Notes on Differential Geometry

with special emphasis on surfaces in $\mathbb{R}^3$

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These notes are an attempt to summarize some of the key mathematical aspects of differential geometry, as they apply in particular to the geometry of surfaces in $\mathbb{R}^3$. The focus is not on mathematical rigor but rather on collecting some bits and pieces of the very powerful machinery of manifolds and "post-Newtonian calculus". Even though the ultimate goal of elegance is a complete coordinate free description, this goal is far from being achieved here—not because such a description does not exist yet, but because the author is far to unfamiliar with it. Most of the geometric aspects are taken from Frankel's book [9], on which these notes rely heavily. For "classical" differential geometry of curves and surfaces Kreyszig book [14] has also been taken as a reference.
The depth of presentation varies quite a bit throughout the notes. Some aspects are deliberately worked out in great detail, others are only touched upon quickly, mostly with the intent to indicate into which direction a particular subject might be followed further.

Contents

1. Some fundamentals of the theory of surfaces

1.1. Basic definitions

1.1.1. Parameterization of the surface

Let U be an (open) subset of $\mathbb{R}^2$ and define the function

$$\vec{r} : \left\{ \begin{array}{ccc} \mathbb{R}^2 \supset U & \to & \mathbb{R}^3 \\ (u^1, u^2) & \mapsto & \vec{r}(u^1, u^2) \end{array} \right. . \tag{1.1}$$

We will assume that all components of this function are sufficiently often differentiable. Define further the vectors[1]

$$\boldsymbol{e}_\mu \equiv \vec{r}_{,\mu} := \frac{\partial \vec{r}}{\partial u^\mu} , \tag{1.2a}$$

$$\text{and} \qquad \vec{n} := \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_2}{|\boldsymbol{e}_1 \times \boldsymbol{e}_2|} . \tag{1.2b}$$

If the $\boldsymbol{e}_\mu$ are everywhere linearly independent[2], the mapping (1.1) defines a smooth surface S embedded in $\mathbb{R}^3$. S is a differentiable submanifold of $\mathbb{R}^3$. The vectors $\boldsymbol{e}_\mu(\vec{r})$ belong to $T_{\vec{r}}S$, the tangent space of S at $\vec{r}$, this is why we use a different notation for them than the "ordinary" vectors from $\mathbb{R}^3$. Note that while $\vec{n}$ is a unit vector, the $\boldsymbol{e}_\mu$ are generally not of unit length.

1.1.2. First fundamental form

The *metric* or *first fundamental form* on the surface S is defined as

$$g_{ij} := \boldsymbol{e}_i \cdot \boldsymbol{e}_j . \tag{1.3}$$

It is a second rank tensor and it is evidently symmetric. If it is furthermore (everywhere) *diagonal*, the coordinates are called *locally orthogonal*.

The *dual* tensor is denoted as g^{ij}, so that we have

$$g_{ij} g^{jk} = \delta_i^k = \left\{ \begin{array}{ll} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{array} \right. , \tag{1.4}$$

where δ_i^k is called the *Kronecker symbol*. Hence, the components of the inverse metric are given by

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \frac{1}{g} \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix} . \tag{1.5}$$

By virtue of Eqn. (1.4) the metric tensor can be used to raise and lower indices in tensor equations. Technically, "indices up or down" means that we are referring to components of tensors which live in the tangent space or the cotangent space, respectively. It requires the additional structure of a metric in the manifold in order to define an isomorphism between these two different vector spaces.

The determinant of the first fundamental form is given by

$$g := \det \boldsymbol{g} \equiv |\boldsymbol{g}| \equiv |g_{ij}| = \frac{1}{2} \varepsilon^{ik} \varepsilon^{jl} g_{ij} g_{kl} , \tag{1.6}$$

where ε^{ik} is the two-dimensional antisymmetric Levi-Cività symbol

$$\varepsilon^{ik} = \begin{vmatrix} \delta_1^i & \delta_2^i \\ \delta_1^k & \delta_2^k \end{vmatrix} = \delta_1^i \delta_2^k - \delta_1^k \delta_2^i \qquad , \qquad \varepsilon_{ik} = \varepsilon^{ik} .$$

[1] $\boldsymbol{e}_\mu = \partial \vec{r} / \partial u^\mu$ is the classical notation. The modern notation simply calls $\boldsymbol{\partial} / \boldsymbol{\partial} u^\mu$ (or even shorter: $\boldsymbol{\partial}_{u^\mu}$) the canonical local coordinate basis belonging to the coordinate system $\{x\}$.

[2] An equivalent requirement is that the differential $\vec{r}_*$ has rank 2 (see Sec. B.1).

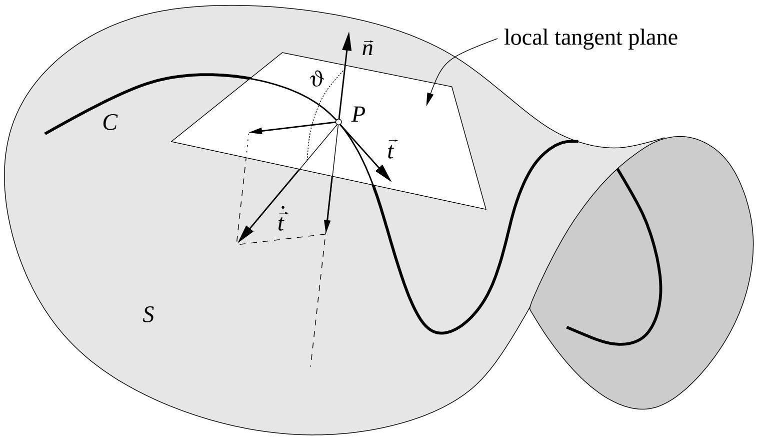


Figure 1.1.: Illustration of the definition of the normal curvature κ_n, Eqn. (1.11), and the geodesic curvature κ_g, Eqn. (1.15). They are essentially given by the projection of $\dot{\vec{t}}$ onto the local normal vector and onto the local tangent plane, respectively.

If φ is the angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, then we have

$$|\boldsymbol{e}_1 \times \boldsymbol{e}_2|^2 = |\boldsymbol{e}_1|^2 |\boldsymbol{e}_2|^2 \sin^2\varphi = g_{11}g_{22}(1-\cos^2\varphi) = g_{11}g_{22} - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2 = g_{11}g_{22} - g_{12}g_{21} = g \ .$$

Hence, we have

$$|\boldsymbol{e}_1 \times \boldsymbol{e}_2| = \sqrt{g} \ .$$

1.1.3. Second fundamental form

Assume that there is some curve C defined on the surface S, which goes through some point P, at which the curve has the tangent vector $\vec{t}$ and principal normal vector $\vec{p} = \dot{\vec{t}}/\kappa$, and at which point the surface has the normal vector $\vec{n}$—see as an illustration Fig. 1.1. We now have the following two equations:

$$\vec{p} \cdot \vec{n} \ = \ \cos\vartheta \qquad\qquad \text{and} \qquad\qquad \dot{\vec{t}} = \kappa\,\vec{p} \ .$$

The first defines the angle ϑ between the two unit vectors $\vec{n}$ and $\vec{p}$, the second defines the curvature of the curve. Combining them, we obtain

$$\kappa\cos\vartheta \ = \ \dot{\vec{t}} \cdot \vec{n} \ . \tag{1.7}$$

If the curve is parameterized as $u^i(s)$, we have

$$\dot{\vec{t}}(s) \ = \ \ddot{\vec{r}}(s) \ = \ \frac{\partial^2}{\partial s^2}\vec{r}(s) \ = \ \frac{\partial}{\partial s}\big(\vec{r}_{,i}\dot{u}^i\big) \ = \ \vec{r}_{,ij}\dot{u}^i\dot{u}^j + \vec{r}_{,i}\ddot{u}^i \ = \ \boldsymbol{e}_{i,j}\dot{u}^i\dot{u}^j + \boldsymbol{e}_i\ddot{u}^i \ .$$

Since $\boldsymbol{e}_i \cdot \vec{n} = 0$, we obtain from this and Eqn. (1.7)

$$\kappa\,\cos\vartheta \ = \ \dot{\vec{t}} \cdot \vec{n} \ = \ \big(\boldsymbol{e}_{i,j} \cdot \vec{n}\big)\,\dot{u}^i\dot{u}^j \ . \tag{1.8}$$

The expression in brackets is independent of the curve and a property of the surface alone. It is called the second fundamental form, and we will term it b_{ij}:

$$b_{ij} := \boldsymbol{e}_{i,j} \cdot \vec{n} \ . \tag{1.9}$$

Since $\boldsymbol{e}_{i,j} = \boldsymbol{e}_{j,i}$, the second fundamental form is symmetric in its two indices. If the second fundamental form is furthermore diagonal, the coordinate lines are called *conjugate*.[3] If first *and* second fundamental form are diagonal, the coordinate lines are orthogonal and they form *lines of curvature*, *i. e.*, they locally coincide with the principal directions of curvature (see below). Differentiating the obvious relation $\boldsymbol{e}_i \cdot \vec{n} = 0$ with respect to u^j shows that $\boldsymbol{e}_{i,j} \cdot \vec{n} + \boldsymbol{e}_i \cdot \vec{n}_{,j} = 0$, from which follows that the second fundamental form is also given by

$$b_{ij} := -\boldsymbol{e}_i \cdot \vec{n}_{,j} \ . \tag{1.10}$$

This expression is usually less convenient, since it involves the derivative of a *unit* vector, and thus the derivative of square-root expressions.

Loosely speaking, the curvature κ of a curve at the point P is partially due to the fact that the curve itself is curved, and partially because the surface is curved. In order to somehow disentangle these two effects, it it useful to define the two concepts *normal curvature* and *geodesic curvature*. We follow Kreyszig [14] in our discussion.

[3] See Ref. [14, paragraph 60] for a more detailed discussion on what this implies

The left hand side of Eqn. (1.8) only depends on the *direction* of the curve at P, *i. e.* $\vec{t}$, but *not* on its curvature. Hence, it is actually a property of the *surface*. It is called the *normal curvature* $\kappa_{\rm n}$ *of the surface in the direction* $\vec{t}$. If we perform a reparameterization of the curve, we find $\dot{u}^i = (\mathrm{d}u^i/\mathrm{d}t)(\mathrm{d}t/\mathrm{d}s) = u'^i/s'$, and from that we find:

$$\kappa_{\rm n} := \kappa \cos\vartheta = \frac{b_{ij}u'^i u'^j}{g_{ij}u'^i u'^j} = \frac{b_{ij}\mathrm{d}u^i\mathrm{d}u^j}{g_{ij}\mathrm{d}u^i\mathrm{d}u^j} . \tag{1.11}$$

The normal curvature is therefore the ratio between the second and the first fundamental form.

Equation (1.8) shows that the normal curvature is a quadratic form of the $\dot{u}^i$, or loosely speaking a quadratic form of the tangent vectors on the surface. It is therefore not necessary to describe the curvature properties of a surface at every point by giving all normal curvatures in all directions. It is enough to know the quadratic form.

It is natural to ask, in which directions the normal curvature is extremal. Rewriting Eqn. (1.11) as

$$\big(b_{ij} - \kappa_{\rm n}g_{ij}\big)v^i v^j = 0 ,$$

and differentiating this expression with respect to v^k (treating $\kappa_{\rm n}$ as a constant, since $\mathrm{d}\kappa_{\rm n} = 0$ is a necessary condition for $\kappa_{\rm n}$ to be extremal), we find

$$\big(b_{ik} - \kappa_{\rm n}g_{ik}\big)v^i = 0 ,$$

or after raising the index k

$$\big(b_i^k - \kappa_{\rm n}\delta_i^k\big)v^i = 0 . \tag{1.12}$$

This is an important result: It shows that the search for extremal curvatures and the corresponding directions leads to an *eigenvalue problem*: The directions along which the normal curvature is extremal are given by the eigenvectors of the matrix b_i^k, and the corresponding eigenvalues are the extremal curvatures. These two eigenvalues are called *principal curvatures*, and we will call them κ_1 and κ_2. This permits us to define the following two important concepts: Mean curvature H and Gaussian curvature K are defined as sum and product of the principal curvatures

$$2\,H := \kappa_1 + \kappa_2 = b_i^i , \tag{1.13a}$$

$$K := \kappa_1\,\kappa_2 = \big|b_i^k\big| = \big|b_{ij}g^{jk}\big| = \big|b_{ij}\big|\,\big|g^{jk}\big| = \frac{b}{g} , \tag{1.13b}$$

where b is the determinant of the second fundamental form:

$$b := \det\boldsymbol{b} \equiv |\boldsymbol{b}| \equiv |b_{ij}| = \frac{1}{2}\varepsilon^{ik}\varepsilon^{jl}b_{ij}b_{kl} , \tag{1.14}$$

Since the definitions of H and K involve the eigenvalues of b_i^j, they are invariant under reparametrizations of the surface. They are intrinsic surface properties.

While the normal curvature measures how the surface bends in space, the so called *geodesic curvature* $\kappa_{\rm g}$ is a measure of how a curve curves on a surface, which is independent of the curvature of the surface itself. While the normal curvature is obtained by projecting the vector $\dot{\vec{t}}$ of the curve onto the local normal vector of the surface, the geodesic curvature is obtained by projecting $\dot{\vec{t}}$ onto the local tangent plane, thereby essentially projecting out any curvature deformations of the surface. Looking at Fig. 1.1, and by a similar argument as the one which lead to Eqn. (1.7), we find

$$\kappa_{\rm g} = \kappa \sin\vartheta . \tag{1.15}$$

1.2. Formulas of Weingarten and Gauss

A key result in the theory of space curves are the *formulas of Frenet*, which express the change of the local coordinate system (tangent vector, normal vector, and binormal vector) upon movements along the curve in terms of this very coordinate system. The analogue of this in the theory of surfaces are the formulas by Weingarten and Gauss, which describe the variation of the local coordinate system upon small movements on the surface.

Since $\vec{n}\cdot\vec{n} = 1$, differentiation with respect to u^μ gives $\vec{n}_{,\mu}\cdot\vec{n} = 0$. This implies that the change in normal vector upon (infinitesimally) moving on the surface is parallel to the surface. It can hence be expressed as a linear combination of the tangent vectors, *i. e.*, we can write $\vec{n}_{,\mu} = A_\mu^\lambda \boldsymbol{e}_\lambda$. A scalar multiplication with $\boldsymbol{e}_\nu$ together with Eqn. (1.10) shows that $A_\mu^\nu = -b_\mu^\nu$, and we thereby obtain the *formula of Weingarten*:

$$\underline{\vec{n}_{,\mu} = -b_\mu^\nu\boldsymbol{e}_\nu} \qquad \text{(Weingarten)} . \tag{1.16}$$

The change of the tangent vectors is generally along all three directions of the local coordinate system, so we may write $\boldsymbol{e}_{\mu,\nu} = A_{\mu\nu}^\sigma\boldsymbol{e}_\sigma + B_{\mu\nu}\vec{n}$. A scalar multiplication with $\vec{n}$ together with Eqn. (1.9) immediately shows that

$B_{\mu\nu} = b_{\mu\nu}$. A scalar multiplication with $\boldsymbol{e}_\lambda$ shows that $A_{\mu\nu\lambda} = \boldsymbol{e}_{\mu,\nu} \cdot \boldsymbol{e}_\lambda$, where we defined $A_{\mu\nu\lambda} = A^\sigma_{\mu\nu} g_{\sigma\lambda}$. Note that this shows that $A_{\mu\nu\lambda}$ is symmetric in its first two indices, because $\boldsymbol{e}_{\mu,\nu} = \vec{r}_{,\mu\nu} = \vec{r}_{,\nu\mu}$.

Let us now look at derivatives of the metric:

$$g_{\mu\lambda,\nu} \;=\; (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\lambda)_{,\nu} \;=\; \boldsymbol{e}_{\mu,\nu} \cdot \boldsymbol{e}_\lambda + \boldsymbol{e}_\mu \cdot \boldsymbol{e}_{\lambda,\nu} \; .$$

Together with the expression for $A_{\mu\nu\lambda}$ and after cyclic permutation, we obtain the three equivalent equations:

$$g_{\mu\lambda,\nu} \;=\; A_{\mu\nu\lambda} + A_{\lambda\nu\mu} \; , \qquad (1.17a)$$
$$g_{\lambda\nu,\mu} \;=\; A_{\lambda\mu\nu} + A_{\nu\mu\lambda} \; , \qquad (1.17b)$$
$$g_{\nu\mu,\lambda} \;=\; A_{\nu\lambda\mu} + A_{\mu\lambda\nu} \; . \qquad (1.17c)$$

If we now add the first two of these equations and from that subtract the third one, *i. e.*, if we form the combination (1.17a)+(1.17b)−(1.17c), and additionally exploit the symmetry of $A_{\mu\nu\lambda}$ in its first two indices, we find

$$A_{\mu\nu\lambda} = \frac{1}{2}\big[g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}\big] \; .$$

This shows that the $A_{\mu\nu\lambda}$ are simply the Christoffel symbols of the first kind—see Eqn. (A.1). And hence, $A^\lambda_{\mu\nu}$ are the Christoffel symbols of the second kind—see Eqn. (A.2). We then arrive at the *formula of Gauss*:

$$\underline{\boldsymbol{e}_{\mu,\nu} \;=\; \Gamma^\sigma_{\mu\nu}\boldsymbol{e}_\sigma + b_{\mu\nu}\vec{n}} \qquad \text{(Gauss)} \; . \qquad (1.18)$$

Finally we remark that using the concept of covariant differentiation, the formulas of Weingarten and Gauss can be rewritten as

$$\nabla_k \vec{n} \;=\; -b^i_k \boldsymbol{e}_i \; , \qquad (1.19a)$$
$$\text{and} \qquad \nabla_k \boldsymbol{e}_i \;=\; b_{ik}\vec{n} \; , \qquad (1.19b)$$

which is about as close as we will get to an analogue of the formulas of Frenet. We want to mention, though, that there exists a very elegant theory of the description of manifolds using "(co)moving orthonormal frames", which includes the formulas of Frenet for curves and the formulas of Weingarten and Gauss for surfaces as special cases. The theory goes back to a large extend to work by Cartan, and is described in Ref. [9] and in a little more detail in Ref. [7].

Problem 1.1 *Using the covariant version of the formulas of Weingarten and Gauss, verify the identity*

$$\vec{n} \;=\; \frac{1}{2}\nabla^a\Big[\big(\vec{r}\cdot\boldsymbol{e}_a\big)\,\vec{n} - \big(\vec{r}\cdot\vec{n}\big)\,\boldsymbol{e}_a\Big] \; . \qquad (1.20)$$

This formula shows that the normal vector can be written as a surface divergence. (Jemal Guven, personal communication.)

1.3. Integrability conditions

As stated above, the formulas of Weingarten and Gauss are the surface analogue of the formulas by Frenet for curves. However, while in the one-dimensional case every prescribed curvature and torsion function gives rise to a well defined curve (up to translations and rotations), the same is not true in the surface case: *Not every prescribed first and second fundamental form describes a surface!* There are *integrability conditions* to be satisfied, which we will now derive. In fact, everything follows from the required identity $\boldsymbol{e}_{i,jk} = \boldsymbol{e}_{i,kj}$. Using the formulas of Gauss and Weingarten, this can be rewritten as

$$\begin{aligned}
\boldsymbol{e}_{i,jk} \;&=\; \boldsymbol{e}_{i,kj} \\
\big(\Gamma^l_{ij}\boldsymbol{e}_l + b_{ij}\vec{n}\big)_{,k} \;&=\; \big(\Gamma^l_{ik}\boldsymbol{e}_l + b_{ik}\vec{n}\big)_{,j} \\
\Gamma^l_{ij,k}\boldsymbol{e}_l + \Gamma^l_{ij}\boldsymbol{e}_{l,k} + b_{ij,k}\vec{n} + b_{ij}\vec{n}_{,k} \;&=\; \Gamma^l_{ik,j}\boldsymbol{e}_l + \Gamma^l_{ik}\boldsymbol{e}_{l,j} + b_{ik,j}\vec{n} + b_{ik}\vec{n}_{,j} \\
\Gamma^l_{ij,k}\boldsymbol{e}_l + \underbrace{\Gamma^l_{ij}\big(\Gamma^m_{lk}\boldsymbol{e}_m + b_{lk}\vec{n}\big)}_{l\leftrightarrow m} + b_{ij,k}\vec{n} - b_{ij}b^l_k\boldsymbol{e}_l \;&=\; \Gamma^l_{ik,j}\boldsymbol{e}_l + \underbrace{\Gamma^l_{ik}\big(\Gamma^m_{lj}\boldsymbol{e}_m + b_{lj}\vec{n}\big)}_{l\leftrightarrow m} + b_{ik,j}\vec{n} - b_{ik}b^l_j\boldsymbol{e}_l \\
\Gamma^l_{ij,k}\boldsymbol{e}_l + \Gamma^m_{ij}\big(\Gamma^l_{mk}\boldsymbol{e}_l + b_{mk}\vec{n}\big) + b_{ij,k}\vec{n} - b_{ij}b^l_k\boldsymbol{e}_l \;&=\; \Gamma^l_{ik,j}\boldsymbol{e}_l + \Gamma^m_{ik}\big(\Gamma^l_{mj}\boldsymbol{e}_l + b_{mj}\vec{n}\big) + b_{ik,j}\vec{n} - b_{ik}b^l_j\boldsymbol{e}_l \\
\boldsymbol{e}_l\big(\Gamma^l_{ij,k} + \Gamma^m_{ij}\Gamma^l_{mk} - b_{ij}b^l_k\big) + \vec{n}\big(\Gamma^m_{ij}b_{mk} + b_{ij,k}\big) \;&=\; \boldsymbol{e}_l\big(\Gamma^l_{ik,j} + \Gamma^m_{ik}\Gamma^l_{mj} - b_{ik}b^l_j\big) + \vec{n}\big(\Gamma^m_{ik}b_{mj} + b_{ik,j}\big) \; .
\end{aligned}$$

However, since the vectors $\boldsymbol{e}_l$ and $\vec{n}$ are linearly independent, the prefactors in front of the $\boldsymbol{e}_l$ and the $\vec{n}$ must be separately equal! One thereby gets the following two equations:

$$\begin{aligned} b_{ij,k} - b_{ik,j} &= \Gamma^m_{ik} b_{mj} - \Gamma^m_{ij} b_{mk} \ , && (1.21) \\ \text{and} \qquad b_{ij} b^l_k - b_{ik} b^l_j &= R^l_{ikj} \ . && (1.22) \end{aligned}$$

The condition (1.21) is sometimes called the *equation of Mainardi-Codazzi.* Using the concept of covariant differentiation (see Sec. C.3) this equation can also be written as

$$\nabla_k b_{ij} - \nabla_j b_{ik} \ = \ 0 \qquad \text{or fancy:} \quad \nabla_{\left[k\right.} b_{\left.j\right] i} \ = \ 0 \qquad \text{or:} \quad \text{“}\nabla_k b_{ij} \text{ is totally symmetric” .} \tag{1.23}$$

For Eqn. (1.22) we have introduced the abbreviation

$$R^l_{ikj} := \Gamma^l_{ij,k} - \Gamma^l_{ik,j} + \Gamma^m_{ij} \Gamma^l_{mk} - \Gamma^m_{ik} \Gamma^l_{mj} \ . \tag{1.24}$$

Note that the left hand side of Eqn. (1.22) is a tensor. Hence, the symbol R^l_{ikj} defined in Eqn. (1.24) is also a tensor[4]—even though the explicit expression doesn't look like it at all! This tensor is called the (mixed) *Riemann curvature tensor* and it plays a fundamental role in Riemannian geometry. From the definition (1.24) we see that the Riemann tensor is skew-symmetric with respect to the last two indices.

The covariant Riemann tensor is defined as

$$R_{likj} \ = \ g_{ls} R^s_{ikj} \ = \ \Gamma_{ijl,k} - \Gamma_{ikl,j} + \Gamma^m_{ik} \Gamma_{ljm} - \Gamma^m_{ij} \Gamma_{lkm} \ , \tag{1.25}$$

where the second equation follows from using identities like

$$g_{ls} \Gamma^s_{ij,k} \ = \ g_{ls} \Big(\frac{\partial}{\partial u^k} \Gamma^s_{ij} \Big) \ = \ \frac{\partial}{\partial u^k} \Big(g_{ls} \Gamma^s_{ij} \Big) - \Gamma^s_{ij} \frac{\partial g_{ls}}{\partial u^k} \ = \ \Gamma_{ijl,k} - \Gamma^s_{ij} \Big(\Gamma_{lks} + \Gamma_{skl} \Big) \ . \tag{1.26}$$

Using (1.25), Eqn. (1.22) can also be written as

$$b_{ij} b_{kl} - b_{ik} b_{jl} \ = \ R_{likj} \ . \tag{1.27}$$

Of course, also the covariant Riemann tensor is skew-symmetric with respect to the last two indices. But Eqn. (1.27) shows that it is additionally skew-symmetric with respect to the *first* two indices.[5]

As a fourth rank tensor, the curvature tensor R_{likj} generally has d^4 components. However, due to the skew-symmetry, only those components are nonzero, for which $l \neq i$ and $k \neq j$. In the case of surfaces, this leaves only the following four components which are nonzero:

$$\begin{aligned} R_{1212} &= R_{2121} \overset{(1.27)}{=} b_{22} b_{11} - (b_{12})^2 \overset{(1.14)}{=} b \ , \\ R_{2112} &= R_{1221} \overset{(1.27)}{=} (b_{12})^2 - b_{22} b_{11} \overset{(1.14)}{=} -b \ . \end{aligned} \tag{1.28}$$

As an immediate consequence follows

> **Gauss' Theorema Egregium**: *The Gaussian curvature is an intrinsic surface property,* i. e., *it does only depend on the first fundamental form.*

The proof is via inspection:

$$K \overset{(1.13b)}{=} \frac{b}{g} \overset{(1.28)}{=} \frac{R_{1212}}{g} \ ,$$

and the right hand side indeed only depends on the metric and not on the second fundamental form!

The relevance of this theorem becomes evident if one considers it in the context of "isometric mappings". A mapping of a portion of a manifold M to a portion of a manifold N is called *isometric*, if the length of any curve on

[4]This proof, making reference to b_{ij}, only works in two dimensions. However, formula (C.8) shows that R^l_{ikj} is indeed a tensor in *any* dimension.

[5]This actually holds in *any* dimension, not just in two, even though in this case we used the Gauss equation to prove it, which only makes sense in two dimensions. As a brute force proof one may just take the definition and work it out. A more elegant way is to use local tangent coordinates—see Sec. 1.4.

N is the same as the length of its pre-image on M. It can be proved [14] that a mapping is isometric if and only if at corresponding points of the two manifolds the coefficients of the metric, when referred to the same coordinates[6], are identical. As a consequence of the theorema egregium, isometric surfaces have the same Gaussian curvature at corresponding point. This for instance shows that there cannot be an isometric mapping from the sphere to the plane, since these two surfaces have different Gaussian curvatures.

We can also obtain the components of the mixed Riemann curvature tensor by calculating

$$R^1_{112} = g^{11}R_{1112} + g^{12}R_{2112} = -g^{12}b \stackrel{(1.5)}{=} g_{12}\frac{b}{g} = g_{12}K \ .$$

We thereby obtain:

$$\begin{aligned} -R^2_{112} = R^2_{121} = g_{11}K \qquad &, \qquad R^1_{112} = -R^1_{121} = g_{12}K \ , \\ -R^2_{212} = R^2_{221} = g_{21}K \qquad &, \qquad R^1_{212} = -R^1_{221} = g_{22}K \ . \end{aligned} \tag{1.29}$$

The *Ricci* tensor is defined as the nontrivial contraction of the Riemann tensor. First note that a contraction with respect to the second index gives the result zero, since the covariant tensor is skew-symmetric with respect to the first two indices. A contraction with the third index gives a nontrivial result. And since the Riemann tensor is skew symmetric also with respect to the last two indices, contracting the first and the fourth gives just the negative of contracting the first and the third. Unfortunately, this gives rise to sign confusions, since apparently there is no generally accepted convention which indices to take: the third or the fourth? We will define the Ricci tensor as the contraction with respect to the *third* index:

$$R_{ij} \ := \ R^l_{ilj} \ = \ \Gamma^l_{ij,l} - \Gamma^l_{il,j} + \Gamma^m_{ij}\Gamma^l_{ml} - \Gamma^m_{il}\Gamma^l_{mj} \ . \tag{1.30}$$

As a consequence of the various symmetries of the Riemann tensor, the Ricci tensor is symmetric in its two indices. In two dimensions we have $R_{ij} = R^1_{i1j} + R^2_{i2j}$. Using Eqns. (1.29), we immediately see

$$R_{ij} \ = \ K\,g_{ij} \qquad\qquad \text{(only in two dimensions!)} \ . \tag{1.31}$$

Therefore, by contracting the Gauss equation (1.22), we obtain the useful result

$$K\,g_{ij} \ = \ R_{ij} \ = \ R^l_{ilj} \ \stackrel{(1.22)}{=} \ b_{ij}b^l_l - b_{il}b^l_j \ \stackrel{(1.13a)}{=} \ 2H\,b_{ij} - b_{il}b^l_j \ ,$$

which, after rearrangement, gives

$$\underline{b_{il}b^l_j \ = \ 2H\,b_{ij} - K\,g_{ij}} \ . \tag{1.32}$$

Contracting the Ricci tensor again, we obtain the *Ricci scalar curvature*:

$$R \ := \ g^{ij}R_{ij} \ . \tag{1.33}$$

For two dimensions this becomes

$$R \ \stackrel{(1.31)}{=} \ g^{ij}g_{ij}\,K \ = \ g^i_iK \ = \ 2K \qquad\qquad \text{(only in two dimensions!)} \ . \tag{1.34}$$

Hence, contracting Eqn. (1.32) once more, we obtain

$$\underline{b^j_lb^l_j} \ = \ g^{ij}b_{il}b^l_j \ \stackrel{(1.32)}{=} \ g^{ij}\big(2H\,b_{ij} - K\,g_{ij}\big) \ = \ 2H\,b^i_i - K\,g^i_i \ = \ \underline{4H^2 - 2K} \ . \tag{1.35}$$

Using the Ricci scalar, Eqn. (1.28) can be neatly rewritten as

$$R_{ijkl} \ = \ \frac{R}{2}\Big(g_{ik}g_{jl} - g_{jk}g_{il}\Big) \qquad \text{(for } d = 2\text{)} \ , \tag{1.36}$$

reconfirming the observation that *in two dimensions* the Riemann tensor is uniquely determined by the value of the Ricci Scalar (*i. e.*, the Gaussian Curvature), which generally depends on position. It is worth pointing out that *for totally symmetric spaces* the Riemann tensor is also of this form, see Eqn. (B.22), but additionally in this case the Ricci scalar is a *constant* over the whole space.

[6] If we have coordinates u^i on M and a mapping F from M to N, then this introduces a natural set of coordinates on N.

1.4. Bianchi Identities

All relations in this section are valid in arbitrary dimensions.

Let us study the properties of the Riemann tensor a bit further. For this, it turns out to be advantageous to choose local tangent coordinates, see Sec. A.3. Recall that in these coordinates all first partial derivatives of the metric tensor vanish. Hence, from Eqn. (1.25) we find

$$R_{likj} \stackrel{\text{ltc}}{=} \Gamma_{ijl,k} - \Gamma_{ikl,j} \;=\; \frac{1}{2}\Big[g_{lj,ik} + g_{ik,lj} - g_{lk,ij} - g_{ij,lk}\Big] \;. \tag{1.37}$$

From this equation it is easy to infer the skew-symmetry of the Riemann tensor with respect to the first and second pair of indices. We can also see that it is symmetric with respect to a swapping of the first and second *pair* of indices:

$$R_{likj} \;=\; R_{kjli} \tag{1.38}$$

Note that we have not written "ltc" over the equality sign. Even though the equation has been derived in a locally tangential coordinate system, it is proper tensor equations, and hence holds in *any* coordinate system!

It is equally straightforward to check the following:

$$R_{likj} + R_{ljik} + R_{lkji} \;=\; 0 \;. \tag{1.39}$$

This equation is called the *first Bianchi identity.* Observe the placements of the indices: The first remains always at its position, while the three others are cycled. Hence, the same identity also holds for the mixed Riemann tensor:

$$R^l_{ikj} + R^l_{jik} + R^l_{kji} \;=\; 0 \;. \tag{1.40}$$

A second identity can be checked just as easily in local tangent coordinates:

$$R_{likj,m} + R_{limk,j} + R_{lijm,k} \stackrel{\text{ltc}}{=} 0 \;.$$

This equation is not valid in every frame, but we can easily get one that is. Since in local tangent coordinates the Christoffel symbols vanish, a *first order* covariant derivative is the same as a first order partial derivative.[7] We may hence substitute the partial derivatives by covariant ones:

$$\nabla_m R_{likj} + \nabla_j R_{limk} + \nabla_k R_{lijm} \;=\; 0 \;.$$

Since this again is a proper tensor equation, it holds in any coordinate system. It is called the *second Bianchi identity.* Note again the placements of the indices: The first two indices remain unchanged, while the last two cycle with the derivative. We may thus raise the first index and get the second Bianchi identity for the mixed Riemann tensor:

$$\nabla_m R^l_{ikj} + \nabla_j R^l_{imk} + \nabla_k R^l_{ijm} \;=\; 0 \;.$$

If we contract the last expression with respect to k and l, exploit the symmetry properties of the Riemann tensor and use the definition (1.30) for the Ricci tensor, we get what is called the *contracted (second) Bianchi identity*:

$$\nabla_m R_{ij} - \nabla_j R_{im} + \nabla_k R^k_{ijm} \;=\; 0 \;.$$

If we now contract the left hand side again over i and j and use the definition (1.33) for the Ricci scalar curvature, we obtain:

$$0 \;=\; g^{ij}\Big(\nabla_m R_{ij} - \nabla_j R_{im} + \nabla_k R^k_{ijm}\Big) \;=\; \nabla_m R - \nabla^i R_{im} - \nabla^k R_{km} \;=\; \nabla^i g_{im} R - 2\nabla^i R_{im} \;.$$

If we define the twofold covariant tensor G_{ij} according to

$$G_{ij} \;:=\; R_{ij} - \frac{1}{2} g_{ij} R \;,$$

the doubly contracted second Bianchi identity can be written as

$$\nabla^i G_{ij} \;=\; 0 \;. \tag{1.41}$$

[7] Convince yourself that this does *not* hold for second order derivatives! For instance, it would be incorrect to argue that one obtains the Riemann tensor in an arbitrary coordinate system by replacing the commas in Eqn. (1.37) by semicolons.

The tensor G_{ij} is called the *Einstein tensor*, and the relation (1.41) is therefore referred to as the second Bianchi identity of either the Ricci tensor or the Einstein tensor.[8] The Einstein tensor appears in Einstein's famous field equation of general relativity, which reads

$$G_{ij} \; = \; \frac{8\pi\mathscr{G}}{c^4} T_{ij} \; ,$$

where $\mathscr{G}$ is the gravitational constant, c the speed of light and T_{ij} the energy-momentum tensor. Just as the Einstein tensor, the energy momentum tensor is also divergence free (which is related to the conservation of energy and momentum), and this might have initially suggested to Einstein that this is the way in which space time and matter couple.

Sidenote: Observe that Einstein's field equations only determine the Ricci tensor, not the Riemann tensor. In four dimensions the Riemann tensor has 20 independent components, and only half of them are known if one knows the Ricci tensor. The other half are given if one also knows the *Weyl tensor*, which is essentially the Riemann tensor with all its "traces" removed.

[8]Observe that as a consequence of Eqn. (1.31) not only the divergence of the Einstein tensor, but actually the Einstein tensor itself vanishes in two dimensions.

2. Some important parameterizations of surfaces

In order to actually describe a surface, one has to give a *parameterization*, which is a concrete version of the mapping (1.1). In this chapter we study a few frequently encountered parameterizations.

2.1. Monge parameterization

2.1.1. Definition and properties

The Monge parameterization is the most straightforward one: A surface is defined by giving its height h over some plane (usually) as a function of orthonormal coordinates x and y in the plane:

$$h : \left\{ \begin{array}{ccc} \mathbb{R}^2 \supset U & \to & \mathbb{R}^3 \\ (x,y) & \mapsto & h(x,y) \end{array} \right. . \tag{2.1}$$

An illustration is given in Fig. 2.1.

One disadvantage of the Monge parameterization is that it is unable to describe "overhangs". However, if one is predominantly interested in describing surfaces which deviate only weakly from a flat plane, then this parameterization is very useful, particularly because of the existence of a simple small gradient expansion (see Sec. 2.1.3).

The position vector $\vec{r}$ and the two tangent vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are given by

$$\vec{r} = \begin{pmatrix} x \\ y \\ h(x,y) \end{pmatrix} \quad , \quad \boldsymbol{e}_x = \frac{\partial \vec{r}}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ h_x \end{pmatrix} \quad , \quad \boldsymbol{e}_y = \frac{\partial \vec{r}}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ h_y \end{pmatrix} ,$$

where an index "x" or "y" on h means partial differentiation of h with respect to this index. Hence, the metric and its determinant are given by

$$g_{ij} = \begin{pmatrix} 1+h_x^2 & h_x h_y \\ h_x h_y & 1+h_y^2 \end{pmatrix} \qquad \Rightarrow \qquad g = \left| g_{ij} \right| = 1 + h_x^2 + h_y^2 . \tag{2.2}$$

Note that even though the coordinates in the *underlying plane* are orthogonal, the metric is generally not diagonal, and hence the coordinate curves on the surface are generally not orthogonal.

The inverse metric is then

$$g^{ij} = \frac{1}{1+h_x^2+h_y^2} \begin{pmatrix} 1+h_y^2 & -h_x h_y \\ -h_x h_y & 1+h_x^2 \end{pmatrix} .$$

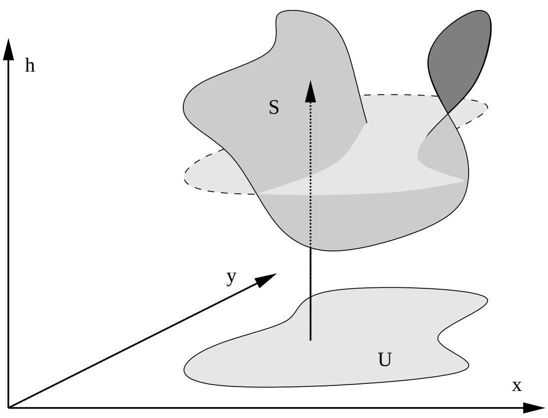


Figure 2.1.: Illustration of the Monge parameterization.

The *upward* normal vector $\vec{n}$ is found via

$$\boldsymbol{e}_x \times \boldsymbol{e}_y = \begin{pmatrix} -h_x \\ -h_y \\ 1 \end{pmatrix} \qquad \Rightarrow \qquad \vec{n} \; = \; \frac{\boldsymbol{e}_x \times \boldsymbol{e}_y}{\sqrt{g}} \; = \; \frac{1}{\sqrt{1+h_x^2+h_y^2}} \begin{pmatrix} -h_x \\ -h_y \\ 1 \end{pmatrix} .$$

From $b_{ij} = \boldsymbol{e}_{i,j} \cdot \vec{n}$ we immediately find the second fundamental form

$$b_{ij} \; = \; \frac{1}{\sqrt{1+h_x^2+h_y^2}} \begin{pmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{pmatrix} . \tag{2.3}$$

The mixed second fundamental form is then found to be

$$b_i^j \; = \; b_{ik} g^{kj} \; = \; \frac{1}{(1+h_x^2+h_y^2)^{3/2}} \begin{pmatrix} h_{xx}(1+h_y^2) - h_{xy}h_x h_y & h_{xy}(1+h_x^2) - h_{xx}h_x h_y \\ h_{xy}(1+h_y^2) - h_{yy}h_x h_y & h_{yy}(1+h_x^2) - h_{xy}h_x h_y \end{pmatrix} . \tag{2.4}$$

From this follow mean and Gaussian curvature:

$$H \; = \; \frac{1}{2} b_i^i \; \stackrel{(2.4)}{=} \; \frac{h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_{xy}h_x h_y}{2\,(1+h_x^2+h_y^2)^{3/2}} , \tag{2.5a}$$

$$K \; = \; \frac{\det b_{ij}}{g} \; \stackrel{(2.2),(2.3)}{=} \; \frac{h_{xx}h_{yy} - (h_{xy})^2}{(1+h_x^2+h_y^2)^2} . \tag{2.5b}$$

2.1.2. Formal expression in terms of $\nabla_{\|}$

In the underlying plane, which is sometimes referred to as the "base manifold", there exists the two-dimensional nabla operator $\nabla_{\|}$, which for instance in terms of the local Cartesian coordinates has the form

$$\nabla_{\|} \; = \; \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \qquad \text{(in Cartesian coordinates)} . \tag{2.6}$$

It is possible to rewrite a few of the formulas from the preceding section in terms of this differential operator. This is convenient, because in this way they are covariant with respect to changes of the coordinate system of the base manifold, because $\nabla_{\|}$ transforms (essentially...) like a vector. In other words, the expressions remain formally the same if one changes for instance to polar coordinates in the underlying plane, even though the coordinate expression (2.6) for the nabla operator is different.

The metric determinant is easily seen to be expressible as

$$g \; = \; 1 + \big(\nabla_{\|} h\big)^2 ,$$

and the normal vector is given by

$$\vec{n} \; = \; \frac{1}{\sqrt{1+\big(\nabla_{\|} h\big)^2}} \begin{pmatrix} -\nabla_{\|} h \\ 1 \end{pmatrix} .$$

The inverse metric can be represented, somewhat formally, as

$$g^{ij} \; = \; \mathbb{I}\big[1 + \big(\nabla_{\|} h\big)^2\big] - \nabla_{\|} h \otimes \nabla_{\|} h .$$

Finally, the mean curvature can be written as

$$2H \; = \; \nabla_{\|} \cdot \frac{\nabla_{\|} h}{\sqrt{1+\big(\nabla_{\|} h\big)^2}} . \tag{2.7}$$

The proof is via direct calculation:

$$\begin{aligned}
\nabla_\parallel \cdot \frac{\nabla_\parallel h}{\sqrt{1+\left(\nabla_\parallel h\right)^2}} &= \partial_x \frac{h_x}{\sqrt{}} + \partial_y \frac{h_y}{\sqrt{}} \\
&= -\frac{1}{2\sqrt{}^3}\big(2h_x h_{xx} + 2h_y h_{yx}\big)h_x + \frac{1}{\sqrt{}}h_{xx} - \frac{1}{2\sqrt{}^3}\big(2h_x h_{xy} + 2h_y h_{yy}\big)h_y + \frac{1}{\sqrt{}}h_{yy} \\
&= -\frac{1}{\sqrt{}^3}\Big(h_x^2 h_{xx} + h_x h_y h_{yx} - (1+h_x^2+h_y^2)h_{xx} + h_x h_y h_{xy} + h_y^2 h_{yy} - (1+h_x^2+h_y^2)h_{yy}\Big) \\
&= \frac{1}{\sqrt{}^3}\Big((1+h_y^2)h_{xx} + (1+h_x^2)h_{yy} - 2h_x h_y h_{xy}\Big) \ .
\end{aligned}$$

This expression should be $2H$, and comparing with Eqn. (2.5a) we see that it indeed is.

2.1.3. Small gradient expansion

Very frequently one is faced with (or restricts oneself to) a situation in which the gradients h_x and h_y are small. In this limit mean and Gaussian curvature are approximately given by

$$H = \frac{1}{2}(h_{xx} + h_{yy}) + \mathscr{O}\big((\nabla_\parallel h)^2\big) \ , \tag{2.8a}$$

$$K = h_{xx}h_{yy} - (h_{xy})^2 + \mathscr{O}\big((\nabla_\parallel h)^2\big) \ . \tag{2.8b}$$

If we use the symbol $\boldsymbol{h}$ to describe the matrix of second derivatives of h (i.e., the "Hessian")

$$(\boldsymbol{h})_{ij} = h_{ij} = \frac{\partial^2 h}{\partial u^i \partial u^j} \ , \tag{2.9}$$

we can write the two above approximate equations as

$$H = \frac{1}{2}\mathrm{Tr}(\boldsymbol{h}) + \mathscr{O}\big((\nabla_\parallel h)^2\big) \ , \tag{2.10a}$$

$$K = \det(\boldsymbol{h}) + \mathscr{O}\big((\nabla_\parallel h)^2\big) \ . \tag{2.10b}$$

Using the formal expression for the mean curvature from the preceding section, we can write Eqn. (2.8a) also as

$$H = \frac{1}{2}\nabla_\parallel^2 h + \mathscr{O}\big((\nabla_\parallel h)^2\big) \ . \tag{2.11}$$

2.2. Cylindrically symmetric surfaces

2.2.1. General case

A very general expression for a cylindrical surface is obtained if one defines a two-dimensional curve and rotates it around some axis. In order to forbid self intersections of the surface we will forbid that the curve intersects itself. Then, a parameterization can be given by

$$\vec{r}: \begin{cases} \mathbb{R}^2 \supset [0;2\pi]\times[a;b] \ \to \ \mathbb{R}^3 \\ (\varphi,t) \ \mapsto \ \vec{r}(\varphi,t) = \begin{pmatrix} r(t)\cos\varphi \\ r(t)\sin\varphi \\ z(t) \end{pmatrix} \end{cases} . \tag{2.12}$$

Thus, the surface is specified by two functions, $r(t)$ and $z(t)$, which together define a curve in the (r,z) plane, and afterwards this curve is rotated about the z-axis. For obvious reasons we will also require $r \geq 0$. See Fig. 2.2 for an illustration.

The tangent vectors are now given by

$$\boldsymbol{e}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{e}_t = \frac{\partial \vec{r}}{\partial t} = \begin{pmatrix} \dot{r}\cos\varphi \\ \dot{r}\sin\varphi \\ \dot{z} \end{pmatrix} \ ,$$

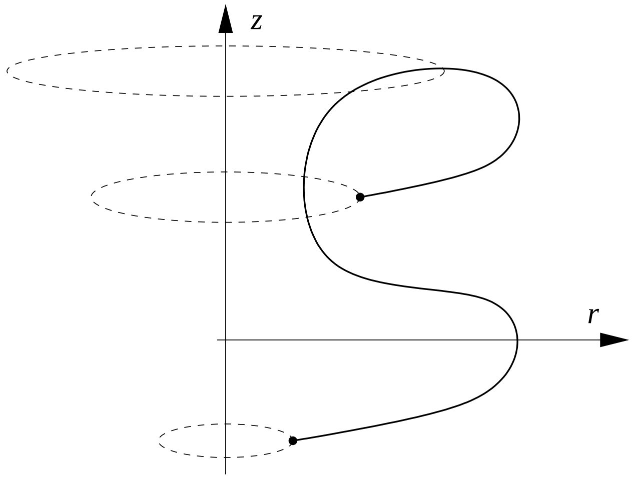


Figure 2.2.: Illustration of a parameterization of a cylindrically symmetric surface. A curve in the (r, z) plane is defined and then rotated about the z-axis.

where we dropped the t-dependence of the functions $r(t)$ and $z(t)$ for notational simplicity, and where a dot is meant to indicate a partial derivative with respect to t. From the tangent vectors we find the metric and its determinant:

$$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & \dot r^2 + \dot z^2 \end{pmatrix} \qquad \Rightarrow \qquad g = \big|g_{ij}\big| = r^2\big(\dot r^2 + \dot z^2\big) . \tag{2.13}$$

The inverse metric is then simply

$$g^{ij} = \frac{1}{r^2\big(\dot r^2 + \dot z^2\big)} \begin{pmatrix} \dot r^2 + \dot z^2 & 0 \\ 0 & r^2 \end{pmatrix} = \begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/\big(\dot r^2 + \dot z^2\big) \end{pmatrix} .$$

The normal vector[1] $\vec n$ is found via

$$\boldsymbol{e}_\varphi \times \boldsymbol{e}_t = \begin{pmatrix} r\dot z \cos\varphi \\ r\dot z \sin\varphi \\ -r\dot r \end{pmatrix} \qquad \Rightarrow \qquad \vec n = \frac{\boldsymbol{e}_\varphi \times \boldsymbol{e}_t}{\sqrt{g}} = \frac{1}{\sqrt{\dot r^2 + \dot z^2}} \begin{pmatrix} \dot z \cos\varphi \\ \dot z \sin\varphi \\ -\dot r \end{pmatrix} .$$

From this follows the second fundamental form

$$b_{ij} = \frac{1}{\sqrt{\dot r^2 + \dot z^2}} \begin{pmatrix} -r\dot z & 0 \\ 0 & \dot z \ddot r - \ddot z \dot r \end{pmatrix} , \tag{2.14}$$

and its mixed version

$$b_i^j = b_{ik} g^{kj} = \frac{1}{\sqrt{\dot r^2 + \dot z^2}} \begin{pmatrix} -\dfrac{\dot z}{r} & 0 \\ 0 & \dfrac{\dot z \ddot r - \ddot z \dot r}{\dot r^2 + \dot z^2} \end{pmatrix} . \tag{2.15}$$

From this follow the two principal curvatures

$$\kappa_1 = -\frac{\dot z}{r\sqrt{\dot r^2 + \dot z^2}} \qquad \text{and} \qquad \kappa_2 = \frac{\dot z \ddot r - \ddot z \dot r}{\big(\dot r^2 + \dot z^2\big)^{3/2}} .$$

The principal direction belonging to the eigenvalue κ_1 is around the axis, *i. e.*, in the direction of $\boldsymbol{e}_\varphi$, while the principal direction belonging to κ_2 is orthogonal to that, *i. e.*, in the direction of $\boldsymbol{e}_t$. Mean and Gaussian curvature follow directly as the arithmetic mean and the product of these principal curvatures.

2.2.2. Special case 1: Arc length parameterization

The formulas given above simplify, if we parameterize the curve $(r(t), z(t))^\top$ which specifies the profile by its *arc length* s. If this is not already the case, it can always be achieved by a smooth reparameterization. The key advantage of this is that the tangent vector on the curve becomes the unit vector, and hence $\dot r^2 + \dot z^2 = 1$, which evidently simplifies the formulas from Sec. 2.2.1 quite a bit.

[1]Note that the direction of the normal vector depends on the direction in which the curve is followed, and no natural rule can be given here. One should always check the particular parameterization one sets up or is confronted with.

object	special case 1, Sec. 2.2.2	special case 2, Sec. 2.2.3	special case 3, Sec. 2.2.4
derivative $'$ is with respect to	arc length s	radial distance r	height z
metric g_{ij}	$\begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} r^2 & 0 \\ 0 & 1+z'^2 \end{pmatrix}$	$\begin{pmatrix} r^2 & 0 \\ 0 & 1+r'^2 \end{pmatrix}$
inverse metric g^{ij}	$\begin{pmatrix} 1/r^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/(1+z'^2) \end{pmatrix}$	$\begin{pmatrix} 1/r^2 & 0 \\ 0 & 1/(1+r'^2) \end{pmatrix}$
second fundamental form b_{ij}	$\begin{pmatrix} -rz' & 0 \\ 0 & z'r''-z''r' \end{pmatrix}$	$\frac{1}{\sqrt{1+z'^2}}\begin{pmatrix} -rz' & 0 \\ 0 & -z'' \end{pmatrix}$	$\frac{1}{\sqrt{1+r'^2}}\begin{pmatrix} -r & 0 \\ 0 & r'' \end{pmatrix}$
azimuthal principal curvature	$-\frac{z'}{r} \equiv -\frac{\sin\psi}{r}$	$-\frac{z'}{r\sqrt{1+z'^2}}$	$-\frac{1}{r\sqrt{1+r'^2}}$
radial principal curvature	$z'r''-z''r' \equiv -\psi'$	$-\frac{z''}{(1+z'^2)^{3/2}}$	$\frac{r''}{(1+r'^2)^{3/2}}$

Table 2.1.: Summary of the three special cases of parameterizing a cylindrically symmetric surface.

In this case it is also advantageous to specify the curve itself by a different kind of parameterization, namely, by giving the angle ψ which it has against the horizontal r-axis as a function of arc length s. One can indeed think of the parameterization $\psi(s)$ as a Frenet type parameterization. Note that $\psi = \arctan(z'/r')$, where by the prime we now mean a differentiation with respect to the arc length. The two principal curvatures are now found to be

$$\kappa_1 \;=\; -\frac{\sin\psi}{r} \qquad \text{and} \qquad \kappa_2 \;=\; -\psi' \; .$$

See Tab. 2.1 for a collection of other important expressions.

This particular parameterization has for instance been used extensively in the numerical study of the shape of vesicles [13, 22].

2.2.3. Special case 2: Height is a function of axial distance

The special case of Sec. 2.2.2 did not limit the *kind* of surfaces which can be described, it only posed a restriction on the parameterization of the boundary curve. In this and the following section we shall study two cases which restrict the kinds of surfaces which can be described. One obvious restriction is to look at cases where the height z of the profile can be written as a function of the radial distance. Formally, this just means that $r(t) \equiv t$, and we will use r rather than t as the appropriate variable. This is simply a Monge parameterization in cylindrical coordinates. Obviously, we now cannot describe overhangs which are such that at a given radial distance there are two values of z. See Tab. 2.1 for a summary of curvature related expressions.

2.2.4. Special case 3: Axial distance is a function of height

This special case is in some sense the conjugate one to the case of Sec. 2.2.3. Instead of assuming the the height can be specified as a function of the radial distance, we assume that the radial distance can be specified as a function of the height. This forbids overhangs which are such that at a given height z there are several corresponding radial distances. Formally, this parameterization implies that in the general case of Sec. 2.2.1 $z(t) \equiv t$, and we will use z as the variable. See Tab. 2.1 for a summary of curvature related expressions. This parameterization has for instance been used in Ref. [18] in a numerical study of the budding behavior of vesicles.

3. Variation of a surface

The following section is based on papers by Zhong-can and Helfrich [25] and Lenz and Lipowsky [16]. Calculations found in both papers are repeated and presented in a slightly different (namely: a bit more covariant) way. The covariant presentation is partly inspired by an article by Capovilla and Guven [4].

3.1. Definition of the variation

We may vary the position vector of the surface according to

$$\vec{r} \to \vec{r} + \delta\vec{r} \tag{3.1}$$

and express the variation $\delta\vec{r}$ in local coordinates:

$$\delta\vec{r} = \phi^i \boldsymbol{e}_i + \psi\vec{n} \ . \tag{3.2}$$

The three functions ϕ^1, ϕ^2 and ψ generally depend on the position vector on the surface and describe the local variation,which we assume to be small ("of first order"). Note that ψ is a scalar field on S, while ϕ^i are the components of a vector field $\boldsymbol{\phi}$ on (the tangent bundle of) S.

From this, we find the variation of the tangent vectors:

$$\begin{aligned}
\delta\boldsymbol{e}_i \quad &= \quad \frac{\partial\vec{r}+\delta\vec{r}}{\partial u^i} - \frac{\partial\vec{r}}{\partial u^i} \ = \ \frac{\partial\delta\vec{r}}{\partial u^i} \ = \ \left[\phi^k\boldsymbol{e}_k + \psi\vec{n}\right]_{,i} \ = \ \phi^k_{,i}\boldsymbol{e}_k + \phi^k\boldsymbol{e}_{k,i} + \psi_{,i}\vec{n} + \psi\vec{n}_{,i} \\
&\overset{(1.16),(1.18)}{=} \quad \phi^k_{,i}\boldsymbol{e}_k + \phi^k\left(\Gamma^l_{ki}\boldsymbol{e}_l + b_{ki}\vec{n}\right) + \psi_{,i}\vec{n} - \psi b^k_i\boldsymbol{e}_k \ = \ \left(\phi^k_{,i} + \phi^l\Gamma^k_{li} - \psi b^k_i\right)\boldsymbol{e}_k + \left(\phi^k b_{ki} + \psi_{,i}\right)\vec{n} \\
&\overset{(C.6)}{=} \quad \underline{\left(\nabla_i\phi^k - \psi b^k_i\right)\boldsymbol{e}_k + \left(\phi^k b_{ki} + \nabla_i\psi\right)\vec{n}} \quad =: \ \underline{{U_i}^k\boldsymbol{e}_k + V_i\vec{n}} \ .
\end{aligned} \tag{3.3}$$

Observe that since $\nabla_i\phi_k$ is not symmetric, we need to distinguish in the definition of U, which index is the first one and which is the second one.

3.2. Variation of the first fundamental form

3.2.1. Metric

We define the variation of the metric according to

$$\delta g_{ij} = \delta g_{ij}(\vec{r}) = g_{ij}(\vec{r}+\delta\vec{r}) - g_{ij}(\vec{r}) \ .$$

Inserting the definition of the metric (1.3) and of the variation (3.1), we find

$$\begin{aligned}
\delta g_{ij} \quad &= \quad \frac{\partial(\vec{r}+\delta\vec{r})}{\partial u^i}\cdot\frac{\partial(\vec{r}+\delta\vec{r})}{\partial u^j} - \frac{\partial\vec{r}}{\partial u^i}\cdot\frac{\partial\vec{r}}{\partial u^j} \quad = \quad \underbrace{\frac{\partial\delta\vec{r}}{\partial u^i}\cdot\frac{\partial\vec{r}}{\partial u^j} + \frac{\partial\vec{r}}{\partial u^i}\cdot\frac{\partial\delta\vec{r}}{\partial u^j}}_{\text{first order}} \ + \ \underbrace{\frac{\partial\delta\vec{r}}{\partial u^i}\cdot\frac{\partial\delta\vec{r}}{\partial u^j}}_{\text{second order}} \\
&= \quad \left(\delta\boldsymbol{e}_i\cdot\boldsymbol{e}_j + \boldsymbol{e}_i\cdot\delta\boldsymbol{e}_j\right) \ + \ \delta\boldsymbol{e}_i\cdot\delta\boldsymbol{e}_j \\
&=: \quad \delta^{(1)}g_{ij} + \delta^{(2)}g_{ij} \ .
\end{aligned}$$

This is exact. The variation of the metric terminates after second order. Using Eqn. (3.3), we now readily find

$$\begin{aligned}
\delta^{(1)}g_{ij} \quad &= \quad \delta\boldsymbol{e}_i\cdot\boldsymbol{e}_j \ + \ \boldsymbol{e}_i\cdot\delta\boldsymbol{e}_j \\
&\overset{(3.3)}{=} \quad \left[\left(\nabla_i\phi^k - \psi b^k_i\right)\boldsymbol{e}_k + \left(\phi^k b_{ki} + \psi_{,i}\right)\vec{n}\right]\cdot\boldsymbol{e}_j + \left[\left(\nabla_j\phi^k - \psi b^k_j\right)\boldsymbol{e}_k + \left(\phi^k b_{kj} + \psi_{,j}\right)\vec{n}\right]\cdot\boldsymbol{e}_i \\
&= \quad \left(\nabla_i\phi^k - \psi b^k_i\right)g_{kj} + \left(\nabla_j\phi^k - \psi b^k_j\right)g_{ki} \\
&= \quad \underline{\nabla_i\phi_j + \nabla_j\phi_i - 2\psi b_{ij}} \ \overset{(D.5)}{=} \ \underline{\mathscr{L}_{\boldsymbol{\phi}}g_{ij} - 2\psi b_{ij}} \ .
\end{aligned} \tag{3.4}$$

The fact that *the tangential variation is given by the Lie-derivative* appears quite plausible [4].

For the second order we find

$$\begin{aligned}
\delta^{(2)} g_{ij} &= \big[(\nabla_i\phi^k - \psi b_i^k)\boldsymbol{e}_k + (\phi^s b_{si} + \psi_{,i})\vec{n}\big]\cdot\big[(\nabla_j\phi^l - \psi b_j^l)\boldsymbol{e}_l + (\phi^s b_{sj} + \psi_{,j})\vec{n}\big] \\
&= (\nabla_i\phi^k - \psi b_i^k)(\nabla_j\phi^l - \psi b_j^l)g_{kl} + (\phi^k b_{ki} + \psi_{,i})(\phi^l b_{lj} + \psi_{,j}) \\
&= \big(\nabla_i\phi^k\big)\big(\nabla_j\phi_k\big) - \psi\big[(\nabla_i\phi^k)b_{jk} + (\nabla_j\phi^k)b_{ik}\big] + \psi^2 b_i^k b_{jk} \\
&\quad + \phi^k\phi^l b_{ik}b_{jl} + \phi^k\big[b_{ik}\psi_{,j} + b_{jk}\psi_{,i}\big] + \psi_{,i}\psi_{,j}\ . \qquad (3.5)
\end{aligned}$$

Note that the tangential part of this contains terms which depend on the mean curvature. Hence, the second tangential variation of the metric tensor cannot be written as a second order Lie derivative, because this would have to yield a purely intrinsic result. This is related to the fact that at second order a tangential variation is no longer equivalent to a pure reparametrization (which would indeed be intrinsic).

In Sec. 3.2.3 we will need the trace of the variation of the metric. For the first order we find

$$\mathrm{Tr}\big[\delta^{(1)}\boldsymbol{g}\big] \ = \ g^{ij}\,\delta^{(1)} g_{ij} \ \overset{(1.13a)}{=} \ 2\nabla_i\phi^i - 4H\psi\ .$$

The trace of the second order expression is

$$\begin{aligned}
\mathrm{Tr}\big[\delta^{(2)}\boldsymbol{g}\big] \ = \ g^{ij}\,\delta^{(2)} g_{ij} &= \big(\nabla_i\phi^k\big)\big(\nabla^i\phi_k\big) + 2b_k^i\big(\phi^k\nabla_i\psi - \psi\nabla_i\phi^k\big) + \big(\nabla_i\psi\big)\big(\nabla^i\psi\big) \\
&\overset{(1.35)}{+} \ \psi^2\big(4H^2 - 2K\big) \ \overset{(1.32)}{+} \ \phi^k\phi^l\big(2Hb_{kl} - Kg_{kl}\big)\ .
\end{aligned}$$

3.2.2. Inverse metric

We will later also need the variation of the *inverse* metric. We will now show, how its variation can be obtained from the known variation of the "normal" metric. First note that after the variation g^{ij} must still be the inverse of g_{ij}. This means

$$\begin{aligned}
\delta_k^i &= \Big(g^{ij} + \frac{\partial g^{ij}}{\partial\lambda}\Delta\lambda + \frac{1}{2}\frac{\partial^2 g^{ij}}{\partial\lambda^2}(\Delta\lambda)^2 + \mathscr{O}\big((\Delta\lambda)^3\big)\Big)\Big(g_{jk} + \frac{\partial g_{jk}}{\partial\lambda}\Delta\lambda + \frac{1}{2}\frac{\partial^2 g_{jk}}{\partial\lambda^2}(\Delta\lambda)^2 + \mathscr{O}\big((\Delta\lambda)^3\big)\Big) \\
&= \delta_k^i + \Big(\frac{\partial g^{ij}}{\partial\lambda}g_{jk} + g^{ij}\frac{\partial g_{jk}}{\partial\lambda}\Big)\Delta\lambda + \Big(\frac{1}{2}\frac{\partial^2 g^{ij}}{\partial\lambda^2}g_{jk} + \frac{\partial g^{ij}}{\partial\lambda}\frac{\partial g_{jk}}{\partial\lambda} + g^{ij}\frac{1}{2}\frac{\partial^2 g_{jk}}{\partial\lambda^2}\Big)(\Delta\lambda)^2 + \mathscr{O}\big((\Delta\lambda)^3\big)\ .
\end{aligned}$$

Since the powers of $\Delta\lambda$ are linearly independent, their coefficients must equate individually. From this we get the relation

$$\delta^{(1)} g^{ij} \ = \ -g^{ik}g^{jl}\big(\delta^{(1)} g_{kl}\big)\ , \qquad (3.6)$$

$$\delta^{(2)} g^{ij} \ = \ -g^{ik}g^{jl}\big(\delta^{(2)} g_{kl} - g^{mn}\delta^{(1)} g_{km}\delta^{(1)} g_{nl}\big)\ . \qquad (3.7)$$

While the first variation of the inverse metric differs from the first variation of the "normal" metric only by a minus sign and the position of the indices, the situation is a bit more complicated for the second variation, which also includes a product of first variations. Using Eqns. (3.4) and (3.5), we find

$$\begin{aligned}
\delta^{(2)} g^{ij} &= 3\psi^2\big(2Hb^{ij} - Kg^{ij}\big) - \psi\big(b_k^j\nabla^i\phi^k + b_k^i\nabla^j\phi^k\big) - 2\psi\big(b_k^j\nabla^k\phi^i + b_k^i\nabla^k\phi^j\big) - \phi^k\phi^l b_k^i b_l^j \\
&\quad - \phi^k\big(b_k^i\nabla^j\psi + b_k^j\nabla^i\psi\big) - \nabla^i\psi\nabla^j\psi + \nabla^i\phi^k\nabla_k\phi^j + \nabla^k\phi^i\nabla_k\phi^j + \nabla^k\phi^i\nabla^j\phi_k \qquad (3.8)
\end{aligned}$$

Observe finally that while the variation of the metric terminates at second order, the variation of the inverse metric has contributions to all orders.

3.2.3. Determinant of the metric

First, it is helpful to recall a few general algebraic facts about matrices. Given an $n \times n$ matrix $\boldsymbol{A}$, the *adjoint matrix* $\tilde{\boldsymbol{A}}$ is defined as the matrix of *cofactors* $\tilde{A}^{ij}$ of $\boldsymbol{A}$. The cofactor $\tilde{A}^{ij}$ is defined to be $(-1)^{i+j}$ times the determinant of the $(n-1)\times(n-1)$ matrix which results from deleting the i-th row and the j-th column from $\boldsymbol{A}$. The determinant of $\boldsymbol{A}$ can now be calculated according to the *Laplace expansion theorem*:

$$|\boldsymbol{A}| \ = \ \sum_{i=1}^{n} A_{ij}\tilde{A}^{ij} \ = \ \frac{1}{n}\sum_{i,j=1}^{n} A_{ij}\tilde{A}^{ij}$$

which can be written very succinctly as

$$|\boldsymbol{A}|\,\mathbb{I} \;=\; \boldsymbol{A}\tilde{\boldsymbol{A}}^\top \;,$$

where $\tilde{\boldsymbol{A}}^\top$ is the transposed of the matrix $\tilde{\boldsymbol{A}}$. Multiplying from the left by $\boldsymbol{A}^{-1}$, we find

$$\tilde{\boldsymbol{A}}^\top \;=\; |\boldsymbol{A}|\,\boldsymbol{A}^{-1} \qquad \text{or in components:} \quad \tilde{A}^{ij} \;=\; |\boldsymbol{A}|\,(\boldsymbol{A}^{-1})_{ji} \;.$$

For the derivative of the metric determinant with respect to the metric itself this implies

$$\frac{\partial g}{\partial g_{ij}} \;=\; \tilde{g}^{ij} \;=\; g\,(\boldsymbol{g}^{-1})_{ji} \;=\; g\,g^{ij} \;. \tag{3.9}$$

Using the relation $\tilde{g}^{ij} = \varepsilon^{ik}\varepsilon^{jl}g_{lk}$, which is specific for 2×2 matrices, we can further work out the second derivative of the meric determinant with res[ect to the metric itself:

$$\frac{\partial^2 g}{\partial g_{ij}\partial g_{kl}} \;=\; \frac{\partial}{\partial g_{kl}}\varepsilon^{im}\varepsilon^{jn}g_{nm} \;=\; \varepsilon^{im}\varepsilon^{jn}\delta^k_n\delta^l_m \;=\; \varepsilon^{il}\varepsilon^{jk} \;.$$

Obviously, higher order derivatives will vanish. This of course is due to the fact that the determinant of a 2×2 matrix is a quadratic function of its components. Using the above results, we can now write the following (exact!) expansion of the variation of the determinant of the metric:

$$\delta g \;=\; \frac{\partial g}{\partial g_{ij}}\delta g_{ij} + \frac{1}{2}\frac{\partial^2 g}{\partial g_{ij}\partial g_{kl}}\delta g_{ij}\delta g_{kl} \;=\; gg^{ij}\delta g_{ij} + \frac{1}{2}\varepsilon^{il}\varepsilon^{jk}\delta g_{ij}\delta g_{kl} \;=\; \underline{gg^{ij}\delta g_{ij} + |\delta\boldsymbol{g}|} \;.$$

So far this expression is exact, but it is very complicated—in particular the last term. However, if we are only interested in the variation *up to quadratic order*, the following observation is very helpful: The determinant is in a sense a "quadratic operator", because it consists of products which contain two terms. Now, any contribution to one of its entries which is quadratic in the perturbations ϕ^i or ψ, will ultimately yield a term which is at least of cubic order in the final result. Looking at the expressions for $\delta\boldsymbol{g}$ which we have derived above, the lowest order terms which come up in $|\delta\boldsymbol{g}|$ are quadratic in the perturbations, and they stem from the linear order $\delta g^{(1)}_{ij}$! Hence, we find immediately

$$\underline{\delta g \;=\; g\,\mathrm{Tr}\big[\delta^{(1)}\boldsymbol{g}\big] \;+\; g\,\mathrm{Tr}\big[\delta^{(2)}\boldsymbol{g}\big] \;+\; \big|\delta^{(1)}\boldsymbol{g}\big| \quad+\; \mathscr{O}(3)\;,}$$

where "$\mathscr{O}(3)$" is an abbreviation for all kinds of third order terms, like ψ^3 or $\psi\phi_i\phi^i$.

We will now evaluate the remaining term which is unknown, namely $|\delta\boldsymbol{g}^{(1)}|$:

$$\begin{aligned}
\big|\delta^{(1)}\boldsymbol{g}\big| \;&=\; \frac{1}{2}\varepsilon^{ij}\varepsilon^{kl}\delta^{(1)}g_{ik}\delta^{(1)}g_{jl} \\
&=\; \frac{1}{2}\varepsilon^{ij}\varepsilon^{kl}\Big[\nabla_i\phi_k + \nabla_k\phi_i - 2\psi b_{ik}\Big]\Big[\nabla_j\phi_l + \nabla_l\phi_j - 2\psi b_{jl}\Big] \\
&=\; \frac{1}{2}\varepsilon^{ij}\varepsilon^{kl}\Big[\Big(\nabla_i\phi_k + \nabla_k\phi_i\Big)\Big(\nabla_j\phi_l + \nabla_l\phi_j\Big) - 2\psi b_{ik}\Big(\nabla_j\phi_l + \nabla_l\phi_j\Big) \\
&\qquad \underbrace{-\,2\psi b_{jl}\Big(\nabla_i\phi_k + \nabla_k\phi_i\Big)}_{i\leftrightarrow j\;,\;k\leftrightarrow l} + 4\psi^2 b_{ik}b_{jl}\Big] \\
&=\; \big|\nabla_i\phi_k + \nabla_k\phi_i\big| + 4\psi^2|b_{ij}| - 2\psi\varepsilon^{ij}\varepsilon^{kl}\Big[b_{ik}\Big(\nabla_j\phi_l + \nabla_l\phi_j\Big)\Big] \\
&=\; g\,\big|\nabla_i\phi^k + \nabla^k\phi_i\big| \overset{(1.13b)}{+} 4g\psi^2 K - 2g\psi\varepsilon^{ij}\varepsilon_{kl}\Big[b^k_i\Big(\nabla_j\phi^l + \nabla^l\phi_j\Big)\Big] \;,
\end{aligned}$$

where in the last step we used the identity $\varepsilon^{ij}\varepsilon^{kl}a_{ik}b_{jl} = g\,\varepsilon^{ij}\varepsilon_{kl}a^k_i b^l_j$ twice.

The sum of all the relevant terms will also contain the expression $(\nabla_i\phi^k)(\nabla^i\phi_k) + |\nabla_i\phi^k + \nabla^k\phi_i|$. This combination, containing in particular the nasty determinant, can be greatly simplified, but I have not found a way to show this other than by *brute force*, *i. e.*, writing out the expressions in *individual* components:

$$\begin{aligned}
\big(\nabla_i\phi^k\big)\big(\nabla^i\phi_k\big) + \big|\nabla_i\phi^k + \nabla^k\phi_i\big| \;&=\; \nabla_1\phi^1\nabla^1\phi_1 + \nabla_1\phi^2\nabla^1\phi_2 + \nabla_2\phi^1\nabla^2\phi_1 + \nabla_2\phi^2\nabla^2\phi_2 \\
&\quad + \big(\nabla_1\phi^1 + \nabla^1\phi_1\big)\big(\nabla_2\phi^2 + \nabla^2\phi_2\big) - \big(\nabla_1\phi^2 + \nabla^2\phi_1\big)\big(\nabla_2\phi^1 + \nabla^1\phi_2\big) \\
&=\; \nabla_1\phi^1\nabla^1\phi_1 + \nabla_1\phi^2\nabla^1\phi_2 + \nabla_2\phi^1\nabla^2\phi_1 + \nabla_2\phi^2\nabla^2\phi_2 \\
&\quad + \nabla_1\phi^1\nabla_2\phi^2 + \nabla_1\phi^1\nabla^2\phi_2 + \nabla^1\phi_1\nabla_2\phi^2 + \nabla^1\phi_1\nabla^2\phi_2 \\
&\quad - \nabla_1\phi^2\nabla_2\phi^1 - \nabla_1\phi^2\nabla^1\phi_2 - \nabla^2\phi_1\nabla_2\phi^1 - \nabla^2\phi_1\nabla^1\phi_2 \;.
\end{aligned}$$

It now can be seen that the second and third term in the first line cancels the second and third term in the third line. Next, we can combine the first term in the first line with the second term in the second time as well as the fourth term in the first line with the third term in the second line. We then find:

$$\begin{aligned}\big(\nabla_i\phi^k\big)\big(\nabla^i\phi_k\big)+\big|\nabla_i\phi^k+\nabla^k\phi_i\big| &= \nabla_1\phi^1\big(\nabla^1\phi_1+\nabla^2\phi_2\big)+\nabla_2\phi^2\big(\nabla^1\phi_1+\nabla^2\phi_2\big)\\ &\quad+\big(\nabla_1\phi^1\nabla_2\phi^2-\nabla_1\phi^2\nabla_2\phi^1\big)+\big(\nabla^1\phi_1\nabla^2\phi_2-\nabla^1\phi_2\nabla^2\phi_1\big)\ .\end{aligned}$$

The two expressions in brackets in the first line are identical and equal to $\nabla^i\phi_i$. Hence, the whole first line becomes equal to $(\nabla_k\phi^k)(\nabla^i\phi_i)=(\nabla_i\phi^i)^2$. And the two expressions in the second line are the determinants of $\nabla_i\phi^k$ and $\nabla^i\phi_k$, which are of course identical. We thereby find the final result

$$\big(\nabla_i\phi^k\big)\big(\nabla^i\phi_k\big)+\big|\nabla_i\phi^k+\nabla^k\phi_i\big| \;=\; \big(\nabla_i\phi^i\big)^2+2\,\big|\nabla_i\phi^k\big|\ . \tag{3.10}$$

Of course, this awkward component-proof leaves open the question whether or not this formula holds in more than two dimensions.

Collecting all bits and pieces, and also using Eqn. (1.32) for $\phi_k\phi^l b^k_s b^s_l$, we finally end up at

$$\boxed{\begin{aligned}\frac{\delta g}{g} &= 2\,\nabla_i\phi^i-4H\psi\\ &\quad+\big(\nabla_i\phi^i\big)^2+2\,\big|\nabla_i\phi^k\big|+2\,b^i_k\big(\phi^k\nabla_i\psi-\psi\nabla_i\phi^k+H\phi_i\phi^k\big)+\psi^2\big(4H^2+2K\big)-K\,\phi_k\phi^k\\ &\quad+\big(\nabla_i\psi\big)\big(\nabla^i\psi\big)-2\,\psi\,\varepsilon^{ij}\varepsilon_{kl}\Big[b^k_i\Big(\nabla_j\phi^l+\nabla^l\phi_j\Big)\Big]\quad+\quad\mathscr{O}(3)\ .\end{aligned}}$$

The first two terms are the variation in linear order, the rest is the quadratic contribution.

3.2.4. Area form

In the surface integrals which we intend to vary, the metric occurs in the form of the square root of its determinant. If we expand this up to quadratic order, we find

$$\delta\sqrt{g} \;=\; \left(\frac{\partial\sqrt{g}}{\partial g}\right)\delta g+\frac{1}{2}\left(\frac{\partial^2\sqrt{g}}{\partial g^2}\right)(\delta g)^2+\mathscr{O}\Big((\delta g)^3\Big) \;=\; \sqrt{g}\left[\frac{\delta g}{2g}-\frac{1}{2}\left(\frac{\delta g}{2g}\right)^2+\mathscr{O}\Big((\delta g/g)^3\Big)\right]\ .$$

The expression $\delta g/g$ has been worked out in the previous section. Note that up to quadratic order the only terms which have to be considered in the second term $(\delta g/g)^2$ are the ones which are first order, *i. e.*

$$\left(\frac{\delta g}{2g}\right)^2 \;=\; \Big(\nabla_i\phi^i-2H\psi\Big)^2+\mathscr{O}(3) \;=\; \big(\nabla_i\phi^i\big)^2-4H\psi\nabla_i\phi^i+4H^2\psi^2+\mathscr{O}\big(\phi^3,\psi^3\big)\ .$$

Thereby we obtain the variation of the square root of the determinant of the metric as

$$\begin{aligned}\delta\sqrt{g} &= \sqrt{g}\,\Big\{\nabla_i\phi^i-2H\psi+\big|\nabla_i\phi^k\big|+b^i_k\big(\phi^k\nabla_i\psi-\psi\nabla_i\phi^k+H\phi_i\phi^k\big)+K\Big(\psi^2-\frac{1}{2}\phi_i\phi^i\Big)\\ &\quad+\frac{1}{2}\big(\nabla_i\psi\big)\big(\nabla^i\psi\big)+2H\psi\nabla_i\phi^i-\psi\,\varepsilon^{kl}\varepsilon_{ij}\Big[b^i_k\Big(\nabla_l\phi^j+\nabla^j\phi_l\Big)\Big]\quad+\quad\mathscr{O}(3)\Big\}\ .\end{aligned}$$

Problem 3.1 *Verify (for instance by a component proof similar to the one which led to Eqn. (3.10)) that*

$$b^i_k\nabla_i\phi^k-b^i_i\nabla^k\phi_k+\varepsilon^{kl}\varepsilon_{ij}\Big[b^i_k\Big(\nabla_l\phi^j+\nabla^j\phi_l\Big)\Big] \;=\; 2H\nabla_k\phi^k-b^i_k\nabla_i\phi^k\ . \tag{3.11}$$

Our expression for $\delta\sqrt{g}$ can be simplified further by making use of Eqn. (3.11). The final result up to quadratic order is then

$$\boxed{\delta\sqrt{g} \;=\; \sqrt{g}\,\Big\{\nabla_i\phi^i-2H\psi+\big|\nabla_i\phi^k\big|+b^i_k\Big[\nabla_i\big(\psi\phi^k\big)+H\phi_i\phi^k\Big]+K\Big(\psi^2-\frac{1}{2}\phi_i\phi^i\Big)+\frac{1}{2}\big(\nabla_i\psi\big)\big(\nabla^i\psi\big)-2H\psi\nabla_i\phi^i+\mathscr{O}(3)\Big\}\ .} \tag{3.12}$$

It is quite remarkable that all three terms from the quadratic term in the expansion of $\sqrt{g}$ cancel some quadratic terms in the first order part of the expansion of $\sqrt{g}$. One might thus wonder whether there is a quicker way to see this. Note also that Eqn. (3.12) gives a cleaner expression than Eqn. (27) in Ref. [16].[1]

We also want to point out that *if the variation is purely normal*, the expression simplifies considerably:

$$\delta_\perp\sqrt{g} \;=\; \sqrt{g}\,\Big[-2H\psi + K\psi^2 + \frac{1}{2}\big(\nabla_i\psi\big)\big(\nabla^i\psi\big) \;+\; \mathscr{O}(3)\Big]\;. \tag{3.13}$$

By using Eqn. (A.12), as well as the fact that for planes the Gaussian curvature is half the Ricci scalar curcature (1.34), we can rewrite the second order normal variation of some area A as

$$\delta_\perp^{(2)}A \;=\; \frac{1}{2}\int \mathrm{d}A\,\psi(R-\nabla^2)\psi \;\;+\;\; \text{boundary term}\;. \tag{3.14}$$

This coincides with Eqn. (64) of Ref. [5], up to a prefactor of 1/2 which is merely due to the different conventions used for the second order variation.

3.3. Variation of the normal vector

Since the normal vector $\vec{n}$ is proportional to the cross product of the two tangent vectors $\boldsymbol{e}_i$, we need to calculate the cross product of the new tangent vectors.

Problem 3.2 *Show that up to linear order the cross product of the varied tangent vectors satisfies*

$$\frac{\mathrm{d}\vec{r}^{\,\prime 1}\times\mathrm{d}\vec{r}^{\,\prime 2}}{\mathrm{d}u^1\mathrm{d}u^2} \;=\; \sqrt{g}\,\Big\{\vec{n}\Big(1+\nabla_i\phi^i-2H\psi\Big) - \boldsymbol{e}_i\Big(\phi^k b_k^i + \nabla^i\psi\Big) \;+\; \mathscr{O}(2)\Big\}\;. \tag{3.15}$$

If we want to do this calculation up to second order, the more succinct notation for the variation of the tangent vectors in Eqn. (3.3), using the abbreviations ${U_i}^k \;=\; \nabla_i\phi^k - \psi b_i^k$ and $V^i \;=\; \phi^k b_k^i + \nabla^i\psi$, is useful. With the help of Eqn. (3.3) we then find

$$\begin{aligned}
\boldsymbol{e}_1'\times\boldsymbol{e}_2' \;&=\; \big(\boldsymbol{e}_1+\delta\boldsymbol{e}_1\big)\times\big(\boldsymbol{e}_2+\delta\boldsymbol{e}_2\big)\\
&=\; \big(\boldsymbol{e}_1+{U_1}^k\boldsymbol{e}_k+V_1\vec{n}\big)\times\big(\boldsymbol{e}_2+{U_2}^k\boldsymbol{e}_k+V_2\vec{n}\big)\\
&=\; \sqrt{g}\,\Big\{\vec{n}\Big(1+{U_k}^k+\big|{U_i}^k\big|\Big)-\boldsymbol{e}_i\Big(V^i+\varepsilon^{kl}\varepsilon_{mn}{U_k}^m V_l g^{ni}\Big)\Big\}\;.
\end{aligned}$$

This expression has to be divided by the square root of the determinant of the varied metric, $\sqrt{g'}$, for which we can use Eqn. (3.12). We can rewrite this in terms of the abbreviations ${U_i}^k$ and V_i by making use of

$$\varepsilon^{ij}\varepsilon_{kl}\big[\big(\nabla_i\phi^k\big)b_j^l\big] + b_k^i\nabla_i\phi^k - 2H\nabla_i\phi^i \;=\; 0\;,$$

(which may be checked by an ugly calculation similar to Eqn. (3.11)) as well as

$$\begin{aligned}
V_iV^i \;&=\; \big(\phi^k b_{ki}+\nabla_i\psi\big)\big(\phi^l b_l^i+\nabla^i\psi\big) \;=\; \phi^k\phi^l b_{ki}b_l^i + \phi^k b_{ki}\nabla^i\psi + \phi^l b_l^i\nabla_i\psi + \big(\nabla_i\psi\big)\big(\nabla^i\psi\big)\\
&\overset{(1.32)}{=}\; 2\Big[H\phi_k\phi^l b_l^k - \frac{1}{2}K\phi_i\phi^i + b_k^i\phi^k\nabla_i\psi + \frac{1}{2}\big(\nabla_i\psi\big)\big(\nabla^i\psi\big)\Big]\;.
\end{aligned} \tag{3.16}$$

Furthermore, the following will also help:

$$\begin{aligned}
-V^i{U_k}^k+\varepsilon^{kl}\varepsilon_{mn}{U_k}^m V_l g^{ni} \;&=\; -V^i{U_1}^1 - V^i{U_2}^2 + {U_1}^1V_2g^{2i} - {U_1}^2V_2g^{1i} - {U_2}^1V_1g^{2i} + {U_2}^2V_1g^{1i}\\
=\; {U_1}^1\big(V_2g^{2i}+V_1g^{1i}-V^i-V_1g^{1i}\big) &+ {U_2}^2\big(V_1g^{1i}+V_2g^{2i}-V^i-V_2g^{2i}\big) - {U_1}^2V_2g^{1i} - {U_2}^1V_1g^{2i}\\
=\; -\Big[V_1\big({U_1}^1g^{1i}+{U_2}^1g^{2i}\big) &+ V_2\big({U_1}^2g^{1i}+{U_2}^2g^{2i}\big)\Big] \;=\; -\big(V_1U^{i1}+V_2U^{i2}\big)\\
=\; -V_kU^{ik} \;&=\; \underline{-V^k{U^i}_k}\;.
\end{aligned}$$

[1] Incidentally, I also believe that the expression given in Ref. [16] is incorrect. The authors seem to make a mistake with their matrix Λ_{ij}, which should be the first order variation of the metric, $\delta g_{ij}^{(1)}$ in the present notation. However, then their Eqn. (19) is incorrect, presumably because they incorrectly swapped the lowering of an index and a partial derivative. However, it is hard to check if this is just a typographical error, because the rest of their expressions is presented in a very "non-covariant" way, resulting in lengthy expressions with a lot of partial derivatives and Christoffel symbols.

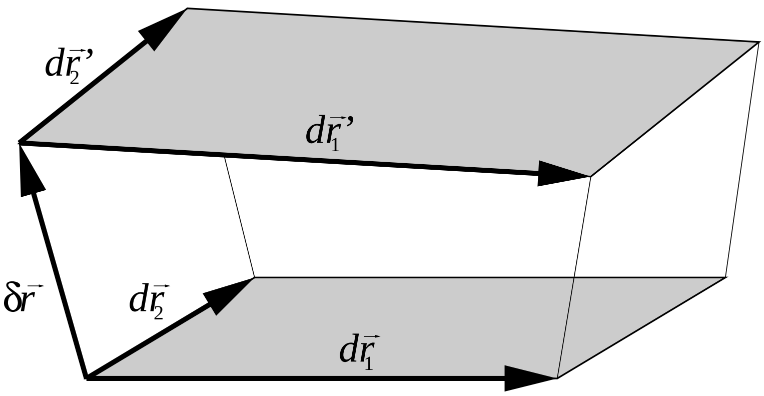


Figure 3.1.: Illustration for the heuristic calculation of the volume variation.

Using the previous three equations, the variation of the normal vector is found to be

$$
\begin{aligned}
\delta\vec{n} &= \frac{\boldsymbol{e}_1'\times\boldsymbol{e}_2'}{\sqrt{g'}} - \vec{n} \;=\; \frac{\sqrt{g}\left\{\vec{n}\Big(1+U_k{}^k+\big|U_i{}^k\big|\Big) - \boldsymbol{e}_i\Big(V^i - V^kU^i{}_k + V^iU_k{}^k\Big)\right\}}{\sqrt{g}\left\{1+U_k{}^k+\big|U_i{}^k\big| + \frac{1}{2}V_kV^k + \mathscr{O}(3)\right\}} - \vec{n} \\
&\overset{\star}{=} \left\{\vec{n}\Big(1+U_k{}^k+\big|U_i{}^k\big|\Big) - \boldsymbol{e}_i\Big(V^i - V^kU^i{}_k + V^iU_k{}^k\Big)\right\}\cdot\left\{1 - U_k{}^k - \big|U_i{}^k\big| - \frac{1}{2}V_kV^k + \big(U_k{}^k\big)^2 + \mathscr{O}(3)\right\} - \vec{n} \\
&= \underline{-\frac{1}{2}V_kV^k\vec{n} - \boldsymbol{e}_i\big(V^i - V^kU^i{}_k\big) \;+\; \mathscr{O}(3)}\;, \qquad (3.17)
\end{aligned}
$$

where at $\star$ we used the geometric series expansion $1/(1+x) = 1 - x + x^2 + \mathscr{O}(x^3)$. Observe that in linear order the variation of $\vec{n}$ is purely tangential. This is due to the fact that $\vec{n}$ is a unit vector, so we have $\vec{n}^2 = 1$ and thus $0 = \delta^1(\vec{n}^2) = 2(\delta^1\vec{n})\cdot\vec{n}$, *i. e.*, the first order variation must be perpendicular to $\vec{n}$.

3.4. Variation of the volume

The variation of the surface in $\mathbb{R}^3$ also implies a change in the volume which lies on one either side of the surface. For instance, if the surface is closed, the interior volume generally changes upon variation of its boundary. In this section we present two different approaches for calculating this change.

3.4.1. Heuristic approach

The present section gives a very heuristic derivation of the volume change up to quadratic order in the surface variation. The idea is presented in Ref. [16], and we refer the reader to this work for some details we will leave out.

The infinitesimal area element is spanned by the two vectors $\mathrm{d}\vec{r}^1 = \boldsymbol{e}_1\mathrm{d}u^1$ and $\mathrm{d}\vec{r}^2 = \boldsymbol{e}_2\mathrm{d}u^2$. Upon variation each corner of this parallelogram is moved. In particular, the origin is moved by the vector $\delta\vec{r} = \phi^k\boldsymbol{e}_k + \psi\vec{n}$, see Fig. 3.1. If the variation were constant across the surface, the infinitesimal volume change would simply be $\delta V = \delta\vec{r}\cdot(\mathrm{d}\vec{r}^1\times\mathrm{d}\vec{r}^2)$. It is claimed in Ref. [16] that up to quadratic order the volume variation can be found by taking the *average* between this value and the value obtained by using instead the varied area element, *i. e.* $\delta\vec{r}\cdot(\mathrm{d}\vec{r}'^1\times\mathrm{d}\vec{r}'^2)$. Using the linear order variation worked out in Problem 3.2, Eqn. (3.15), the total volume change, up to *quadratic* order in the variation, is given by

$$
\begin{aligned}
\delta V &= \frac{1}{2}\delta\vec{r}\cdot\int\big(\mathrm{d}\vec{r}^1\times\mathrm{d}\vec{r}^2 + \mathrm{d}\vec{r}'^1\times\mathrm{d}\vec{r}'^2\big) \;+\; \mathscr{O}(3) \\
&= \frac{1}{2}\int\mathrm{d}u^1\mathrm{d}u^2\;\sqrt{g}\big(\phi^k\boldsymbol{e}_k + \psi\vec{n}\big)\left\{\vec{n}\Big(2+\nabla_i\phi^i - 2H\psi\Big) - \boldsymbol{e}_i\Big(\phi^j b^i_j + \nabla^i\psi\Big)\right\} \;+\; \mathscr{O}(3) \\
&= \underline{\int\mathrm{d}A\left\{\psi - H\psi^2 + \frac{1}{2}\Big(\psi\nabla_i\phi^i - \phi^i\nabla_i\psi - \phi^i\phi^j b_{ij}\Big)\right\} \;+\; \mathscr{O}(3)}\;. \qquad (3.18)
\end{aligned}
$$

We again remark that if the variation is purely normal, this expression simplifies quite a bit:

$$
\delta_\perp V \;=\; \int\mathrm{d}A\,\Big(\psi - H\psi^2\Big)\;. \qquad (3.19)
$$

Let us check this formula in a very simple case: Take a sphere of radius R, choose the outward normal, and perform a purely normal surface variation of magnitude ψ, constant over the entire surface (*i. e.*, increase the radius by ψ). What is the volume change up to quadratic order?

$$\Delta V \;=\; \frac{4}{3}\pi\big(R+\psi\big)^3 - \frac{4}{3}\pi R^3 \;=\; \frac{4}{3}\pi\big(R^3+3R^2\psi+3R\psi^2+\psi^3-R^3\big) \;=\; 4\pi R^2\big(\psi+\frac{1}{R}\psi^2+\mathscr{O}(\psi^3)\big)\ .$$

Since for the sphere with outward pointing normal $H=-1/R$, we see that this result confirms Eqn. (3.19) up to quadratic order.

3.4.2. Formal approach

Instead of relying on heuristic arguments such as the ones presented in Sec. 3.4.1, we can also use a formal approach. Due to Gauss' Theorem we can write the volume of some object as a surface integral over the local normal vector dotted into the parameterization:

$$V \;=\; \int_V \mathrm{d}V \;=\; \int_V \mathrm{d}V\,\underbrace{\frac{\nabla\cdot\vec{r}}{3}}_{=1} \;\stackrel{\text{Gauss}}{=}\; \frac{1}{3}\int_{\partial V}\mathrm{d}A\,\vec{n}\cdot\vec{r}\,. \tag{3.20}$$

Generalizations of this to higher dimensions are obvious.

Since we thus have V as a simple integral over the surface, we can apply the formal variational scheme we've used so far. Up to first order the volume can not depend on tangential variations, because those only correspond to a reparametrization.[2] We thus find

$$\begin{aligned}
\delta V \;=\; \delta_\perp V &\;=\; \delta_\perp\,\frac{1}{3}\int_{\partial V}\mathrm{d}A\,\vec{n}\cdot\vec{r}\\
&\;=\; \frac{1}{3}\int_{\partial V}\mathrm{d}u^1\,\mathrm{d}u^2\,\Big\{(\delta_\perp\sqrt{g})\vec{n}\cdot\vec{r}+\sqrt{g}(\delta_\perp\vec{n})\cdot\vec{r}+\sqrt{g}\vec{n}\cdot(\delta_\perp\vec{r})\Big\}\\
&\;=\; \frac{1}{3}\int_{\partial V}\mathrm{d}u^1\,\mathrm{d}u^2\,\Big\{(-2H\psi\sqrt{g})\vec{n}\cdot\vec{r}+\sqrt{g}(-\boldsymbol{e}_i\nabla^i\psi)\cdot\vec{r}+\sqrt{g}\vec{n}\cdot(\psi\vec{n})\Big\}\\
&\;=\; \frac{1}{3}\int_{\partial V}\mathrm{d}A\Big\{-2H\psi\vec{n}\cdot\vec{r}-\underbrace{\boldsymbol{e}_i\cdot\vec{r}\,\nabla^i\psi}_{\text{int. by parts}}+\psi\Big\}\\
&\;=\; \frac{1}{3}\int_{\partial V}\mathrm{d}A\Big\{-2H\psi\vec{n}\cdot\vec{r}+\big[\underbrace{(\nabla^i\boldsymbol{e}_i)}_{=b^i_i\vec{n}=2H\vec{n}}\cdot\vec{r}+\underbrace{\boldsymbol{e}_i\cdot\nabla^i\vec{r}}_{=g^i_i=2}\big]\psi+\psi\Big\}\\
&\;=\; \int_{\partial V}\mathrm{d}A\,\psi
\end{aligned} \tag{3.21}$$

In order to obtain the second variation, we can make use of the nice result that it can be obtained as $\frac{1}{2}$ times the first variation of the first variation:

$$\delta^{(2)}f \;=\; \frac{1}{2}\delta^{(1)}\big[\delta^{(1)}f\big] \;=\; \frac{1}{2}\delta^{(1)}f^{(1)}\ . \tag{3.22}$$

However, when varying Eqn. (3.21) once more, we have to remember that $\delta\vec{r}$ is our fundamental variation, and not merely ψ, which is given by $\psi=\vec{n}\cdot\delta\vec{r}$. Hence, the integrand ψ contains the normal vector and must be varied as well. We therefore obtain

$$\begin{aligned}
\delta^{(2)}V &\;=\; \frac{1}{2}\delta^{(1)}\big[\delta^{(1)}V\big] \;=\; \frac{1}{2}\delta^{(1)}\int\mathrm{d}A\,\vec{n}\cdot\delta\vec{r}\\
&\;=\; \frac{1}{2}\int\Big\{\mathrm{d}A^{(1)}\,\vec{n}\cdot\delta\vec{r}+\mathrm{d}A\,\vec{n}^{(1)}\cdot\delta\vec{r}\Big\}\\
&\;=\; \frac{1}{2}\int\mathrm{d}A\,\Big\{(\nabla_i\phi^i-2H\psi)\psi+(-b^i_k\phi^k-\nabla^i\psi)\underbrace{\boldsymbol{e}_i\cdot\delta\vec{r}}_{\phi_i}\Big\}\\
&\;=\; \int\mathrm{d}A\,\Big\{-H\psi^2+\frac{1}{2}\Big[\psi\nabla_i\phi^i-\phi^i\nabla_i\psi-b_{ik}\phi^i\phi^k\Big]\Big\}\ .
\end{aligned} \tag{3.23}$$

[2] In fact, one can easily check that contributions from the tangential variations just cancel.

This coincides with the second order variation calculated in Sec. 3.4.1. Observe also that by partially integrating the third term in the integrand, this may alternatively be written as (up to boundary terms, as usual)

$$\delta^{(2)}V \;=\; \int \mathrm{d}A\,\Big\{-H\psi^2 + \psi\nabla_i\phi^i - \frac{1}{2}b_{ik}\phi^i\phi^k\Big]\Big\} \;. \tag{3.24}$$

3.5. Variation of the extrinsic geometry

3.5.1. Second fundamental form

Now we want to find the variation of b_{ij}. The covariant version (1.19) of equation of Gauss shows that $b_{ij} = \vec{n}\cdot\nabla_i\boldsymbol{e}_j$. Using this, we find

$$\delta b_{ij} \;=\; \delta\vec{n}\cdot\big(\nabla_i\boldsymbol{e}_j\big) + \vec{n}\cdot\big(\nabla_i\,\delta\boldsymbol{e}_j\big) + \delta\vec{n}\cdot\big(\nabla_i\,\delta\boldsymbol{e}_j\big)\;.$$

Using the covariant version of the equations of Gauss and Weingarten (1.19), the variation of the tangent vector (3.3), and using the expressions derived above, this can be rewritten as

$$\delta b_{ij} \;=\; \underbrace{b_{ik}U_j{}^k + \nabla_i V_j}_{\text{first order}}\underbrace{-\frac{1}{2}b_{ij}V_kV^k - V_k\nabla_i U_j{}^k + b_i^kV_jV_k}_{\text{second order}} \;=:\; \delta^{(1)}b_{ij} + \delta^{(2)}b_{ij}\;. \tag{3.25}$$

We now have to reintroduce the abbreviations $U_j{}^k$ and V_j. For the first order we then find

$$\begin{aligned}
\delta^{(1)}b_{ij} \;&=\; b_{ik}\big(\nabla_j\phi^k - \psi b_j^k\big) + \nabla_i(\phi^k b_{kj} + \nabla_j\psi)\\
&\overset{(1.23)}{=}\; \nabla_i\nabla_j\psi - b_{ik}b_j^k\psi + \phi^k\nabla_k b_{ij} + b_{kj}\nabla_i\phi^k + b_{ik}\nabla_j\phi^k\\
&=\; \underline{\nabla_i\nabla_j\psi - b_{ik}b_j^k\psi + \mathscr{L}_{\boldsymbol{\phi}}b_{ij}}\;.
\end{aligned} \tag{3.26}$$

Just as in the case of the first fundamental form, the tangential variation in first order is found to be given by the Lie derivative [4].

3.5.2. Mean curvature

Since $2H = g^{ij}b_{ij}$, we have in first order

$$\begin{aligned}
\delta^{(1)}H \;&=\; \frac{1}{2}\Big[\big(\delta g^{ij}\big)b_{ij} + g^{ij}\big(\delta b_{ij}\big)\Big]\\
&=\; \frac{1}{2}\Big[-\big(\nabla^i\phi^j + \nabla^j\phi^i - 2\psi b^{ij}\big)b_{ij} + g^{ij}\big(\nabla_i\nabla_j\psi - b_{ik}b_j^k\psi + \phi^k\nabla_k b_{ij} + b_{kj}\nabla_i\phi^k + b_{ik}\nabla_j\phi^k\big)\Big]\\
&=\; -\frac{1}{2}\big(\nabla^i\phi^j + \nabla^j\phi^i\big)b_{ij} + \psi\big(4H^2 - 2K\big) + \frac{1}{2}\Big[\nabla^2\psi - \big(4H^2 - 2K\big)\psi + 2\phi^k\nabla_k H + 2b_i^k\nabla_k\phi^i\Big]\\
&=\; \underline{\psi\big(2H^2 - K\big) + \frac{1}{2}\nabla^2\psi + \phi^k\nabla_k H}\;.
\end{aligned} \tag{3.27}$$

Let us also look at the first variation of $\sqrt{g}H$. It is given by

$$\delta^{(1)}\big(\sqrt{g}H\big) \;=\; H\delta^{(1)}\sqrt{g} + \sqrt{g}\,\delta^{(1)}H \;\overset{(3.12,3.27)}{=}\; \sqrt{g}\Big[H\nabla_i\phi^i - K\psi + \frac{1}{2}\nabla^2\psi + \phi^i\nabla_i H\Big]\;. \tag{3.28}$$

Let M be the integral over the mean curvature over some portion of the manifold. The first variation of M is the integral of Eqn. (3.28) over this portion. The term proportional to $\nabla^2\psi$ will only contribute a boundary term, and the last term will upon partial integration (see Eqn. (A.9)) cancel the first term, again up to a boundary term. Hence we find

$$\delta^{(1)}M \;=\; \delta^{(1)}\int\mathrm{d}A\;H \;=\; \int\mathrm{d}^2u\;\;\delta^{(1)}\big(\sqrt{g}H\big) \;\overset{(3.28,A.9)}{=}\; -\int\mathrm{d}A\;K\psi \;\;+\text{boundary terms}\;. \tag{3.29}$$

Quite remarkably, the first variation of the integral over the *extrinsic* curvature depends only on the *intrinsic* curvature! (This extends Eqn. (50) in Ref. [5], which is only written for the *normal* variation—but we see that the tangential part only contributes boundary terms.)

The second variation of the mean curvature is given by

$$\delta^{(2)}H \;=\; \frac{1}{2}\Big[\big(\delta^{(2)}g^{ij}\big)b_{ij} + \big(\delta^{(1)}g^{ij}\big)\big(\delta^{(1)}b_{ij}\big) + g^{ij}\big(\delta^{(2)}b_{ij}\big)\Big]\;. \tag{3.30}$$

Let us calculate each of these three terms in turn. For the first term we find, using Eqns. (3.8),

$$\begin{aligned}\big(\delta^{(2)}g^{ij}\big)b_{ij} \;&=\; 6H\psi^2\big(4H^2-3K\big)-\big(4H^2-K\big)b_i^k\phi^i\phi_k+2HK\phi_i\phi^i-4Hb_i^k\phi^i\nabla_k\psi+2K\phi_i\nabla^i\psi-b_{ij}\nabla^i\psi\nabla^j\psi\\ &\quad+b_{ij}\big(\nabla^i\phi^k\nabla_k\phi^j+\nabla^k\phi^i\nabla_k\phi^j+\nabla^k\phi^i\nabla^j\phi_k\big)-12H\psi b_i^k\nabla^i\phi_k+6K\psi\nabla_i\phi^i \qquad (3.31)\end{aligned}$$

For the second term Eqns. (3.4), (3.6), and (3.26) yield

$$\begin{aligned}\big(\delta^{(1)}g^{ij}\big)\big(\delta^{(1)}b_{ij}\big) \;&=\; -\big(\nabla^i\phi^j+\nabla^j\phi^i\big)\big(\nabla_i\nabla_j\psi\big)+2\psi b^{ij}\nabla_i\nabla_j\psi+12H\psi b_{ij}\nabla^i\phi^j-6K\psi\nabla_i\phi^i-4H\psi^2\big(4H^2-3K\big)\\ &\quad-2\phi^k\big(\nabla_k b_{ij}\big)\big(\nabla^i\phi^j\big)+2\psi b^{ij}\phi^k\nabla_k b_{ij}-2\big(\nabla^i\phi^j\big)b_{kj}\nabla_i\phi^k-2\big(\nabla^i\phi^j\big)b_{ik}\nabla_j\phi^k\ . \qquad (3.32)\end{aligned}$$

The third term is the most complicated one. Using Eqn. (3.25), it is found to be

$$g^{ij}\big(\delta^{(2)}b_{ij}\big) \;=\; -HV_kV^k-V_k\nabla^iU_i{}^k+b_i^kV^iV_k\ , \qquad (3.33)$$

but this expression has to be translated back in terms of ϕ^k and ψ. The first term in Eqn. (3.33) has been worked out in Eqn. (3.16). For the second term we find

$$\begin{aligned}V_k\nabla^iU_i{}^k \;&=\; \big(\phi^lb_{lk}+\nabla_k\psi\big)\nabla^i\big(\nabla_i\phi^k-\psi b_i^k\big) \;=\; \big(\phi^lb_{lk}+\nabla_k\psi\big)\big(\nabla^2\phi^k\overset{(1.23)}{-}2\psi\nabla^kH-b_i^k\nabla^i\psi\big)\\ &=\; b_{lk}\phi^l\nabla^2\phi^k-2b_{lk}\psi\phi^l\nabla^kH+\nabla_k\psi\nabla^2\phi^k-2\psi\nabla_k\psi\nabla^kH-\phi^l\big(2Hb_{il}-Kg_{il}\big)\nabla^i\psi-b_i^k\nabla^i\psi\nabla_k\psi\ ,\end{aligned}$$

and the third term is given by

$$\begin{aligned}b_i^kV^iV_k \;&=\; b_i^k\big(\phi^lb_l^i+\nabla^i\psi\big)\big(\phi^mb_{mk}+\nabla_k\psi\big)\\ &\overset{(1.32),(1.35)}{=}\; \big(4H^2-K\big)b_i^k\phi^i\phi_k-2HK\phi_k\phi^k+4Hb_i^k\phi^i\nabla_k\psi-2K\phi^k\nabla_k\psi+b_i^k\big(\nabla^i\psi\big)\big(\nabla_k\psi\big)\ .\end{aligned}$$

Combining these expressions, we find for the third term in Eqn. (3.30)

$$\begin{aligned}g^{ij}\big(\delta^{(2)}b_{ij}\big) \;&=\; \big(2H^2-K\big)b_i^k\phi^i\phi_k-HK\phi_k\phi^k+4Hb_i^k\phi^i\nabla_k\psi-H\big(\nabla_i\psi\big)\big(\nabla^i\psi\big)-b_{ik}\phi^i\nabla^2\phi^k\\ &\quad+2b_i^k\psi\phi^i\nabla_kH-\big(\nabla_k\psi\big)\nabla^2\phi^k+2\psi\big(\nabla_k\psi\big)\big(\nabla^kH\big)-3K\phi^k\nabla_k\psi+2b_i^k\big(\nabla^i\psi\big)\big(\nabla_k\psi\big)\ . \qquad (3.34)\end{aligned}$$

If we now insert the results from Eqns. (3.31), (3.32), and (3.34), into Eqn. (3.30), many terms cancel. The final result is then

$$\boxed{\begin{aligned}\delta^{(2)}H \;&=\; H\psi^2\big(4H^2-3K\big)-H^2b_i^k\phi^i\phi_k-Hb_i^k\phi^i\nabla_k\psi-K\phi^i\nabla_i\psi+b_{ij}\psi\phi^i\nabla^jH+\psi\nabla_k\psi\nabla^kH\\ &\quad+\psi b_{ij}\nabla^i\nabla^j\psi-\phi^k\big(\nabla_kb_{ij}\big)\nabla^i\phi^j+\psi b^{ij}\phi^k\nabla_kb_{ij}-\frac{1}{2}\big(\nabla^i\phi^j+\nabla^j\phi^i\big)\nabla_i\nabla_j\psi\\ &\quad+\frac{1}{2}H\big(K\phi_i\phi^i-\nabla_i\psi\nabla^i\psi\big)-\frac{1}{2}b_{ij}\big(\nabla^k\phi^i\nabla_k\phi^j+\phi^i\nabla^2\phi^j-\nabla^i\psi\nabla^j\psi\big)-\frac{1}{2}\nabla_k\psi\nabla^2\phi^k\ .\end{aligned}}$$

As usual, this expression simplifies considerably if the variation is purely normal:

$$\delta_\perp^{(2)}H \;=\; H\psi^2\big(4H^2-3K\big)+\psi\nabla_k\psi\nabla^kH+\psi b_{ij}\nabla^i\nabla^j\psi-\frac{1}{2}\big(H\nabla_i\psi\nabla^i\psi-b_{ij}\nabla^i\psi\nabla^j\psi\big)\ .$$

This again coincides with Eqn. (69) of Ref. [5], up to a prefactor 1/2 due to different conventions for the second variation.

4. Some applications to problems involving the first area variation

4.1. Minimal surfaces

Roughly speaking, minimal surfaces have the property that locally one cannot deform them without increasing their area. They can often be visualized as soap films suspended between wire frames (see for instance the beautiful book by Isenberg [12]). The classical minimal surfaces – plane, catenoid and helicoid – date back into the 18th century. Many important analytical properties of such surfaces have been uncovered, most importantly their relation with holomorphic functions discovered by Weierstrass. Still, not many explicit examples of minimal surfaces have been found until the beginning of the 1980s, where the rapid increase in computer power boosted this field of mathematics.

In this section we will only touch upon a few very simple problems and a few minimal surfaces without getting into any mathematical study of their properties.

4.1.1. Defining property

Consider a set of surfaces S_i which all have the same closed curve C as their boundary. Which of these surfaces has the smallest area? We will skip the usual mathematical intricacies related to questions of whether such a minimum exists, but we will ask the question of how to find or characterize such a *minimal* surface. The area A_i of any such surface S_i is evidently given by

$$A_i \;=\; \int_{S_i} \mathrm{d}A \;=\; \int_{U_i} \mathrm{d}u^1 \mathrm{d}u^2 \,\sqrt{g}\;,$$

where U_i is the coordinate patch describing surface S_i and $\sqrt{g}$ is the metric of that surface. A minimal surface will be such that its first variation $\delta^{(1)}A$ of the area vanishes, so a necessary condition for S to be minimal is

$$\delta^{(1)}A \;=\; \int_{U_i} \mathrm{d}u^1 \mathrm{d}u^2 \,\delta^{(1)}\sqrt{g} \;\stackrel{(3.13)}{=}\; \int_{U_i} \mathrm{d}u^1 \mathrm{d}u^2 \,\sqrt{g}\big(-2H\psi(u^1,u^2)\big) \;=\; 0\;,$$

where we restricted to normal variations ψ, which is enough if we leave the boundary untouched. Since the variation ψ is arbitrary (except that we require it to vanish at the boundary), this integral can only vanish if $H \equiv 0$. We thus see: *A necessary condition for a surface to be minimal is that its mean curvature vanishes at every point.* This is intuitively clear, because if a surface had a region in which H would for instance be positive, this region would be a little “bump”, and the total area could be reduced by flattening the bump. Of course, this condition is not sufficient. We only have established a criterion for the surface to be *stationary*. Neither do we know whether the solution of the (differential!) equation $H = 0$ is unique (it needn’t be), nor do we know whether the solution corresponds to a minimum, a maximum or a saddle point in the “space of surfaces”.

Note that since $2H = \kappa_1 + \kappa_2$ and $K = \kappa_1\kappa_2$, the Gaussian curvature of a minimal surface necessarily satisfies $K \le 0$ everywhere.

Formula (C.12) in Appendix C.4 shows that the coordinate functions of a minimal surface are harmonic.

Problem 4.1 *Show, that the following surface given in Monge parameterization by the function*

$$h : \begin{cases} \big(-\frac{\pi}{2},\frac{\pi}{2}\big)\times\big(-\frac{\pi}{2},\frac{\pi}{2}\big) & \to \quad \mathbb{R} \\ (x,y) & \mapsto \quad h(x,y) = \log\dfrac{\cos x}{\cos y} \end{cases} \tag{4.1}$$

is minimal. This surface is called “Scherk’s surface”. An illustration is given in Fig. 4.1. Not restricting the region of x and y to the central square $(-\frac{\pi}{2},\frac{\pi}{2})\times(-\frac{\pi}{2},\frac{\pi}{2})$, we see that this surface is actually doubly periodic.

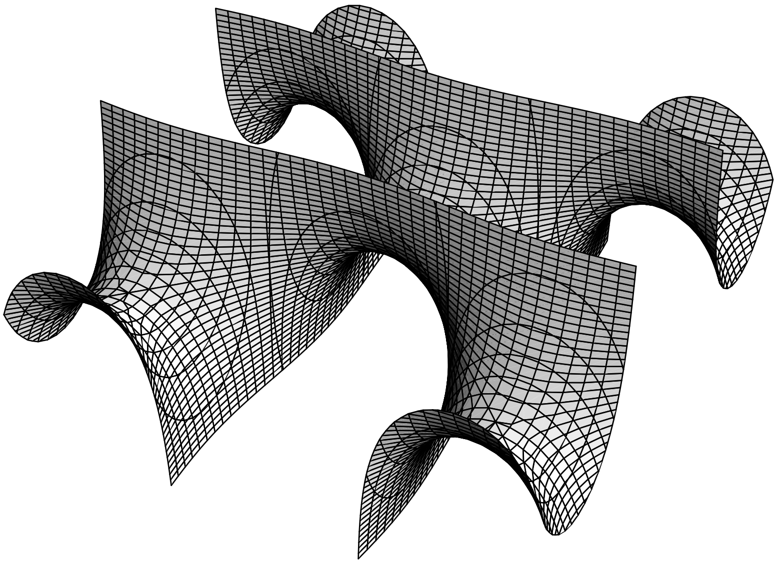


Figure 4.1.: Illustration of a central portion of Scherk's surface—see Problem 4.1. It is named after Heinrich Ferdinand Scherk, who discovered this minimal surface in 1835 and provided thereby the third nontrivial example of a minimal surface (after the catenoid and the helicoid, discovered by Jean Baptiste Marie Meusnier in 1776). Scherk's surface is double periodic, and it links a collection of parallel planes to a collection of perpendicular parallel planes. One thus might speculate that a defect surface in a lamellar phase of microemulsions looks like essentially Scherk's surface (and would thus not cost any bending energy!).

4.1.2. Example 1: Soap film between two circles

Consider a soap film suspended between two parallel, coaxial circles of equal radius [1, Example 17.2.2]. The surface tension of the soap film will attempt to give it a shape which minimizes its area and therefore render the mean curvature zero.[1] Using the parameterization discussed in Sec. 2.2.4 (see also Table. 2.1), the condition for such a surface to be minimal can be written as

$$0 \stackrel{!}{=} \kappa_1 + \kappa_2 = -\frac{1}{r\sqrt{1+r'^2}} + \frac{r''}{\left(1+r'^2\right)^{3/2}} = \frac{-(1+r'^2) + r\,r''}{r\left(1+r'^2\right)^{3/2}} \ .$$

This equation is "satisfied" in the pathological[2] case $r' \equiv \infty$, which just corresponds to a plane. If r' is different from infinity, we will require the nominator to vanish. This gives the ordinary second order nonlinear differential equation

$$r\,r'' - \left(r'\right)^2 = 1 \ ,$$

and one can easily check that this has the general solution

$$r(z) = c_1 \cosh\frac{z - c_2}{c_1} \ . \tag{4.2}$$

A surface from this two-parameter family of surfaces is called a *catenoid*. Since the metric in this particular parameterization is given by $r^2(1+r'^2)$, the metric determinant for the catenoid is $\sqrt{g} = c_1\cosh^2[(z-c_2)/c_1]$. The surface area S of the symmetric portion of a catenoid of height d is therefore given by

$$S = \int_{-d/2}^{d/2} \mathrm{d}z \int_o^{2\pi} \mathrm{d}\varphi\; c_1 \cosh^2\frac{z}{c_1} = \pi c_1\Big(d + c_1 \sinh\frac{d}{c_1}\Big) \ . \tag{4.3}$$

It turns out that a stable film can only develop if the two rings are not too far apart. Looking at Eqn. (4.2), it is clear that the boundary value problem to solve is

$$c_1 \cosh\frac{d}{2c_1} = R \qquad \text{or:} \qquad \frac{\cosh x}{x} = \frac{2R}{d} \qquad \text{with} \quad x := \frac{d}{2c_1} \ . \tag{4.4}$$

[1] Physically, one may worry that there are also small but nonzero *bending* moments of the soap film, *i. e.*, that there is a contribution to the energy due to the curvature. However, if this energy per unit area is given by a Helfrich Hamiltonian, see Eqn. (5.1), and if the spontaneous curvature of the film is zero, then the surfaces considered here (for which the pressure on both sides is zero) *automatically* also satisfy the more complicated differential equation (5.3) which minimizes curvature *and* tension energy, and which we will derive in Sec. (5.1).

[2] A plane is strictly speaking outside the realm of this particular parameterization. Still, it leaves its traces here in the sense that it comes up as a "physicist's solution" of the problem.

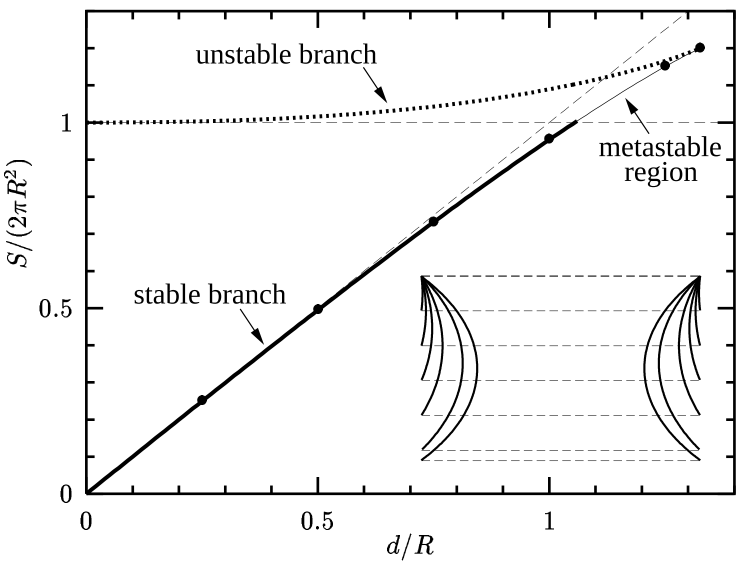


Figure 4.2.: Area of a stationary catenoid suspended between two parallel coaxial rings of radius R as a function of their distance d. Below a maximum distance $d_{\max}/R \simeq 1.32549$ two branches exist, the one with the lower area being the stable solution. The inset shows the profiles of a sequence of stable catenoids at the distances $d/R \in \{0.25, 0.5, 0.75, 1.0, 1.25, 1.32549\}$, also indicated by the "•" symbols.

However, the function $(\cosh x)/x$ has a *minimum* at some value of x:

$$0 \stackrel{!}{=} \frac{\partial}{\partial x}\frac{\cosh x}{x} \quad \Rightarrow \quad \cosh x - x \sinh x = 0 \quad \Rightarrow \quad \frac{d}{2c_1} = x_{\min} \simeq 1.19968 \ .$$

Clearly, Eqn. (4.4) only has a solution if the radius R is larger than the minimum value which the left hand side assumes. Inserting the value for x at the minimum, this gives the condition

$$d \leq d_{\max} \simeq 1.32549 \times R \ .$$

For values of d smaller than this, Eqn. (4.4) has two solutions, see Fig. 4.2. If $d/R \gtrsim 1.05539$ the area of the catenoid is larger than the combined area of two soap films which separately cover the two rings, and the catenoid becomes metastable. At the critical value $d_{\max}$ the area of the soap film is $S \simeq 7.5378\,R^2$, which is about 20% larger than the area spanned by the two rings individually, which is $2\pi R^2 \simeq 6.2832\,R^2$. Beyond $d_{\max}$ the solution consisting of two separately covered rings is the only stable one. This was first shown analytically by Goldschmidt in 1831, and the equilibrium solution consisting of two separate films spanning the two circles individually has become known as the "Goldschmidt discontinuous solution" [1, 12]

The inset in Fig. 4.2 shows the shapes of soap films suspended between two circles as the two circles are gradually being pulled apart. This situation is somewhat peculiar because there does not seem to be anything "forbidding" about the limiting profile: It does not touch in the middle, it does not have infinite slopes or sharp edges. Why then, physically, does the soap film snap?

4.1.3. Example 2: Helicoid

Take two straight lines which intersect at a right angle. Now move the second line along the first one while simultaneously rotating it with a constant angular velocity which points along the first line. The second line then traces[3] out a helical surface which is given by the following parameterization:

$$\vec{r}: \begin{cases} \mathbb{R}^2 \quad \rightarrow \quad \mathbb{R}^3 \\ (r,z) \quad \mapsto \quad \vec{r}(r,z) = \begin{pmatrix} r\cos z \\ r\sin z \\ cz \end{pmatrix} \end{cases} \tag{4.5}$$

This surface is called a *helicoid*, see Fig. 4.3. The constant c determines the pitch of the helix.

[3] A surface which is generated by a line moving through space is called a "ruled surface", and the lines are called the "generators" of the surface. A "developable" surface is a special ruled surface, which has the additional property that it has the same tangent plane on all points of one and the same generator. It can be proved that a portion of a surface is developable if and only if its Gaussian curvature is zero everywhere. It can also be proved that a (sufficiently small) portion of a surface can be isometrically mapped to a plane if and only if it is developable [14].

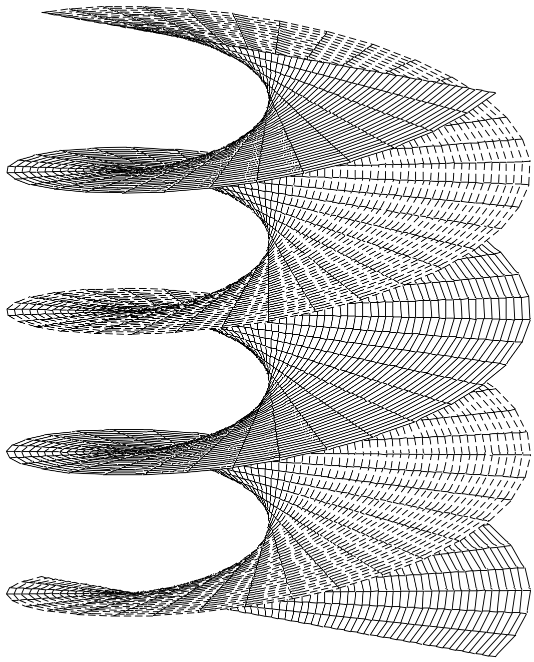


Figure 4.3.: Illustration of two turns of a helicoid. Note that when "walking upwards" on one of the "stairs" for one complete rotation, one has actually moved up *two* turns. This is just like the famous spiral staircase in the Vatican museum in Rome.

A helicoid is actually a minimal surface, as we will now verify. The tangent vectors are given by

$$\boldsymbol{e}_r \;=\; \begin{pmatrix} \cos z \\ \sin z \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{e}_z \;=\; \begin{pmatrix} -r\sin z \\ r\cos z \\ c \end{pmatrix} \;.$$

From this we find the metric and its determinant (take $u^1 = r$, $u^2 = z$)

$$g_{ij} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & c^2 + r^2 \end{pmatrix} \qquad \text{and} \qquad g \;=\; c^2 + r^2 \;.$$

This also shows that the coordinate representation is orthogonal. The normal vector is given by

$$\vec{n} \;:=\; \frac{\boldsymbol{e}_r \times \boldsymbol{e}_z}{\sqrt{g}} \;=\; \frac{1}{\sqrt{c^2+r^2}} \begin{pmatrix} c\sin z \\ -c\cos z \\ r \end{pmatrix} \;.$$

For the second fundamental form we need the second partial derivatives

$$\boldsymbol{e}_{r,r} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{e}_{z,z} \;=\; \begin{pmatrix} -r\cos z \\ -r\sin z \\ 0 \end{pmatrix} \quad , \text{ and } \quad \boldsymbol{e}_{r,z} \;=\; \boldsymbol{e}_{z,r} \;=\; \begin{pmatrix} -\sin z \\ \cos z \\ 0 \end{pmatrix} \;.$$

So the second fundamental b_{ij} and its determinant b are given by

$$b_{ij} \;:=\; \boldsymbol{e}_{i,j} \cdot \vec{n} \;=\; -\frac{c}{\sqrt{c^2+r^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad b \;=\; -\frac{c^2}{c^2+r^2} \;.$$

The Gaussian curvature K is the ratio between the determinant of first and second fundamental form, and is given by

$$K \;=\; \frac{b}{g} \;=\; -\frac{c^2}{(c^2+r^2)^2} \;<\; 0 \;. \tag{4.6}$$

Twice the mean curvature, $2H$, is the complete contraction of b_{ij} with g^{ij} and given by

$$2H \;=\; b_{ij} g^{ij} \;=\; b_{rr} g^{rr} + b_{zz} g^{zz} \;=\; 0 \;. \tag{4.7}$$

This proves that the helicoid is a minimal surface.

In cylindrical coordinates (r, z, φ) the helicoid is given by $z = c\varphi$.

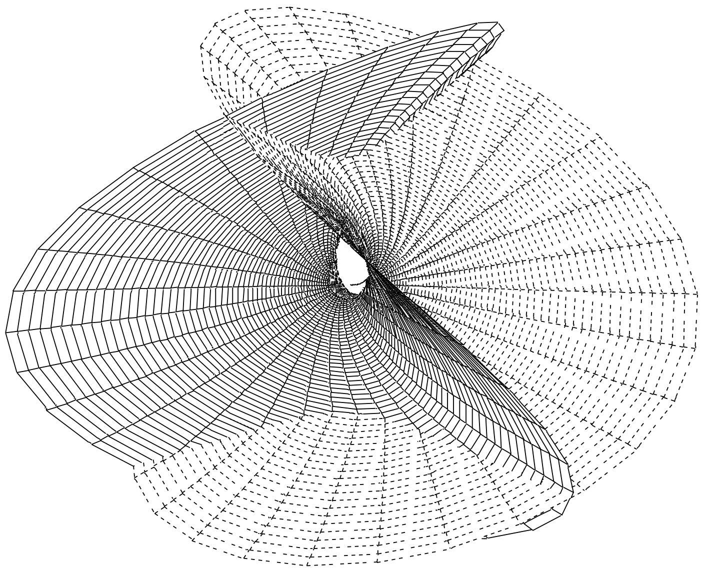


Figure 4.4.: Illustration of the central portion of Enneper's minimal surface (r has been plotted out up to $r_{\max} = 2.5$). If one were to look from farther away (and plotting the surface for larger values of $r_{\max}$), the Enneper surface looks like a plane which winds up three times before it meets with itself.

4.1.4. Example 3: Enneper's minimal surface

In 1863 the German mathematician Alfred Enneper discovered that the surface given by the parameterization

$$\vec{r}: \begin{cases} \mathbb{R}_0^+ \times [0; 2\pi) & \to \qquad \mathbb{R}^3 \\ (r, \phi) & \mapsto \quad \vec{r}(r,\phi) = \begin{pmatrix} r\cos\phi - \frac{1}{3}r^3\cos(3\phi) \\ -r\sin\phi - \frac{1}{3}r^3\sin(3\phi) \\ r^2\cos(2\phi) \end{pmatrix} \end{cases} \tag{4.8}$$

is a minimal surface. It cannot be embedded in R^3 because (for $r \geq \sqrt{3}$) it develops self-intersections, which for $z > 0$ lie in the $y-z$-plane and for $z < 0$ in the $x-z$-plane. However, it is still a proper immersion. It also contains two straight lines in the plane $z = 0$ which intersect orthogonally, namely if the coordinate ϕ takes the values $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$. See Fig. 4.4 for an illustration.

Let's check that indeed the mean curvature vanishes everywhere. The tangent vectors are given by

$$\boldsymbol{e}_r = \begin{pmatrix} \cos\phi - r^2\cos(3\phi) \\ -\sin\phi - r^2\sin(3\phi) \\ 2r\cos(2\phi) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{e}_\phi = \begin{pmatrix} -r\sin\phi + r^3\sin(3\phi) \\ -r\cos\phi - r^3\cos(3\phi) \\ -2r^2\sin(2\phi) \end{pmatrix} .$$

From this we immediately get the metric ($u^1 = r$, $u^2 = \phi$) and the metric determinant

$$g_{ij} = (1+r^2)^2 \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad \text{and} \qquad g = r^2(1+r^2)^4 .$$

This also shows that the coordinate representation is orthogonal. The normal vector is given by

$$\vec{n} := \frac{\boldsymbol{e}_r \times \boldsymbol{e}_\phi}{\sqrt{g}} = \frac{1}{1+r^2} \begin{pmatrix} 2r\cos\phi \\ 2r\sin\phi \\ r^2 - 1 \end{pmatrix} .$$

We finally need to know the second partial derivatives:

$$\boldsymbol{e}_{r,r} = \begin{pmatrix} -2r\cos(3\phi) \\ -2r\sin(3\phi) \\ 2\cos(2\phi) \end{pmatrix} , \qquad \boldsymbol{e}_{\phi,\phi} = \begin{pmatrix} -r\cos\phi + 3r^3\cos(3\phi) \\ r\sin\phi + 3r^3\sin(3\phi) \\ -4r^2\cos(2\phi) \end{pmatrix} ,$$

$$\text{and} \qquad \boldsymbol{e}_{r,\phi} = \boldsymbol{e}_{\phi,r} = \begin{pmatrix} -\sin\phi + 3r^2\sin(3\phi) \\ -\cos\phi - 3r^2\cos(3\phi) \\ -4r\sin(2\phi) \end{pmatrix} .$$

From this we now get the second fundamental form b_{ij} and its determinant b:

$$b_{ij} \;:=\; \boldsymbol{e}_{i,j} \cdot \vec{n} \;=\; \begin{pmatrix} -2\cos(2\phi) & 2r\sin(2\phi) \\ 2r\sin(2\phi) & 2r^2\cos(2\phi) \end{pmatrix} \qquad \text{and} \qquad b \;=\; -4r^2 \;.$$

The Gaussian curvature K is the ratio between the determinant of first and second fundamental form, and is given by

$$K \;=\; \frac{b}{g} \;=\; -\frac{4}{(1+r^2)^4} \;<\; 0 \;. \tag{4.9}$$

Twice the mean curvature, $2H$, is the complete contraction of b_{ij} with g^{ij} and given by

$$2H \;=\; b_{ij}g^{ij} \;=\; b_{rr}g^{rr} + b_{\phi\phi}g^{\phi\phi} \;=\; -\frac{2\cos(2\phi)}{(1+r^2)^2} + \frac{2r^2\cos(2\phi)}{r^2(1+r^2)^2} \;=\; 0 \;. \tag{4.10}$$

This proves that the Enneper surface is a minimal surface.

Tore Nordstrand gives the following implicit representation of Enneper's surface:

$$\left[\frac{y^2-x^2}{2z} + \frac{2}{9}z^2 + \frac{2}{3}\right]^3 - 6\left[\frac{y^2-x^2}{4z} - \frac{1}{4}\left(x^2+y^2+\frac{8}{9}z^2\right) + \frac{2}{9}\right]^2 \;=\; 0 \;.$$

4.2. Laplace's formula

Take a closed surface $S = \partial V$ (using again an outward pointing normal) with some internal pressure p_i inside a medium with (external) pressure p_e. Let there be a surface energy σ per unit area. What can we say about the shape of this surface? The energy is given by

$$E \;=\; \int_{\partial V} \mathrm{d}A\,\sigma + (p_\mathrm{e} - p_\mathrm{i})\int_V \mathrm{d}V \;,$$

where $\mathrm{d}A$ and $\mathrm{d}V$ are the area and volume form. In equilibrium, the first variation of this energy has to vanish. For a closed surface it is enough to restrict to normal variations, since tangential variations will only effectively reparameterize the surface. We therefore get from Eqns. (3.13) and (3.19)

$$0 \;\stackrel{!}{=}\; \delta^1 E \;=\; \int_{\partial V} \mathrm{d}A\Big\{ -2H\sigma\psi + (p_\mathrm{e} - p_\mathrm{i})\psi \Big\} \;.$$

From this follows the *formula of Laplace*

$$\boxed{\; p_\mathrm{i} - p_\mathrm{e} \;=\; -2H\sigma \;. \;} \tag{4.11}$$

In particular, if we have a *spherical bubble* with radius R, this reduces to

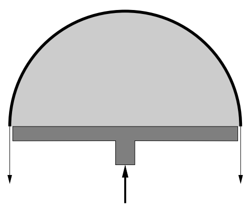
$$\Delta p \;:=\; p_\mathrm{i} - p_\mathrm{e} \;=\; \frac{2\sigma}{R} \;. \tag{4.12}$$

Thus, the pressure inside a spherical bubble is *larger* than outside, and this difference is more pronounced the smaller the bubble is.[4]

Remark: Whether a closed surface immersed in $\mathbb{R}^3$ with constant mean curvature must be a round sphere is known as the *Hopf conjecture*—and it is a quite nontrivial question! See for instance Ref. [6, 17]. The answer is "yes", if one has for instance either one of the following additional conditions: (*i*) the surface has genus 0 (proved by Hopf) or (*ii*) if the immersion is actually an embedding (proved by Alexandrov).[5] This essentially implies that under "ordinary physical circumstances" the Laplace law indeed forces bubbles to be spherical.

---

[4] A very simple derivation of this result works like this: Take a bubble and "cut" it in two halfs (see picture on the right). One now has to apply two additional forces in order to keep the system at equilibrium. First, everywhere along the rim one has to pull downward with the line tension σ. This gives the total force $F_\downarrow = 2\pi R \times \sigma$. Second, the excess interior pressure Δp of the bulk interior has to be supported by a piston, which pushes upwards with the force $F_\uparrow = \pi R^2 \times \Delta p$. In equilibrium these forces have to balance, and equating them gives Eqn. (4.12).



[5] Some further remarks on terminology seem appropriate [7, 15]: A C^∞ mapping F from a subset U of a manifold M to a manifold

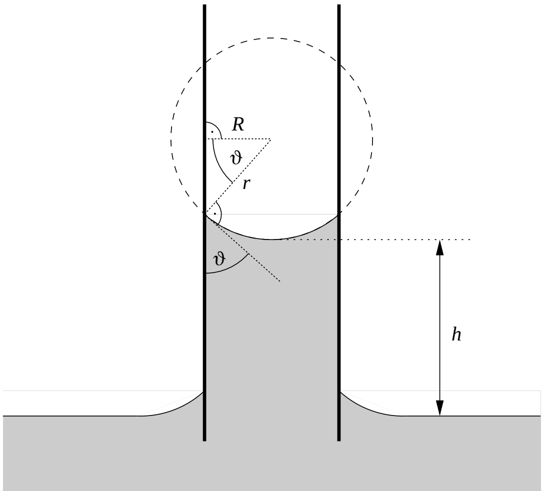


Figure 4.5.: Illustration for the calculation of the capillary rise.

With the formula of Laplace we can for instance explain the phenomenon of *capillary rise* [21]. If a liquid wets a substrate, it is well known that it then will rise in a capillary of small inner diameter. Why? Look at Fig. 4.5. Since the surface of the liquid makes a nonzero contact angle ϑ with the substrate, it can generally not be flat. If $\vartheta < \pi/2$, the situation is such as depicted, and the surface is curved downward in the capillary. However, Laplace's formula then tells us that the pressure below this curved surface has to be *smaller* than the pressure below the flat surface far away from the capillary. This imbalance is remedied by the drop in hydrostatic pressure which follows the rise of the liquid in the capillary. If we assume that the capillary is thin, the surface will be approximately spherical and from Fig. 4.5 we see that its curvature radius is then $r = R/\cos\vartheta$. If $\Delta\rho$ is the density difference between the liquid and the vapor above it, and g is the gravitational acceleration, we find

$$\Delta p \;=\; \frac{\Delta\rho\,\pi R^2\,h\,g}{\pi R^2} \qquad \Rightarrow \qquad h \;=\; \frac{\ell_{\rm c}^2\cos\vartheta}{R} \;\le\; \frac{\ell_{\rm c}^2}{R} \qquad\qquad \text{with} \quad \ell_{\rm c} \;:=\; \sqrt{\frac{2\sigma}{g\,\Delta\rho}}\;.$$

The rise of liquid is thus inversely proportional to the inner radius of the capillary. If one does not use a cylindrical capillary but two planes at a distance $2R$, the liquid only rises half as high, since the curved surface is now a cylinder and the mean curvature thus only half as big. The length $\ell_{\rm c}$ is called *capillary length* or *capillary constant.* For water at 0° C it has the value 3.9 mm, and it falls steadily to zero at the critical point.[6]

4.3. Stability analysis for the isoperimetric problem

A problem very close to the one we've discussed in Sec. 4.2 is the isoperimetric problem: Which shape encloses a given volume with the smallest possible surface? Essentially, we again have to look at the integral

$$S \;=\; \int_{\partial V} {\rm d}A \;\; + \lambda\left[\,\int_V {\rm d}V \;\; - V_0\right]\,,$$

where S is the surface area and the variable λ now enters as a Lagrange multiplier intended to fix the volume constraint. Setting the first variation to zero, $\delta^1 S = 0$, gives the result $2H = \lambda = \text{const.}$, showing that the surface

N is called an *immersion* if at every point $u \in M$ its differential ${\rm d}F(u)$ is injective, *i. e.* if it has rank $\dim(M)$. Note that neither F itself nor the map $u \mapsto {\rm d}F(u)$ are required to be injective. The map F is called a *submersion*, if at every point its differential is surjective, *i. e.* if it has rank $\dim(N)$. An immersion is called an *embedding*, if furthermore the following condition holds: For every open set $P \subset M$ there exists an open set $Q \subset N$ such that $F(U) \cap Q = F(P)$. This condition in particular forbids self-intersections. Indeed, the famous counter-example for a surface of constant mean curvature which is not a sphere, the "Wente torus" [24], has self-intersections.

[6] That the capillary length vanishes at the critical point is not obvious. Certainly, the surface tension must vanish when the notion of a surface ceases to be meaningful, but also the density difference between fluid and gas phase vanishes there. In fact, at the critical point the surface tension vanishes like $\sigma \sim (1-T/T_{\rm c})^\mu$, where the critical exponent μ has the value 3/2 in mean field theory, and the actual value $\mu \approx 1.26$ in three dimensions. Contrary to that, the density difference vanishes as $\Delta\rho \sim (1-T/T_{\rm c})^\beta$, with a mean field value $\beta = 1/2$ and an actual value of $\beta \approx 0.32$ in three dimensions. From the definition $\ell_{\rm c} := \sqrt{2\sigma/g\Delta\rho}$ we hence see $\ell_{\rm c} \sim (1-T/T_{\rm c})^{(\mu-\beta)/2}$, where the exponent is 1/2 in mean field theory and about 0.47 in three dimensions. Thus, $\ell_{\rm c}$ indeed vanishes at the critical point in a power law behavior. A detailed discussion can be found in Chapter 9 of the book by Rowlinson and Widom [21].

essentially has to be a sphere. From this follows $H = -1/R$ and $K = 1/R^2$, with $\frac{4}{3}\pi R^3 = V_0$. Using this, as well as the result for the Lagrange parameter, $\lambda = 2H$, the second variation of S is then found to be

$$\delta^2 S \;=\; \int_{\partial V} \mathrm{d}A \, \Big\{ \Big(K\psi^2 + \frac{1}{2}(\nabla_i\psi)(\nabla^i\psi) \Big) - \lambda H \psi^2 \Big\} \;\overset{\text{(A.12)}}{=}\; \int_{\partial V} \mathrm{d}A \, \Big\{ -\frac{\psi^2}{R^2} - \frac{1}{2}\psi\nabla^2\psi \Big\} \; . \tag{4.13}$$

We shall not (and need not) enforce the constraint of fixed volume exactly, but will instead contend ourselves with satisfying it up to quadratic order, which means that we will demand

$$0 \;\overset{!}{=}\; \delta^2 V \;=\; \int_{\partial V} \mathrm{d}A \, \Big\{ \psi + \frac{1}{R}\psi^2 \Big\} \; . \tag{4.14}$$

The essential question is now: Is there any way that the second variation of the area, under the constraint of fixed volume, is negative? So, in other words, we are searching for specific variations $\psi(u^i)$ which might lower the surface area even more. This is best done by expanding the general variation ψ in some convenient basis. Since Eqn. (4.13) shows that we would have to calculate the (covariant) Laplacian of the variation, it is most natural to expand in eigenfunctions of the Laplacian. On the sphere those are the spherical harmonics. Using the usual spherical coordinates, we can therefore write the expansion as

$$\psi(\vartheta, \varphi) \;=\; \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{lm} Y_{lm}(\vartheta, \varphi) \; , \tag{4.15}$$

where $\psi_{l,-m} = (-1)^m \psi^*_{lm}$, in order to make the result real.[7] For a summary of important properties of the spherical harmonics Y_{lm} (a few of which we are going to use) see for instance [1].

Let us first turn to the volume constraint (4.14). Inserting (4.15) yields

$$0 \;=\; \int_{\partial V} \mathrm{d}A \, \Big\{ \sum_{lm} \psi_{lm} Y_{lm} + \frac{1}{R} \sum_{lm,l'm'} \psi_{lm}\psi^*_{l'm'} Y_{lm} Y^*_{l'm'} \Big\} \;=\; \sqrt{4\pi}\, R^2 \, \psi_{00} + R \sum_{lm} \big|\psi_{lm}\big|^2 \; .$$

From this we get a quadratic equation for ψ_{00}:

$$\psi_{00} \;=\; -\frac{1}{\sqrt{4\pi}\, R} \sum_{lm} \big|\psi_{lm}\big|^2 \;=\; -\frac{1}{\sqrt{4\pi}\, R} \Big[\psi_{00}^2 + {\sum_{lm}}' \big|\psi_{lm}\big|^2 \Big] \; ,$$

where the prime on the second sum indicates that the term with $l = 0$ is left out. Note also that $|\psi_{00}|^2 = \psi_{00}^2$, since $Y_{00} = 1/\sqrt{4\pi}$ is real. Solving the quadratic equation and subsequently expanding for small values of the $|\psi_{lm}|^2$, we find

$$\psi_{00} \;=\; \sqrt{\pi}\, R \left\{ -1 + \sqrt{1 - \frac{1}{\pi R^2} {\sum_{lm}}' \big|\psi_{lm}\big|^2} \right\} \;=\; -\frac{1}{2\sqrt{\pi}\, R} {\sum_{lm}}' \big|\psi_{lm}\big|^2 \;+\; \mathscr{O}(3) \; .$$

This result shows that the constraint of volume conservation (up to quadratic order) requires the expansion coefficient ψ_{00} to be a *second order* function of all the other expansion coefficients.

Let us now insert the expansion (4.15) into the second variation (4.13). Using the fact that the Y_{lm} are eigenfunctions of the Laplacian with eigenvalues $-l(l+1)/R^2$, we find

$$\begin{aligned}
\delta^2 S \;&=\; \frac{1}{R^2} \int_{\partial V} \mathrm{d}A \, \Big\{ -\sum_{lm,l'm'} \psi_{lm}\psi^*_{l'm'} Y_{lm} Y^*_{l'm'} + \frac{1}{2} \sum_{lm,l'm'} l(l+1) \psi_{lm}\psi^*_{l'm'} Y_{lm} Y^*_{l'm'} \Big\} \\
&=\; \sum_{lm} \big|\psi_{lm}\big|^2 \Big[-1 + \frac{l(l+1)}{2} \Big] \;=\; -\psi_{00}^2 \;+\; {\sum_{lm}}' \big|\psi_{lm}\big|^2 \Big[-1 + \frac{l(l+1)}{2} \Big] \\
&=\; \underline{{\sum_{lm}}' \big|\psi_{lm}\big|^2 \Big[-1 + \frac{l(l+1)}{2} \Big] \;+\; \mathscr{O}(3)} \; .
\end{aligned} \tag{4.16}$$

In the last step we used the fact that ψ_{00} is already of quadratic order in the other variations. Note that this essentially comes down to taking out the mode $l = 0$, which changes volume to first order! And indeed, the condition on ψ_{00} was precisely derived from the desire to keep the volume constant.

[7] Since $Y_{l,-m} = (-1)^m Y^*_{lm}$, we can then have $\psi_{l,-m} Y_{l,-m} = \psi^*_{lm} Y^*_{lm}$. Hence, if we have a sum over all m, we can replace m by $-m$ and thereby change to the complex conjugate expression. This is clever if done in one term of a *double sums*, because it generates $\psi_{lm}\psi^*_{l'm'}$ as well as $Y_{lm} Y^*_{l'm'}$, and the latter is a product of two functions which satisfy an orthogonality relation when being integrated.

Equation (4.16) says the following: The second variation vanishes for the three modes with $l = 1$. This is due to the fact that these modes simply describe a *translation* of the whole surface, which of course does not change its area, and which we can therefore forget about. However, all the other modes contribute *positively* to the second variation. Hence, the spherical shape is in fact *stable*, in the sense that there exists no possible variation of the surface which reduces its area any further.

4.4. The Plateau-Rayleigh-instability

Let us repeat the above calculation in a "slightly" different geometry: Let us not look at spheres but at *cylinders*. Of course, we know that a cylinder does *not* minimize the area for a given volume being enclosed, but *if we forget about the inevitable problems at the two caps*, a cylinder is a surface with constant mean curvature! Hence, it makes the area minimization problem as discussed above stationary. The question now is, whether this stationary state is an equilibrium state. Since we already know that the cylinder is not the optimal answer, we might indeed expect an instability, and indeed this is what we will find.

The difference in the cylindrical case, as compared to the case above, is manifest only in a few points: First, the area element is $\mathrm{d}A = R\mathrm{d}\varphi\mathrm{d}z$. Second, the Gaussian curvature is $K = 0$. From this we see that the second order variation $\delta^2 S$ is given by

$$\delta^2 S \;=\; \int_0^L \int_0^{2\pi} R\,\mathrm{d}\varphi\,\mathrm{d}z\,\Big\{-\frac{1}{2}\frac{\psi^2}{R^2} - \frac{1}{2}\psi\nabla^2\psi\Big\}\;, \tag{4.17}$$

where we assumed that the cylinder has a length L. Third, the surface Laplacian, using cylindrical coordinates (R, ϕ, z), is $\nabla^2 = R^{-2}\partial^2/\partial\varphi^2 + \partial^2/\partial z^2$. A set of orthogonal eigenfunctions is therefore given by

$$Z_{nm}(\varphi, z) \;:=\; \mathrm{e}^{\mathrm{i}m\varphi}\,\mathrm{e}^{2\pi\mathrm{i}nz/L} \qquad n, m \in \mathbb{N}_0 \qquad , \qquad \int_0^L \int_0^{2\pi} R\,\mathrm{d}\varphi\,\mathrm{d}z\; Z_{nm}Z^*_{n'm'} \;=\; 2\pi RL\,\delta_{nn'}\delta_{mm'}\;, \tag{4.18}$$

and the eigenvalues are

$$\nabla^2 Z_{nm} \;=\; -\frac{1}{R^2}\Big[m^2 + \Big(\frac{2\pi R}{L}\Big)^2 n^2\Big]\,Z_{nm}\;.$$

We will expand the normal variation in these eigenfunctions according to

$$\psi(\varphi, z) \;=\; \sum_{n,m} \psi_{nm} Z_{nm}(\varphi, z)$$

with $\psi_{-n,-m} = \psi^*_{nm}$. Inserting these expansions into the constraint of volume conservation, and noting that this time the mean curvature is not $-1/R$ but $-1/2R$, we find

$$\begin{aligned} 0 \;\stackrel{!}{=}\; \delta^2 V \;&=\; \int_0^L \int_0^{2\pi} R\,\mathrm{d}\varphi\,\mathrm{d}z\,\Big\{\sum_{nm}\psi_{nm}Z_{nm} + \frac{1}{2R}\sum_{nm,n'm'}\psi_{nm}\psi^*_{n'm'}Z_{nm}Z^*_{n'm'}\Big\} \\ &=\; 2\pi RL\,\psi_{00} + \pi L\Big[\psi_{00}^2 + {\sum_{nm}}' |\psi_{nm}|^2\Big]\;, \end{aligned}$$

where the prime in the sum again indicates that the mode $n = m = 0$ is left out. And from this we can again obtain the amplitude ψ_{00} of this mode which changes volume in first order:

$$\psi_{00} \;=\; R\left\{-1 + \sqrt{1 - \frac{1}{R^2}{\sum_{nm}}' |\psi_{nm}|^2}\right\} \;=\; -\frac{1}{2R}{\sum_{nm}}' |\psi_{nm}|^2 \;+\; \mathscr{O}(3)\;.$$

Just as in the spherical case, the condition of volume conservation forces the zero mode ψ_{00} to be of quadratic order in the other modes. For the second variation $\delta^2 S$ we therefore find

$$\begin{aligned} \delta^2 S \;&=\; \frac{1}{2R^2}\int_{\partial V}\mathrm{d}A\,\Big\{-\sum_{nm,n'm'}\psi_{nm}\psi^*_{n'm'}Z_{nm}Z^*_{n'm'} + \sum_{nm,n'm'}\psi_{nm}\psi^*_{n'm'}Z_{nm}Z_{n'm'}\Big[m^2 + \Big(\frac{2\pi R}{L}\Big)^2 n^2\Big]\Big\} \\ &=\; \frac{\pi L}{R}\sum_{nm}|\psi_{nm}|^2\Big[-1 + m^2 + \Big(\frac{2\pi R}{L}\Big)^2 n^2\Big] \;=\; -\frac{\pi L}{R}\psi_{00}^2 \;+\; \frac{\pi L}{R}{\sum_{nm}}'|\psi_{nm}|^2\Big[-1 + m^2 + \Big(\frac{2\pi R}{L}\Big)^2 n^2\Big] \\ &=\; \underline{\frac{\pi L}{R}{\sum_{nm}}'|\psi_{nm}|^2\Big[-1 + m^2 + \Big(\frac{2\pi R}{L}\Big)^2 n^2\Big] \;+\; \mathscr{O}(3)\;.} \end{aligned} \tag{4.19}$$

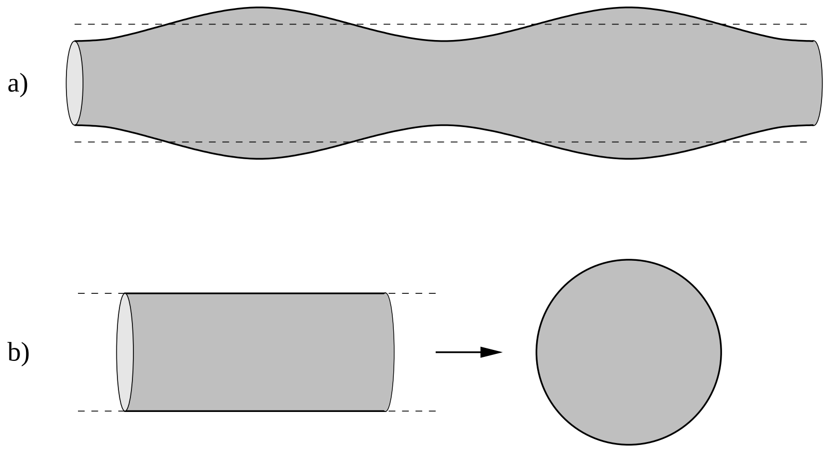


Figure 4.6.: a) Illustration of a critical wavelength fluctuation $\lambda_c = 2\pi R$ in the Plateau-Rayleigh problem. b) Simple approximative argument for the critical wavelength based on equating volume and area of a piece of the cylinder with its resulting sphere (see Eqn. (4.20)).

Contrary to the case of the second variation of the spherical surface, Eqn. (4.16), the above expression is *not* always positive. It can become negative for $m = 0$ and sufficiently large L. Indeed, the earliest mode which "blows up" is $n = 1$, from which we see that the cylindrical surface becomes unstable if it gets longer than its circumference. Alternatively, we may think of the expression $q := 2\pi n/L$ as a *wave vector*, and we then see that a long cylinder is unstable with respect to undulation modes with a wave vector $q < 1/R$ or wave length $\lambda = 2\pi/q > 2\pi R$.

This of course has the following physical significance: Think of a "cylinder of water", as it comes for instance out of a tap, or – a little more controlled – as one might pull it under zero gravity out of a droplet of water. The water volume is conserved, and the surface tension tries to minimize the area. The above calculation shows that if the cylinder becomes too long, it will be unstable against long wavelength fluctuations. Plateau was the first to study this problem experimentally, and he found our result that *a cylinder of fluid subject to surface tension is stable against fluctuations which break the symmetry, but unstable against symmetry conserving undulations which have a wave length exceeding the circumference of the cylinder.* Rayleigh was the first to give a theoretical explanation of this phenomenon (see for instance Ref. [20]), which is now referred to as the *Plateau-Rayleigh-instability.* An illustration of the critical fluctuation mode is shown in Fig. 4.6.

Observe that the reason for the system to "search" for lower areas is the surface tension of the liquid. However, it's precise value is immaterial, since all what is needed is *some* driving force toward a state of lower area. Hence, the Plateau-Rayleigh instability happens at arbitrarily small values of the surface tension.

Wave vectors $q < 1/R$ are unstable, but in order to understand which wave vector grows *fastest* if the instability sets in, one has to do a dynamical calculation, which is also due to Rayleigh (see again Ref. [20]). For an incompressible non-viscous fluid the calculation is comparatively simple. There exists a potential for the velocity field of the fluid which satisfies the Laplace equation. One searches for a solution which matches the boundary of the specific mode one is interested in. Integrating over the volume of the fluid one finds the kinetic energy. Together with the potential energy calculated above one has the Lagrange function and thus the equation of motion for every mode. One then sees that for $m = 0$ and $qR < 1$ the perturbation is exponentially growing, with a growth constant c given by [20]

$$c = \frac{\sigma}{\rho R^3}\frac{qR\,\mathrm{I}_1(qR)}{\mathrm{I}_0(qR)}\big(1-(qR)^2\big) .$$

where ρ is the density of the liquid and I_0 and I_1 are modified Bessel functions. The wave vector which has the largest c, *i. e.*, the fastest growing mode, is given by $qR \simeq 0.697019$.

We finally want to show that there is a simple argument leading to the same physics, but with somewhat wrong prefactors: Assume we chop a cylinder of water having radius R into smaller cylinders. What does the length L of these cylinders have to be such that if they collapse into spheres of radius R', their initial outer surface is equal to the surface of the sphere? In order to answer that one simply has to solve the two equations which fix volume and area:

$$\left.\begin{array}{ll}\text{area:} & 2\pi RL = 4\pi R'^2 \\[2ex] \text{volume:} & \pi R^2 L = \frac{4}{3}\pi R'^3\end{array}\right\} \quad\Rightarrow\quad \left\{\begin{array}{l} L = \frac{9}{2}R \\[2ex] R' = \frac{3}{2}R\end{array}\right. . \tag{4.20}$$

One can now convince oneself that for cylinders which are longer than this value, the chopping up will result in an area reduction of the total surface. The estimate $L_{\text{crit}} = 4.5R$ is about 40% smaller than the exact answer $L_{\text{crit}} = 2\pi R$ derived above.

5. Vesicles

Vesicles are objects consisting essentially of a closed membrane. Soap bubbles are a good example. In biology the membrane is invariably a *phospholipid bilayer.* Much ought to be said about the interesting physics of such systems, but here we will restrict to the set of problems which arises when the membrane is described as a two-dimensional fluid[1] elastic. This means, there exists (*i*) a surface tension and (*ii*) a *bending* energy involved with "out-of-the-plane" motions of the membrane. Following Helfrich [11], the (free) energy per unit area of membrane is then given by

$$e \;=\; \sigma + 2\,k_{\rm c}\big(H-c_0\big)^2 + \bar{k}_{\rm c}K\;. \tag{5.1}$$

The first term is the tension. The second and third term describe energy contributions due to the curvature of the membrane. The parameter c_0 is called the *spontaneous curvature*, for which we use a slightly different convention than usual. If c_0 is zero, the bending energy is found to be proportional, through the two elastic moduli $k_{\rm c}$ and $\bar{k}_{\rm c}$, to the square of the mean curvature and the Gaussian curvature. Those are two convenient quadratic invariants of the curvature tensor. And since there are no more than two quadratic invariants, the description is complete. For nonzero spontaneous curvature the first term gets somewhat "renormalized", in the sense of "what is the mean curvature at which the bending energy vanishes"? The expansion of the elastic strain is performed about a pre-curved ground state.

5.1. Shape equation

Given a closed surface ∂V surrounded by an elastic fluid membrane, its elastic energy due to the Helfrich Hamiltonian (5.1) is given by

$$\mathscr{H} \;=\; \int_{\partial V}{\rm d}A\,\Big[2k_{\rm c}\big(H-c_0\big)^2 + \bar{k}_{\rm c}K + \sigma\Big] - P\int_V {\rm d}V\;,$$

The variables σ and P are surface tension and (excess interior) pressure. Alternatively, they can be viewed as Lagrange multipliers which may be used to fix a constraint of constant area or volume, respectively. This essentially depends on the ensemble one wishes to study. We now want to look at the variation of this energy. First it is helpful to realize that due to the Gauss-Bonnet theorem [9, 14] variations which leave the boundary and the topology unchanged will not change the integral over K, since this is a topological invariant. This term does therefore not contribute to variation of $\mathscr{H}$. Furthermore, for the sake of finding the shape equations, it is sufficient to restrict to normal variations. The first variation of the term $(H-c_0)^2$ is given by

$$\delta^{(1)}(H-c_0)^2 \;=\; 2(H-c_0)\delta^{(1)}H \;\stackrel{(3.27)}{=}\; 2(H-c_0)\big[(2H^2-K)\psi + \frac{1}{2}\nabla^2\psi\big]\;,$$

and using our expression for the variation of the square root of the metric determinant (3.13) we then find

$$\begin{aligned}
\delta^{(1)}\big(\sqrt{g}\,(H-c_0)^2\big) \;&=\; \big(\delta^{(1)}\sqrt{g}\big)(H-c_0)^2 + \sqrt{g}\big(\delta^{(1)}(H-c_0)^2\big) \\
&=\; \sqrt{g}\,\Big(-2H\psi(H-c_0)^2 + 2(H-c_0)\big[(2H^2-K)\psi + \frac{1}{2}\nabla^2\psi\big]\Big) \\
&=\; \underline{\sqrt{g}\,\Big(2(H-c_0)\big[\psi\big(H^2-K+c_0H\big) + \frac{1}{2}\nabla^2\psi\big]\Big)}\;.
\end{aligned} \tag{5.2}$$

From this, and additionally Eqn. (3.19), we can readily work out the first variation of the Helfrich Hamiltonian:

$$\delta^{(1)}\mathscr{H} \;=\; \int_{\partial V}{\rm d}A\Big\{2k_{\rm c}\big[2(H-c_0)(H^2-K+c_0H) + \nabla^2 H\big] - 2H\sigma - P\Big\}\,\psi\;,$$

[1]By "fluid" we mean that the in-plane shear modulus is zero. The opposite would be "tethered". Since in tethered membranes local distances are fixed, the metric is fixed, and hence, by Gauss' theorema egregium, the Gaussian curvature is fixed. A piece of paper would be an example of a tethered membrane. In fluid membranes the Gaussian curvature can vary, and it hence makes sense to add an elastic contribution proportional to it.

where we partially integrated the term $\nabla^2\psi$ twice by making use of Eqn. (A.12). Setting this variation to zero gives the *shape equation for vesicles* [25][2]

$$\boxed{P \;=\; 2k_{\rm c}\Big[\nabla^2 H + 2\big(H-c_0\big)\big(H^2-K+c_0 H\big)\Big] \;-\; 2H\,\sigma\;. \qquad (5.3)}$$

This nonlinear partial differential equation is *extremely complicated to solve*, and we will not make any attempt at it. However, note that for $\kappa = 0$ Eqn. (5.3) reduces to the formula of Laplace, Eqn. (4.11). The beauty of Eqn. (5.3) is that it is entirely coordinate free, since mean and Gaussian curvature are properties of a surface which can be defined irrespective of coordinates. Much research has been done with essentially the aim to understand the implication of this equation on the stationary shapes of vesicles. For a review see for instance the paper by Seifert [23].

5.2. Stability of free cylindrical vesicles

In Sec. 4.4 we studied the surface tension driven instability of a cylinder of liquid. In this section we want to study the same problem for liquid cylinders additionally coated by a membrane, *i. e.*, a cylindrical vesicle.

It is easy to see that the shape equation (5.3) permits a solution which describes a cylinder of radius R. In this case, $H = -\frac{1}{2R} = \text{const}$ and $K = 0$, and the shape equation reduces to

$$P \;=\; -\frac{k_{\rm c}}{2R^3}\Big[1-\big(2c_0R\big)^2\Big] + \frac{\sigma}{R}\;. \qquad (5.4)$$

However, for the case of *free* cylindrical vesicles it is crucial to realize that there is a second variable in the problem which also has to be equilibrated, namely the *length* L of the vesicle. Neglecting end-effects, the energy of the cylindrical vesicle is given by

$$\mathscr{H} \;=\; \Big[2\,k_{\rm c}\Big(-\frac{1}{2R}-c_0\Big)^2 + \sigma\Big]\,2\pi RL - P\,\pi R^2 L\;.$$

Requiring $\partial E/\partial L = 0$ implies the additional equilibrium condition

$$P \;=\; \frac{k_{\rm c}}{R^3}\big(1+2c_0R\big)^2 + \frac{2\sigma}{R}\;. \qquad (5.5)$$

Equations (5.4) and (5.5) determine the required value of pressure *and* surface tension

$$P \;=\; -\frac{2k_{\rm c}}{R^3}\big(1+2c_0R\big)\;, \qquad (5.6a)$$

$$\sigma \;=\; -\frac{k_{\rm c}}{2R^2}\big(1+2c_0R\big)\big(3+2c_0R\big)\;. \qquad (5.6b)$$

Both P and σ are uniquely determined by the radius. This is in contrast to the case of a spherical vesicle, where one of the two variables can be chosen freely.

Even though we've found that cylindrical vesicles can solve the shape equation (neglecting problems at the ends, of course), this does not answer the question whether this solution is stable against small perturbations. In order to answer this question, we will now perform a *linear stability analysis* for the cylinder.

We first require the second variation of $(H-c_0)^2$, which in the cylindrical case is given by

$$\begin{aligned}\delta^{(2)}(H-c_0)^2 &= 2(H-c_0)\delta^{(2)}H + \big(\delta^{(1)}H\big)^2 \\ &= (1+2c_0R)\Big[\frac{1}{2R^4}\psi^2 - \frac{1}{R}b^{ij}\psi\nabla_i\nabla_j\psi - \frac{1}{2R}b^{ij}\nabla_i\psi\nabla_j\psi - \frac{1}{4R^2}\nabla_i\psi\nabla^i\psi\Big] \\ &\quad + \frac{1}{4R^4}\psi^2 + \frac{1}{2R^2}\psi\nabla^2\psi + \frac{1}{4}\big(\nabla^2\psi\big)^2\;.\end{aligned}$$

[2]When comparing Eqn. (5.3) with the expression in the paper by Helfrich and Zhong-can [25, Eqn. (31)], one has to bear in mind that these authors use a different convention for the spontaneous curvature and for the mean curvature.

From this we then obtain the second variation of $\sqrt{g}\,(H-c_0)^2$ as

$$\begin{aligned}\delta^{(2)}\big(\sqrt{g}\,(H-c_0)^2\big) &= \big(\delta^{(2)}\sqrt{g}\big)\big((H-c_0)^2\big) + \big(\delta^{(1)}\sqrt{g}\big)\big(\delta^{(1)}(H-c_0)^2\big) + \sqrt{g}\big(\delta^{(2)}(H-c_0)^2\big) \\ &= \sqrt{g}\,\bigg\{\frac{1}{8R^2}(1+2c_0R)^2\nabla_i\psi\nabla^i\psi + \frac{1}{4R^4}\psi^2 + \frac{1}{2R^2}\psi\nabla^2\psi + \frac{1}{4}\big(\nabla^2\psi\big)^2 \\ &\quad -(1+2c_0R)\Big(\frac{1}{2R^2}\psi\nabla^2\psi + \frac{1}{R}b^{ij}\psi\nabla_i\nabla_j\psi + \frac{1}{2R}b^{ij}\nabla_i\psi\nabla_j\psi + \frac{1}{4R^2}\nabla_i\psi\nabla^i\psi\Big)\bigg\}\ .\end{aligned}$$

From this we can finally work out the second variation of the energy of the cylindrical vesicle:

$$\begin{aligned}\delta^{(2)}\mathscr{H} &= \int_{\partial V}\mathrm{d}A\,\bigg\{k_\mathrm{c}\bigg[\frac{1}{4R^2}(1+2c_0R)^2\underline{\nabla_i\psi\nabla^i\psi} + \frac{1}{2R^4}\psi^2 + \frac{1}{R^2}\psi\nabla^2\psi + \frac{1}{2}\big(\nabla^2\psi\big)^2 \\ &\quad -(1+2c_0R)\Big(\frac{1}{R^2}\psi\nabla^2\psi + \frac{2}{R}b^{ij}\underline{\psi\nabla_i\nabla_j\psi} + \frac{1}{R}b^{ij}\nabla_i\psi\nabla_j\psi + \frac{1}{2R^2}\underline{\nabla_i\psi\nabla^i\psi}\Big)\bigg] \\ &\quad \overset{(5.6b)}{+}\Big(-\frac{k_\mathrm{c}}{2R^2}(1+2c_0R)(3+2c_0R)\Big)\Big(\frac{1}{2}\underline{\nabla_i\psi\nabla^i\psi}\Big) \overset{(5.6a)}{-}\Big(-\frac{2k_\mathrm{c}}{R^3}(1+2c_0R)\Big)\Big(\frac{1}{2R}\psi^2\Big)\bigg\}\ .\end{aligned}$$

In this expression the four underlined terms are now integrated by parts using Eqn. (A.12). Also, in cylindrical coordinates the only nonzero element of b^{ij} is $b^{\varphi\varphi}=-1/R^3$. Using this, one finally ends up at

$$\delta^{(2)}\mathscr{H} = \frac{k_\mathrm{c}}{R^4}\int_{\partial V}\mathrm{d}A\,\Big[\frac{3}{2}\psi^2 + \psi R^2\nabla^2\psi + \frac{1}{2}\big(R^2\nabla^2\psi\big)^2 - (1+2c_0R)\big(\partial_\varphi\psi\big)^2 + 2c_0R\psi^2\Big]\ .$$

The rest runs completely analogous to the Plateau-Rayleigh problem from Sec. 4.4. We expand ψ in terms of the eigenfunctions Z_{nm} of the cylindrical Laplacian, Eqn. (4.18), and exploit their orthogonality. We then immediately find the following mode-expansion of the second variation of the energy [25]:

$$\boxed{\delta^{(2)}\mathscr{H} = \frac{\pi k_\mathrm{c}L}{R^3}\sum_{nm}\big|\psi_{nm}\big|^2\bigg\{3-4m^2-2\Big(\frac{2\pi R}{L}\Big)^2n^2 + \Big[m^2+\Big(\frac{2\pi R}{L}\Big)^2n^2\Big]^2 + 4c_0R(1-m^2)\bigg\}\ .} \tag{5.7}$$

Let us introduce the abbreviation $A=2\pi R/L$. We first study the stability of modes which preserve the cylindrical symmetry, *i. e.*, the case $m=0$. Such a mode becomes unstable if

$$0 \ \geq\ 3-2A^2n^2+(A^2n^2)^2+4c_0R \ =\ (A^2n^2-1)^2+2(1+2c_0R)\ .$$

A necessary condition for this equation to have a real solution is

$$c_0 \ \leq\ -\frac{1}{2R}\ . \tag{5.8}$$

Looking back at the definition of the Hamiltonian and the sign convention of the curvature, we find that a cylindrical vesicle can only become unstable against undulations along its axis if the spontaneous curvature *favors* an *inward* bending. If Inequality (5.8) holds, we obtain

$$A^2n^2 \ \leq\ 1+\sqrt{-2(1+2c_0R)}\ .$$

If the cylinder is essentially infinitely long, the least stable mode is characterized by $2c_0R=-1$ and $An=1$. Hence, the wavelength of this undulation is

$$\lambda_\mathrm{crit} \ =\ \frac{L}{n} \ =\ \underline{2\pi R}\ ,$$

which happens to be exactly the same wavelength as the one for the Plateau-Rayleigh problem.

For $m=1$ the second variation of the energy is

$$\delta^{(2)}\mathscr{H} \overset{m=1}{=} \frac{\pi k_\mathrm{c}L}{R^3}\sum_n\big|\psi_{n1}\big|^2A^4n^4\ .$$

For $n = 0$ this corresponds to a small translation of the cylinder perpendicular to its axis, which of course costs no energy, while for larger n this deformation describes bending modes of the cylinder, and they require energy. In fact, one can see that for $m \geq 1$ no unstable modes exist.

A "pearling" instability of *pinned* cylindrical vesicles has recently been observed experimentally by Bar-Ziv and Moses [2]. These authors also give an explanation of the phenomenon similar to the analysis performed above (the energy of undulations is worked out numerically). A very different theoretical explanation is proposed in Refs. [10, 19]. It is argued that one needs *dynamical* considerations in order to understand the phenomenon, statics would not suffice. I have not yet followed their arguments well enough to comment on this.

A. Christoffel symbols

A.1. Definition and transformation law

The *Christoffel symbols of the first kind* are defined by

$$\Gamma_{ijk} = \frac{1}{2}\left[g_{kj,i} + g_{ik,j} - g_{ij,k}\right] . \tag{A.1}$$

Note that Γ_{ijk} is symmetric in the first two indices. The *Christoffel symbols of the second kind* are obtained by formally raising the last index:

$$\Gamma^l_{ij} = \Gamma_{ijk} g^{kl} . \tag{A.2}$$

Note that Γ^l_{ij} is symmetric in the two lower indices.

In Sec. 1.2 we have seen that these symbols occur in the Gauss equation (1.18), which describes the change of the local tangent coordinate vectors $\boldsymbol{e}_i$ upon movements on the surface. Here we will collect a few properties of these symbols.

One of the most important things to appreciate is that *the Christoffel symbols are no tensors!* To see this, let us derive their transformation behavior (see [14]). From the equation of Gauss we have in local coordinates u^1, u^2:

$$\boldsymbol{e}_{i,j} \;=\; \frac{\partial \vec{r}}{\partial u^i \partial u^j} \;=\; \Gamma^k_{ij}\boldsymbol{e}_k + b_{ij}\vec{n} . \tag{A.3}$$

Or, in a different coordinate system $\bar{u}^p, \bar{u}^q$:

$$\boldsymbol{e}_{\bar{p},\bar{q}} \;=\; \frac{\partial \vec{r}}{\partial \bar{u}^p \partial \bar{u}^q} \;=\; \Gamma^{\bar{r}}_{\bar{p}\bar{q}}\boldsymbol{e}_{\bar{r}} + b_{\bar{p}\bar{q}}\vec{n} . \tag{A.4}$$

On the other hand, since

$$\boldsymbol{e}_i \;=\; \boldsymbol{e}_{\bar{p}}\frac{\partial \bar{u}^p}{\partial u^i} ,$$

we get by differentiation with respect to j

$$\Gamma^k_{ij}\boldsymbol{e}_k + b_{ij}\vec{n} \overset{\text{(A.3)}}{=} \boldsymbol{e}_{i,j} \;=\; \boldsymbol{e}_{\bar{p}\bar{q}}\frac{\partial \bar{u}^p}{\partial u^i}\frac{\partial \bar{u}^q}{\partial u^j} + \boldsymbol{e}_{\bar{r}}\frac{\partial^2 \bar{u}^r}{\partial u^i \partial u^j} \overset{\text{(A.4)}}{=} \left[\Gamma^{\bar{r}}_{\bar{p}\bar{q}}\boldsymbol{e}_{\bar{r}} + b_{\bar{p}\bar{q}}\vec{n}\right]\frac{\partial \bar{u}^p}{\partial u^i}\frac{\partial \bar{u}^q}{\partial u^j} + \boldsymbol{e}_{\bar{r}}\frac{\partial^2 \bar{u}^r}{\partial u^i \partial u^j} .$$

Since the vectors $\boldsymbol{e}_i$ and $\vec{n}$ are linearly independent, the two terms on the left and the right side of this equation have to be equal *individually*. Comparing first the prefactors of $\vec{n}$, we find

$$b_{ij} \;=\; b_{\bar{p}\bar{q}}\frac{\partial \bar{u}^p}{\partial u^i}\frac{\partial \bar{u}^q}{\partial u^j} ,$$

proving that the second fundamental form b_{ij} is a (twofold covariant) tensor. Comparing the rest, we see

$$\Gamma^k_{ij}\boldsymbol{e}_k \;=\; \Gamma^{\bar{r}}_{\bar{p}\bar{q}}\boldsymbol{e}_{\bar{r}}\frac{\partial \bar{u}^p}{\partial u^i}\frac{\partial \bar{u}^q}{\partial u^j} + \boldsymbol{e}_{\bar{r}}\frac{\partial^2 \bar{u}^r}{\partial u^i \partial u^j} .$$

Inserting the inverse transform

$$\boldsymbol{e}_{\bar{r}} \;=\; \boldsymbol{e}_k\frac{\partial u^k}{\partial \bar{u}^r} ,$$

and comparing the prefactors of the two linearly independent vectors $\boldsymbol{e}_k$, we finally arrive at

$$\underline{\Gamma^k_{ij} \;=\; \Gamma^{\bar{r}}_{\bar{p}\bar{q}}\frac{\partial \bar{u}^p}{\partial u^i}\frac{\partial \bar{u}^q}{\partial u^j}\frac{\partial u^k}{\partial \bar{u}^r} + \frac{\partial^2 \bar{u}^r}{\partial u^i \partial u^j}\frac{\partial u^k}{\partial \bar{u}^r}} . \tag{A.5}$$

Were it not for the second inhomogeneous term, the Christoffel symbols would transform as tensors. Indeed, if one restricts to *affine linear* transformations, the second derivative term vanishes and the Christoffel symbols transform as tensors with respect to this restricted set of transformations.

A.2. Some identities

Just as the Christoffel symbols are given by combinations of the derivative of the metric tensor, these derivatives can be written in terms of the Christoffel symbols:

$$
\begin{aligned}
g_{ij,k} &= \Gamma_{ikj} + \Gamma_{jki} = g_{lj}\Gamma^l_{ik} + g_{li}\Gamma^l_{jk} , && \text{(A.6a)}\\
g^{ij}_{,k} &= -g^{lj}\Gamma^i_{lk} - g^{li}\Gamma^j_{lk} , && \text{(A.6b)}
\end{aligned}
$$

where the second equation (A.6b) follows from differentiating the identity $g_{ij}g^{jk} = \delta^k_i$ and inserting the first equation (A.6a).

From the expression of the derivative of the metric determinant in Eqn. (3.9) we also find

$$
\frac{\partial g}{\partial u^k} \overset{(3.9)}{=} g g^{ij} g_{ij,k} \overset{(A.6a)}{=} g g^{ij}\big(\Gamma_{ikj} + \Gamma_{jki}\big) = g\big(\Gamma^i_{ik} + \Gamma^j_{jk}\big) = 2g\Gamma^i_{ik} .
$$

From this we obtain

$$
\Gamma^i_{ik} = \frac{1}{2g}\frac{\partial g}{\partial u^k} = \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial u^k} = \frac{\partial}{\partial u^k}\log\sqrt{g} . \tag{A.7}
$$

The above formula can be extended in the following way:

$$
\frac{(X^k\sqrt{g})_{,k}}{\sqrt{g}} = \frac{X^k_{,k}\sqrt{g} + X^k\sqrt{g}_{,k}}{\sqrt{g}} \overset{(A.7)}{=} X^k_{,k} + X^k\Gamma^i_{ik} = X^k_{,k} + X^i\Gamma^k_{ki} \overset{(C.6)}{=} \nabla_k X^k , \tag{A.8}
$$

where in the last step we used the concept of covariant differentiation, to be introduced in Sec. C.3.

This equation can be turned into a useful integration formula (in the following we'll assume that *the manifold has no boundary*!):

$$
\int_M \mathrm{d}A\, X\nabla_i Y^i = \int_M \mathrm{d}^n x\, \sqrt{g} X\nabla_i Y^i \overset{(A.8)}{=} \int_M \mathrm{d}^n x\, X\big(\sqrt{g}Y^i\big)_{,i} \overset{\text{part. int.}}{=} -\int_M \mathrm{d}^n x\, X_{,i}\sqrt{g}Y^i = -\int_M \mathrm{d}A\, Y^i\nabla_i X . \tag{A.9}
$$

If in particular we choose X as the unit tensor, we see that the surface integral of $\nabla_i Y^i$ over a manifold without boundary vanishes:

$$
\int_M \mathrm{d}A\, \nabla_i Y^i = 0 \qquad (\text{for } \partial M = 0) .
$$

We can further extend the expression (A.8) to a twofold contravariant tensor, but then the expression becomes more messy:

$$
\frac{(X^{ij}\sqrt{g})_{,i}}{\sqrt{g}} = X^{ij}_{,i} + X^{ij}\Gamma^k_{ki} = \big(X^{ij}_{,i} + X^{kj}\Gamma^i_{ki} + X^{ik}\Gamma^j_{ki}\big) - X^{ik}\Gamma^j_{ki} \overset{(C.7)}{=} \nabla_i X^{ij} - X^{ik}\Gamma^j_{ik} . \tag{A.10}
$$

We can again use this to derive an integration formula over manifolds (and we again will assume for simplicity that the (n-dimensional) manifold M we are integrating over has no boundary, *i. e.*, $\partial M = 0$.):

$$
\begin{aligned}
\int_M \mathrm{d}A\, \Big[X^{ij}\nabla_i Y_j\Big] &= \int_M \mathrm{d}^n x\, \sqrt{g}\, \Big[X^{ij}\big(Y_{j,i} - \Gamma^k_{ji}Y_k\big)\Big] \overset{\text{part. int.}}{=} -\int_M \mathrm{d}^n x\, \sqrt{g}\, \Big[\frac{(X^{ij}\sqrt{g})_{,i}}{\sqrt{g}} Y_j + X^{ij}\Gamma^k_{ji}Y_k\Big] \\
&\overset{(A.10)}{=} -\int_M \mathrm{d}^n x\, \sqrt{g}\, \Big[Y_j\nabla_i X^{ij} - X^{ik}\Gamma^j_{ik}Y_j + X^{ij}\Gamma^k_{ij}Y_k\Big] = -\int_M \mathrm{d}A\, \Big[Y_j\nabla_i X^{ij}\Big] . \qquad \text{(A.11)}
\end{aligned}
$$

In particular, if we set $Y_i = \nabla_i\psi$ and $X^{ij} = g^{ij}\psi$, with ψ being some (scalar) function on M, we find

$$
\int_M \mathrm{d}A\, \big(\nabla_i\psi\big)\big(\nabla^i\psi\big) = -\int_M \mathrm{d}A\, \big(\psi\nabla^2\psi\big) \qquad (\text{for } \partial M = 0) , \tag{A.12}
$$

where $\nabla^2 = \nabla_i\nabla^i = g^{ij}\nabla_i\nabla_j$ is the covariant Laplace operator on the manifold.

A.3. Local tangent coordinates

Let u^k_0 be the coordinates of some point P_0 in the manifold. Define a new set of coordinates u^{*k} via the relation

$$
u^k = u^k_0 + u^{*k} - \frac{1}{2}\big(\Gamma^k_{ij}\big)_0 u^{*i}u^{*j} , \tag{A.13}
$$

where the Christoffel symbol is evaluated in the old coordinates, at point P_0. Noting that at P_0 we have $u_0^k = 0$, we find by differentiation that at P_0 we have

$$\left(\frac{\partial u^k}{\partial u^{*p}}\right)_0 \;=\; \delta_p^k \;, \tag{A.14a}$$

and

$$\left(\frac{\partial^2 u^k}{\partial u^{*p}\partial u^{*q}}\right)_0 \;=\; -(\Gamma^k_{pq})_0 \;. \tag{A.14b}$$

Eqn. (A.14a) confirms that (A.13) is indeed an allowable coordinate transformation. Inserting both equations into the expression of the Christoffel symbol in the new coordinates (simply invert Eqn. (A.5)), we find:

$$\Gamma^{*r}_{pq} \;=\; \Gamma^k_{ij}\,\frac{\partial u^i}{\partial u^{*p}}\frac{\partial u^j}{\partial u^{*q}}\frac{\partial u^{*r}}{\partial u^k} + \frac{\partial^2 u^k}{\partial u^{*p}\partial u^{*q}}\frac{\partial u^{*r}}{\partial u^k} \;=\; \Gamma^k_{ij}\,\delta^i_p\delta^j_q\delta^r_k - \Gamma^k_{pq}\,\delta^r_k \;=\; \Gamma^r_{pq} - \Gamma^r_{pq} \;=\; 0 \;.$$

Hence, in the new coordinate system *the Christoffel symbols vanish at the point* P_0! We will refer to such coordinates as *local tangent coordinates*, and to the system as a *locally tangential coordinate system at* P_0. This terminology stems from the fact that we essentially use coordinates in which the deviations from the manifold become second order, *i. e.*, we use coordinates which at P_0 coincide with the coordinates of the tangential plane to the manifold.

An equivalent way to express the vanishing of the Christoffel symbols is to say that one can always introduce coordinates in which the first partial derivative of the metric vanishes. From Eqn. (A.6) we see that if one makes the Christoffel symbols vanish, the first derivatives of the metric vanish too. Conversely, if the first derivatives of the metric vanish, Eqn. (A.1) shows that then the Christoffel symbols will also vanish. Note, however, that it is generally impossible to make the Christoffel symbols vanish *everywhere*, because otherwise every manifold could be equipped with a set of coordinates in which it looks flat. In fact, we have Riemann's theorem: *A Riemannian manifold can be equipped with a local Euclidean metric if and only if the Riemann tensor vanishes.* For a proof, see [9, Chapter 9.7].

Local tangent coordinates are sometimes very useful in deriving particular results. As long as one makes sure that the final equations obtained that way are still tensor equations, they will continue to hold in *any* coordinate system!

Sidenote: In general relativity a locally tangential coordinate system is more conventionally referred to as a *local inertial frame*, or a *free falling reference system.* In such a frame all first order effects of gravity vanish. However, quadratic effects (like tidal forces!) remain. It is generally impossible to transform into a coordinate system in which *all* effects due to gravity vanish. (In fact, a change of coordinates is a Lorentz transformation, which has 10 parameters. However, the Riemann tensor, describing the curvature of space time, has 20 independent parameters!)

B. Mappings

B.1. Differentials and and pull-backs

For the following see for instance Ref. [9].

Let $F : M^n \to V^m$ be a smooth map from an n-dimensional manifold M^n to an m-dimensional manifold V^m, and let $\vec{y} = F\vec{x}$. The *differential* $\mathrm{d}F \equiv F_*$ of the mapping F at $\vec{x}$ is the linear map which sends vectors in $T_{\vec{x}}M^n$ to vectors in $T_{\vec{y}}V^m$:

$$F_* : \begin{cases} T_{\vec{x}}M^n & \to \quad T_{\vec{y}}V^m \\ \boldsymbol{v} & \mapsto \quad \boldsymbol{w} = F_*(\boldsymbol{v}) \end{cases} . \tag{B.1}$$

The mapping itself is defined such that it takes tangent vectors of curves to the corresponding tangent vectors of the image curves: Take a smooth curve $\vec{x}(t)$ such that $\vec{x}(0) = \vec{x}_0$ and $\dot{\vec{x}}(0) := (\mathrm{d}\vec{x}/\mathrm{d}t)(0) = \boldsymbol{v}$, for example, the straight line $\vec{x}(t) = \vec{x}_0 + t\boldsymbol{v}$. The image of this curve $\vec{y}(t) = F(\vec{x}(t))$ has a tangent vector $\boldsymbol{w}$ at $\vec{y}_0 = F(\vec{x}_0)$ given by the chain rule:

$$w^\alpha = \dot{y}^\alpha = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(\vec{x}_0)\, \dot{x}^i(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(\vec{x}_0)\, v^i \ .$$

The above calculation shows that in terms of the bases $\{\boldsymbol{\partial}/\boldsymbol{\partial} x\}$ and $\{\boldsymbol{\partial}/\boldsymbol{\partial} y\}$ in the two tangent spaces, the differential is simply given by the Jacobian matrix

$$(F_*)^\mu_\nu = \frac{\partial F^\mu}{\partial x^\nu}(\vec{x}) = \frac{\partial y^\mu}{\partial x^\nu}(\vec{x}) \qquad\qquad \boldsymbol{w} = F_*\boldsymbol{v} \quad \Leftrightarrow \quad w^\mu = \frac{\partial F^\mu}{\partial x^\nu} v^\nu = \frac{\partial y^\mu}{\partial x^\nu} v^\nu \ . \tag{B.2}$$

If in particular the mapping F is a change of coordinates, then (B.2) expresses the fact that, in physicist's language, "the vector $\boldsymbol{v}$ transforms *contravariantly* during a change of coordinates".

Having defined the differential, which maps vectors, we now introduce a similar operation which acts on *covectors*: Define the *pull-back* via

$$F^*(\beta)(\boldsymbol{v}) := \beta(F_*(\boldsymbol{v})) \ , \tag{B.3}$$

where $\beta \in T^*_{\vec{y}}V^m$ and $\boldsymbol{v} \in T_{\vec{x}}M^n$. Hence, the covector $\boldsymbol{\beta}$ defined in $T^*_{\vec{y}}V^m$ is "pulled back" to a covector at $T^*_{\vec{x}}M^n$.

For completeness, we also define the pull-back of a function by

$$(F^* f)(\vec{x}) = (f \circ F)(x) = f(\vec{y}(\vec{x})) \ . \tag{B.4}$$

Hence, we have

$$\boldsymbol{v}(F^* f) = \boldsymbol{v}\{f[y(x)]\} = v^i \frac{\partial}{\partial x^i}\{f[y(x)]\} = v^i \left(\frac{\partial f}{\partial y^j}\right)\left(\frac{\partial y^j}{\partial x^i}\right) = \left[v^i \left(\frac{\partial y^j}{\partial x^i}\right)\frac{\partial}{\partial y^j}\right] f \overset{(B.2)}{=} (F_*\boldsymbol{v})(f) = \mathrm{d}f(F_*\boldsymbol{v}) \ , \tag{B.5}$$

where the last equality is simply the definition[1] of the differential of a function.

We now want to express the definition (B.3) in local coordinates. Let x^i and y^j be local coordinates near $\vec{x}$ and $\vec{y}$, respectively. The bases for $T_{\vec{x}}M^n$ and $T_{\vec{x}}V^m$ are $\{\boldsymbol{\partial}/\boldsymbol{\partial} x^i\}$ and $\{\boldsymbol{\partial}/\boldsymbol{\partial} y^j\}$. We then have

$$F^*\beta = \sum_i F^*(\beta)\Big(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\Big)\boldsymbol{d}x^i \overset{(B.3)}{=} \sum_i \beta\Big(F_*\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^i}\Big)\boldsymbol{d}x^i = \sum_i \beta\Big(\sum_j \Big(\frac{\partial y^j}{\partial x^i}\Big)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^j}\Big)\boldsymbol{d}x^i = \sum_{i,j}\Big(\frac{\partial y^j}{\partial x^i}\Big)\underbrace{\beta\Big(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} y^j}\Big)}_{\beta_j}\boldsymbol{d}x^i \ .$$

We therefore have in local coordinates

$$[F^*\beta]_i = \sum_j \beta_j\Big(\frac{\partial y^j}{\partial x^i}\Big) \ .$$

[1] The definition $\mathrm{d}f(\boldsymbol{v}) = \boldsymbol{v}(f)$ is unusual, but coordinate free. Using a local coordinate system $\{\boldsymbol{\partial}_{x^i}\}$ we find $\mathrm{d}f(v^i\boldsymbol{\partial}_{x^i}) = v^i\partial f/\partial x^i$. If in particular we use a coordinate function, we find $\mathrm{d}x^i(\boldsymbol{\partial}_{x^j}) = \partial x^i/\partial x^j = \delta^i_j$, showing that the $\{\mathrm{d}x^i\}$ is the dual basis to $\{\boldsymbol{\partial}_{x^i}\}$. Hence, any 1-form $\alpha \in T^*_{\vec{x}}M^n$ can be written as $\alpha = \alpha_i\mathrm{d}x^i = \alpha(\boldsymbol{\partial}_{x^i})\mathrm{d}x^i$. In particular, for the 1-form $\mathrm{d}f$ we find thereby $\mathrm{d}f = \mathrm{d}f(\boldsymbol{\partial}_{x^i})\mathrm{d}x^i = (\partial f/\partial x^i)\mathrm{d}x^i$, *i. e.*, the standard coordinate expression of the differential of a function f.

Hence, in local coordinates the pull-back is also given by the Jacobi-matrix, but it acts on the *rows* of β_j from the right.

Again, if the mapping is a change of coordinates, the physicist's language is that "the vector β transforms *covariantly* during a change of coordinates".

At this point the following **warning** from Ref. [9, page 54] is appropriate:

> Let $F : M^n \to V^m$ and let $\boldsymbol{v}$ be a vector *field* on M. It may very well be that there are two distinct points $\vec{x}$ and $\vec{x}'$ that get mapped by F to the same point $\vec{y} = F(\vec{x}) = F(\vec{x}')$. Usually we have $F_*(\boldsymbol{v}(\vec{x})) \neq F_*(\boldsymbol{v}(\vec{x}'))$, since the field $\boldsymbol{v}$ need have no relation to the map F. In other words, $F_*(\boldsymbol{v})$ does not yield a well defined vector field on V^m (does one pick $F_*(\boldsymbol{v}(\vec{x}))$ or $F_*(\boldsymbol{v}(\vec{x}'))$?). *$F_*$ does not take vector fields into vector fields.* (There is an exception if $n = m$ and F is 1:1.) On the other hand, *if β is a covector field on V^m, then $F^*\beta$ is always a well defined covector field on M^n*; $F^*(\beta(\vec{y}))$ yields a definite covector at each point $\vec{x}$ such that $F(\vec{x}) = \vec{y}$. [...]

A simple application of pull-backs occurs in the theory of integration. Assume we have a smooth mapping $F : M^n \to V^m$ and a p-form α^p defined on W. Assume further that we have an oriented p-subset σ, and that we want to integrate α^p over the image $F(\sigma)$ of this subset. Then we have

$$\int_{F(\sigma)} \alpha^p = \int_\sigma F^*\alpha^p \ .$$

So, using F^* we just pull the p-form back to the manifold M on which σ is defined.

B.2. Conformal and isometric mappings

A mapping from a portion of a manifold M to a portion of a manifold N is called *conformal*, if it is *angle preserving*, *i. e.*, the angle between any pair of tangent vectors on N is equal to the angle of their pre-images. A necessary and sufficient condition for a mapping to be conformal is that if on M and N the same coordinates are introduced, the metric on N is proportional to the metric on M with a prefactor that may depend on the position:

$$g_{ij}^{(N)} \;=\; \eta(u)\, g_{ij}^{(M)} \ . \tag{B.6}$$

For a proof see [14]. The factor η is called the *conformal factor*. If $\eta \equiv 1$, the mapping is called an *isometry*, since it does not only leave angles, but even lengths invariant. A metric which is related to the Euclidean metric by a conformal mapping is called *conformally flat*.

Note that Eqn. (B.6) can also be written as

$$\frac{\partial \bar{u}^m}{\partial u^k}\frac{\partial \bar{u}^n}{\partial u^l} g_{mn}\big(\bar{u}(u)\big) \;=\; \eta(u) g_{kl}(u) \ , \tag{B.7}$$

where the Jacobi matrices accomplish the conformal transformation.

B.3. Killing fields

A great deal of this Section is based on the beautiful online script of Norbert Dragon on relativity theory [8].

B.3.1. Killing equation

Consider a continuous 1-parameter family Φ_t of conformal transformations such that $\phi_0 = \mathrm{id}$. The derivative of this transformation with respect to the coordinates evaluated at $t = 0$ is given by

$$\frac{\partial\big(\Phi(u)\big)^i}{\partial u^j}\bigg|_{t=0} \;=\; \frac{\partial \bar{u}^i}{\partial u^j}\bigg|_{t=0} \;=\; \frac{\partial u^i}{\partial u^j}\bigg|_{t=0} \;=\; \delta^i_j \ . \tag{B.8}$$

Moreover, the derivative of this expression with respect to the transformation parameter, also evaluated at $t = 0$, is given by

$$\frac{\partial}{\partial t}\frac{\partial\big(\Phi(u)\big)^i}{\partial u^j}\bigg|_{t=0} \;=\; \frac{\partial}{\partial u^j}\frac{\partial\big(\Phi(u)\big)^i}{\partial t}\bigg|_{t=0} \;=\; \frac{\partial}{\partial u^j}X^i \;=\; X^i_{,j} \ , \tag{B.9}$$

where the vector field X^i is the generating field of the flow Φ_t.

Now look at the Eqn. (B.7) and rewrite the conformal factor as

$$\eta_t(u) \;=:\; \mathrm{e}^{\lambda_t(u)} \;.$$

The (logarithm of the) conformal factor, λ_t, of course depends on the parameter t, but for $t = 0$ it has the value 0, since the transformation becomes the identity there.

Let us now differentiate Eqn. (B.7) with respect to the parameter t of the family Φ_t of conformal transformations and evaluate at $t = 0$:

$$\begin{aligned} 0 \quad &= \quad \frac{\partial}{\partial t}\bigg|_{t=0} \left[\frac{\partial\big(\Phi_t(u)\big)^m}{\partial u^k} \frac{\partial\big(\Phi_t(u)\big)^n}{\partial u^l} g_{mn}\big(\Phi_t(u)\big) - \mathrm{e}^{\lambda_t(u)} g_{kl}(u) \right] \\ &\overset{(B.8),(B.9)}{=} \quad X^m_{,k}\delta^n_l g_{mn}(u) + \delta^m_k X^n_{,l} g_{mn}(u) + \delta^m_k \delta^n_l g_{mn,i}(u) X^i - \frac{\partial\lambda_t(u)}{\partial t}\bigg|_{t=0} g_{kl}(u) \\ &= \quad X^m_{,k} g_{ml}(u) + X^n_{,l} g_{kn}(u) + g_{kl,i}(u) X^i + \varepsilon g_{kl}(u) \;, \end{aligned}$$

where we used the abbreviation $\varepsilon = -\partial\lambda_t/\partial t\big|_{t=0}$. We see that the first three terms are just the Lie derivative (see Appendix D) of the metric with respect to the vector field X^i which generates the conformal transformation. We thereby obtain the *conformal Killing equation*

$$\boxed{\big(\mathscr{L}_{\boldsymbol{X}} + \varepsilon\big) g_{ij} \;=\; 0 \;. \qquad \text{(B.10)}}$$

Vector fields $\boldsymbol{X}$ which are solutions of this differential equation are called *konformal Killing fields* of the metric g_{ij}. In more than 2 dimensions the conformal Killing equation usually has no nonvanishing solution.

In the special case where the vector X^i field does not just generate a conformal transformation but actually an isometry, the conformal factor is constant and therefore $\varepsilon \equiv 0$. We then obtain the (usual) Killing equation

$$\boxed{\mathscr{L}_{\boldsymbol{X}}\, g_{ij} \;=\; 0 \;. \qquad \text{(B.11)}}$$

Vector fields $\boldsymbol{X}$ satisfying this differential equations are called *Killing fields* of the metric g_{ij}. The Killing equation restricts *both* the vector field $\boldsymbol{X}$, namely, to be an infinitesimal isometry of the metric, *and* the metric, namely to have such a symmetry in the first place. Note that it basically states that the metric remains unchanged when dragged along the flow of a continuous isometry.

Note that due to Eqn. (D.5) the Killing equation (B.11) can also be written as

$$\nabla_i X_j + \nabla_j X_i \;=\; 0 \;. \qquad \text{(B.12)}$$

The length of a Killing vector $\boldsymbol{X}$ is constant along the flow which it generates. Since $X^k\nabla_k$ is the directional derivative along the flow, we easily see

$$X^k\nabla_k\left(g^{ij}X_iX_j\right) = X^k\underbrace{\big(\nabla_k g^{ij}\big)}_{=0}X_iX_j + \underbrace{X^k g^{ij}\big(\nabla_k X_i\big)X_j}_{k\leftrightarrow j} + \underbrace{X^k g^{ij}X_i\big(\nabla_k X_j\big)}_{k\leftrightarrow i} = X^iX^j\big(\nabla_j X_i + \nabla_i X_j\big) \;=\; 0.$$

The covariant derivative of the metric vanishes due to Ricci's lemma (C.9), and the symmetrized covariant derivative of X_i vanishes because $\boldsymbol{X}$ is a Killing vector.

B.3.2. Number of Killing fields

Recall from Eqn. (C.8) that the commutator of covariant derivatives is proportional to the Riemann tensor. If in particular we calculate the commutator of the derivatives of a Killing field, we find

$$-R^l_{kij}X_l \;=\; \nabla_i\nabla_j X_k - \nabla_j\nabla_i X_k \;=\; \nabla_i\nabla_j X_k \overset{(B.12)}{+} \nabla_j\nabla_k X_i \;. \qquad \text{(B.13)}$$

We now use the first Bianchi identity (1.40), according to which the sum over the cyclic permutation of the three lower indices of the Riemann tensor vanishes. Applying this to the above equation, we find

$$
\begin{aligned}
0 &= -\big(R^l_{kij} + R^l_{ijk} + R^l_{jki}\big) X_l \\
&\overset{\text{(B.13)}}{=} \nabla_i\nabla_j X_k + \nabla_j\nabla_k X_i + \nabla_j\nabla_k X_i + \nabla_k\nabla_i X_j + \nabla_k\nabla_i X_j + \nabla_i\nabla_j X_k \\
&= 2\big(\nabla_i\nabla_j X_k + \nabla_j\nabla_k X_i + \nabla_k\nabla_i X_j\big) \;=\; 2\big(-R^l_{kij} X_l + \nabla_k\nabla_i X_j\big) \;,
\end{aligned}
$$

from which we get the equation

$$\nabla_k\nabla_i X_j \;=\; R^l_{kij} X_l \;. \tag{B.14}$$

This equation determines the second derivatives of X_j; and via further differentiation one gets all higher derivatives. Hence, in the Taylor series of a Killing field around some point P only the coefficients $X_i\big|_P$ and the *antisymmetric* derivatives[2] $\nabla_i X_j - \nabla_j X_i$ are free. For a d-dimenaional manifold this leaves $d + \frac{1}{2}d(d-1)$ coefficients. Hence, on such a manifold there are at most $\frac{1}{2}d(d+1)$ independent Killing fields! For instance, in the Euclidean plane this gives three Killing fields, namely

$$
\begin{matrix}
\boldsymbol{e}_x \\
\boldsymbol{e}_y \\
x\boldsymbol{e}_y - y\boldsymbol{e}_x
\end{matrix}
\quad , \tag{B.15}
$$

i. e., the generators of translation (two) and rotation (one), which are the (continuous!) isometries of the plane. In three dimensions we have at most six Killing fields, and in Euclidean space those are the generators for translation (three) and rotation (three more).

B.3.3. Killing vectors along geodesics

Let $\boldsymbol{T}$ be the tangent vector to a geodesic (see Appendix C) and let $\boldsymbol{X}$ be a Killing field. Then we have the following remarkable result: *The scalar product of the two is constant along the geodesic*, or in equations

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}s}\, T^i X_i \;=\; 0 \;,} \tag{B.16}$$

where $\mathrm{D}/\mathrm{D}s = T^k\nabla_k$ is the (covariant) derivative along the geodesic. The proof is by a direct calculation:

$$\frac{\mathrm{D}}{\mathrm{D}s}\, T^i X_i \;=\; \frac{\mathrm{D}T^i}{\mathrm{D}s}\, X_i + T^i \frac{\mathrm{D}X_i}{\mathrm{D}s} \;=\; T^j(\nabla_j T^i) X_i + T^i T^j \nabla_j X_i \;=\; X_i\big(\boldsymbol{\nabla_T T}\big)^i + \frac{1}{2} T^i T^j \big(\nabla_i X_j + \nabla_j X_i\big) \;. \tag{B.17}$$

In the last step we used (i) Eqn. (C.5), which introduced the concept of the covariant differentiation of a vector along some other vector, as well as (ii) the fact that $T^i T^j$ is symmetric and we can thus replace $\nabla_i X_j$ by its symmetrized version. However, $\boldsymbol{\nabla_T T}$ is zero, because the tangent vector of a geodesic is parallel transported along the geodesic,[3] and the term $\nabla_i X_j + \nabla_j X_i = \mathscr{L}_{\boldsymbol{X}} g_{ij}$ vanishes because $\boldsymbol{X}$ is a Killing vector.

Let us consider a simple example: In the Euclidean plane the three vector fields (B.15) are the Killing fields, and geodesics are just straight lines. It is clear that the projection of the (unit) tangent vector of any straight line on either $\boldsymbol{e}_x$ or $\boldsymbol{e}_y$ is constant. It is not readily clear (but equally simple) that the same applies for the third Killing field $x\boldsymbol{e}_y - y\boldsymbol{e}_x$. If the straight line passes to through the origin, then the scalar product of the unit tangent of the geodesic with the third Killing field is of course zero, since being the generator of a rotation, the Killig field is everywhere tangential to circles with the origin as a center. If the straight line does not pass through the center, we can parameterize it as

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \;,$$

and the scalar product of the tangent vector with the third local Killing field is therefore

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \left[\begin{pmatrix} -c_2 \\ c_1 \end{pmatrix} + t \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}\right] \;=\; c_1 v_2 - c_2 v_1 \;=\; \text{const.} \tag{B.18}$$

Fig. B.1 gives a graphical illustration of this fact.

[2]The symmetric derivatives are zero by the Killing equation!

[3]A little bit more explicit, we can see this as follows: $T^j(\nabla_j T^i) = T^j(T^i_{,j} + \Gamma^i_{kj}T^k) \overset{\text{(C.6)}}{=} \frac{\mathrm{d}u^j}{\mathrm{d}s}\frac{\partial T^i}{\partial u^j} + \Gamma^i_{kj}T^kT^j = \dot{T}^i \overset{\text{(C.1)}}{-} \dot{T}^i = 0.$

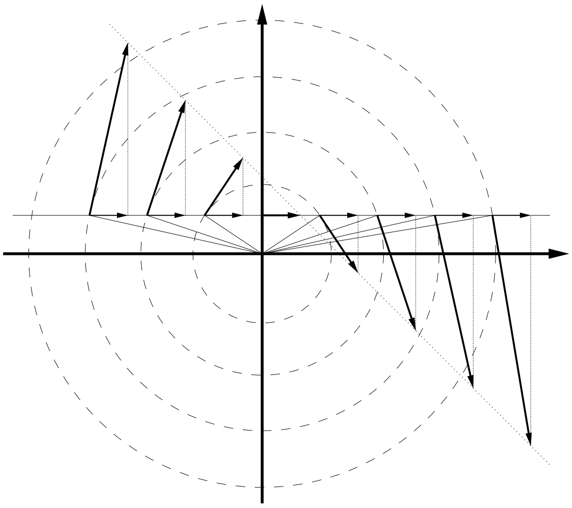


Figure B.1.: Graphical illustration of Eqn. (B.18). The oblique arrows in the Figure are representatives of the Killing vector field which generates a (clockwise) rotation in the plane about the origin. It is seen that if these generators are located on a line (which does not necessarily pass through the origin), their projection onto this line is constant. (Note that in this picture the Killing vectors belong to the field $\boldsymbol{y e}_x - \boldsymbol{x e}_y$, which is the negative of the third field in Eqn. (B.15).)

Equation (B.16) can in fact be read as a Noether identity: Geodesics are the solutions of a variational problem using the Lagrangian $g_{ij}T^iT^j$. The Noether theorem states that every continuous symmetry of the Lagrangian gives rise to a conservation law, and in this case the symmetry is the isometry of the metric, which implies a conservation law for the projection of the Killing vector onto the tangent vector of the geodesic.

B.3.4. Maximally symmetric spaces

We have seen that on a d-dimensional manifold there are at most $\frac{1}{2}d(d+1)$ independent Killing fields. However, generally a manifold has less, and it actually needn't have any. Therefore, manifolds which have the maximum number of possible Killing fields are very special. Since a large number of independent Killing fields means that a large number of independent isometries exists, a manifold with the maximum possible number of independent Killing fields is called *maximally symmetric*.

If we apply the commutator $[\nabla_i, \nabla_j]$ on $\nabla_k X_l$, we obtain with the help of Eqn. (C.8) and the fact that the commutator of a derivation is again a derivation (*i. e.*, we may apply the product rule!)

$$[\nabla_i, \nabla_j]\nabla_k X_l \;=\; -R^r_{kij}\nabla_r X_l - R^r_{lij}\nabla_k X_r \;.$$

On the other hand, if X_l is a Killing field, the left hand side by virtue of Eqn. (B.14) is given by

$$[\nabla_i, \nabla_j]\nabla_k X_l \;=\; \nabla_i\big(R^r_{jkl}X_r\big) - \nabla_j\big(R^r_{ikl}X_r\big) \;=\; \big(\nabla_i R^r_{jkl}\big)X_r + R^r_{jkl}\nabla_i X_r - \big(\nabla_j R^r_{ikl}\big)X_r - R^r_{ikl}\nabla_j X_r \;.$$

Hence, X_i and its first derivatives satisfy the following linear homogeneous system of equations:

$$\Big[\nabla_i R^r_{jkl} - \nabla_j R^r_{ikl}\Big]X_r + \Big[R^s_{jkl}\delta^r_i - R^s_{ikl}\delta^r_j + R^r_{kij}\delta^s_l \overset{*}{-} R^r_{lij}\delta^s_k\Big]\nabla_r X_s \;=\; 0 \;, \tag{B.19}$$

where at $*$ we used the fact that $\nabla_r X_s = -\nabla_s X_r$ because $\boldsymbol{X}$ is a Killing field. Since we are only interested in the antisymmetric part of $\nabla_r X_s$ (the symmetric one vanishes), the above system of equations is of the form $\boldsymbol{A}\vec{x} = 0$, where $\vec{x}$ is a vector consisting of the d components X_r and the $\frac{1}{2}d(d-1)$ antisymmetric components of $\nabla_r X_s$, or, in other words, we have a linear homogeneous system of equations in $\frac{1}{2}d(d+1)$ dimensions. The solution of this system of equations gives the expansion coefficients for the Taylor series of the Killing fields, see Sec. B.3.2. We now see that if this system of equations restricts its solution to a lowerdimensional subspace, we cannot have the maximum number of independent Killing fields. In fact, we will only have the full number $\frac{1}{2}d(d+1)$, if the Kernel of the above system of equations has this dimension. However, since the dimension of the system *itself* is $\frac{1}{2}d(d+1)$, this implies that its rank has to be zero. Therefore, the manifold can only be maximally symmetric if the two expressions in square brackets in Eqn. (B.19) vanish identically.

Taking the second bracket and summing with δ^i_r then gives

$$\begin{aligned} (d-1)R^s_{jkl} &= R_{lj}\delta^s_k - R_{kj}\delta^s_l \\ \text{lower } s\text{:} \qquad (d-1)R_{sjkl} &= R_{lj}g_{ks} - R_{kj}g_{ls} \end{aligned} \tag{B.20}$$

Since the covariant Riemann tensor is antisymmetric in the first pair and the second pair of indices, we can swap both of them simulataneously. If we afterwards contract with g^{jl}, we find:

$$\begin{aligned} (d-1)R_{jslk}g^{jl} &= \big(R_{lj}g_{ks} - R_{kj}g_{ls}\big)g^{jl} \\ (d-1)R_{sk} &= R\,g_{ks} - R_{ks} \;. \end{aligned}$$

Since the Ricci tensor is symmetric, this equation finally implies

$$R_{ks} = \frac{1}{d} R\, g_{ks} \ . \tag{B.21}$$

This shows that *in maximally symmetric spaces the Ricci-tensor is proportional to the metric.*[4] However, more than that holds: We will now show that *the Ricci scalar has to be a constant.* For this we make use of the fact that also the first square bracket in Eqn. (B.19) has to vanish. Let us first insert Eqns. (B.20) and (B.21) into it:

$$\begin{aligned}
\nabla_i R^r_{jkl} - \nabla_j R^r_{ikl} &= \frac{1}{d-1}\Big[\nabla_i\big(R_{lj}g_{kr} - R_{kj}g_{lr}\big) - \nabla_j\big(R_{li}g_{kr} - R_{ki}g_{lr}\big)\Big] \\
&= \frac{1}{d-1}\Big[g_{kr}\nabla_i R_{lj} - g_{lr}\nabla_i R_{kj} - g_{kr}\nabla_j R_{li} + g_{lr}\nabla_j R_{ki}\Big] \\
&= \frac{1}{d(d-1)}\Big[g_{kr}g_{lj}\nabla_i R - g_{lr}g_{kj}\nabla_i R - g_{kr}g_{li}\nabla_j R + g_{lr}g_{ki}\nabla_j R\Big] \ .
\end{aligned}$$

This expression has to vanish, and in fact it does so if R is constant. However, we will also show that it *only* vanishes if R is constant. To see this, contract this equation with g^{ki} and g^{lj} and set it to zero:

$$\begin{aligned}
0 &= \frac{1}{d(d-1)} g^{ki} g^{lj}\Big[g_{kr}g_{lj}\nabla_i R - g_{lr}g_{kj}\nabla_i R - g_{kr}g_{li}\nabla_j R + g_{lr}g_{ki}\nabla_j R\Big] \\
&= \frac{1}{d(d-1)}\Big[\delta^i_r\delta^j_j\nabla_i R - \delta^j_r\delta^i_j\nabla_i R - \delta^i_r\delta^j_i\nabla_j R + \delta^j_r\delta^i_i\nabla_j R\Big] \;=\; \frac{2}{d}\,\nabla_r R \ ,
\end{aligned}$$

And from this we see that R has to be constant if the expression is to vanish.[5] In maximally symmetric spaces the Riemann tensor is therefore given by

$$\boxed{R_{ijkl} = \frac{R}{d(d-1)}\Big(g_{ik}g_{jl} - g_{jk}g_{il}\Big) \qquad , \qquad R = \text{const.}} \tag{B.22}$$

Such spaces are completely classified by two numbers: The scalar curvature R and the signature of the metric.

[4] The prefactor then of course must be R/d, for otherwise the contraction of the Ricci tensor would not yield the Ricci scalar.

[5] One might be a little bit worried that we divided by $d-1$ at some point in the calculation, which is not permitted at $d=1$. However, the theory becomes almost trivial for $d=1$, and the Riemann tensor is just a number anyway.

C. Geodesics, parallel transport and covariant differentiation

C.1. Geodesics

There are various ways in which a straight line in usual Euclidean geometry can be characterized. For instance, it has zero curvature everywhere, all its tangent vectors are parallel, or it is the solution of the simple first order linear differential equation $\dot{\vec{v}}(t) = \vec{v}_0$. Neither of these characterizations can be immediately transferred to the case of curves within a Riemannian manifold—but the following definition is generalizable: A straight line between two points is the curve which minimizes the distance between these points. Since in a Riemannian metric we have the notion of "length", we can use this to define what a "straight line" in a curved space is. Such "straight lines" are called "*geodesics*".

The square of the "infinitesimal length element" in a Riemannian manifold is:

$$\mathrm{d}s^2 \;=\; \big(\vec{r}(u^i + \mathrm{d}u^i) - \vec{r}(u^i)\big)^2 \;=\; \Big(\frac{\partial \vec{r}}{\partial u^i}\,\mathrm{d}u^i\Big)^2 \;=\; \frac{\partial \vec{r}}{\partial u^i}\cdot\frac{\partial \vec{r}}{\partial u^j}\,\mathrm{d}u^i\mathrm{d}u^j \;=\; g_{ij}\,\mathrm{d}u^i\mathrm{d}u^j \;.$$

Given a curve C with parametrization $u^i(t)$, the length ℓ of this curve between the two points $P_1 = \vec{r}(u(t_1))$ and $P_2 = \vec{r}(u(t_2))$ is then seen to be

$$\ell[u, C] \;=\; \int_{t_1}^{t_2} \mathrm{d}t\,\sqrt{g_{ij}\frac{\mathrm{d}u^i}{\mathrm{d}t}\frac{\mathrm{d}u^j}{\mathrm{d}t}} \;=:\; \int_{t_1}^{t_2} \mathrm{d}t\,\sqrt{g_{ij}\dot{u}^i\dot{u}^j} \;.$$

This length now has to be minimized variationally. However, instead of using the "Lagrangian" $L := \sqrt{g_{ij}\dot{u}_i\dot{u}^j}$, it is equivalent, but much easier, to minimize the functional corresponding to its square, L^2. We thereby obtain:

$$0 \;\stackrel{!}{=}\; \delta\int_{t_1}^{t_2}\mathrm{d}t\; g_{ij}\dot{u}^i\dot{u}^j \;=\; \int_{t_1}^{t_2}\mathrm{d}t\Big\{g_{ij,k}\delta u^k\dot{u}^i\dot{u}^j + g_{ij}\delta\dot{u}^i\dot{u}^j + g_{ij}\dot{u}^i\delta\dot{u}^j\Big\} \;.$$

Partially integrating the last two terms, and noting that the integrated out parts vanish because we assume that there will be no variation at the end points, we obtain

$$\begin{aligned} 0 \;&=\; \int_{t_1}^{t_2}\mathrm{d}t\Big\{g_{ij,k}\delta u^k\dot{u}^i\dot{u}^j - \underbrace{\big(g_{ij,k}\dot{u}^k\dot{u}^j + g_{ij}\ddot{u}^j\big)\delta u^i}_{i\leftrightarrow k} - \underbrace{\big(g_{ij,k}\dot{u}^k\dot{u}^i + g_{ij}\ddot{u}^i\big)\delta u^j}_{j\leftrightarrow k}\Big\} \\ &=\; \int_{t_1}^{t_2}\mathrm{d}t\Big\{\big(g_{ij,k} - g_{kj,i} - g_{ik,j}\big)\dot{u}^i\dot{u}^j - 2g_{ik}\ddot{u}^i\Big\}\delta u^k \;=\; -2\int_{t_1}^{t_2}\mathrm{d}t\Big\{\Gamma_{ijk}\dot{u}^i\dot{u}^j + g_{ik}\ddot{u}^i\Big\}\delta u^k \;. \end{aligned}$$

Since the variation δu^k is arbitrary, the expression in curly brackets has to vanish. Raising the index k, we then obtain the *geodesic equation*

$$\boxed{\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j \;=\; 0 \;. \qquad \text{(C.1)}}$$

Remarks:

- The fundamental theorem of differential equations assures that the geodesic equation (C.1) has a unique solution for given initial conditions $u^k(0)$ and $(\mathrm{d}u^k/\mathrm{d}t)(0)$. Thus, there is a unique geodesic at every point of the manifold in every direction. However, since the geodesic equation is nonlinear, this solution may not exist for all parameter values t. If all geodesics go on indefinitely (*i. e.*, if they are isometric to the real line), the Riemannian space (or its defining metric) is called *complete*. For instance, any metric on a compact manifold is complete.

- If $u^k(t)$ are the coordinates of a geodesic, and $s \mapsto t(s)$ is a reparameterization of the curve, the new curve $\tilde{u}^k(s) = u^k(t(s))$ is generally *not* a geodesic, unless the transformation is of the form $t = a + bs$. This is easily seen by inserting the reparameterization into the geodesic equation.

- A different characterization of a geodesic is the following: A curve on a surface is geodesic if and only if the normal vector to the *curve* is everywhere parallel to the local normal vector of the *surface*. This goes back to Johann Bernoulli (1697!).

- Yet another way to characterize a geodesic is the following: Geodesics are curves along which the geodesic curvature κ_{g} vanishes (see Eqn. (1.15)). This is of course where the geodesic curvature has its name from. Recall that the geodesic curvature measures the projection of the curvature of a curve onto the local tangent plane, *i. e.*, the part which is in some sense independent of the curvature of the surface. Hence, requiring $\kappa_{\mathrm{g}} = 0$ means, loosely speaking, geodesics have no curvature other than the inevitable curvature which is due to the bending of the surface itself.

- The original condition we started out with was *global*, the final condition we ended up with is a differential equation, *i. e.*, it is *local*. This has the consequence that a solution of the geodesic equation may not satisfy our initial global aim of finding the curve yielding the smallest distance. Without further investigation, all we can be sure of is that the length is *stationary*. However, over sufficiently small distances every solution of Eqn. (C.1) is indeed length-minimizing.

- As a continuation: Whether or not a geodesic between two points P_1 and P_2 is minimizing is related to the existence of certain *Jacobi-fields* between these points. A Jacobi field along a geodesic is a vector field $\boldsymbol{Y}$ which is invariant under the local flow generated by the tangent vector $\boldsymbol{T}$ to the geodesic, *i. e.*, the Lie derivative $\mathscr{L}_{\boldsymbol{T}}\boldsymbol{Y}$ vanishes along the geodesic. Assume that we have found a nontrivial Jacobi field which vanishes at P_1 *and also at some point P' on the geodesic between P_1 and P_2*. Then P' is called *conjugate* to P_1. This is important because of the following Theorem: *A geodesic containing a point which is conjugate to its initial point is not minimizing.* This may be stated even more precisely in the following way: A conjugate point P' is said to have (Morse) index λ, if there exist exactly λ linearly independent Jacobi fields which vanish both at P_1 and P'. Now let $P^{(1)}$, $P^{(2)}$, …, $P^{(n)}$ be all points on a geodesic from P_1 to P_2 which are conjugate to P_1, and let $\lambda^{(i)}$ be the Morse index of $P^{(i)}$. Then there exist exactly $\sum_i \lambda^{(i)}$ independent variations which strictly reduce the length of the geodesic. This is known as the “Morse index theorem”.

- If we can joint two points on a manifold by a curve, we can also join them by a geodesic, since one can shorten the curve up to a point where additional modifications do no longer reduce the length.

- Two closed curves on a surface which can be smoothly transformed into each other are called *homotopic*. Since homotopy is an equivalence relation, it divides the set of all closed curves on surfaces into classes, called *homotopy classes*. For instance, on a torus there are infinitely many homotopy classes, which can be labeled by how many times the curve cycles around each of the two “circles” before closing upon itself. In each nontrivial homotopy class of a closed manifold M^n there exists at least one closed geodesic. Closed geodesics are for instance interesting from the point of view of mechanics, since they correspond to periodic motion in phase space.

- If the manifold is not Riemannian but pseudo-Riemannian, things are a little bit more complicated. For instance, geodesics may be *maximizing* rather than minimizing (think for instance of general relativity, where the proper time is maximized along the path of a free falling particle), and there may also be *null geodesics*, for which the tangent vector at each point has length zero, and which therefore cannot be parameterized by their arc length.

A deeper discussion of these topics can be found in the book of Frankel [9].

C.2. Parallel displacement of Levi-Cività

Contrary to Euclidean space, it does not make sense on a general manifold to ask whether two vectors at *different* points are parallel, since these vectors live in different tangent spaces. Therefore, in order to re-introduce the concept of parallelism on a general manifold, we have to do some more work. Ultimately we will find that we cannot globally reestablish the concept, but locally we can.

There are many very abstract ways of doing this, but here we will restrict to the special case of a Riemannian manifold, *i. e.*, when we have a metric. Since then we also have a scalar product, we can speak out the following definition:

A vector $\boldsymbol{X}$ *is* parallel transported *along a curve* C*, if the scalar product of* $\boldsymbol{X}$ *with the tangent vector* $\boldsymbol{T}$ *of the curve at that point is the same everywhere on the curve.*

Let the curve be parameterized by $u^i(t)$. Then, in local coordinates, this condition requires

$$0 \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{u}^i X_i\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(g_{ij}\dot{u}^i X^j\big) \qquad \text{everywhere on } u^i(t)\ . \tag{C.2}$$

Now, parallel transporting a vector in this way along some arbitrary curve does not yield much useful information about the concept of parallelism. However, *if the curve is a geodesic*, we might expect to gain something: Since a geodesic is "as straight as one can get" on a curved manifold, transport along a geodesic might help to transfer this straightness into a "conservation of parallelity". These are of course very fluffy words. Note, however, that this definition reduces to the common notion of parallelity in Euclidean space and that, if nothing else, it is a viable attempt to extend it to curved space.

If $u^i(t)$ is a geodesic, it has to satisfy the geodesic equation (C.1), and we may use this information to reformulate the condition (C.2):

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{u}^k X_k\big) = \ddot{u}^k X_k + \dot{u}^k \dot{X}_k \stackrel{(C.1)}{=} -\Gamma^k_{ij}\dot{u}^i\dot{u}^j X_k + \dot{u}^i\dot{X}_i = \underline{\big(\dot{X}_i - \Gamma^k_{ij}\dot{u}^j X_k\big)\dot{u}^i}\ . \tag{C.3a}$$

Or, a little bit longer for contravariant components:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\big(g_{ij}\dot{u}^j X^i\big) = g_{ij,k}\dot{u}^k\dot{u}^j X^i + g_{ij}\ddot{u}^j X^i + g_{ij}\dot{u}^j\dot{X}^i \\ &\stackrel{(A.6a),(C.1)}{=} \big(\Gamma_{ikj} + \Gamma_{jki}\big)\dot{u}^k\dot{u}^j X^i - g_{ij}\Gamma^j_{kl}\dot{u}^k\dot{u}^l X^i + g_{ij}\dot{u}^j\dot{X}^i \\ &= \Gamma_{ikj}\dot{u}^k\dot{u}^j X^i + g_{ij}\dot{u}^j\dot{X}^i = \underline{\big(\dot{X}^i + \Gamma^i_{jk}\dot{u}^k X^j\big)\dot{u}_i}\ . \end{aligned} \tag{C.3b}$$

Hence, parallel transport is achieved if we set the expressions in parentheses in Eqns. (C.3a) and (C.3b) to zero:

$$\dot{X}^i + \Gamma^i_{jk}\dot{u}^k X^j = 0\ , \tag{C.4a}$$
$$\dot{X}_i - \Gamma^j_{ik}\dot{u}^k X_j = 0\ . \tag{C.4b}$$

A vector whose components are transported according to these differential equations is said to be subject to the *parallel displacement of Levi-Cività*.

Remarks:

- Looking back at the geodesic equation, we see that it can also be read in the following way: *The tangent vector to a geodesic is parallel displaced along the geodesic.*
- Furthermore, since this parallel transport by construction preserves scalar products, it will preserve the length of transported vectors. Hence, a geodesic being the solution of (C.1) will be automatically parameterized by arc length if the initial tangent vector had unit length. This also explains why a nonlinear reparameterization destroys the geodesic property: If the tangent vector was initially constant – as it must be for a valid solution of the geodesic equation – it would not remain constant along the way.
- From the definition of parallel transport, it is not immediately obvious that the expression on the left hand side of Eqns. (C.4) is indeed a tensor—but it is.
- *Parallel transport depends on the path*! This is the most important difference between parallel transport in an Euclidean space and in a general Riemannian manifold. It implies that the concept of parallelism cannot be globally extended, since whether or not a vector at Q is parallel to a vector at P does not simply depend on P and Q alone. An equivalent way of saying this is that a vector $\boldsymbol{X}$ at a point P, parallel transported around a *closed* curve, will generally not be parallel to a copy of itself left at P for reference purposes. This can be nicely "visualized" in the simple case of the surface of a sphere, where the geodesic are great circles—see Fig. C.1.

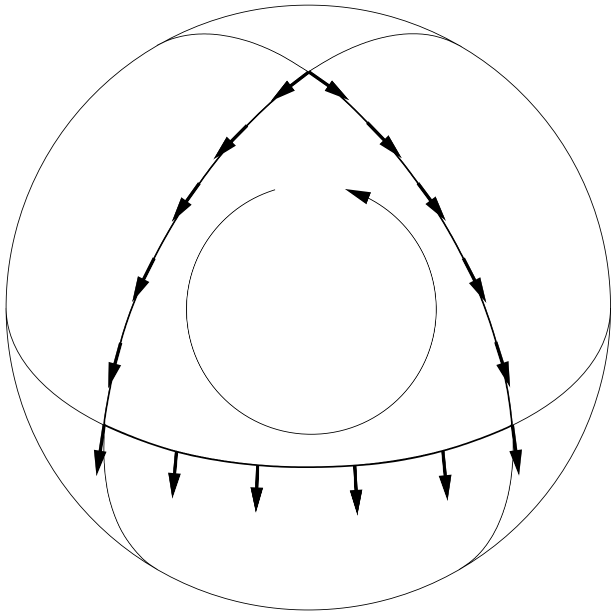


Figure C.1.: Illustration that a parallel transport around a closed loop generally will change the direction of the vector from what it originally was. A vector initially placed at the "north pole" is parallel displaced along its initial direction up to the equator. From there it is again parallel displaced along the equator for a certain while. Finally it is again moved back to the north pole, where it is found that the vector does no longer point in the same direction in which it originally pointed.

C.3. Covariant differentiation

The partial derivative of a tensor with respect to a coordinate is itself not a tensor. In order to obtain a tensor, one has to use the *absolute* or *covariant* derivative, which we will now define.

Look at the two equations (C.4) which define parallel transport. They can be rewritten as

$$\big(X^i_{,k} + \Gamma^i_{jk} X^j\big)\dot{u}^k \;=:\; \big(\nabla_k X^i\big)T^k \;\equiv\; \big(\boldsymbol{\nabla_T X}\big)^i \;=\; 0\,, \tag{C.5a}$$

$$\big(X_{i,k} - \Gamma^j_{ik} X_j\big)\dot{u}^k \;=:\; \big(\nabla_k X_i\big)T^k \;\equiv\; \big(\boldsymbol{\nabla_T X}\big)_i \;=\; 0\,. \tag{C.5b}$$

These expressions clearly make sense without explicit reference to the geodesic. They are called the *covariant derivative of the vector* $\boldsymbol{X}$ *with respect to the vector* $\boldsymbol{T}$, and the left hand side defines their components. With this terminology the concept of parallel transport can be restated as: *A vector is parallel transported along a geodesic* C*, if its covariant derivative with respect to the local tangent vector vanishes everywhere on* C*.*

Let us collect these definitions again. First, for the sake of completeness, we define the covariant derivative of a scalar to be identical to the partial derivative:

$$\nabla_k\phi \;\equiv\; \phi_{;k} \;:=\phi_{,k}\;.$$

The covariant derivative of a contravariant vector is given by

$$\nabla_k a^i \;\equiv\; a^i_{;k} \;:=\; a^i_{,k} + a^j\Gamma^i_{jk}\;. \tag{C.6}$$

The covariant derivative of a covariant vector is given by

$$\nabla_k a_i \;\equiv\; a_{i;k} \;:=\; a_{i,k} - a_j\Gamma^j_{ik}\;.$$

This extends in the natural way to an arbitrary tensor:

$$\boxed{\begin{aligned}\nabla_\nu T^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_m} \;=\;& T^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_m,\nu}\\ &+ T^{\gamma\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_m}\Gamma^{\alpha_1}_{\gamma\nu} + T^{\alpha_1\gamma\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_m}\Gamma^{\alpha_2}_{\gamma\nu} + \cdots + T^{\alpha_1\alpha_2,\ldots\gamma}_{\beta_1\beta_2\ldots\beta_m}\Gamma^{\alpha_n}_{\gamma\nu}\\ &- T^{\alpha_1\alpha_2\ldots\alpha_n}_{\gamma\beta_2\ldots\beta_m}\Gamma^{\gamma}_{\beta_1\nu} - T^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\gamma\ldots\beta_m}\Gamma^{\gamma}_{\beta_2\nu} - \cdots - T^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\gamma}\Gamma^{\gamma}_{\beta_m\nu}\;.\end{aligned}} \tag{C.7}$$

It is important to realize that ∇_i *and* ∇_j *generally do not commute*. We rather have

$$\big(\nabla_i\nabla_j - \nabla_j\nabla_i\big)T_k \;=\; -R^l_{kij}T_l \qquad\text{and}\qquad \big(\nabla_i\nabla_j - \nabla_j\nabla_i\big)T^k \;=\; R^k_{lij}T^l\;. \tag{C.8}$$

The *Lemma of Ricci* states that the covariant derivative of the metric and the metric determinant are zero:

$$\boxed{\nabla_k g_{ij} = 0 \quad , \quad \nabla_k g^{ij} = 0 \quad , \quad \nabla_k g \; = \; 0 \; . \qquad \text{(C.9)}}$$

The proof of the first two equations is via straightforward calculation. The proof of the third equation follows from the fact that the metric determinant is just a sum over products over coefficients g_{ij}. Ricci's lemma implies that *raising and lowering indices commutes with the process of covariant differentiation*, because the metric tensor acts like a constant with respect to ∇_k. This is computationally extremely advantageous. This fact also implies that it makes sense to talk about an absolute contravariant differentiation $\nabla^i = g^{ij}\nabla_j$, because the index can always be raised or lowered without interfering with the process of differentiation. Note that this commutation property does *not* hold for the usual partial derivative.

C.4. Laplace Operator

The Laplace operator is defined as the trace of the covariant derivative:

$$\boxed{\nabla^2 \; = \; g^{ij}\nabla_i\nabla_j \; = \; \nabla_i\nabla^i \; , \qquad \text{(C.10)}}$$

Even though ∇_i and ∇_j generally do *not* commute, the above definition is independent of the order of i and j, because g^{ij} is symmetric.

From Eqn. (A.8) we can immediately get a nice formula for the Laplacian of a scalar function in terms of the metric and the metric determinant. Setting $A^j = \nabla^j\phi = g^{ij}\phi_{,j}$, we obtain

$$\nabla^2\phi \; = \; \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^i}\Big(\sqrt{g}g^{ij}\frac{\partial\phi}{\partial u^j}\Big) \; . \qquad \text{(C.11)}$$

The following observation is worth pointing out: While Eqn. (C.10) is very general, entirely covariant, and almost coordinate free, the same does not hold for Eqn. (C.11). Most importantly, this expression *only* describes the effect of the Laplacian on a *scalar* field! It would be *wrong* to use the same coordinate expression for a vector or tensor field. The reason is easy to understand: Eqn. (C.11) relies on Eqn. (A.8), whose simple form breaks down if one has tensors which are of higher than first rank. Note for instance that its extension to tensors of second degree, Eqn. (A.10), has additional Christoffel symbols flying around.

We will now give a nice application of the Laplacian to the problem of surfaces, which is an almost direct consequence of the Gauss equation. If we look at the functions $(\vec{r})^i$ which describe the surface, we find by applying the Laplacian to them

$$\nabla^2\vec{r} = g^{ij}\nabla_i\nabla_j\vec{r} = g^{ij}\nabla_i\boldsymbol{e}_j \overset{(1.19b)}{=} g^{ij}b_{ij}\vec{n} \overset{(1.13a)}{=} \underline{2H\vec{n}} \; . \qquad \text{(C.12)}$$

Problem C.1 *Verify Eqn. (C.12) for the special case of the surface of a sphere in spherical polar coordinates.*

Surfaces which have zero mean curvature everywhere are called *minimal surfaces*, see Sec. 4.1. Eqn. (C.12) then shows that for such surfaces the coordinate functions $\vec{r}$ are *harmonic*, *i. e.*, they satisfy the Laplace equation $\nabla^2\vec{r} = 0$.

C.5. Example: The Poincaré plane

The Poincaré plane is an example of a space with constant negative sectional[1] curvature. Such spaces are called *hyperbolic* and the concomitant geometry is also called hyperbolic. The reader can find a short introduction into these concepts in Ref. [3].

[1] The sectional curvature is a higherdimensional generalization of the Gaussian curvature and measures the rate of geodesic deviation.

C.5.1. Metric and Christoffel symbols

The Poincaré plane is defined to be the twodimensional Riemannian manifold $\mathbb{R} \times \mathbb{R}^+$ (*i. e.*, all pairs (x, y) with $x, y \in \mathbb{R}$ and $y > 0$) endowed with the following metric:

$$g_{ij} = \frac{1}{y^2} \mathbb{I} = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} .$$

Note that within the classification introduced in Sec. B.2 this metric is *conformally flat*.

The only partial derivatives of the metric which are nonzero are $g_{xx,y} = g_{yy,y} = -2/y^3$. From this we get the following nonzero Christoffel symbols of the first kind:

$$\Gamma_{xxy} = y^{-3} \qquad \text{and} \qquad \Gamma_{xyx} = \Gamma_{yxx} = \Gamma_{yyy} = -y^{-3} ,$$

and of the second kind

$$\Gamma^y_{xx} = y^{-1} \qquad \text{and} \qquad \Gamma^x_{xy} = \Gamma^x_{yx} = \Gamma^y_{yy} = -y^{-1} . \tag{C.13}$$

C.5.2. Parallel transport

Let us now look at two examples of parallel transport. For this we first have to define a curve along which we transport the vector. We will look at the two sets of coordinate curves $x = \text{const}$ and $y = \text{const}$ and then solve the equation for parallel transport, Eqn. (C.4).

Take the set of curves

$$u^i(t) = \begin{pmatrix} x_0 \\ y_0 + t \end{pmatrix} \qquad \Rightarrow \quad \dot{u}^i(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{C.14}$$

The equation of parallel transport for a vector (with covariant components) X^j are

$$\dot{X}^j = -\Gamma^j_{ik} \dot{u}^i X^k = -\Gamma^j_{yk} X^k ,$$

from which we get by using Eqns. (C.13)

$$\begin{aligned} \dot{X}^x &= -\Gamma^x_{yk} X^k = \frac{1}{y} X^x , \\ \dot{X}^y &= -\Gamma^y_{yk} X^k = \frac{1}{y} X^x . \end{aligned}$$

These two equations decouple, and therefore are easily integrated to give

$$X^i(t) = \mathrm{e}^{t/y} X^i(0) . \tag{C.15}$$

Thus, vectors are not rotated while being moved upwards, but their length gets changed along this particular curve.

Let us now look at the set of horizontal curves

$$u^i(t) = \begin{pmatrix} x_0 + t \\ y_0 \end{pmatrix} \qquad \Rightarrow \quad \dot{u}^i(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{C.16}$$

In this case, using Eqn. (C.4) we find

$$\dot{X}^x = -\frac{1}{y} X^y \qquad \text{and} \qquad \dot{X}^y = \frac{1}{y} X^x ,$$

which can be succinctly written as

$$\dot{\vec{X}} = \frac{1}{y} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{X} ,$$

where by $\vec{X}$ we mean the tupel with components X^i and not the true tangent vector $X^i \boldsymbol{e}_i$. This differential equation can again be integrated (almost) immediately:

$$\vec{X}(t) = \exp\left\{ \frac{1}{y} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t \right\} \vec{X}(0) = \begin{pmatrix} cos \frac{t}{y} & \sin \frac{t}{y} \\ -\sin \frac{t}{y} & \cos \frac{t}{y} \end{pmatrix} \vec{X}(0) . \tag{C.17}$$

Hence, a vector being moved to the right is rotated clockwise with an angular frequence being inversely proportional to the coordinate y—see Fig. C.2

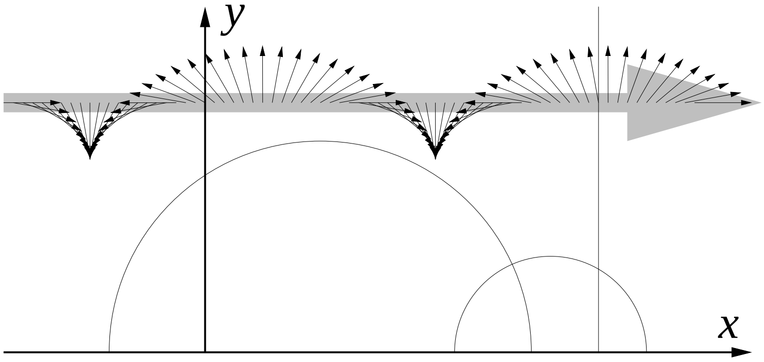


Figure C.2.: The Poincaré plane. The sequence of arrows indicates how a tangent vector is rotated upon parallel transport along the curve given by Eqn. (C.16). Vertical lines are geodesics, as are all semicircles which intersect the horizontal axis at a right angle.

C.5.3. Geodesics

The equations for the geodesics in the Poincaré plane are

$$\ddot{u}^x \;=\; \frac{2}{y}\dot{u}^x\dot{u}^y \qquad \text{and} \qquad \ddot{u}^y \;=\; \frac{1}{y}\left[(\dot{u}^y)^2 - (\dot{u}^x)^2\right] \;.$$

The geodesics follow from solving these equations, but this is somewhat hard. It is much easier to make use of some of the concepts we have learned so far. We follow [9, problem 20.1(4)].

First note that $(\mathrm{d}x^2 + \mathrm{d}y^2)/y^2 \geq \mathrm{d}y^2/y^2$, hence the vertical lines $\mathrm{d}x = 0$ are minimizing geodesics. All the other geodesics can be found by the following clever calculation: Since the metric is independent of x, $\boldsymbol{\partial}_x$ is a Killing field (since it gives rise to an isometry). Hence, if $\boldsymbol{T}$ is the unit tangent vector to a geodesic, its scalar product with $\boldsymbol{\partial}_x$ remains constant. If α is the angle between $\boldsymbol{T}$ and $\boldsymbol{\partial}_x$, we have

$$\text{const.} \;=\; k \;:=\; \boldsymbol{T}\cdot\boldsymbol{\partial}_x \;=\; |\boldsymbol{T}|\,|\boldsymbol{\partial}_x|\;\cos\alpha \;=\; \frac{1}{y}\cos\alpha \;. \tag{C.18}$$

In the last step we used that $|\boldsymbol{T}| = 1$ and $|\boldsymbol{\partial}_x| = \sqrt{g_{xx}} = 1/y$.

Next we make use of the fact that the Poincaré metric is conformally flat. This means that the angle α which we just calculated is in fact identical to the angle in the Euclidean coordinate patch (*i. e.*, the angle which we would find in a Figure like Fig. C.2). However, in the Euclidean metric we have $\mathrm{d}y/\mathrm{d}s = \sin\alpha$! Since we furthermore have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}s} \;=\; \frac{\mathrm{d}\alpha}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}s} \;=\; -\frac{k}{\sqrt{1-(ky)^2}}\sin\alpha \;=\; -\frac{k}{\sin\alpha}\sin\alpha \;=\; -k \;=\; \text{const.}\;,$$

we see that the geodesics are lines of constant Euclidean curvature $-k$. Hence, the geodesics are (downward curved) arcs of a *circle*. If the geodesic is not a vertical line, α is not equal to $\pi/2$, and from Eqn. (C.18) we then see that $k \neq 0$. Hence, the geodesic is not straight (in the Euclidean metric). At the highest point y_0 the angle is $\alpha = 0$ and thus, again using Eqn. (C.18), $k = 1/y_0$. Also, the expression $\frac{1}{y}\cos\alpha$ can only remain constant in the limit $y \to 0$ if the cosinus goes to zero, *i. e.*, if $\alpha \to \pm\pi/2$. From this follows the final result: *The geodesics of the Poincaré metric are Euclidean circles (or vertical lines) that meet the x axis orthogonally.*

The above calculation illustrates quite vividly that the knowledge of a Killing field can be extremely advantageous, for instance in the process of finding geodesics. The whole idea is useful because (just like in this case) it is often easier to “see” a symmetry of the metric rather than seeing a solution of the (nonlinear!) geodesic differential equation.

Problem C.2 *Show explicitly that the following vector fields are Killing fields of the Poincaré metric:*

$$\textit{(a)}\quad \boldsymbol{\partial}_x \qquad\qquad\qquad \textit{(b)}\quad x\boldsymbol{\partial}_x + y\boldsymbol{\partial}_y \;.$$

C.5.4. Finding all Killing fields of the Poincaré metric

If we write the covariant components K_i of a Killing field as

$$K_i \;=\; \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} \;,$$

the Killing equation $\mathscr{L}_{\boldsymbol{K}}\boldsymbol{g} = 0$ yields the following three differential equations:

$$a_x - b/y = 0 \,, \tag{C.19a}$$
$$b_y + b/y = 0 \,, \tag{C.19b}$$
$$b_x + a_y + 2a/y = 0 \,. \tag{C.19c}$$

The second equation (C.19b) is readily seen to have the general solution

$$b(x,y) = c'(x)/y \,, \tag{C.20}$$

where $c'(x)$ is the derivative of some as yet unspecified function of x. Inserting this into Eqn. (C.19a), we can integrate with respect to x and get

$$a(x,y) = c(x)/y^2 + d(y) \,, \tag{C.21}$$

where $d(y)$ is an as yet unspecified function of y. If we insert these two equations into Eqn. (C.19c), we obtain

$$c''(x) + y\,d'(y) + 2d(y) = 0 \,.$$

This partial differential equation is separable, therefore $c''(x)$ has to be equal to a constant, say $2A$, and this can be readily integrated:

$$c''(x) = 2A \qquad \Rightarrow \qquad c(x) = Ax^2 + Bx + C \,,$$

where B and C are two further integration constants. The "other part" of the separable differential equation then reads

$$y\,d'(y) + 2d(y) = -2A \,.$$

An obvious particular solution of this equation is $d(y) = -A$, and the homogeneous solution is solved by $d(y) = D/y^2$, where D is yet another integration constant.

Now that we solved for $c(x)$ and $d(y)$, we get $a(x,y)$ and $b(x,y)$ from Eqns. (C.20) and (C.21):

$$a(x,y) = \frac{Ax^2 + Bx + C}{y^2} + \frac{D}{y^2} - A \,, \tag{C.22a}$$
$$b(x,y) = \frac{Ax + B}{y} \,. \tag{C.22b}$$

We can see that the two integration constants C and D in Eqn. (C.22a) really serve the same purpose and we can just dispense with one of them, say D, by setting $D = 0$ without loss of generality.

The contravariant components of the Killing fields are just given by the covariant ones times y^2, so we finally have the following expression for a general Killing field within the Poincaré metric:

$$K^i = A\begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} + B\begin{pmatrix} x \\ y \end{pmatrix} + C\begin{pmatrix} 1 \\ 0 \end{pmatrix} \,. \tag{C.23}$$

This says that every Killing field on the Poincaré metric can be written as a linear combination of the three evidently independent Killing fields

$${}^1K^i = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix} \quad , \quad {}^2K^i = \begin{pmatrix} x \\ y \end{pmatrix} \quad , \text{ and } \quad {}^3K^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,. \tag{C.24}$$

In Sec. (B.3.2) we have seen that a two dimensional manifold can have *at most* 3 independent Killing fields, and this manifold indeed has this maximum number. In Sec. B.3.4 we have seen that such manifolds are very special and are called *maximally symmetric*. They come as close as one can get to the Euclidean notion of a homogeneous and isotropic space.

The three Killing fields give rise to continuous isometries of the Poincaré plane. Let us finally find the flow lines corresponding to these fields/isometries. The simplest flow pattern is that of ${}^3\boldsymbol{K}$, which obviously just corresponds to horizontal lines and which indeed is an evident isometry of the Poincaré plane, since the metric does not even depend on x. Indeed, in Sec. C.5.3 we made use of the fact that we could guess this Killing field in order to work out the geodesics. The flow belonging to ${}^2\boldsymbol{K}$ is equally simple, since it just describes lines diverging radially from the origin at $x = 0$, $y = 0$. The only flow which requires a bit more work is the one corresponding to ${}^1\boldsymbol{K}$. if $x(t)$ and $y(t)$ is a parameterization of a flow line, then we have to solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x^2(t) - y^2(t) \\ 2x(t)y(t) \end{pmatrix} \,.$$

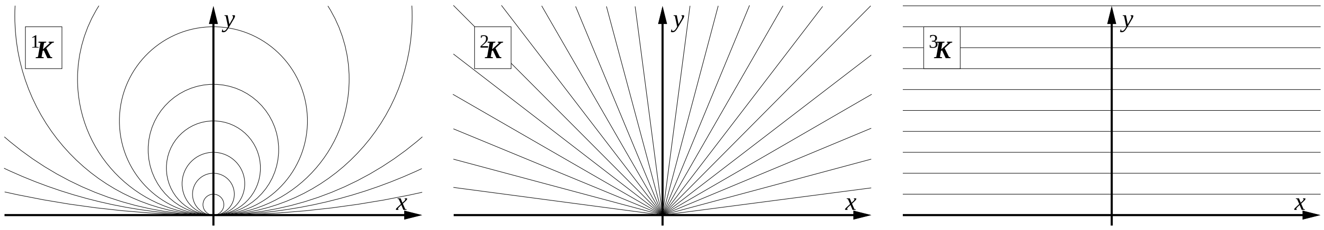


Figure C.3.: Illustration of the flow lines belonging to the three Killing fields of the Poincareé metric (see Eqn. (C.24)).

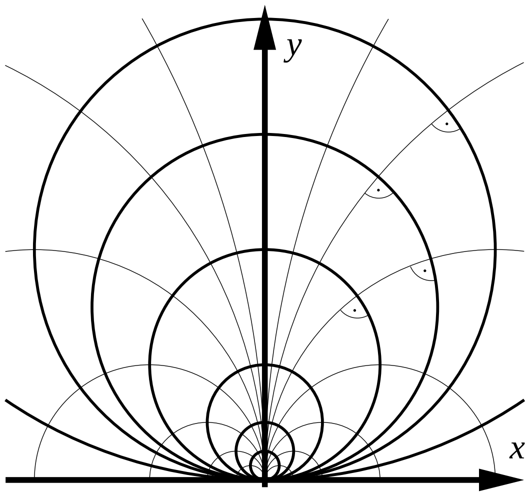


Figure C.4.: Graphical illustration of the fact that the flow belonging to the Killing field ${}^1\boldsymbol{K}$ of the Poincaré metric (bold lines) is everywhere orthogonal to the set of geodesics (thin lines) which pass through the point (at infinity) through which also all lines of the flow pass. Recall that since the Poincaré metric is conformally flat, the angles in this picture are identical to the "real" angles in hyperbolic space.

This equation can be solved by the following nice trick: If we introduce the complex variable $z(t) := x(t) + \mathrm{i}y(t)$, the differential equation can be simply written as $\dot{z}(t) = z^2(t)$, which can be directly solved by separating the variables, leading to

$$z(t) \;=\; \left(\frac{1}{z_0} - t\right)^{-1} ,$$

with the integration constant $z_0 = a_0 + \mathrm{i}b_0 = a(0) + \mathrm{i}b(0)$. All one now needs to do is to separate this up again into its real and imaginary part. The final answer then is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \;=\; \frac{1}{(1-a_0t)^2 + (b_0t)^2} \begin{pmatrix} a_0 - (a_0^2 + b_0^2)t \\ b_0 \end{pmatrix} .$$

If we specialize to flow lines which start with at $a_0 = 0$ (since all the other flow patterns can be obtained by shifting), we find

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \;=\; \frac{b_0}{1 + (b_0t)^2} \begin{pmatrix} -b_0t \\ 1 \end{pmatrix} .$$

It is a straightforward exercise to check that these flow lines describe circles of radius $b_0/2$ which touch the x-axis at $x = 0$. An illustration of all three flow patterns can be found in Fig. C.3.

The flow-lines of the three Killing fields can also be characterized with respect to their relation to the geodesics in the Poincaré plane. The flow of ${}^3\boldsymbol{K}$ is obviously everywhere orthogonal to the set of all vertical geodesics. The flow of ${}^2\boldsymbol{K}$ is everywhere orthogonal to the set of geodesics encircling the point from which the flow emanates. And finally, the flow of ${}^1\boldsymbol{K}$ is everywhere orthogonal to the set of geodesics which pass through the point at which the flow touches the x-axis, as is illustrated in Fig. C.4.

C.5.5. Curvature

The Ricci tensor and the Ricci scalar curvature for the Poincaré plane can be worked out with the help of Formula 1.30 and the Cristoffel symbols (C.13) from above:

$$\left.\begin{aligned} R_{xx} &= \Gamma^i_{xx,i} - \Gamma^i_{xi,x} + \Gamma^i_{xx}\Gamma^j_{ij} - \Gamma^j_{xi}\Gamma^u_{jx} \;=\; -y^{-2} \\ R_{yy} &= \Gamma^i_{yy,i} - \Gamma^i_{yi,y} + \Gamma^i_{yy}\Gamma^j_{ij} - \Gamma^j_{yi}\Gamma^i_{jy} \;=\; -y^{-2} \\ R_{xy} &= R_{yx} \;=\; 0 \end{aligned}\right\} \quad\Rightarrow\quad R_{ij} \;=\; -g_{ij} \quad \text{and} \quad R = -2\;. \tag{C.25}$$

This illustrates our proof from Sec. B.3.4, namely that for maximally symmetric spaces the Ricci tensor has to be proportional to the metric with a constant prefactor. In this case the prefactor is negative and the corresponding geometry is called "hyperbolic". Unlike in the case of a maximally symmetric two-dimensional space with *positive* curvature (namely, the surface of a sphere), the case with negative curvature cannot be isometrically embedded in Euclidean $\mathbb{R}^3$, as proved by Hilbert in 1901. However, it *can* be isometrically embedded in Euclidean $\mathbb{R}^6$, and a specific embedding has been given by Blanusa (for which I don't have the reference, though).

D. Lie Derivative

The concept of a *Lie derivative* is introduced in order to quantify how tensors change along some given "direction" in the manifold, specified by a vector field $\boldsymbol{X}$.

Contrary to the covariant derivative, it will be seen that the Lie derivative does *not* change the order of the tensor it differentiates.

D.1. Lie derivative of a function, i. e., a scalar

We simply define the Lie derivative of a function f defined on a manifold M^n along a vector field $\boldsymbol{X}$ likewise defined on M^n as the action of $\boldsymbol{X}$ on f:

$$\mathscr{L}_{\boldsymbol{X}} f \ := \ \boldsymbol{X} f \ .$$

In local coordinates this becomes

$$\mathscr{L}_{\boldsymbol{X}} f \ = \ X^i f_{,i} \ = \ X^i \nabla_i f \ ,$$

which is indeed just the directional derivative of f "along" $\boldsymbol{X}$.

D.2. Lie Derivative of a vector field

For the following see for instance Ref. [9], page 125ff.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vector fields on a manifold M^n, and let Φ_t be the local *flow*[1] generated by X. Then $\Phi_t\vec{r}$ shifts the point $\vec{r}$ for a "time" t along the flow. One may now look at two different tangent vectors at the point $\Phi_t\vec{r}$:

$\boldsymbol{Y}_{\Phi_t\vec{r}}$	:	The value of the vector field $\boldsymbol{Y}$ at the *new* point
$(\Phi_t)_*\boldsymbol{Y}_{\vec{r}}$	:	The value of $\boldsymbol{Y}$ at the *old* point, pushed forward to the *new* point via the differential $(\Phi_t)_*$

We can use the difference between the two to define the *Lie derivative of* $\boldsymbol{Y}$ *with respect to* $\boldsymbol{X}$:

$$[\mathscr{L}_{\boldsymbol{X}}\boldsymbol{Y}]_{\vec{r}} = \lim_{t\to 0}\frac{\boldsymbol{Y}_{\Phi_t\vec{r}} - (\Phi_t)_*\boldsymbol{Y}_{\vec{r}}}{t} = \lim_{t\to 0}(\Phi_t)_*\frac{(\Phi_{-t})_*\boldsymbol{Y}_{\Phi_t\vec{r}} - \boldsymbol{Y}_{\vec{r}}}{t} = \lim_{t\to 0}\frac{(\Phi_{-t})_*\boldsymbol{Y}_{\Phi_t\vec{r}} - \boldsymbol{Y}_{\vec{r}}}{t} \ .$$

The last inequality follows because $(\Phi_0)_*$ is the identity. Note that we can also write this as

$$[\mathscr{L}_{\boldsymbol{X}}\boldsymbol{Y}]_{\vec{r}} = \left\{\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_{-t})_*\boldsymbol{Y}_{\Phi_t\vec{r}}\right\}_{t=0}$$

In Ref. [9] it is proved that this can be expressed in a coordinate free way as

$$\mathscr{L}_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X},\boldsymbol{Y}] \ ,$$

where $[\boldsymbol{X},\boldsymbol{Y}]$ is the *Lie bracket* between the two vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$. In local coordinates, this can be expressed as

$$[\mathscr{L}_{\boldsymbol{X}}\boldsymbol{Y}]^\mu = [\boldsymbol{X},\boldsymbol{Y}]^\mu = X^\nu Y^\mu_{,\nu} - Y^\nu X^\mu_{,\nu} \ . \tag{D.1}$$

D.3. Lie Derivative for a 1-form

The Lie derivative of a scalar can also be written as

$$\mathscr{L}_{\boldsymbol{X}} f = \frac{\mathrm{d}}{\mathrm{d}t} f(\Phi_t\vec{r})\Big|_{t=0} \overset{\text{(B.4)}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\big[(\Phi_t)^* f\big]\Big|_{t=0} \ ,$$

[1] Roughly speaking, this means that Φ_t is the solution of the differential equation $(\mathrm{d}/\mathrm{d}t)(\Phi_t\vec{x}) = \boldsymbol{X}_{\vec{x}}$, which states that the tangent vector to the flow at $\vec{x}$ always coincides with the vector $\boldsymbol{X}_{\vec{x}}$ at that point.

where Φ_t is the flow of $\boldsymbol{X}$ introduced in the last section and $(\Phi_t)^*$ is its *pull-back*. Since the pull-back can also act on forms, we will use this to define the Lie derivative of a p-form β^p as

$$\mathscr{L}_{\boldsymbol{X}}\beta^p := \frac{\mathrm{d}}{\mathrm{d}t}\big[(\Phi_t)^*\beta^p\big]\Big|_{t=0} .$$

We can make this even a bit more explicit. Take vectors $\boldsymbol{Y}^{(1)}, \dots \boldsymbol{Y}^{(p)}$ located at x. Then

$$\Big[\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_t)^*\beta^p\Big](\boldsymbol{Y}^{(1)}, \dots \boldsymbol{Y}^{(p)}) = \frac{\mathrm{d}}{\mathrm{d}t}\big[(\Phi_t)^*\beta^p(\boldsymbol{Y}^{(1)}, \dots \boldsymbol{Y})\big] = \frac{\mathrm{d}}{\mathrm{d}t}\beta^p\big((\Phi_t)_*\boldsymbol{Y}^{(1)}, \dots, (\Phi_t)_*\boldsymbol{Y}^{(p)}\big) .$$

In particular, one may extend the vectors $\boldsymbol{Y}^{(i)}$ to be *invariant fields under the flow*m, such that they satisfy $(\Phi_t)_*\boldsymbol{Y}^{(i)}_{\vec{x}} = \boldsymbol{Y}^{(i)}_{\Phi_t\vec{x}}$. Then we can write

$$\mathscr{L}_{\boldsymbol{X}}\beta^p(\boldsymbol{Y}^{(1)}, \dots \boldsymbol{Y}^{(p)}) = \frac{\mathrm{d}}{\mathrm{d}t}\big[\beta^p_{\Phi_t\vec{x}}(\boldsymbol{Y}^{(1)}, \dots \boldsymbol{Y}^{(p)})\big]\Big|_{t=0} .$$

In words: *$\mathscr{L}_{\boldsymbol{X}}\beta^p$ measures the derivative (as one moves along the orbit of $\boldsymbol{X}$) of the value of β^p evaluated on a p-tuple of vector fields $\boldsymbol{Y}$ that are invariant under the flow generated by $\boldsymbol{X}$ [9, page 132].*

The Lie derivative commutes with exterior differentiation:

$$\mathrm{d} \circ \mathscr{L}_{\boldsymbol{X}} = \mathscr{L}_{\boldsymbol{X}} \circ \mathrm{d} .$$

For a proof, see [9, page 133-134]. As a special application, we find the Lie derivative of a coordinate differential

$$\mathscr{L}_{\boldsymbol{X}}(\mathrm{d}x^i) = \mathrm{d}(\mathscr{L}_{\boldsymbol{X}}x^i) = \mathrm{d}(\boldsymbol{X}x^i) = \mathrm{d}\Big(X^j \underbrace{\frac{\partial}{\partial x^j}x^i}_{\delta^{ij}}\Big) = \mathrm{d}X^i .$$

The Lie derivative is a special case of a *derivation*. If α and β are two forms, the Lie derivative hence satisfies the following differentiation rule:

$$\mathscr{L}_{\boldsymbol{X}}(\alpha \otimes \beta) = (\mathscr{L}_{\boldsymbol{X}}\alpha) \otimes \beta + \alpha \otimes (\mathscr{L}_{\boldsymbol{X}}\beta)$$

Applying this to the special case of a 1-form $\alpha = \alpha_i \mathrm{d}x^i$, we find[2]

$$\begin{aligned}\mathscr{L}_{\boldsymbol{X}}\alpha &= \mathscr{L}_{\boldsymbol{X}}(\alpha_i\mathrm{d}x^i) = (\mathscr{L}_{\boldsymbol{X}}\alpha_i)\mathrm{d}x^i + \alpha_i\mathscr{L}_{\boldsymbol{X}}\mathrm{d}x^i = (\boldsymbol{X}\alpha_i)\mathrm{d}x^i + \underbrace{\alpha_i\mathrm{d}X^i}_{i\to j} \\ &= X^j\frac{\partial\alpha_i}{\partial x^j}\mathrm{d}x^i + \alpha_j\frac{\partial X^j}{\partial x^i}\mathrm{d}x^i = \Big[X^j\frac{\partial\alpha_i}{\partial x^j} + \alpha_j\frac{\partial X^j}{\partial x^i}\Big]\mathrm{d}x^i\end{aligned}$$

We thereby see that in local coordinates the formula for the Lie-derivative of a <u>1-form</u> is given by

$$[\mathscr{L}_{\boldsymbol{X}}\alpha]_i = X^j\alpha_{i,j} + \alpha_j X^j_{,i} . \tag{D.2}$$

D.4. Lie derivative of a general tensor field

Let T be an n-fold contravariant and m-fold covariant tensor, with components in local coordinates $T^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_m}$. Extending formulas (D.1) and (D.2), we find the general formula in local coordinates

$$\boxed{\begin{aligned}(\mathscr{L}_{\boldsymbol{X}}\mathrm{T})^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_m} = \;& T^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_m,\gamma}X^\gamma \\ & -T^{\gamma\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\beta_m}X^{\alpha_1}_{,\gamma} - T^{\alpha_1\gamma\dots\alpha_n}_{\beta_1\beta_2\dots\beta_m}X^{\alpha_2}_{,\gamma} - \dots - T^{\alpha_1\alpha_2\dots\gamma}_{\beta_1\beta_2\dots\beta_m}X^{\alpha_n}_{,\gamma} \\ & +T^{\alpha_1\alpha_2\dots\alpha_n}_{\gamma\beta_2\dots\beta_m}X^\gamma_{,\beta_1} + T^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\gamma\dots\beta_m}X^\gamma_{,\beta_2} + \dots + T^{\alpha_1\alpha_2\dots\alpha_n}_{\beta_1\beta_2\dots\gamma}X^\gamma_{,\beta_m} .\end{aligned}} \tag{D.3}$$

This should be compared to the general formula for the covariant derivative, (C.7).

It can be checked in an essentially straightforward way, that *in the above formula of the Lie-derivative the partial differentiations can be replaced by covariant ones.* If this is done, the first term will create a whole lot of additional Christoffel-symbol-terms, but each one of them is canceled by the terms which each of the other covariant derivatives of X^γ spawn. This observation also readily shows that the Lie derivative does indeed produce a tensor!

[2] Note that if f is a 0-form and α is a p-form, one usually abbreviates $f \otimes \alpha \to f\alpha$.

D.5. Special case: Lie derivative of the metric

Applying Eqn. (D.3) to the metric $g_{\mu\nu}$, we find

$$\mathscr{L}_{\boldsymbol{X}} g_{\mu\nu} = g_{\mu\nu,\lambda} X^\lambda + g_{\lambda\nu} X^\lambda_{,\mu} + g_{\mu\lambda} X^\lambda_{,\nu} \ . \tag{D.4}$$

From Ricci's Lemma $\nabla_\lambda g_{\mu\nu} = 0$ as well as the general formula (C.7) for the covariant derivative we find

$$g_{\mu\nu,\lambda} = g_{\kappa\nu} \Gamma^\kappa_{\mu\lambda} + g_{\mu\kappa} \Gamma^\kappa_{\nu\lambda} \ .$$

Inserting this into Eqn. (D.4), we find

$$\begin{aligned}
\mathscr{L}_{\boldsymbol{X}} g_{\mu\nu} &= \Big(g_{\kappa\nu} \Gamma^\kappa_{\mu\lambda} + g_{\mu\kappa} \Gamma^\kappa_{\nu\lambda} \Big) X^\lambda + g_{\lambda\nu} X^\lambda_{,\mu} + g_{\mu\lambda} X^\lambda_{,\nu} \\
&= \Big(g_{\lambda\nu} \Gamma^\lambda_{\mu\kappa} + g_{\mu\lambda} \Gamma^\lambda_{\nu\kappa} \Big) X^\kappa + g_{\lambda\nu} X^\lambda_{,\mu} + g_{\mu\lambda} X^\lambda_{,\nu} \\
&= g_{\lambda\nu} \Big(X^\lambda_{,\mu} + X^\kappa \Gamma^\lambda_{\mu\kappa} \Big) + g_{\mu\lambda} \Big(X^\lambda_{,\nu} + X^\kappa \Gamma^\lambda_{\nu\kappa} \Big) \\
&= g_{\lambda\nu} \nabla_\mu X^\lambda + g_{\mu\lambda} \nabla_\nu X^\lambda \\
&= \underline{\nabla_\mu X_\nu + \nabla_\nu X_\mu} \ .
\end{aligned} \tag{D.5}$$

Remember that Ricci's lemma stated that the *covariant* derivative of the metric is zero.

We want to point out that this calculation can be significantly abbreviated by making use of the fact that the partial derivatives in the formula (D.3) can be replaced by covariant ones:

$$\mathscr{L}_{\boldsymbol{X}} g_{\mu\nu} \ = \ \big(\nabla_\gamma g_{\mu\nu} \big) X^\gamma + g_{\gamma\nu} \nabla_\mu X^\gamma + g_{\mu\gamma} \nabla_\nu X^\gamma \ = \ \underline{\nabla_\mu X_\nu + \nabla_\nu X_\mu} \ .$$

The Lie derivative of the metric plays a key role in the theory of Killing fields (see Sec. B.3), which are generators of continuous isometries. A vector field is a Killing field, if the Lie derivative of the metric with respect to this field vanishes, *i. e.*, if the metric remains unchanged when dragged along the flow generated by the field.

E. Solutions to problems

Problem 1.1

Note first that $\boldsymbol{e}_a = \vec{r}_{,a} = \vec{r}_{;a} = \nabla_a \vec{r}$. The identity can now be checked by a straightforward calculation:

$$
\begin{aligned}
&\frac{1}{2}\nabla^a\Big[\big(\vec{r}\cdot\boldsymbol{e}_a\big)\,\vec{n} - \big(\vec{r}\cdot\vec{n}\big)\,\boldsymbol{e}_a\Big] \;=\\
&\frac{1}{2}\Big[\big(\vec{r}^{;a}\cdot\boldsymbol{e}_a\big)\,\vec{n} + \big(\vec{r}\cdot\boldsymbol{e}_a^{;a}\big)\,\vec{n} + \big(\vec{r}\cdot\boldsymbol{e}_a\big)\,\vec{n}^{;a} - \big(\vec{r}^{;a}\cdot\vec{n}\big)\,\boldsymbol{e}_a - \big(\vec{r}\cdot\vec{n}^{;a}\big)\,\boldsymbol{e}_a - \big(\vec{r}\cdot\vec{n}\big)\,\boldsymbol{e}_a^{;a}\Big] \;=\\
&\frac{1}{2}\Big[\big(\underbrace{\boldsymbol{e}^a\cdot\boldsymbol{e}_a}_{=\delta^a_a=2}\big)\,\vec{n} + \underline{b^a_a\big(\vec{r}\cdot\vec{n}\big)\,\vec{n}} - \underline{\underline{b^{ab}\big(\vec{r}\cdot\boldsymbol{e}_a\big)\,\boldsymbol{e}_b}} - \big(\underbrace{\boldsymbol{e}^a\cdot\vec{n}}_{=0}\big)\,\boldsymbol{e}_a + \underline{\underline{b^{ab}\big(\vec{r}\cdot\boldsymbol{e}_b\big)\,\boldsymbol{e}_a}} - \underline{b^a_a\big(\vec{r}\cdot\vec{n}\big)\,\vec{n}}\Big] \;=\\
&\vec{n} \qquad \text{...as we intended to show.}
\end{aligned}
$$

Problem 3.1

Using ugly component notation, we can calculate

$$
\begin{aligned}
& b^i_k\nabla_i\phi^k - b^i_i\nabla^k\phi_k + \varepsilon^{kl}\varepsilon_{ij}\Big[b^i_k\Big(\nabla_l\phi^j + \nabla^j\phi_l\Big)\Big] \quad =\\
=\quad & \Big[b^1_1\nabla_1\phi^1 + b^2_2\nabla_2\phi^2 + b^1_1\nabla_2\phi^1 + b^1_2\nabla_1\phi^2\Big] \;-\; \Big[b^1_1\nabla^1\phi_1 + b^2_2\nabla^2\phi_2 + b^1_1\nabla^2\phi_2 + b^2_2\nabla^1\phi_1\Big]\\
& + b^1_1(\nabla_2\phi^2 + \nabla^2\phi_2) + b^2_2(\nabla_1\phi^1 + \nabla^1\phi_1) - b^2_1(\nabla_2\phi^1 + \nabla^1\phi_2) - b^1_2(\nabla_1\phi^2 + \nabla^2\phi_1)\\
=\quad & (b^1_1 + b^2_2)(\nabla_1\phi^1 + \nabla_2\phi^2) \;-\; \Big[b^1_1\nabla^1\phi_1 + b^2_2\nabla^2\phi_2 + b^2_1\nabla^1\phi_2 + b^1_2\nabla^2\phi_1\Big]\\
=\quad & b^i_i\nabla_k\phi^k - b^k_i\nabla^i\phi_k \quad = \quad \underline{2H\nabla_k\phi^k - b^i_k\nabla_i\phi^k}\;.
\end{aligned}
$$

Note that the ultimate effect of the determinant-like expression is to change the sign of the two other terms!

Problem 3.2

Using Eqn. (3.3) for the variation of the tangent vector, we can work out their product up to linear order:

$$
\begin{aligned}
\frac{\mathrm{d}\vec{r}'^1\times\mathrm{d}\vec{r}'^2}{\mathrm{d}u^1\mathrm{d}u^2} \quad &= \quad \big[\boldsymbol{e}_1 + (\nabla_1\phi^i - \psi b^i_1)\boldsymbol{e}_i + (\phi^i b_{1i} + \psi_{,1})\vec{n}\big] \times \big[\boldsymbol{e}_2 + (\nabla_2\phi^j - \psi b^j_2)\boldsymbol{e}_j + (\phi^j b_{2j} + \psi_{,2})\vec{n}\big]\\
&= \quad \boldsymbol{e}_1\times\boldsymbol{e}_2\\
&\qquad + \boldsymbol{e}_1\times\boldsymbol{e}_j(\nabla_1\phi^j - \psi b^j_1) + \boldsymbol{e}_i\times\boldsymbol{e}_2(\nabla_1\phi^i - \psi b^i_1)\\
&\qquad + \boldsymbol{e}_1\times\vec{n}(\phi^j b_{2j} + \psi_{,2}) + \vec{n}\times\boldsymbol{e}_2(\phi^i b_{1i} + \psi_{,1}) \;+\; \mathscr{O}(2)\\
&= \quad \vec{n}\sqrt{g}\big(1 + \nabla_i\phi^i - 2H\psi\big) + \boldsymbol{e}_1\times\vec{n}(\phi^j b_{2j} + \psi_{,2}) + \vec{n}\times\boldsymbol{e}_2(\phi^i b_{1i} + \psi_{,1}) \;+\; \mathscr{O}(2)\;.
\end{aligned} \tag{E.1}
$$

We now rewrite the vector products as follows:

$$
\begin{aligned}
\boldsymbol{e}_1\times\vec{n} \quad &= \quad \frac{\boldsymbol{e}_1\times(\boldsymbol{e}_1\times\boldsymbol{e}_2)}{\sqrt{g}} \;=\; \frac{(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)\boldsymbol{e}_1 - (\boldsymbol{e}_1\cdot\boldsymbol{e}_1)\boldsymbol{e}_2}{\sqrt{g}} \;=\; \frac{g_{12}\boldsymbol{e}_1 - g_{11}\boldsymbol{e}_2}{\sqrt{g}} \;\overset{(1.5)}{=}\; -\sqrt{g}g^{2i}\boldsymbol{e}_i\;,\\
\vec{n}\times\boldsymbol{e}_2 \quad &= \quad \frac{(\boldsymbol{e}_1\times\boldsymbol{e}_2)\times\boldsymbol{e}_2}{\sqrt{g}} \;=\; \frac{(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)\boldsymbol{e}_2 - (\boldsymbol{e}_2\cdot\boldsymbol{e}_2)\boldsymbol{e}_1}{\sqrt{g}} \;=\; \frac{g_{12}\boldsymbol{e}_2 - g_{22}\boldsymbol{e}_1}{\sqrt{g}} \;\overset{(1.5)}{=}\; -\sqrt{g}g^{1i}\boldsymbol{e}_i\;.
\end{aligned}
$$

Inserting this into Eqn. (E.1) we obtain Eqn. (3.15).

Problem 4.1

Eqn. (4.1) defines the function $h(x,y)$. All we need to do is to verify that the mean curvature, which in Monge parameterization is given by Eqn. (2.5a), vanishes vor all x and y. Let's first calculate the partial derivatives of first and second order:

$$
h_x \;=\; -\tan x \quad,\quad h_y \;=\; \tan y \quad,\quad h_{xx} \;=\; -\cos^{-2}x \quad,\quad h_{yy} \;=\; \cos^{-2}y \quad,\quad h_{xy} \;=\; h_{yx} \;=0\;.
$$

From this we find that the nominator of the expression for H in Eqn. (2.5a) is given by

$$
h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_{xy}h_xh_y = -\frac{1+\tan^2 y}{\cos^2 x} + \frac{1+\tan^2 x}{\cos^2 y} = \frac{-\cos^2 y\,(1+\tan^2 y) + \cos^2 x\,(1+\tan^2 x)}{\cos^2 x\;\cos^2 y} = 0\;.
$$

Problem C.1

The surface of a sphere of radius R in spherical coordinates $u^1 = \vartheta$ and $u^2 = \varphi$ is parameterized by

$$\vec{r} = \begin{pmatrix} R\sin\vartheta\cos\varphi \\ R\sin\vartheta\sin\varphi \\ R\cos\vartheta \end{pmatrix} \quad , \quad \boldsymbol{e}_\vartheta = \frac{\partial\vec{r}}{\partial\vartheta} = \begin{pmatrix} R\cos\vartheta\cos\varphi \\ R\cos\vartheta\sin\varphi \\ -R\sin\vartheta \end{pmatrix} \quad , \quad \boldsymbol{e}_\varphi = \frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix} -R\sin\vartheta\sin\varphi \\ R\sin\vartheta\cos\varphi \\ 0 \end{pmatrix} .$$

From this we also get the metric and its inverse

$$g_{ij} \;=\; \boldsymbol{e}_i\cdot\boldsymbol{e}_j \;=\; \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\vartheta \end{pmatrix} \quad , \quad g^{ij} \;=\; \frac{1}{R^4\sin^2\vartheta}\begin{pmatrix} R^2\sin^2\vartheta & 0 \\ 0 & R^2 \end{pmatrix} \;=\; \begin{pmatrix} 1/R^2 & 0 \\ 0 & 1/(R^2\sin^2\vartheta) \end{pmatrix} . \tag{E.2}$$

The *outward* normal vector $\vec{n}$ is found via

$$\boldsymbol{e}_\vartheta\times\boldsymbol{e}_\varphi = R^2\sin\vartheta\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \qquad \Rightarrow \qquad \vec{n} = \frac{\boldsymbol{e}_\vartheta\times\boldsymbol{e}_\varphi}{|\boldsymbol{e}_\vartheta\times\boldsymbol{e}_\varphi|} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} = \frac{\vec{r}}{|\vec{r}|} .$$

From Eqn. (C.11) and Eqn. (E.2) we get the covariant Laplacian, which is of course just the angular part of the Laplacian in spherical curvilinear coordinates:

$$\nabla^2 = \frac{1}{R^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\Big(\sin\vartheta\frac{\partial}{\partial\vartheta}\Big) + \frac{1}{R^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} .$$

Applying this operator on the three components of $\vec{r}$ we find:

$$\begin{aligned}
\nabla^2(R\sin\vartheta\cos\varphi) \;&=\; \frac{1}{R\sin\vartheta}\partial_\vartheta\big[\sin\vartheta\;\partial_\vartheta(\sin\vartheta\cos\varphi)\big] + \frac{1}{R\sin^2\vartheta}\partial_\varphi^2\sin\vartheta\cos\varphi \\
&=\; \frac{\cos\varphi}{R\sin\vartheta}(\cos^2\vartheta-\sin^2\vartheta) - \frac{\cos\varphi}{R\sin\vartheta} \;=\; -\frac{2}{R}\sin\vartheta\cos\varphi \;, \\[1ex]
\nabla^2(R\sin\vartheta\sin\varphi) \;&=\; \frac{1}{R\sin\vartheta}\partial_\vartheta\big[\sin\vartheta\;\partial_\vartheta(\sin\vartheta\sin\varphi)\big] + \frac{1}{R\sin^2\vartheta}\partial_\varphi^2\sin\vartheta\sin\varphi \\
&=\; \frac{\sin\varphi}{R\sin\vartheta}(\cos^2\vartheta-\sin^2\vartheta) - \frac{\sin\varphi}{R\sin\vartheta} \;=\; -\frac{2}{R}\sin\vartheta\sin\varphi \;, \\[1ex]
\nabla^2(R\cos\vartheta) \;&=\; \frac{1}{R\sin\vartheta}\partial_\vartheta\big[\sin\vartheta\;\partial_\vartheta\cos\vartheta\big] \;=\; -\frac{1}{R\sin\vartheta}\partial_\vartheta\sin^2\vartheta \;=\; -\frac{2}{R}\cos\vartheta \;.
\end{aligned}$$

We thus obtain

$$\nabla^2\,\vec{r} \;=\; -\frac{2}{R}\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \;=\; -\frac{2}{R}\vec{n} \;=\; 2H\vec{n} \;.$$

Indeed, the two principal curvatures of a sphere are both $1/R$, so twice the mean curvature is $2H = 2/R$, and the minus sign is due to the fact that the normal vector points *outward* and the sphere bends away from it.

Problem C.2

(a) In components we have $\boldsymbol{\partial}_x = K^i\boldsymbol{\partial}_i$, and thus $K^x = 1$ and $K^y = 0$. The covariant components are given by $K_i = g_{ij}K^j$, so we have $K_x = 1/y^2$ and $K_y = 0$. In order to see that we have a Killing field, we have to verify Eqn. (B.11), which means (by virtue of Eqn. (D.5)) we have to show that all components of the symmetric tensor $\nabla_iK_j + \nabla_jK_i$ vanish. Using the values of the Christoffel symbols from Eqn. (C.13), we find

$$\begin{aligned}
(\mathscr{L}_{\boldsymbol{\partial}_x}\boldsymbol{g})_{xx} \;&=\; 2\big(K_{x,x} - \Gamma^i_{xx}K_i\big) \;=\; 0 - 2\frac{1}{y}K_y \;=\; 0 \;, \\
(\mathscr{L}_{\boldsymbol{\partial}_x}\boldsymbol{g})_{yy} \;&=\; 2\big(K_{y,y} - \Gamma^i_{yy}K_i\big) \;=\; 0 - 2\Big(-\frac{1}{y}\Big)K_y \;=\; 0 \;, \\
(\mathscr{L}_{\boldsymbol{\partial}_x}\boldsymbol{g})_{xy} \;&=\; \big(K_{x,y} - \Gamma^i_{xy}K_i\big) + \big(K_{y,x} - \Gamma^i_{yx}K_i\big) \;=\; -\frac{2}{y^3} - \Big(-\frac{1}{y}\Big)\frac{1}{y^2} + 0 - \Big(-\frac{1}{y}\Big)\frac{1}{y^2} \;=\; 0 \;,
\end{aligned}$$

and this proves that $\boldsymbol{\partial}_x$ is indeed a Killing field.

(b) In this case the components of the vector field are $K^x = x$ and $K^y = y$. We now find

$$\begin{aligned}
(\mathscr{L}_{x\boldsymbol{\partial}_x+y\boldsymbol{\partial}_y}\boldsymbol{g})_{xx} \;&=\; 2\big(K_{x,x} - \Gamma^i_{xx}K_i\big) \;=\; 2\Big(\frac{1}{y^2} - \frac{1}{y}\frac{1}{y}\Big) \;=\; 0 \;, \\
(\mathscr{L}_{x\boldsymbol{\partial}_x+y\boldsymbol{\partial}_y}\boldsymbol{g})_{yy} \;&=\; 2\big(K_{y,y} - \Gamma^i_{yy}K_i\big) \;=\; 2\Big(-\frac{1}{y^2} - \Big(-\frac{1}{y}\Big)\frac{1}{y}\Big) \;=\; 0 \;, \\
(\mathscr{L}_{x\boldsymbol{\partial}_x+y\boldsymbol{\partial}_y}\boldsymbol{g})_{xy} \;&=\; \big(K_{x,y} - \Gamma^i_{xy}K_i\big) + \big(K_{y,x} - \Gamma^i_{yx}K_i\big) \;=\; -\frac{2x}{y^3} - \Big(-\frac{1}{y}\Big)\frac{x}{y} + 0 - \Big(-\frac{1}{y}\Big)\frac{x}{y^2} \;=\; 0 \;,
\end{aligned}$$

showing that $x\boldsymbol{\partial}_x + y\boldsymbol{\partial}_y$ is also a Killing field. Note that it describes a "stretching" of the plane away from the "origin" at $x = 0, y = 0$. It may be surprising that this is actually an *isometry*, *i. e.*, leaves hyperbolic lengths unchanged.

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