The shape of a straight fluid meniscus

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(Dated: March 15, 2004)

Fluid interfaces approach solid surfaces at a particular wetting angle, determined from a balance between the involved interfacial free energies. This generally implies a deformation of the interface close to the solid, which is termed "meniscus". The prototypical case of an asymptotically horizontal surface in contact with a vertical plane is studied in these notes.

Angle–arc-length description

Consider a straight fluid meniscus as illustrated in Fig. 1. The interface between two incompressible liquids of density $\rho_>$ and $\rho_< < \rho_>$ rises at a vertical plane to an equilibrium height y_0, since it wants to establish some prescribed wetting angle α. We want to derive simple equations describing the shape of this meniscus.

Far away from the vertical plane the interface is flat, and there is no pressure difference across it. However, near the vertical plane the interface has risen a distance y above the asymptotic level, and a hydrostatic excess pressure $\Delta P = (\rho_> - \rho_<)gy$ acts on its upper surface (g is the gravitational acceleration). According to the Young-Laplace law this excess pressure must equal twice the interfacial tension σ times the mean curvature of the surface. Using the parameterization indicated in Fig. 1, which specifies the angle $\psi(s)$ against the horizontal as a function of the arc-length s along the profile, the mean curvature is (up to the purely conventional sign) readily seen to be $\frac{1}{2}\dot{\psi}$, where the dot indicates a derivative with respect to s. Choosing the sign such that the excess pressure acts indeed on the concave side, we arrive at the following two differential equations for the profile:

$$\dot{\psi} \;=\; -\frac{y}{\ell^2}\;, \tag{1a}$$

$$\dot{y} \;=\; -\sin\psi\;, \tag{1b}$$

where we also introduced the so-called *capillary length*

$$\ell \;:=\; \sqrt{\frac{\sigma}{g\,(\rho_> - \rho_<)}}\;. \tag{2}$$

For the water-air interface we have $\sigma \simeq 80\,\mathrm{mN/m}$, which gives $\ell \simeq 2.8\,\mathrm{mm}$.

From the chain rule we have $\dot{\psi} = \frac{\mathrm{d}\psi}{\mathrm{d}y}\dot{y}$. Eqns. (1a, 1b) can thus be rewritten as $y = \ell^2\frac{\mathrm{d}\psi}{\mathrm{d}y}\sin\psi$, a single differential equation which can easily be solved by separation of variables. Integrating from the asymptotic boundary, where $\psi = 0$ and $y = 0$, we find $y^2 = 2\ell^2(1-\cos\psi)$. Hence, as long as $0 \le \psi \le 2\pi$, we get

$$y(\psi) \;=\; 2\ell\,\sin\frac{\psi}{2}\;. \tag{3}$$

At contact we have $\psi(s=0) = \psi_0 = \frac{\pi}{2} - \alpha$, from which we readily get the height of the meniscus:

$$\frac{y_0}{\ell} \;=\; 2\,\sin\frac{\pi - 2\alpha}{4} \quad \left(\stackrel{\alpha\to 0}{\longrightarrow}\;\sqrt{2}\;\right)\;. \tag{4}$$

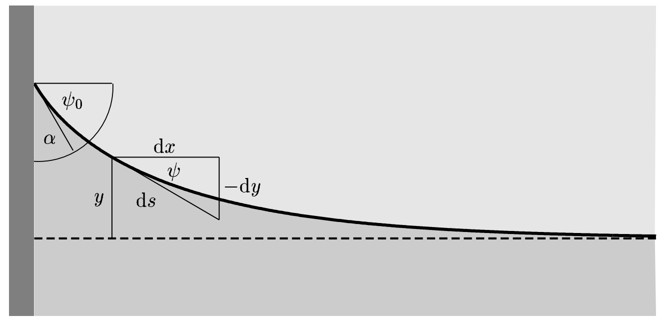

FIG. 1: Parameterization of a simple planar meniscus in terms of the angle ψ against the horizontal as a function of arc length s along the profile ($s = 0$ at contact).

In order to completely solve the shape equations, we differentiate Eqn. 3 with respect to s:

$$-2\sin\frac{\psi}{2}\cos\frac{\psi}{2} \;=\; -\sin\psi \;\stackrel{(1b)}{=}\; \dot{y} \;\stackrel{(3)}{=}\; 2\ell\,\cos\frac{\psi}{2}\,\frac{\dot{\psi}}{2}\;. \tag{5}$$

This differential equation can again be solved easily by separation of variables. The solution is

$$\psi(s) \;=\; 4\,\arctan\left[\tan\frac{\psi_0}{4}\,\mathrm{e}^{-s/l}\right]\;. \tag{6}$$

This formula gives the angle $\psi(s)$ of the profile as a function of the arc-length s. To get the actual position x and y, one needs to integrate $\cos\psi(s)$ and $-\sin\psi(s)$ from the contact up to the arc-length in question:

$$\frac{x(s)}{\ell} \;=\; \frac{\frac{s}{\ell}\cosh\frac{s}{\ell} + \left(\frac{s}{\ell}\cos\frac{\psi_0}{2} - (1-\cos\psi_0)\right)\sinh\frac{s}{l}}{\cosh\frac{s}{l} + \cos\frac{\psi_0}{2}\sinh\frac{s}{\ell}} \tag{7a}$$

$$\frac{y(s)}{\ell} \;=\; \frac{2\,\sin\frac{\psi_0}{2}}{\cosh\frac{s}{l} + \cos\frac{\psi_0}{2}\sinh\frac{s}{\ell}}\;. \tag{7b}$$

x versus y

Equations (7a) and (7b) constitute a parametric representation of the meniscus. If one wishes a description entirely in terms of x and y, one needs to eliminate s between them. This is unfortunately a bit tricky, but it can be accomplished in the following (a bit convoluted) way:

We first specialize these equations for the case $\psi_0 = \pi$, in which case they simplify considerably:

$$\frac{x(s,\psi_0=\pi)}{\ell} = \frac{s}{\ell} - 2\,\tanh\frac{s}{\ell}\ , \tag{8a}$$

$$\frac{y(s,\psi_0=\pi)}{\ell} = 2\,\mathrm{sech}\frac{s}{\ell}\ . \tag{8b}$$

In this form Eqn. (8b) can be solved for s, with the result $\frac{s}{\ell} = \mathrm{arcosh}\frac{2\ell}{y}$. Reinserting into Eqn. (8a) gives

$$\frac{x(y,\psi_0=\pi)}{\ell} = \mathrm{arcosh}\frac{2\ell}{y} - 2\sqrt{1-\Big(\frac{y}{2\ell}\Big)^2}\ . \tag{9}$$

At the maximum height $y/\ell = \sqrt{2}$ (see Eqn. (4)) the right hand side has the value $\mathrm{arcosh}\sqrt{2} - \sqrt{2}$. Hence, subtracting this value shifts the x position to zero, and thus describes the profile for a wetting angle of $\alpha = 0$. Other profiles are then obtained by simply shifting the x-coordinate relative to the complete wetting case. After rewriting $\mathrm{arcosh}(x) = \log\big[x+\sqrt{x^2-1}\big]$ we find

$$\frac{x(y)-x_0}{\ell} = \log\frac{\frac{2\ell}{y} - \sqrt{\Big(\frac{2\ell}{y}\Big)^2-1}}{\sqrt{2}-1} - 2\sqrt{1-\Big(\frac{y}{2\ell}\Big)^2} + \sqrt{2}\ . \tag{10}$$

Why is this funny trick allowed? Eqn. (6) shows that the influence of ψ_0 on the shape is rather limited: Looking at $\psi(s)$ over the entire parameter range $-\infty < s < +\infty$ shows that it describes one unique curve, and ψ_0 merely specifies the angle which is reached at the particular parameter value $s = 0$. Since we want to eliminate s anyway, we can choose ψ_0 as convenient as we wish, and later shift the curve to the correct location (thereby also using translational invariance in the x direction).

In any case, it appears quite obvious that the profile description of Eqn. (10) is still rather complicated, and not much appears to be gained compared to the rather explicit parametric description (7a, 7b), let alone the angle–arc-length description from Eqn. (6).

Two mathematical close relatives

There is a close correspondence between the shape of a meniscus and a well known problem from classical mechanics: the planar mathematical pendulum. A Lagrangian for this system is given by

$$\mathscr{L}' = \frac{1}{2}m(L\dot{\psi})^2 - mgL\cos\psi\ , \tag{11}$$

where m is the pendulum mass, L its length, g the gravitational acceleration, ψ the angle of the pendulum against the vertical (here measured from the *unstable* direction pointing upward), and where the dot indicates a derivative with respect to time. By rescaling the energy and introducing the classical low amplitude frequency $\omega := \sqrt{g/L}$, we can rewrite the problem in terms of a dimensionless Lagrangian $\mathscr{L} := \frac{\mathscr{L}'}{mgL} = \frac{\dot{\psi}^2}{2\omega^2} - \cos\psi$. The momentum canonically conjugate to ψ is $y := \frac{\partial\mathscr{L}}{\partial\dot{\psi}} = \dot{\psi}/\omega^2$, and the Hamiltonian corresponding to $\mathscr{L}$ is

$$\mathscr{H} := \dot{\psi}y - \mathscr{L} = \frac{1}{2}\omega^2y^2 + \cos\psi\ . \tag{12}$$

Its equations of motion are

$$\dot{\psi} = \frac{\partial\mathscr{H}}{\partial y} = \omega^2 y\ , \tag{13a}$$

$$\dot{y} = -\frac{\partial\mathscr{H}}{\partial\psi} = \sin\psi\ . \tag{13b}$$

Up to an irrelevant overall sign (which we can get rid of by reversing the direction of time) these equations are exactly the same as Eqns. (1a, 1b), with the (inverse) frequency ω replacing the capillary length, and (negative) time replacing arc-length.

There's also a second system which can be brought into correspondence with the meniscus, and this is an elastic rod. Kirchhoff pointed out first that the equations governing the equilibrium shape of elastic rods are equivalent to the equations of motion of a spinning top in a gravitational field. We have just seen that the meniscus problem is equivalent to a pendulum (which is just a "non-spinning" top). "No rotation" maps to "no twist elasticity", and in this case planar solutions exist. In fact, Eqns. (7a, 7b) precisely describe the shape of a twist-free infinitely long elastic rod with bending modulus A which is pulled at its two ends at a constant force F and which is planar and has precisely one loop (see the illustration in Fig. 2). The characteristic length is in this case $\ell = \sqrt{A/F}$. In the pendulum language this corresponds to the degenerate motion in which the pendulum starts at time $t = -\infty$ in the position pointing upwards, swinging once, passing downwards at $t = 0$, and reaching upwards at $t = +\infty$.

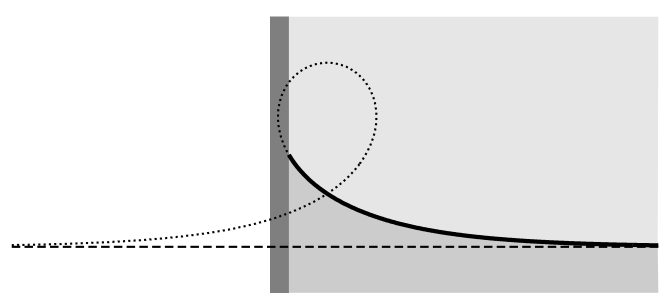

FIG. 2: Correspondence between the simple planar meniscus (solid line) and the elastic rod with bending stiffness A, which is pulled at its ends at a constant force F and which has exactly one loop (doted curve). The shapes overlap if the capillary length ℓ equals $\sqrt{A/F}$.