for all α, $\lim_\alpha Y_\alpha = Y \succ 0$, and $\liminf_\alpha P(X_\alpha) \leqslant 0$.

Delbaen and Schachermayer [3] give a version of Proposition 5 for stochastic processes that relies on a condition that is weaker than no free lunch but still stronger than no arbitrage. We will not pursue that condition here.

Proposition 8. *If there is an arbitrage opportunity, then there is a free lunch.*

Proof. Suppose that there exists an arbitrage opportunity. Then there is an X in the linear span of $\mathscr{X}$ with $P(X) \leqslant 0$ and $X \succ 0$. Let $\aleph = \mathbb{Z}^+$ in Definition 4, $X_\alpha = X$ for all α, and $Y_\alpha = X - 1/\alpha$ for all α. Then $X_\alpha \succ Y_\alpha$ for all α, $\lim_\alpha Y_\alpha = X \succ 0$, and $\liminf_\alpha P(X_\alpha) = P(X) \leqslant 0$. □

The converse of Proposition 8 is false as illustrated in Example 7.

Example 7. In Example 6, let $\aleph = \mathbb{Z}^+$,

$$X_\alpha(n) = \left(\frac{1}{n}\right) I_{\{1,\ldots,\alpha\}}(n) + \left(\frac{1}{\alpha}\right) I_{\{\alpha+1,\ldots\}}(n),$$

and $Y_\alpha = Y$ for all α. Since $P(X_\alpha) = 1/\alpha$, this is a free lunch.

Requiring that there be no free lunch requires that prices be a countably additive linear functional.

Proposition 9. *If prices are merely finitely additive, then there is a free lunch.*

Proof. If prices are merely finitely additive, then there exists a sequence $\{Z_n\}_{n=1}^{\infty}$ and Z such that $Z_n \leqslant Z$ for all n, $\lim_n Z_n = Z$, but $\lim_n P(Z_n) < P(Z)$. Let $c \leqslant P(Z) - \lim_n P(Z_n)$. Let $\aleph = \mathbb{Z}+$. For each $\alpha \in \aleph$, let $Y_\alpha = Z_\alpha - Z + c/2$ and $X_\alpha = Y_\alpha + c/4$. Then $X_\alpha \succ Y_\alpha$ for all α, $\lim_\alpha Y_\alpha = c/2 \succ 0$, and $\lim_\alpha P(X_\alpha) \leqslant -c/4$. □

For more discussion of free lunch when probabilities are countably additive, see [5].

4. Finitely additive probability

An alternative method of extending coherent previsions is provided by the Hahn–Banach theorem. Suppose that $\mathscr{X}$ consists of a collection of bounded gambles and $\mathscr{Y} \supseteq \mathscr{X}$ is a larger set of bounded gambles with larger linear span than $\mathscr{X}$. The Hahn–Banach theorem guarantees the existence of an extension of a positive linear functional L on the linear span of $\mathscr{X}$ to a linear functional L' on the linear span of $\mathscr{Y}$. We can make sure that L' is positive using the fundamental theorem of prevision. We have not been able to apply the same reasoning to asset prices and arbitrage without additional conditions, as we explain following the proof of Proposition 10 below.

Let $\mathscr{Y}$ contain $\mathscr{X}$ and all indicators I_A for sets A in some collection of subsets of Ω. For example, we could include all subsets or just those in some field or some σ-field. When we extend our previsions to the linear span of $\mathscr{Y}$ and then restrict the extension to just the collection of indicators, we have a finitely additive probability.

To be specific, let $\mu(A) = P(I_A)$. Then $\mu(\Omega) = 1$, $\mu(A) \geqslant 0$ for all A, and $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$. Every finitely additive probability μ has a unique decomposition as $\alpha\mu_c + (1-\alpha)\mu_f$ where $0 \leqslant \alpha \leqslant 1$, μ_c is countably additive, and μ_f is purely finitely additive. (See [6].)

Definition 5. A probability v is *purely finitely additive* if, for every $\epsilon > 0$, there exists a countable partition $\{A_n\}_{n=1}^{\infty}$ of Ω such that $\sum_{n=1}^{\infty} v(A_n) < \epsilon$. A probability v is *strongly finitely additive* if there exists a partition such that $\sum_{n=1}^{\infty} v(A_n) = 0$. We call a probability v *weakly finitely additive* if, no set A with $v(A) > 0$ equals a countable union of sets each with 0 probability.

Proposition 10. *Suppose that P is a strictly positive linear functional defined on the linear span of $\mathscr{X}$. Let $\mathscr{Y} \supset \mathscr{X}$ include indicators for all events in a class $\mathscr{C}$ such that $\{\omega : Y(\omega) > x\} \in \mathscr{C}$ for every $Y \in \mathscr{Y}$ and every real x. Let P' be an extension of P to $\mathscr{Y}$ and let $\mu(A) = P'(I_A)$ for each $A \in \mathscr{C}$. Assume that the null sets are the sets C with $\mu(C) = 0$, and assume that μ is weakly finitely additive. Then P' is strictly positive.*

Proof. Let $Y \in \mathscr{Y}$ be such that $Y \succ 0$, that is, $Y(\omega) \geqslant 0$ for all ω with strict inequality for $\omega \in A$ a non-null set. Write $A = \bigcup_{i=1}^{\infty} A_i$ where $A_i = \{\omega\colon Y(\omega) \geqslant 1/i\}$. Since $\mu(A) > 0$ and μ is weakly finitely additive, there must exist j such that $\mu(A_j) > 0$. Now, $Y \geqslant I_{A_j}/j$, so $P'(Y) \geqslant \mu(A_j)/j > 0$. □

In the light of the unavoidable arbitrage that results from the coherent previsions in Example 2, where the finitely additive probability is strongly finitely additive, it is necessary to add a restriction about the finitely additive probability appearing in Proposition 10 in order to assure that the extension of P to P' is arbitrage-free. Thus, the existence of arbitrage-free prices on a specific set is insufficient to assure the existence of arbitrage-free extensions to larger sets.

From the lesson of Example 4, we can inquire what kind of integral is possible that does not give equivalent random variables equal prevision. The following discussion indicates how to do this.

Suppose that we have a positive linear functional defined on a linear space $\mathscr{L}$. This space might be the linear span of $\mathscr{X}$ or it might also contain indicators for some events. If X is a simple function, i.e. $X = \sum_{i=1}^{n} a_i I_{A_i}$ where each $I_{A_i} \in \mathscr{L}$, then $P(X) = \sum_{i=1}^{n} a_i \mu(A_i)$. This looks a lot like the first part of the definition of the Lebesgue integral with respect to μ.

Let $X \in \mathscr{L}$ be bounded, and suppose that $X^{-1}(A)$ has its indicator in $\mathscr{L}$ for every interval A. Then, there exist sequences of simple functions $\{\underline{X}_n\}_{n=1}^{\infty}$ and $\{\overline{X}_n\}_{n=1}^{\infty}$ such that, for all n,

- $\underline{X}_n \leqslant X \leqslant \overline{X}_n$,
- $\overline{X}_n - \underline{X}_n \leqslant 1/2^n$,
- $\underline{X}_n \leqslant \underline{X}_{n+1}$, and $\overline{X}_{n+1} \leqslant \overline{X}_n$.

It follows that

$$P(X) = \lim_{n\to\infty} P(\underline{X}_n) = \lim_{n\to\infty} P(\overline{X}_n).$$

This also looks like a part of the definition of the Lebesgue integral.

The general theory of integration with respect to finitely additive measures starts with a finitely additive signed measure defined on a field $\mathscr{F}$ of subsets of Ω. Many interesting functions are not measurable with respect to a typical field. The general definition of finitely additive integral is fraught with measurability considerations.

Without going into details, there are conditions under which a non-measurable function f still has a "uniquely defined" finitely additive integral. In particular, there needs to be a sequence $\{f_n\}_{n=1}^{\infty}$ of integrable simple functions such that the outer absolute measure of $\{\omega: |f_n(\omega) - f(\omega)| > \epsilon\}$ goes to 0 for every $\epsilon > 0$ and the functions are an L^1 Cauchy sequence. See [4, Section III.2] for more detail on the general theory of finitely additive integrals.

An alternative definition of integral begins with a positive linear functional L on a linear space of functions $\mathscr{L}$. Such a functional is a *Daniell integral* if $f_n \downarrow 0$ implies $L(f_n) \to 0$. This last condition is equivalent to countable additivity for indicator functions (using pointwise convergence in $\mathscr{L}$). So, the following definition seems natural.

Definition 6. A positive linear functional L is a *finitely additive Daniell integral*. We call such an L the finitely additive Daniell integral with respect to μ if $L(I_A) = \mu(A)$ for each set A at which μ is defined.

There is a question of whether or not we should add a weaker continuity condition to the definition before calling L a finitely additive integral.

Example 8. Let $\mathscr{F}$ be a field of subsets of Ω, and let μ be a finitely additive probability. Let $\mathscr{L}$ consist of the set of all bounded measurable real-valued functions on Ω. Define $L(f) = \int f\,d\mu$, the finitely additive Daniell integral with respect to μ. Suppose that $f_n \to f$ uniformly. Then, $L(f_n) \to L(f)$.

With the definition of a finitely additive Daniell integral, we have the following rewording of the result in Proposition 4: "Previsions for a collection of gambles are coherent if and only if they are the finitely additive Daniell integrals of the gambles." To put the claim into perspective, recall Example 5.

Example 9. Let $\Omega = \mathbb{Z}^+$ and let $P(I_{\{n\}}) = 2^{-n}$ for all n. Let $X(n) = n$ for all n. Then $\underline{P}(X) = 2$ and $\overline{P}(X) = \infty$ in both Propositions 6 and 7. This time, we have many coherent (and arbitrage-free) choices for $P(X)$. In particular, we could choose $P(X) = 4$, which does not match the countably additive integral of X. However, in the light of Example 4, upon doing so, on pain of incoherence, not all Geometric(1/2) random variables can have the same prevision.

Does $P(X) = 4$ match a finitely additive integral in Example 9? The answer is "yes" according to the fundamental theorem of prevision and the equivalence of coherence with the existence of positive linear functionals. We can even make the linear functional continuous. Of course, every positive linear functional L such that $L(1) = 1$ is continuous in the topology of uniform convergence. But the topology of uniform convergence does not extend nicely to sets with unbounded functions.

Proposition 11. *Suppose that You assign the value $c \geqslant 2$ as $P(X)$ in Example 9. Let $\mathscr{L}$ be the linear span of X and the bounded functions. There exists a norm $\|\cdot\|$ on $\mathscr{L}$ and a positive linear functional L that extends previsions to all $\mathscr{L}$ such that L is continuous with respect to $\|\cdot\|$.*

Proof. Each $f \in \mathscr{L}$ has a unique representation as $f = \alpha X + h$ where h is bounded. Define $L(f) = \alpha c + P(h)$. This L extends P from the bounded functions to $\mathscr{L}$. Let $\|\cdot\|_\infty$ be the L^∞ norm with respect to the (countably additive) measure μ that derives from P. Let $d > 0$. For $f = \alpha X + h$, define

$$\|f\| = \|h\|_\infty + |\alpha| d.$$

It is easy to check that this is a norm. Notice that $\|\alpha_n X + h_n\| \to 0$ if and only if $\alpha_n \to 0$ and $\|h_n\|_\infty \to 0$. But then $L(\alpha_n X + h_n) = \alpha_n c + P(h_n) \to 0$. So, L is continuous with respect to the norm. □

In the topology of Proposition 11, no sequence of bounded functions converges to an unbounded function, although some sequences of unbounded functions do converge to bounded functions. Despite the fact that the underlying measure μ (that derives from P on the bounded functions) is countably additive, the extension of P

to $\mathscr{L}$ is merely finitely additive in the sense of Definition 3. For example, let $X_n = \min\{X, n\}$ for all n and notice that $\lim_n X_n(\omega) = X(\omega)$ for all ω but $\lim_n P(X_n) = 2 < P(X)$. (This limit is pointwise as required by Definition 3, not the limit in the $\|\cdot\|$ topology.)

Choosing $P(X)$ to be greater than the expectation of X puts constraints on the values that we could assign as previsions for other unbounded random quantities.

Proposition 12. *In Example 9, suppose that $P(X) = E(X) + \Delta$, where $E(\cdot)$ stands for expectation and $\Delta > 0$. Let Y be a non-negative random variable defined on the same domain as X:*

- *If $\lim_{n\to\infty} Y(n)/n = \infty$, then $P(Y) = \infty$.*
- *If $\lim_{n\to\infty} Y(n)/n = 0$, then $P(Y) = E(Y)$.*

Proof. For each positive integer k, define $X_k = X$ if $X \leqslant k$ and $X_k = 0$ otherwise. Also define $X'_k = X - X_k$ so that $X = X_k + X'_k$ for all k. Because X_k is bounded, $P(X_k) = E(X_k)$, so $P(X'_k) = E(X'_k) + \Delta$. Next, define $Y_k = Y$ if $X \leqslant k$ and $Y_k = 0$ otherwise. Also define $Y'_k = Y - Y_k$ so that $Y = Y_k + Y'_k$ for all k. If $\lim_{n\to\infty} Y(n)/n = \infty$, then for every $M > 0$ there exists N_M such that for all $n \geqslant N_M$, $Y(n) \geqslant Mn$. For every $M > 0$ and $k \geqslant N_M$, $Y'_k \geqslant MX'_k$, hence

$$P(Y) \geqslant P(Y'_k) \geqslant MP(X'_k) = ME(X'_k) + M\Delta.$$

Since $\Delta > 0$ and M can be chosen arbitrarily large, it follows that $P(Y) = \infty$.

For the second claim, we know that for every $\epsilon > 0$ there exists M_ϵ such that for all $n \geqslant M_\epsilon$, $Y(n) \leqslant \epsilon n$. For every $\epsilon > 0$ and $k \geqslant M_\epsilon$, $Y'_k \leqslant \epsilon X'_k$, hence

$$P(Y) = P(Y_k) + P\left(Y'_k\right) = E(Y_k) + P\left(Y'_k\right) \leqslant E(Y_k) + \epsilon P\left(X'_k\right) \leqslant E(Y_k) + \epsilon P(X).$$

Since $P(X)$ is finite and ϵ can be arbitrarily small,

$$P(Y) \leqslant \lim_{k\to\infty} E(Y_k) = E(Y).$$

Since $P(Y) \geqslant E(Y)$ by coherence, we must have $P(Y) = E(Y)$. □

If two random variables have the same distribution, then they have the same expectation. The same is not true of prevision if the random variables are unbounded, as Example 4 shows.

Suppose that we have two fair coins, and we believe that their flips are independent of each other. Let X be the number of the flip on which the first coin lands heads for the first time. Let Y be the number of the flip on which the second coin lands heads for the first time. Coherence does not require that $P(X) = P(Y)$. Of course, violating $P(X) = P(Y)$ implies finite additivity of the previsions in the sense of Definition 3, as illustrated in Example 4.

5. The numeraire

The finitely additive nature of previsions like those in the previous examples becomes more apparent when we consider a change of numeraire. Results about coherence were stated in terms of random variables and numerical previsions. Implicit in all this is what is meant by a unit. That is, we pay $P(X)$ units to receive $X(\omega)$ units in state ω. If all units are dollars, we can make sense of this. Similarly, if all units are euros, we can make sense of it. If we are willing to contemplate both currencies simultaneously, then we have to consider the exchange rate. In particular, the exchange rate itself can be a random variable.

Example 10. Let X be a function from Ω to $\mathbb{R}$. If we think of X as specifying a number of dollars in each state, this will be different than if X specifies a number of euros in each state. The distinction is caused by the fact that the exchange rate can be random. To keep things straight, let P_D and P_E stand for previsions when the random quantities are assumed to be in units of dollars and euros, respectively.

Suppose that there are three states $\Omega = \{\omega_1, \omega_2, \omega_3\}$ which have equal probabilities in the following sense. When prizes are dollars, each of the acts $X_i = I_{\{\omega_i\}}$ for $i = 1, 2, 3$ has $P_D(X_i) = 1/3$. So, You are willing to pay \$1/3 in order to get \$1 if ω_i occurs and 0 if not, for $i = 1, 2, 3$.

Suppose that the three states have different exchange rates, however. For example, if ω_1 occurs, €1 = \$1.10, if ω_2 occurs, €1 = \$1.20, and if ω_3 occurs €1 = \$1.30. Let $c_1 = 1.1$, $c_2 = 1.2$, $c_3 = 1.3$, and define $C(\omega_i) = c_i$ for $i = 1, 2, 3$. Then C is the random exchange rate in \$/€, and $1/C$ is the random exchange rate in €/\$.

For each gamble Y in dollars, $Y' = Y/C$ is the same gamble reexpressed in units of euros. Similarly, if Y' is a gamble in units of euros, then $Y = Y'C$ is the same gamble in dollars.

The following result shows how to do change of numeraire calculations in cases like Example 10. Although Proposition 13 uses the notation of Example 10, it is very general.

Proposition 13. *Let C be the random exchange rate in \$/€. Assume that $0 < P_D(C) < \infty$:*

- *The marginal exchange rate in \$/€ is $P_D(C) = 1/P_E(1/C)$, and $P_E(1/C)$ is the marginal exchange rate in €/\$.*
- *Let Y be a gamble in dollars with its equivalent $Y' = Y/C$ in euros. Assume that previsions are defined for both gambles in their respective currencies. Then $P_E(Y') = P_D(Y)/P_D(C)$ and $P_D(Y) = P_E(Y')/P_E(1/C)$.*

Proof. Let Y' be the gamble that pays €1 in every state so that $P_E(Y') = 1$. The dollar gamble that pays C dollars in each state is equivalent to the euro gamble Y', so $P_D(C)$ is the amount of dollars that has the same value to You as €1. That is $P_D(C)$ is the marginal exchange rate in \$/€. Similarly, the euro gamble that pays $1/C$ euros in every state is the same as the dollar gamble that pays \$1 in every state. So, $P_E(1/C)$ is the marginal exchange rate in €/\$. The fact that these are reciprocals will follow from the second claim.

For the second claim, let Y be a gamble in dollars with $Y' = Y/C$ its equivalent in euros. Assume that these gambles have finite previsions in their respective currencies. Hence, You are indifferent between Y and Y'. You are also indifferent between Y and $P_D(Y)$ dollars as well as between Y' and $P_E(Y')$ euros. Changing dollars to euros means that you are indifferent between $P_D(Y)P_E(1/C)$ euros and $P_E(Y')$ euros. So, $P_D(Y) = P_E(Y')/P_E(1/C)$. Similarly, changing euros to dollars means that you are indifferent between $P_E(Y')P_D(C)$ dollars and $P_D(Y)$ dollars. So, $P_E(Y') = P_D(Y)/P_D(C)$. Combining the two equations just established for the second claim also shows that the two exchange rates are reciprocals of each other.

Finally, suppose that $P_D(Y) = \infty$. Then You are still indifferent between Y and Y', however, now Y is worth more that every dollar amount x. Hence Y' is worth more than every euro amount $xP_E(1/C)$, and $P_E(Y') = \infty$. A similar argument shows that if $P_E(Y') = \infty$, then $P_E(Y) = \infty$. □

Example 10 illustrates another interesting feature that applies regardless of whether previsions are countably additive or merely finitely additive.

Example 11. Consider again the three gambles (from Example 10) in dollars, $X_i = I_{\{\omega_i\}}$ for $i = 1, 2, 3$. We had $P_D(X_i) = 1/3$ for $i = 1, 2, 3$. Now consider the same three numerical functions as Euro values instead of dollar values. Then

$$P_E(X_i) = \frac{P_D(X_iC)}{P_D(C)} = \frac{c_i/3}{1.2} = \begin{cases} 0.3056 & \text{if } i = 1, \\ 0.3333 & \text{if } i = 2, \\ 0.3611 & \text{if } i = 3. \end{cases}$$

The states have different probabilities when elicited in euros instead of dollars.

The point of Example 11 (extracted with modification from [7]) is that if one interprets the prevision of the indicator of an event as the probability of the event, one must realize that what counts as a unit (the numeraire) makes a difference.

The effect of finitely additive previsions on exchange rate changes can be illustrated by returning to Example 9. There, X had the distribution of the number of tosses of a fair coin until the first head, but we gave X the prevision 4. Suppose that this prevision was in dollars. Suppose that the random exchange rate in \$/€ is $C = X$. Then $P_D(C) = 4$ is the marginal exchange rate. What are the new probabilities for each state $\{n\}$ when elicited in euros?

As in Proposition 13,

$$P_E(I_{\{n\}}) = \frac{P_D(I_{\{n\}}C)}{P_D(C)} = \frac{n}{2^{n+2}},$$

for $n = 1, 2, \ldots$ It is easy to see that $\sum_{n=1}^{\infty} P_E(I_{\{n\}}) = 1/2$. We started with a countably additive probability over the states. Then we performed a change of numeraire which produced a finitely additive probability. The reason is that we had assigned the random exchange rate a finitely, but not countably, additive prevision. Nevertheless, the previsions in one currency are coherent if and only if the previsions in the other currency are coherent, so long as the marginal exchange rate is strictly positive and finite.

6. Summary

Although the coherence and arbitrage-free conditions are similar, they are not identical. Forbidding arbitrage is a stronger requirement than requiring coherence. Each is equivalent to the existence of certain linear functionals that reproduce previsions/prices. Each allows mere finite additivity. Absence of arbitrage does preclude certain finitely additive setups while coherence allows all finitely additive setups. A condition even stronger than being arbitrage-free is "no free lunch" which precludes all mere finite additivity as well as some countably additive setups. Extending coherent previsions to include an additional gamble is always possible. Extending arbitrage-free prices to include an additional gamble is sometimes possible.

Regardless of whether prices are countably or finitely additive, the choice of unit (numeraire) makes a difference in how previsions/prices are interpreted. In particular, changes in numeraire can change probabilities of events. In this sense, a change of numeraire is similar to a change to an equivalent measure. Previsions/prices for unbounded quantities can be merely finitely additive even if probabilities are countably additive. In such cases, a change of numeraire can convert the probabilities to be merely finitely additive, with the consequence that arbitrage-free prices must be different for some equivalent random quantities.

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