Homework 7 (due October 31, 2019)

(1) Biased Random Walk (15 pts)

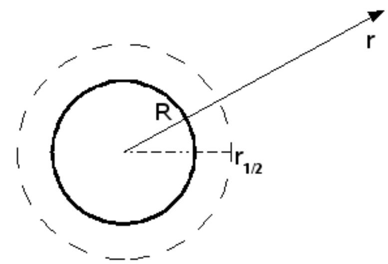
In class, we explored the unbiased, one-dimensional random walk (coin toss determines random steps to the right or left with step size *L*) and found that the mean displacement is $\langle x_N \rangle = 0$ and the variance, $\langle x_N^2 \rangle = NL^2$.

More generally, in a *biased* random walk, the step probabilities may be biased depending on direction, or the step lengths may depend on direction (as for diffusion within a biasing external field): $x_N = x_{N-1} + k_N^j \cdot L$, where $k^j L$ depends on the sign of *k*, either by differences in p_k ("unfair coin") or by differing step sizes that occur with probability p_{k^j} . For example, a unit step forward, k^{+1}, occurs with probability $p_{k^{+1}} = 0.75$ and a unit step back, k^{-1}, with $p_{k^{-1}} = 0.25$. In such cases, $\langle x_N \rangle = NL \langle k^j \rangle$ and $\text{var}(x_N) = NL^2 \cdot \text{var}(k^j)$.

(a) Consider a random walk in which the particle has, at each time step, a 50% chance of moving a distance $(L + a)$ to the right or a distance $-(L - a)$ to the left (with $L > a$). Determine the mean displacement and the variance of the position of the particle after a large number *N* of steps. Describe how this result differs from the unbiased random walk in terms of $\langle x_N \rangle$ and var(x_n).

(b) Consider a random walk in which the particle has, at each time step, a chance $P_{k^{-1}} = 1/3$, $P_{k^0} = 1/6$ and $P_{k^{+1}} = 1/2$ to step to the left by *L*, take no step at all, or step to the right by *L*, respectively. Determine the mean displacement and the variance of the position of the particle after a large number *N* of steps. Again, compare with the unbiased random walk.

(2) Oxygen absorption by a cell **(15 pts)**



A bacterium absorbs dissolved oxygen from the surrounding medium through its cell membrane, as the molecules diffuse in the medium with a diffusion constant D. The O_2 intake rate I is defined as the flux of O_2 molecules through the surface of the whole bacterium. Approximate the bacterium as a sphere of radius R with the coordinate $r = 0$ at its center (see graph). The concentration of O_2 far away is constant $c(\infty) = c_0$, while at the surface of the bacterium $c(R) = 0$ because O_2 concentration is depleted due to the intake.

a) Integrate Fick's law $j = -D(\mathrm{d}c/\mathrm{d}r)$ using these boundary conditions to obtain the concentration profile $c(r)$ of O_2 molecules from $r = R$ outwards. Provide a qualitative plot of $c(r)$ for two bacterium sizes R_1 and R_2, with $R_1 < R_2$.

b) How does $r_{1/2}$, the distance at which $c(r)$ is $c(\infty)/2$, depend on R?

c) The bacterium has metabolic activity A, defined as the oxygen consumption rate E divided by its mass, m. It thrives only if the consumption rate E doesn't exceed the intake rate I which, as you probably found in (a), is linear in R. Sketch E and I as a function of R to visualize the range of sizes that allow survival. With your result for I from (a), calculate the maximal radius R_m at which a spherical bacterium of density ρ can survive.