Fluid lipid membranes:
From differential geometry to curvature stresses

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These notes are based on the following review article:

M. Deserno, Chem. Phys. Lipids 185, 11-45 (2015)

Motivation: How do we describe the surface of a sphere?

ugly!

$$h(x,y) = \pm\sqrt{R^2 - x^2 - y^2}$$

cartesian coordinates are a graceless way to describe anything non-rectangular.

$$\vec{X}(\vartheta,\varphi) = R\begin{pmatrix} \sin\vartheta\,\sin\varphi \\ \sin\vartheta\,\cos\varphi \\ \cos\vartheta \end{pmatrix}$$

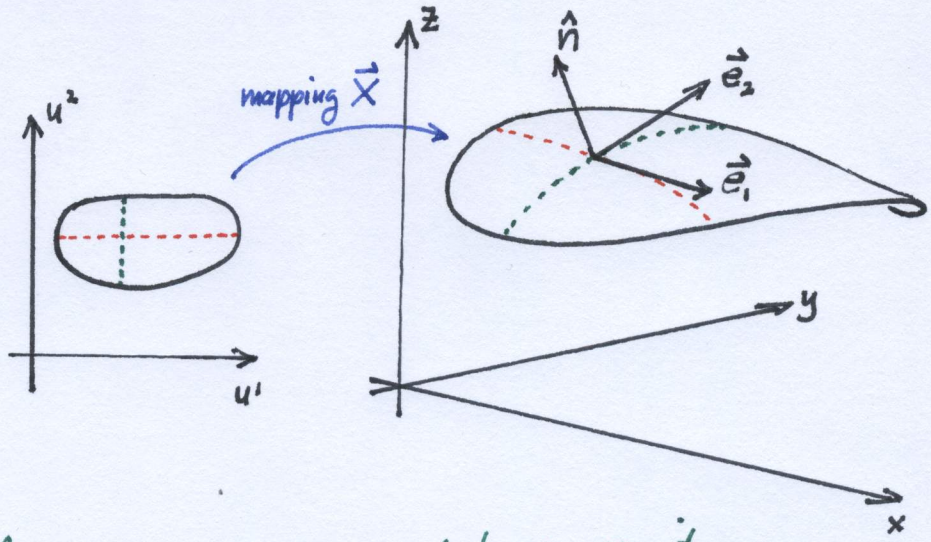
now we're using spherical coordinates, but the expression is still nonlinear!

$$r = R$$

this looks nice!
But it is global, not local!
Can we generalize this in a local way?

General purpose parametrization:

$$\vec{X}(u^1, u^2) = \begin{pmatrix} X(u^1, u^2) \\ Y(u^1, u^2) \\ Z(u^1, u^2) \end{pmatrix}$$



To describe a complete surface, we generally need to cover it with multiple "patches", or multiple <u>maps</u>.

A collection of maps that covers a surface is called an <u>"atlas"</u>.

Where two patches overlap, the connection must be differentiable:

$$\left.\begin{aligned} (u^1, u^2) &\longrightarrow \vec{X}(u^1, u^2) \\ (v^1, v^2) &\longrightarrow \vec{Y}(v^1, v^2) \end{aligned}\right\} \quad \begin{aligned} \vec{Y}^{-1} \circ \vec{X} &: \quad u\text{-space} \to v\text{-space} \\ \vec{X}^{-1} \circ Y &: \quad v\text{-space} \to u\text{-space} \end{aligned}$$

must be differentiable! ("diffeomorphism")

tangent vectors:

$$\vec{e}_a := \frac{\partial \vec{X}}{\partial u^a} \equiv \partial_a \vec{X}$$

$\vec{e}_a$: lower index; u^a: upper index } what does the placement of indices mean?

Excursion: Coordinate transformations

Geometry must be coordinate invariant.

→ We must understand how objects change under coordinate transformations. Only those that do not change are going to have geometrical significance!

[This is the same reason why in physics we care about the behavior of mathematical objects under Galilei or Lorentz transformations!]

Consider a potentially very nonlinear coordinate transformation:

$$u^a \longrightarrow \bar{u}^b(u^a)$$

This changes the coordinate differentials as follows:

$$du^a = \sum_b \underbrace{\left(\frac{\partial u^a}{\partial \bar{u}^b}\right)}_{\text{Jacobian!}} d\bar{u}^b$$

This is a matrix, and the transformation of differentials is linear, even if the underlying coordinate transformation is very nonlinear!

Terminology: every 1-index object that transforms in the same way as the coordinate differentials is called a contravariant vector. They get an upper index.

Note: Not all 1-index objects transform this way! Example:

$$\vec{e}_a = \frac{\partial \vec{x}}{\partial u^a} = \sum_b \frac{\partial \vec{x}}{\partial \bar{u}^b} \frac{\partial \bar{u}^b}{\partial u^a} = \sum_b \left(\frac{\partial \bar{u}^b}{\partial u^a} \right) \vec{e}_{\bar{b}} .$$

inverse of Jacobian!

Terminology: every 1-index object that transforms in the same way as the tangent vector is called a covariant vector. They get a lower index.

At first sight, this difference looks like a big nuisance. However, it offers a great opportunity: Since Jacobians get canceled by inverse Jacobians, we can construct objects out of contravariant and covariant vectors in which all Jacobians cancel upon coordinate transformations. These objects are hence independent of our choice of coordinates. They are geometrical scalars!

Example:

$$\underbrace{\sum_a X^a Y_a}_{\text{old coordinates}} = \sum_a \left[\sum_b \bar{X}^b \left(\frac{\partial u^a}{\partial \bar{u}^b} \right) \right] \left[\sum_c \bar{Y}_c \left(\frac{\partial \bar{u}^c}{\partial u^a} \right) \right]$$

$$= \sum_{b,c} \bar{X}^b \bar{Y}_c \underbrace{\sum_a \left(\frac{\partial \bar{u}^c}{\partial u^a} \frac{\partial u^a}{\partial \bar{u}^b} \right)}_{= \delta_b^c}$$

$$= \sum_{b,c} \bar{X}^b \bar{Y}_c \, \delta_b^c = \underbrace{\sum_b \bar{X}^b \bar{Y}_b}_{\text{new coordinates}}$$

→ same form as in old coordinates!

Since this always requires summing over one covariant and one contravariant index, we can conveniently keep track of this by

i) always keeping $\left\{ \begin{matrix} \text{contravariant} \\ \text{covariant} \end{matrix} \right\}$ indices $\left\{ \begin{matrix} \text{upstairs} \\ \text{downstairs} \end{matrix} \right\}$.

ii) automatically sum over a doubly occurring index, when it occurs as an upstairs-downstairs pair. We then don't need to write the summation symbol: it is <u>implied</u>.

"Einstein summation convention".

Further important geometrical objects

"hat" means: unit vector!

normal vector: $\hat{n} := \dfrac{\vec{e}_1 \times \vec{e}_2}{|\vec{e}_1 \times \vec{e}_2|}$

The normal vector is perpendicular to the surface, and it is by definition normalized to unit length.

In contrast, the tangent vectors are neither normalized, nor necessarily perpendicular to one another. We can summarize their relation to one another by the

ordinary scalar product in the embedding 3-space.

metric tensor: $g_{ab} := \vec{e}_a \cdot \vec{e}_b$

observe that this is obviously symmetric!

Observe that using the metric, we can express scalar products from 3-space via their surface-based coordinates:

$$\vec{V} := V^a \vec{e}_a \qquad \vec{W} := W^b \vec{e}_b$$

$$\vec{V} \cdot \vec{W} = (V^a \vec{e}_a) \cdot (W^b \vec{e}_b) = V^a W^b \vec{e}_a \cdot \vec{e}_b = V^a W^b g_{ab}$$

inverse metric: written as g^{ab}, and defined such that

$$g^{ab} g_{bc} = \delta^a_c = \begin{cases} 1, & \text{if } a = c \\ 0, & \text{otherwise} \end{cases}$$ "Kronecker-delta"

Important: we can use the inverse metric and the metric to raise and lower indices, respectively:

$$V^a = g^{ab} V_b \qquad W_{ab} = g_{ac} g_{bd} W^{cd}$$

The metric tensor therefore provides a connection between the space of contravariant vectors (tensors) and covariant vectors (tensors). In the absence of a metric, these are really independent objects, and the possibility of identifying them is a special property of a manifold that actually has a metric. ("Riemannian manifold")

In this case it therefore makes sense to speak of a vector (or tensor) and its contravariant or covariant <u>components</u>, because after identifying them via the metric, this really is the same geometrical object.

Warning: When you raise or lower an index on a tensor with many indices, you need to keep track which one you raised or lowered — unless the tensor has certain symmetry properties that make it unnecessary to distinguish which index is the first, second, ...

$$R_{ab}{}^{c}{}_{d} = R_{abed}\, g^{ce}$$

lifting the third index!

Metric determinant:

Warning: This looks like a scalar, since it has no indices, but it is actually a scalar tensor density of weight 2. See e.g. Lovelock & Rund, "Tensors, Differential Forms, and Variational Principles", chapter 4.1

$$g := \det(g_{ab}) = g_{11}g_{22} - g_{12}g_{21}$$

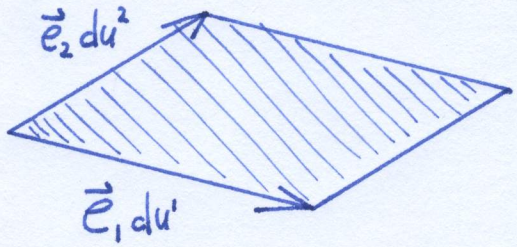
$$= (\vec{e}_1\cdot\vec{e}_1)(\vec{e}_2\cdot\vec{e}_2) - (\vec{e}_1\cdot\vec{e}_2)^2$$

angle between $\vec{e}_1$ and $\vec{e}_2$

$$= |\vec{e}_1|^2\,|\vec{e}_2|^2\left[\,1 - \cos^2\varphi\,\right]$$

$$= |\vec{e}_1|^2\cdot|\vec{e}_2|^2\,\sin^2\varphi$$

$$= |\vec{e}_1\times\vec{e}_2|^2$$



area element:

$$dA = |\vec{e}_1\,du^1\times\vec{e}_2\,du^2| = |\vec{e}_1\times\vec{e}_2|\,du^1du^2 = \sqrt{g}\,du^1du^2$$

Curvature tensor: $K_{ab} = \vec{e}_a \cdot \partial_b \hat{n}$

"how does the projection of the normal vector into the plane changes, as we move along the surface?"

Observe: $\vec{e}_a \cdot \hat{n} = 0 \Rightarrow \partial_b (\vec{e}_a \cdot \hat{n}) = 0$

product rule $\Rightarrow$ $(\partial_b \vec{e}_a) \cdot \hat{n} + \underbrace{\vec{e}_a \cdot \partial_b \hat{n}}_{K_{ab}} = 0$

$$\Rightarrow K_{ab} = \vec{e}_a \cdot \partial_b \hat{n} = -\hat{n} \cdot \partial_b \vec{e}_a = -\hat{n} \cdot \underbrace{\frac{\partial^2 \vec{X}}{\partial u^b \partial u^a}}_{\text{symmetric in } a,b\,!}$$

Normal curvature

Let $\hat{t} = t^a \vec{e}_a$ be a vector tangential to the surface.

If we "cut" the surface with a plane spanned by $\hat{t}$ and $\hat{n}$, we obtain a planar cutting curve.

It's curvature at the point where $\hat{t}$ sits is called the "normal curvature of the surface into the direction $\hat{t}$."

It can be expressed as follows:

$$K_{\hat{t}} = K_{ab}\, t^a t^b \qquad \text{(normal curvature)}$$

Observe that the normal curvature is a <u>quadratic</u> in the components of the associated tangent vector $\hat{t}$!

<u>Question:</u> In which directions are the curvatures extremal?

<u>Answer:</u> Extremize $K_{\hat{t}}$ subject to the constraint $|\hat{t}| = 1$:

Lagrange multiplier enforcing ↑

$$0 = \frac{\partial}{\partial t^c}\left[K_{ab}\, t^a t^b - c\left(g_{ab}\, t^a t^b - 1 \right)\right]$$

$$0 = K_{ab}\left(\underbrace{\frac{\partial t^a}{\partial t^c}}_{\delta^a_c}\, t^b + t^a \underbrace{\frac{\partial t^b}{\partial t^c}}_{\delta^b_c} \right) - c\, g_{ab}\left(\underbrace{\frac{\partial t^a}{\partial t^c}}_{\delta^a_c}\, t^b + t^a \underbrace{\frac{\partial t^b}{\partial t^c}}_{\delta^b_c} \right)$$

$$0 = \underbrace{K_{cb}\, t^b + K_{ac}\, t^a}_{\text{identical, since } K_{ab} = K_{ba}} - c\underbrace{\left(g_{cb}\, t^b + g_{ac}\, t^a \right)}_{\text{identical, since } g_{ab} = g_{ba}} \qquad \| : 2$$

$$0 = \left(K_{ab} - c\, g_{ab} \right) t^b \qquad \| \text{ lift a index}$$

$$0 = \left(K^a_{\ b} - c\, g^a_b \right) t^b$$

need not write $K^a{}_b$, because K_{ab} is symmetric and so the placement of indices is irrelevant!

$g^a_b \equiv \delta^a_b$

This can now be written as:

$$\boxed{K^a_b\, t^b = c\, t^a}$$

This is an eigenvalue equation involving the matrix K^a_b !

$\Rightarrow$ The extremal values of the normal curvature are the eigenvalues of the matrix K^b_a. They are called "principal curvatures". The associated directions are the corresponding eigenvectors — which must be perpendicular because the curvature tensor is symmetric. They are called the "principal directions".

Scalar invariants

Now that we have the two important tensors g_{ab} and K_{ab}, we can go about constructing scalar invariants from them.

At linear order in the curvature tensor, we get

$$K := \mathrm{Tr}(\mathbb{K}) = g^{ab} K_{ab} = c_1 + c_2$$

$\mathbb{K}$: curvature tensor

c_1, c_2: principal curvatures

"curvature" (twice the "mean curvature")

It is well known that a tensor has two quadratic invariants:

The square of the trace and the trace of the square:

1) $\left[\mathrm{Tr}\left(\mathbb{K}\right)\right]^2 = K^2 = \left(c_1 + c_2\right)^2$

2) $\mathrm{Tr}\left(\mathbb{K}^2\right) = \mathrm{Tr}\left(K^b_a K^a_b\right)$

since traces don't change under coordinate transformations, we can go into a coordinate system in which K^b_a is diagonal!

$$= \mathrm{Tr}\left(\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \cdot \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{pmatrix}\right) = c_1^2 + c_2^2$$

Since the curvature tensor is a 2×2 tensor, its determinant is also a quadratic invariant! Is it independent?

3) $K_G := \det\left(K^b_a\right) = c_1 \cdot c_2$ "Gaussian curvature"

$\longrightarrow$ not independent!

could also have been derived from the Hamilton-Cayley theorem.

$$\left[\mathrm{Tr}\left(\mathbb{K}\right)\right]^2 - \mathrm{Tr}\left(\mathbb{K}^2\right) = \left(c_1+c_2\right)^2 - \left(c_1^2+c_2^2\right) = 2c_1c_2 = 2K_G$$

$\longrightarrow$ Typically, one takes K^2 and K_G as the two independent quadratic invariants!

Covariant differentiation

Partial derivatives of tensors generally are not tensors!

Example: How does $\partial_a \vec{e}_b$ transform?

$$\partial_a \vec{e}_b = \frac{\partial}{\partial u^a}\frac{\partial \vec{X}}{\partial u^b} = \frac{\partial}{\partial u^a}\left(\frac{\partial \vec{X}}{\partial \bar{u}^d}\frac{\partial \bar{u}^d}{\partial u^b}\right)$$

$$= \left(\frac{\partial}{\partial \bar{u}^c}\frac{\partial \vec{X}}{\partial \bar{u}^d}\right)\frac{\partial \bar{u}^c}{\partial u^a}\frac{\partial \bar{u}^d}{\partial u^b} + \frac{\partial \vec{X}}{\partial \bar{u}^d}\frac{\partial^2 \bar{u}^d}{\partial u^a \partial u^b}$$

$$= \underbrace{\partial_{\bar{c}} \vec{e}_{\bar{d}}\frac{\partial \bar{u}^c}{\partial u^a}\frac{\partial \bar{u}^d}{\partial u^b}}_{\text{this part constitutes the ordinary transformation}} + \underbrace{\frac{\partial \vec{X}}{\partial \bar{u}^d}\frac{\partial^2 \bar{u}^d}{\partial u^a \partial u^b}}_{\text{... but this extra bit spoils the tensor character!}}$$

The reason for this is that not just the tensors themselves, but also the coordinate system depends on the position, and this drags the arbitrariness of the parametrization into the derivative!

Can we somehow "subtract out" the offending bit?

Yes → "covariant derivative"

Example:

covariant derivative

$$\nabla_a V^b := \vec{e}^{\,b} \cdot \partial_a \vec{V}$$

partially differentiate vector, then project onto tangent direction belonging to component

$$= \vec{e}^{\,b} \cdot \partial_a \left(V^c \vec{e}_c \right)$$

$$= \vec{e}^{\,b} \cdot \left(\vec{e}_c \, \partial_a V^c + V^c \partial_a \vec{e}_c \right)$$

$$= \underbrace{\left(\vec{e}^{\,b} \cdot \vec{e}_c \right)}_{= \delta^b_c} \partial_a V^c + V^c \underbrace{\left(\vec{e}^{\,b} \cdot \partial_a \vec{e}_c \right)}_{\text{what is this ???}}$$

$\rightarrow$ call it Γ^b_{ac} for now

$$= \underbrace{\partial_a V^b}_{\text{not a tensor!}} + \Gamma^b_{ac} V^c$$

so this cannot be a tensor either, if it is supposed to "subtract out" the "non tensorness" of $\partial_a V^b$!

The object Γ^b_{ac} is called the <u>Christoffel-symbol</u> (of the second kind). One may show that it is entirely given by the metric and partial derivatives of it:

$$\Gamma_{acd} = \frac{1}{2}\left(\partial_a g_{cd} + \partial_c g_{da} - \partial_d g_{ac}\right)$$

Christoffel symbol of first kind

$$\Gamma^b_{ac} = g^{bd}\,\Gamma_{acd}$$

Christoffel symbol of second kind

- The Christoffel symbols are purely <u>intrinsic</u>: They are given by the metric alone. At no point do they make a reference to the "embedding space"; in particular, they do not require referring to the normal vector.
- It is virtually always a bad idea to drag Christoffel symbols into any calculation. The goal should always be to work with covariant derivatives and never split them into partial derivatives and a Christoffel term. Thankfully, covariant derivatives are the ones that are geometrically meaningful, and so they will occur naturally. In contrast, partial derivatives almost never show up. (Even in the definition of the curvature tensor: we can replace the partial by the covariant derivative!) (see page 19)

So we have:

$$\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c$$

for a covariant vector, we instead find

$$\nabla_a V_b = \partial_a V_b - \Gamma^c_{ab} V_c$$

Proof: define the scalar $W^b V_b$, use the fact that covariant derivatives of scalars coincide with partial ones, and use the product rule:

$$\nabla_a (W^b V_b) = \partial_a (W^b V_b)$$

$$V_b \nabla_a W^b + W^b \nabla_a V_b = V_b \partial_a W^b + W^b \partial_a V_b$$

$$V_b (\partial_a W^b + \Gamma^b_{ac} W^c) + W^b \nabla_a V_b = V_b \partial_a W^b + W^b \partial_a V_b$$

related: $b \leftrightarrow c$

$$V_c \Gamma^c_{ab} W^b + W^b \nabla_a V_b = W^b \partial_a V_b$$

$$W^b \left(\nabla_a V_b - \partial_a V_b + \Gamma^c_{ab} V_c \right) = 0$$

arbitrary! $\Longrightarrow$ must vanish $\longrightarrow$ q.e.d.

More generally: the covariant derivative of arbitrary tensors is the partial derivative, and then every contravariant index gets a $+\Gamma^{\cdot}_{\cdot a} T^{\dots a \dots}_{\dots\dots}$ term added, and every covariant index gets a $-\Gamma^{c}_{\cdot\cdot} T^{\dots\dots}_{\dots c \dots}$ subtracted.

Lemma of Ricci:

$$\nabla_a g_{bc} = 0 \qquad \nabla_a g^{bc} = 0 \qquad \nabla_a g = 0$$

- Hugely important consequence: Raising and lowering of indices commutes with covariant differentiation.
- This also makes it possible to define $\nabla^a = g^{ab}\nabla_b$.

But: there is one price to pay: Covariant derivatives only commute when acting on a scalar. When acting on vectors and higher order tensors, this is no longer true. For instance, their action on a vector is as follows:

$$[\nabla_a, \nabla_b] V_c = \nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{abcd} V^d$$

Riemann tensor!

→ later!

Very briefly: Integration and Stokes theorem.

Since vectors and tensors strictly "live" in the local tangent (or cotangent) plane ("bundle"), it makes no sense to add vectors defined at different points of a surface. The only objects for which it is obvious that they can be moved around (and hence compared) are scalars. And hence the only obviously meaningful integral is an integral over a scalar.

$$\int_{\mathcal{P}} dA\, \phi = \text{scalar}$$ (and hence geometrically meaningful)

$\mathcal{P}$: patch on surface

dA: area element, e.g. with coordinates $\sqrt{g}\, du^1 du^2$

ϕ: scalar

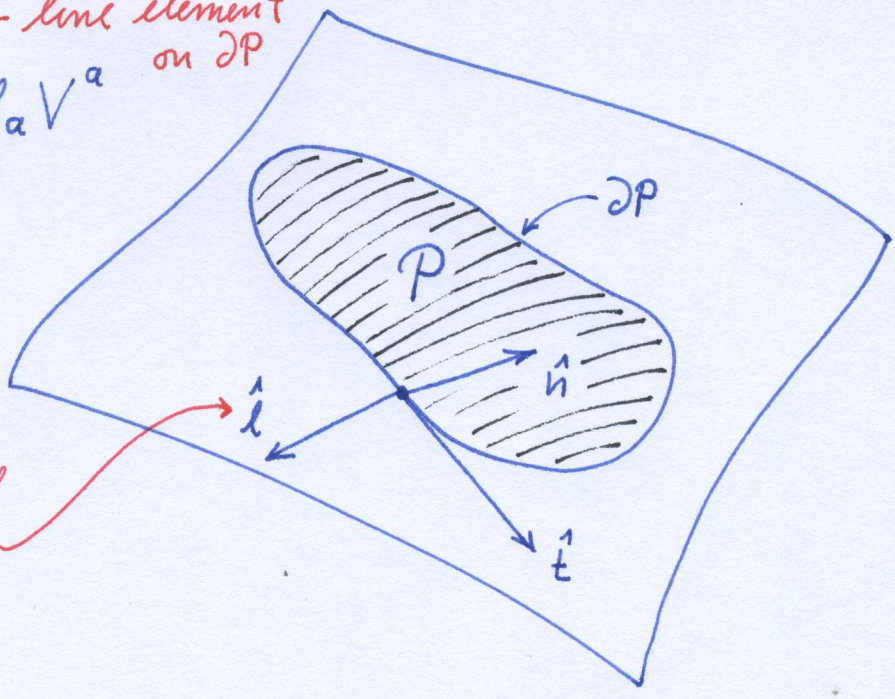
One interesting scalar we can consider is the covariant divergence of a vector field: $\phi = \nabla_a V^a$.

For this object a version of Stokes' theorem exists:

$$\int_{\mathcal{P}} dA\, \nabla_a V^a = \oint_{\partial\mathcal{P}} ds\, l_a V^a$$

line element on $\partial\mathcal{P}$

outward pointing unit vector normal to $\partial\mathcal{P}$ but tangential to the surface: $\hat{l} = l^a \vec{e}_a$



Gauss, Weingarten, Codazzi, Mainardi

There are four more equations that are very useful. The first set is an expression of the change of the local coordinate system upon differentiation, and so they are in some sense the analog of the Frenet-Serret equations for curves:

① $\nabla_a \vec{e}_b = - K_{ab} \hat{n}$ "Gauss"

② $\nabla_a \hat{n} = + K_a^b \vec{e}_b$ "Weingarten"

{ notice that they imply $\hat{n} \cdot \nabla_a \vec{e}_b = - K_{ab}$ $\quad \vec{e}_b \cdot \nabla_a \hat{n} = + K_{ab}$ }

(compare with definition on page ⑨)

The second set of equations are integrability conditions which (as always) follow from the commuting of some partial derivatives:

$$\partial_a \partial_b \vec{e}_c = \partial_b \partial_a \vec{e}_c$$

Working out these derivatives, and then taking the normal and the tangential projection of the resulting identity leads to the following equations:

(3) $\nabla_a K_{bc} = \nabla_b K_{ac}$ "Codazzi-Mainardi"

(4) $\underbrace{K_{ac}K_{bd} - K_{ad}K_{bc}}_{\text{defined extrinsically}} = R_{abcd}$ "Gauss"

R_{abcd} "Riemann tensor"

$$R_{abcd} = \underbrace{\partial_c \Gamma_{bda} - \partial_d \Gamma_{bca} + \Gamma^e_{bc}\Gamma_{ade} - \Gamma^e_{bd}\Gamma_{ace}}_{\text{defined fully intrinsically! Only metric required!}}$$

(In fact, from this definition it is not even obvious that R_{abcd} is a tensor, given that the Christoffel symbols themselves are not tensors!)

Let us contract the Gauss equation once:

$$R_{bd} := g^{ac} R_{abcd} = K K_{bd} - K_{ad} K^{a}_{b}$$

"Ricci tensor"

and let us contract one more time

$$R := g^{bd} R_{bd} = K^2 - K^{b}_{a} K^{a}_{b}$$

see page (12)

$$= (c_1 + c_2)^2 - (c_1^2 + c_2^2)$$

$$= 2 c_1 c_2 = 2 K_G$$

"Ricci scalar"

But this is entirely intrinsic!

!

Gaussian curvature!

To define this via the tensor K_{ab}, we made reference to the embedding space!

=> Gauss' "Theorema Egregium":

The Gaussian curvature is an <u>intrinsic</u> property of the surface.

(never mind that we originally defined it with reference to an embedding!)

The Helfrich Hamiltonian

The goal is to construct a surface energy functional such that the total energy of a membrane can be written as an area integral over an energy density, which is constructed from geometric surface scalars. We can dimensionally order the possible scalars and hence systematically write that energy density as a derivative expansion

If we go up to order $(\text{length})^{-2}$, we find:

order $(\text{length})^{0}$: 1

order $(\text{length})^{-1}$: K

order $(\text{length})^{-2}$: K^2, K_G (see also page (12))

From these scalars we can construct the energy density that largely goes back to Helfrich:

energy ↓ area ↓ energy/area ↓

$$H = \int dA\, \mathcal{H}(K, K_G)$$

[Helfrich, Z. Naturforsch. C 28, 693 (1973)]

energy/area ↓ energy ↓ 1/length ↓ energy ↓

$$\mathcal{H}(K, K_G) = \sigma + \frac{1}{2}\kappa (K - K_0)^2 + \bar{\kappa} K_G$$

σ: surface tension

κ: bending modulus

K_0: spontaneous curvature

$\bar{\kappa}$: Gaussian curvature modulus

↑ This turns out to be a much more subtle concept than what one might initially think. We have no time here to dwell on details.

↑ This is only nonzero if the up-down symmetry of a membrane is broken. It is hard to make such membranes in the lab, but nature does it all the time!

Two questions:

1. What membrane shape minimizes H?
2. What are the stresses in a deformed membrane?

↑ What do we even mean by that?

To answer both questions, we need to investigate, how the membrane energy changes under deformations, especially under "infinitesimal" deformations.

→ We need calculus of variations!

Variation of the surface geometry

The brute force way of doing this is to start with a variation of the embedding functions and then step by step work out, what this implies for the "higher level" geometrical objects. We will not follow this through all the way, but show how it starts:

surface variation

$$\vec{X} \longrightarrow \vec{X} + \delta\vec{X}$$

$$\delta\vec{X} = \underbrace{\phi\,\hat{n}}_{\text{normal part}} + \underbrace{\phi^{a}\vec{e}_{a}}_{\text{tangential part}}$$

It turns out that to first order in the variation we will not have to consider tangential variations, since they merely correspond to reparametrizations.

see e.g.:
[Capovilla, Guven, Santiago, J. Phys. A: Math. Gen. 36, 6281 (2003)]

$\Rightarrow$ it is enough to write: $\delta\vec{X} = \phi\,\hat{n}$

How do such variations change other geometric objects?

$$\delta \vec{e}_a = \delta \nabla_a \vec{X} = \nabla_a \delta \vec{X} = \nabla_a (\phi \hat{n}) = \hat{n} \nabla_a \phi + \phi \nabla_a \hat{n}$$

$$= (\nabla_a \phi) \hat{n} + \phi K_a^b \vec{e}_b$$

from there we can now work out the variation of the metric:

$$\delta g_{ab} = \delta (\vec{e}_a \cdot \vec{e}_b) = \delta \vec{e}_a \cdot \vec{e}_b + \vec{e}_a \cdot \delta \vec{e}_b$$

$$= \left(\nabla_a \phi \, \hat{n} + \phi K_a^c \vec{e}_c \right) \cdot \vec{e}_b + \vec{e}_a \cdot \left(\nabla_b \phi \, \hat{n} + \phi K_b^c \vec{e}_c \right)$$

$$= \phi K_a^c g_{cb} + \phi K_b^c g_{ac}$$

$$= 2 \phi K_{ab}$$

... and so on! One would for instance also find:

$$\delta \hat{n} = (\nabla^a \phi) \vec{e}_a$$

$$\delta g^{ab} = -2 \phi K^{ab}$$

$$\delta g = 2 g K \phi$$

$$\delta K_{ab} = (-\nabla_a \nabla_b - K K_{ab} + K_G g_{ab}) \phi$$

$$\delta K = (-\Delta - K^2 + 2 K_G) \phi$$

$$\delta K_G = -K K_G \phi$$

$\Delta = \nabla_a \nabla^a$ is the covariant Laplace operator on the surface.

Calculating these variations quickly turns into an exercise that is not only tedious, but also unilluminating. The hard bit is that we have to drag the variation through all the various geometric definitions and connections.

But there is a better way: We can explicitly enforce all the geometric connections via additional Lagrange parameter constraints in the Hamiltonian, and then vary all the geometric objects independently!

$\longrightarrow$ [J. Guven, J. Phys. A: Math. Gen. 37, L313 (2004)]

e.g. Helfrich energy density; but really can be any scalar geometric density!

$$\tilde{H} := \int dA \Big\{ \mathcal{H} + \vec{f}^{\,a} \cdot (\vec{e}_a - \nabla_a \vec{X}) + \lambda^a_{\perp} (\vec{e}_a \cdot \hat{n}) + \lambda_n (n^2 - 1) + \lambda^{ab} (g_{ab} - \vec{e}_a \cdot \vec{e}_b) + \Lambda^{ab} (K_{ab} - \vec{e}_a \cdot \nabla_b \hat{n}) \Big\}$$

$\longrightarrow$ now $\vec{X}$, $\vec{e}_a$, $\hat{n}$, g_{ab}, and K_{ab} can be varied independently!

What for instance happens if we vary $\vec{X}$? Notice that this now only occurs at a single spot in the functional, namely, at the point where we enforce the definition of the tangent vectors!

$$\delta_{\vec{x}} \tilde{H} = -\int dA\, \vec{f}^{\,a} \cdot \nabla_a \delta\vec{X}$$

$$= -\int dA \left\{ \nabla_a \left(\vec{f}^{\,a} \cdot \delta\vec{X} \right) - \left(\nabla_a \vec{f}^{\,a} \right) \cdot \delta\vec{X} \right\}$$

total divergence $\Rightarrow$ push to boundary via the Stokes theorem!

$$= \underbrace{-\oint ds\, l_a \vec{f}^{\,a} \cdot \delta X}_{\text{boundary term}} + \underbrace{\int dA \left(\nabla_a \vec{f}^{\,a} \right) \cdot \delta\vec{X}}_{\text{bulk term}}$$

must vanish independently in equilibrium!

Vanishing of the bulk term, by virtue of the fundamental theorem in the calculus of variations, leads to the Euler-Lagrange equation of the system:

$$\boxed{\nabla_a \vec{f}^{\,a} = 0}$$

There exists an object, $\vec{f}^{\,a}$, which is covariantly conserved! This is an exceptionally concise form of the shape equation!

Comment: If the membrane is closed, and there is an excess pressure P on the inside, we instead get:

$$\boxed{\nabla_a \vec{f}^{\,a} = P\hat{n}}$$

Apparently, $\vec{f}^{\,a}$ is an object of considerable interest.
This raises two questions:

1) Can we express $\vec{f}^{\,a}$ in terms of the surface geometry?

2) What is the physical meaning of $\vec{f}^{\,a}$?

The answer to 1) is: yes! All we need to do is to perform the variation with respect to the other geometric objects. This will result in equations that can be solved for $\vec{f}^{\,a}$!

$$\vec{e}_\alpha: \quad \delta_{\vec{e}_\alpha}\tilde{H} = \int dA \left\{ \vec{f}^{\,a}\cdot\delta\vec{e}_\alpha + \lambda_\perp \delta\vec{e}_\alpha\hat{n} - 2\lambda^{ab}\vec{e}_a\,\delta\vec{e}_b - \Lambda^{ab}\underbrace{\nabla_b\hat{n}}_{=K_b^c\vec{e}_c}\cdot\delta\vec{e}_a \right\}$$

$$\Rightarrow \vec{f}^{\,a} = -\lambda_\perp^a\hat{n} + \left(2\lambda^{ab} + \Lambda^{ac}K_c^b\right)\vec{e}_b$$

$$\hat{n}: \Rightarrow 0 = \left(\lambda_\perp^a + \nabla_b\Lambda^{ab}\right)\vec{e}_a + \left(2\lambda_n - \Lambda^{ab}K_{ab}\right)\hat{n}$$

g_{ab}: $\delta_{g_{ab}} \tilde{H} = \delta_{g_{ab}} \int dA \Big\{ \underbrace{\mathcal{H}(g_{ab}, K_{ab})}_{= du^1 du^2 \sqrt{q}\,\mathcal{H}} + \text{constraints} \Big\}$

$$= \int du^1 du^2 \left\{ \frac{\partial \sqrt{q}\,\mathcal{H}}{\partial g_{ab}} \delta g_{ab} + \sqrt{q} \frac{\partial \text{constraints}}{\partial g_{ab}} \delta g_{ab} \right\}$$

$$= \int dA \left\{ \underbrace{\frac{1}{\sqrt{q}} \frac{\partial \sqrt{q}\,\mathcal{H}}{\partial g_{ab}}}_{=: -\frac{1}{2} T^{ab}} + \lambda^{ab} \right\} \delta g_{ab}$$

$$\Rightarrow \quad \lambda^{ab} = \tfrac{1}{2} T^{ab}$$

K_{ab}: $\delta_{K_{ab}} \tilde{H} = \int dA \left\{ \underbrace{\frac{\partial \mathcal{H}}{\partial K_{ab}}}_{=: \mathcal{H}^{ab}} + \Lambda^{ab} \right\} \delta K_{ab}$

$$\Rightarrow \quad \Lambda^{ab} = - \mathcal{H}^{ab}$$

Putting everything together, we therefore find:

$$\boxed{\vec{f}^{\,a} = \left(T^{ab} - \mathcal{H}^{ac} K_c^{\,b}\right)\vec{e}_b - \left(\nabla_b \mathcal{H}^{ab}\right)\hat{n}}$$

For any form of the energy density $\mathcal{H}$, the object $\vec{f}^{\,a}$ can therefore be calculated once we know the geometry of the surface!

$\longrightarrow$ What does $\vec{f}^{\,a}$ look like for the Helfrich Hamiltonian?

To calculate this, it is useful to first consider the pieces; in particular, let us look at the density $\mathcal{H} = K^n$:

$$\mathcal{H}^{ab}(K^n) = \frac{\partial K^n}{\partial K_{ab}} = n K^{n-1} \frac{\partial (K_{ab} g^{ab})}{\partial K_{ab}} = n K^{n-1} g^{ab}$$

The calculation of T^{ab} is slightly more involved, because we need to know what the derivative of the metric determinant with respect to the metric is.

One way to calculate this is as follows:

$$\frac{\partial g}{\partial g_{ab}} = \frac{\partial}{\partial g_{ab}} \det(g) = \frac{\partial}{\partial g_{ab}} \exp\left\{ \log\left[\det(g)\right] \right\} = \frac{\partial}{\partial g_{ab}} \exp\left\{ \mathrm{Tr}\left[\log(g)\right] \right\}$$

diagonalise g to see this!

$$= \left\{ \exp\left[\mathrm{Tr}(\log g)\right] \right\} \cdot \frac{\partial}{\partial g_{ab}} \mathrm{Tr}(\log g)$$

$$= g\, \mathrm{Tr}\left[\frac{\partial}{\partial g_{ab}} \log g \right] = g\, \mathrm{Tr}\left[g^{-1} \frac{\partial}{\partial g_{ab}} g \right]$$

$$= g\left[g^{cd} \frac{\partial g_{dc}}{\partial g_{ab}} \right] = \underline{g\, g^{ab}}$$

Hence:

$$T^{ab}(K^n) = -\frac{2}{\sqrt{g}} \frac{\partial \sqrt{g} K^n}{\partial g_{ab}}$$

$$= -\frac{2}{\sqrt{g}} \left[K^n \frac{\partial \sqrt{g}}{\partial g_{ab}} + \sqrt{g}\, n K^{n-1} \frac{\partial K_{cd} g^{cd}}{\partial g_{ab}} \right]$$

$$\frac{\partial \sqrt{g}}{\partial g_{ab}} = \frac{1}{2\sqrt{g}} \frac{\partial g}{\partial g_{ab}} = \frac{1}{2}\sqrt{g}\, g^{ab}$$

($\frac{\partial g}{\partial g_{ab}} = g\, g^{ab}$)

$$\frac{\partial K_{cd} g^{cd}}{\partial g_{ab}} = K_{cd} \frac{\partial g^{cd}}{\partial g_{ab}} = -K^{ab}$$

($\frac{\partial g^{cd}}{\partial g_{ab}} = -g^{ac} g^{bd}$)

$$= -K^n g^{ab} + 2n K^{n-1} K^{ab}$$

We also need to work out K_G. This is purely intrinsic, but it is maybe easier at this point to rewrite it extrinsically:

$$K_G = \frac{1}{2}\left(K^2 - K_{ab}K^{ab}\right)$$

See page (12)

and from these we get:

$$\mathcal{H}^{ab}(K_G) = Kg^{ab} - K^{ab}$$

$$T^{ab}(K_G) = K_G g^{ab}$$

We now have everything together to calculate $\vec{f}^{\,a}$ for the Helfrich Hamiltonian on page (23):

Notice: no $\vec{x}$ dependence!

$$\boxed{\vec{f}^{\,a} = \left\{\varkappa(K-K_0)\left[K^{ab} - \tfrac{1}{2}(K-K_0)g^{ab}\right] - \sigma g^{ab}\right\}\vec{e}_b - \varkappa\left(\nabla^a K\right)\hat{n}}$$

And since $\nabla_a \vec{f}^{\,a} = P\hat{n}$ is the shape equation, calculating this and projecting on $\hat{n}$ gives the equation

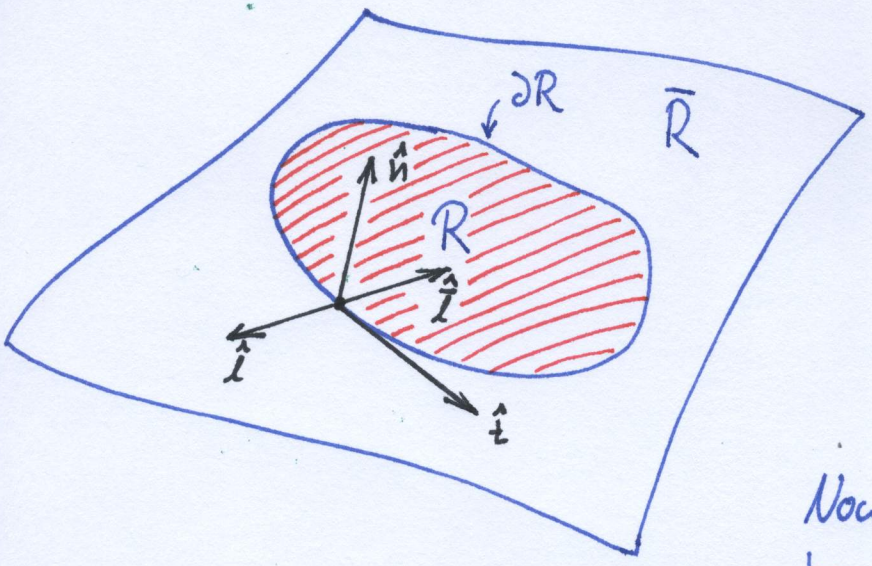
Notice: no $\vec{x}$ dependence!

$$\boxed{\sigma K - \varkappa\left\{\Delta K - \tfrac{1}{2}(K-K_0)\left[(K-K_0)K - 2K^2 + 4K_G\right]\right\} = P}$$

ΔK + "cubic in curvature"

Nothing in this equation suggests that it is actually an expression of a conservation law!

What is the physical meaning of $\vec{f}^{\,a}$?



Consider a region R on a membrane. The shape equation might not hold on R, but we assume that it holds on the complement $\bar{R}$.

Now perform a translation of R by some fixed amount $\delta\vec{a}$. How does the energy change?

We assume R translates rigidly, so there is no change in shape over R, and the only contribution comes from $\bar{R}$ and maybe the boundary ∂R. In fact, we get:

$$\delta_{\delta\vec{a}}\tilde{H} = -\oint_{\partial R} ds\, \bar{l}_a \vec{f}^{\,a}\cdot\delta\vec{a} + \int_{\bar{R}} dA\, \underbrace{\left(\nabla_a \vec{f}^{\,a}\right)}_{=0 \text{ on } \bar{R}}\cdot\delta\vec{a}$$

($\bar{l}_a$: unit vector pointing out of $\bar{R}$; $\delta\vec{a}$: constant!)

$$= -\delta\vec{a}\cdot\oint_{\partial R} ds\, l_a \vec{f}^{\,a} = \delta\vec{a}\cdot\underbrace{\oint_{\partial R} ds\, l_a \vec{f}^{\,a}}_{\vec{F}}$$

($\bar{l}_a = -l_a$)

$$= \delta\vec{a}\cdot\vec{F}$$

The energy change is equal to the translation $\delta\vec{a}$ dotted into an object $\vec{F}$. Hence, $\vec{F}$ must be the <u>force</u> with which the patch R acts on its surroundings!

$$\vec{F} = \oint ds\, l_a \vec{f}^a$$

→ But then this equation states that $\vec{f}^a$ is nothing but the <u>membrane stress tensor</u>! Contracting it with the local outward pointing normal l_a and integrating it around a closed patch gives the force with which this patch acts on its surrounding!

(attention: there are obvious minus sign convention troubles here! Force <u>of</u> patch? Force <u>on</u> patch? It's always good to make sure one understands the sign conventions. Luckily, this is relatively easy if we look at a few formulas later on!)

See also:

[Capovilla & Guven, J. Phys. A: Math. Gen. <u>35</u>, 6233 (2002)]

[Jenkins, SIAM J. Appl. Math <u>32</u>, 755 (1977)]

The force per unit length, $\vec{f}_\perp := l_a \vec{f}^{\,a}$, is sometimes called the traction. It is useful to express it in a coordinate system adjusted to the patch boundary ∂R: $\{\hat{\ell}, \hat{t}, \hat{n}\}$.

$$\vec{f}_\perp = \varkappa(K-K_0)\Big[l_a K^{ab}\underbrace{(\ell_b\ell^c + t_b t^c)}_{=\delta_b^c}\vec{e}_c - \tfrac{1}{2}(K-K_0)\underbrace{l_a g^{ab}\vec{e}_b}_{=\hat{\ell}}\Big]$$

$$- \sigma \underbrace{l_a g^{ab}\vec{e}_b}_{=\hat{\ell}} - \varkappa \underbrace{l_a \nabla^a K}_{=:\nabla_\perp,\ \text{the surface derivative along } \hat{\ell}}\hat{n}$$

$$= \varkappa(K-K_0)\Big[K_\perp \hat{\ell} + K_{\perp\parallel}\hat{t} - \tfrac{1}{2}(K-K_0)\hat{\ell}\Big] - \sigma\hat{\ell} - \varkappa(\nabla_\perp K)\hat{n}$$

$K_\perp$: $K_{ab}\ell^a\ell^b$ (normal curvature in $\hat{\ell}$-direction)

$K_{\perp\parallel}$: $K_{ab}\ell^a t^b$

also: $K_\parallel := K_{ab}t^a t^b$ (normal curvature in $\hat{t}$-direction)

$$\Rightarrow K = K_\perp + K_\parallel$$

$$\boxed{\vec{f}_\perp = \Big\{\tfrac{1}{2}\varkappa\Big[K_\perp^2 - (K_\parallel - K_0)^2\Big] - \sigma\Big\}\hat{\ell} + \varkappa(K-K_0)K_{\perp\parallel}\hat{t} - \varkappa(\nabla_\perp K)\hat{n}}$$

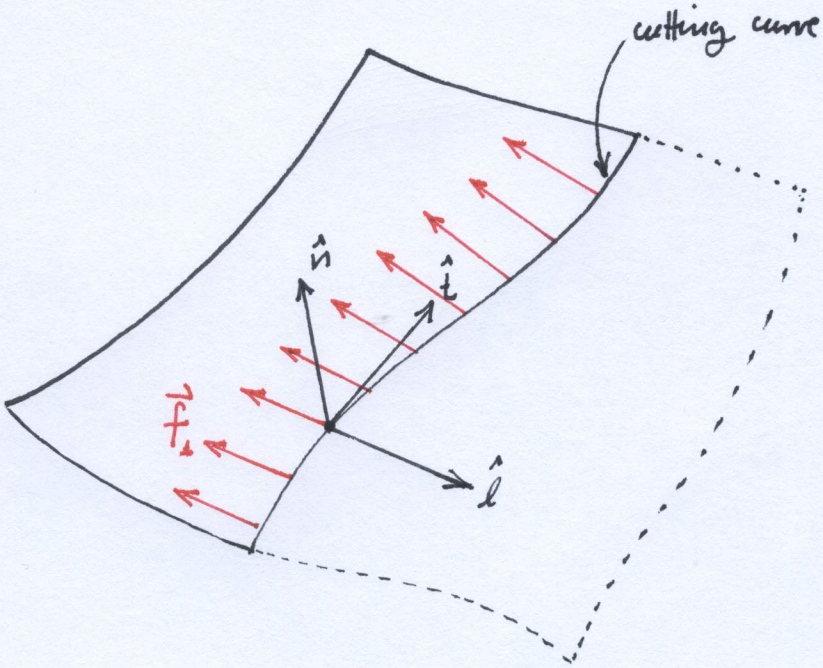
This is the traction in the ∂R adapted coordinate system, which is also called the Darboux-frame of ∂R on the surface.

see picture on page (19) or (33)

In the special case of no bending rigidity, $\varkappa = 0$, (e.g. surface of water, soap films) this strongly simplifies to:

$$\vec{f}_\perp \overset{\varkappa=0}{=\!=} -\sigma \hat{l} \qquad \text{(capillary case)}$$

This is exactly what we mechanically mean by a surface tension: A force that pulls tangentially to the surface, normal to the reference line, and of constant force per unit length σ.



Cut part of a capillary surface, such as a soap film, away. Along the cut the remaining surface will pull with a force σ per unit length, tangential to the surface, and normal to the cutting curve.

In particular, the traction <u>for the capillary case</u>, $\vec{f}_\perp = -\sigma \hat{l}$, is

- constant in magnitude
- isotropic (independent of the direction of the cut)
- tangential to the surface

It is crucial to understand, though, that this is a special case. The traction for the Helfrich Hamiltonian is neither of these:

- not constant (it depends on the local curvature, which need not be the same all over the surface)
- not isotropic (it depends on the tensor K_{ab}, and hence cannot generally be rotationally symmetric. In fact, the tensor singles out two principal directions, the eigenvectors of K^b_a, and these will also play a special role for the stress tensor)
- not tangential (the traction has a normal component; in fact it has such a component if and only if $\nabla_\perp K \neq 0$, showing that a normal component requires the curvature to be changing perpendicularly to the direction of the reference cut)

This shows that if we base our intuition about membrane stresses on the simpler case of surfaces subject only to surface tension, we miss important new pieces of the physics!

A few simple special cases as examples

1) **Plane** $K_\perp = K_\parallel = K_{\perp\parallel} = 0$

"isotropic tension"

$$\Rightarrow \vec{f}_\perp = -\underbrace{\left(\sigma + \tfrac{1}{2}\kappa K_0^2\right)}\hat{e} =: \Sigma\,\hat{e}$$

"spontaneous tension" (Lipowsky, 2013)

But careful: the effect of the spontaneous curvature on membrane stresses does not only come in the form of an effective surface tension.

2) **Sphere** $K_\perp = K_\parallel = \frac{1}{R}$; $K_{\perp\parallel} = 0$; $\nabla_\perp K = 0$

$$\vec{f}_\perp = \left\{\tfrac{1}{2}\kappa\left[K_0(K-K_0)\right] - \sigma\right\}\hat{e} = \left\{\tfrac{1}{2}\kappa K_0 K - \Sigma\right\}\hat{e}$$

$$\overset{K_0=0}{=} -\sigma\hat{e}$$

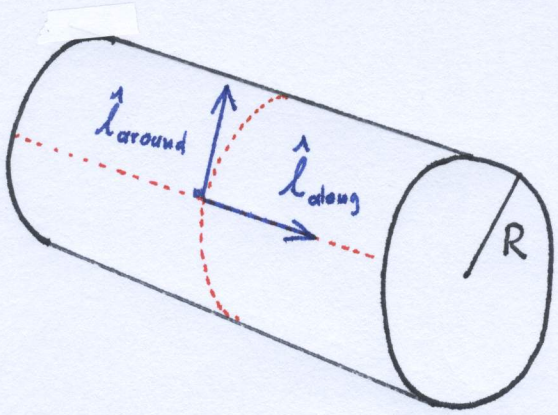
spherical membranes with $K_0 = 0$ do not have any bending stresses!

↓

This might look surprising, but it's true.
Of course, these "spherical vesicles" still have a bending energy.
They also have intrinsic torques.

3) Cylinder

This is an anisotropic case, so we must look at different directions:



1) traction along the cylinder
(having a cutting curve that's just a circular cross section)

2) traction around the cylinder
(having a cutting curve that's just a straight line along the side

along: $K_\perp = 0 \quad K_\parallel = \frac{1}{R} \quad K_{\perp\parallel} = 0 \quad \nabla_\perp K = 0$

$$\vec{f}_\perp = \left[-\tfrac{1}{2}\varkappa \left(\frac{1}{R} - K_0 \right)^2 - \sigma \right] \hat{l}_{along}$$

around: $K_\perp = \frac{1}{R} \quad K_\parallel = 0 \quad K_{\perp\parallel} = 0 \quad \nabla_\perp K = 0$

$$\vec{f}_\perp = \left[\tfrac{1}{2}\varkappa \left(\frac{1}{R^2} - K_0^2 \right) - \sigma \right] \hat{l}_{around}$$

If the cylinder can adjust its radius, then the stress around its circumference must vanish: $\vec{f}_{\perp,\,around} = 0$

$$\Rightarrow 0 = \tfrac{1}{2}\varkappa \left(\frac{1}{R^2} - K_0^2 \right) - \sigma = \tfrac{1}{2}\varkappa \frac{1}{R^2} - \Sigma$$

$$\Rightarrow R = \sqrt{\frac{\varkappa}{2\Sigma}} \overset{K_0 = 0}{=} \sqrt{\frac{\varkappa}{2\sigma}}$$, a well known result.

The force <u>along</u> the cylinder is then seen to be

$$\vec{F} = 2\pi R \, \vec{f}_{\perp,\,\text{along}}$$

$= \frac{1}{2}\varkappa\left(\frac{1}{R^2} - K_0^2\right)$ from "around case"

$$= 2\pi R \left[-\tfrac{1}{2}\varkappa\left(\tfrac{1}{R} - K_0\right)^2 - \sigma \right] \hat{\ell}_{\text{along}}$$

$$= -2\pi\varkappa R \left[\tfrac{1}{2}\left(\tfrac{1}{R^2} - \tfrac{2K_0}{R} + \cancel{K_0^2}\right) + \tfrac{1}{2}\left(\tfrac{1}{R^2} - \cancel{K_0^2}\right) \right] \hat{\ell}_{\text{along}}$$

$$= -2\pi\varkappa \left[\frac{1}{R} - K_0 \right] \hat{\ell}_{\text{along}}$$

pull!

if $R = K_0^{-1}$, the cylinder has the radius it wants and the bending stresses go away, so that no pulling force arises.

using $R = \sqrt{\frac{\varkappa}{2\Sigma}}$, this can also be written as

$$\vec{F} = -2\pi\varkappa \left[\sqrt{\frac{2\Sigma}{\varkappa}} - K_0 \right] \hat{\ell}_{\text{along}}$$

$$= -2\pi \left[\sqrt{2\Sigma\varkappa} - \varkappa K_0 \right] \hat{\ell}_{\text{along}}$$

$$\overset{K_0 = 0}{=} -2\pi\sqrt{2\sigma\varkappa}\, \hat{\ell}_{\text{along}}$$

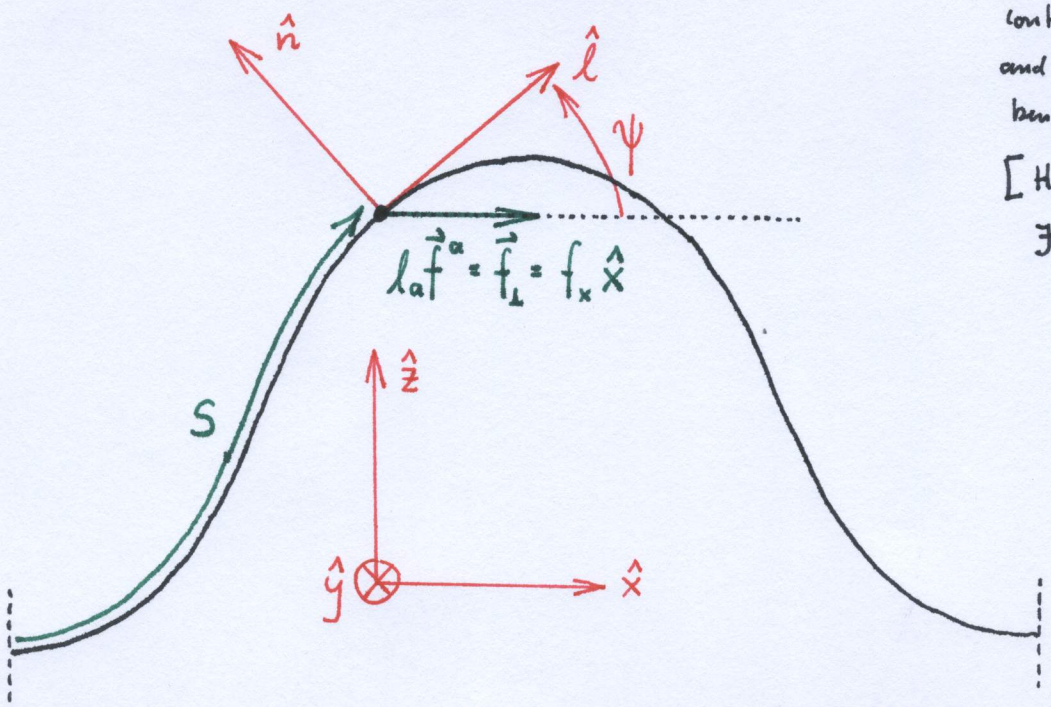
this is again very well known.

Application to some interesting geometries

1) Planar Euler buckle

for a practical application in the context of membrane simulations and the determination of a membrane's bending modulus, see:

[Hu, Diggins, Deserno, J. Chem. Phys. 138, 214110 (2013)]



If we cut the buckle along a line parallel to the ridges ($\hat{y}$-direction), the traction along that line is:

$$\vec{f}_\perp = \left[\tfrac{1}{2} \varkappa K_\perp^2 - \Sigma \right] \hat{l} - \varkappa \left(\nabla_\perp K_\perp \right) \hat{n}$$

Now, for this essentially 1-dimensional problem, the conservation law for the stress tensor, $\nabla_a \vec{f}^a = 0$, essentially implies that $\vec{f}_\perp = \text{const.}$ Hence, it must be equal to the forces applied at the end of the membrane to buckle it:

$$\vec{f}_\perp = \vec{f}_{external} = f_x \hat{x} \,.$$

To get f_x from the traction, we can therefore do two things:
We can take its <u>projection</u> on $\hat{x}$, or we can take its <u>magnitude</u>.

1) projection: $f_x = \vec{f}_\perp \cdot \hat{x} = \left(\frac{1}{2}\varkappa K_\perp^2 - \Sigma\right)\cos\psi + \varkappa\left(\nabla_\perp K_\perp\right)\sin\psi$

2) magnitude: $f_x^2 = \vec{f}_\perp \cdot \vec{f}_\perp = \left(\frac{1}{2}\varkappa K_\perp^2 - \Sigma\right)^2 + \varkappa^2\left(\nabla_\perp K_\perp\right)^2$

Between these two equations, we can eliminate the higher order derivative $\nabla_\perp K_\perp$ and then arrive at

$$\frac{1}{2}\varkappa K_\perp^2 - \Sigma = f_x \cos\psi$$

← In some sense, this is the first integral of the shape equation, which we arrived at by making use of stress conservation!

Since $K_\perp = -\partial\psi(s)/\partial s = -\dot{\psi}$, we get the first order ODE

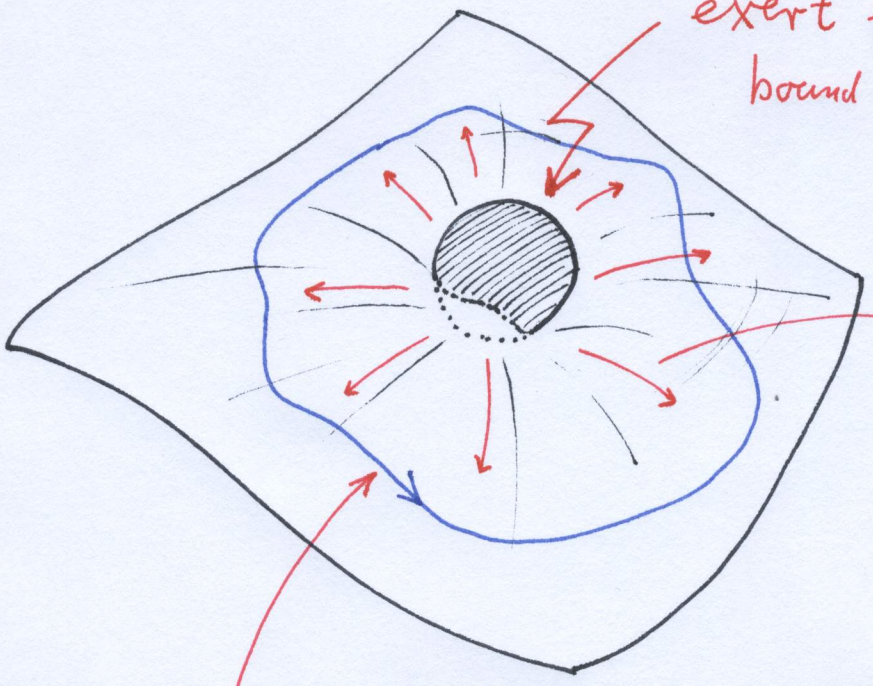
$$\frac{1}{2}\varkappa \dot{\psi}^2 - \Sigma = f_x \cos\psi$$

which can be solved by separation of variables and leads to elliptic integrals and functions.
Also, notice that at the point $s = s_i$ where the curvature vanishes, $\dot{\psi}(s_i) = 0$ (i.e., the inflection point of the buckle), we have $\psi_i = \psi(s_i)$ and

$$\Sigma = f_x \cos\psi_i$$

This shows that the buckling stress f_x and the isotropic tension Σ in the membrane are not the same!

2) Membrane mediated interactions



exert force $\vec{F}$ onto some object bound to a membrane...

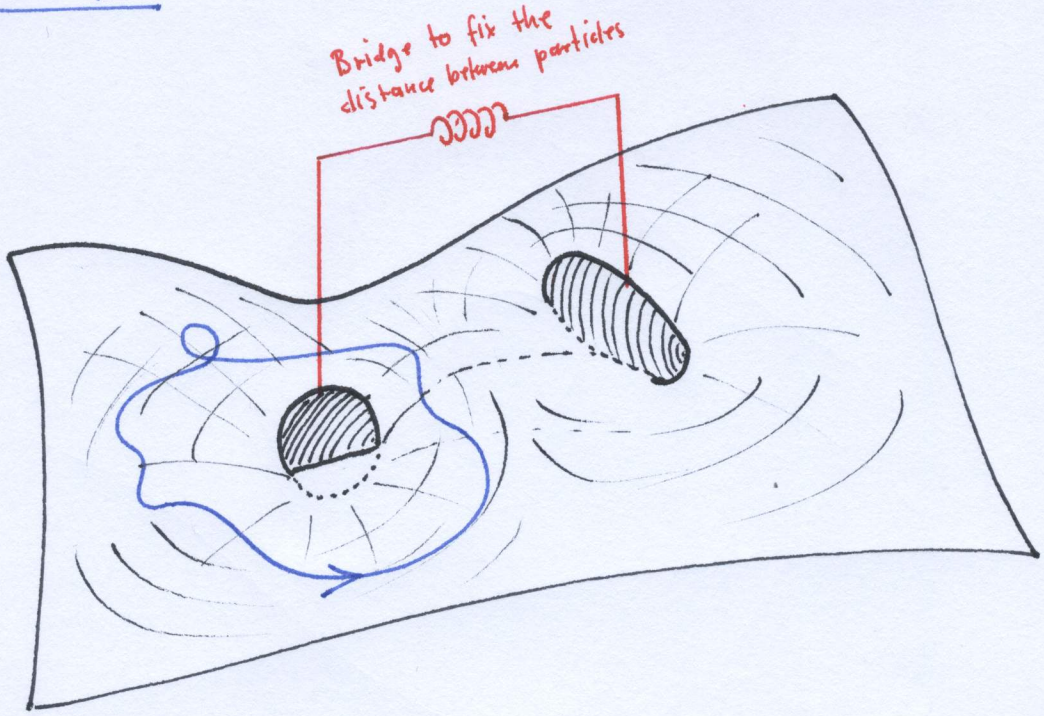
... the resulting stresses are transmitted into the membrane and "spread away" from the particle

... but they can be picked up from a distance: $\vec{F} = \oint ds\, \vec{f}_\perp$

⇒ Closed-loop integrals of the stress tensor pick up the total force exerted onto the closed region.
If we deform the integration contour without changing the force-sources it encloses, the value of the integral does not change. (because $\nabla_a \vec{f}^a = 0$!)

⇒ Closed-loop stress tensor integrals are a good way to get forces, in particular because the contour can often be adjusted to conform to existing symmetries!

Example: membrane-mediated force between two particles



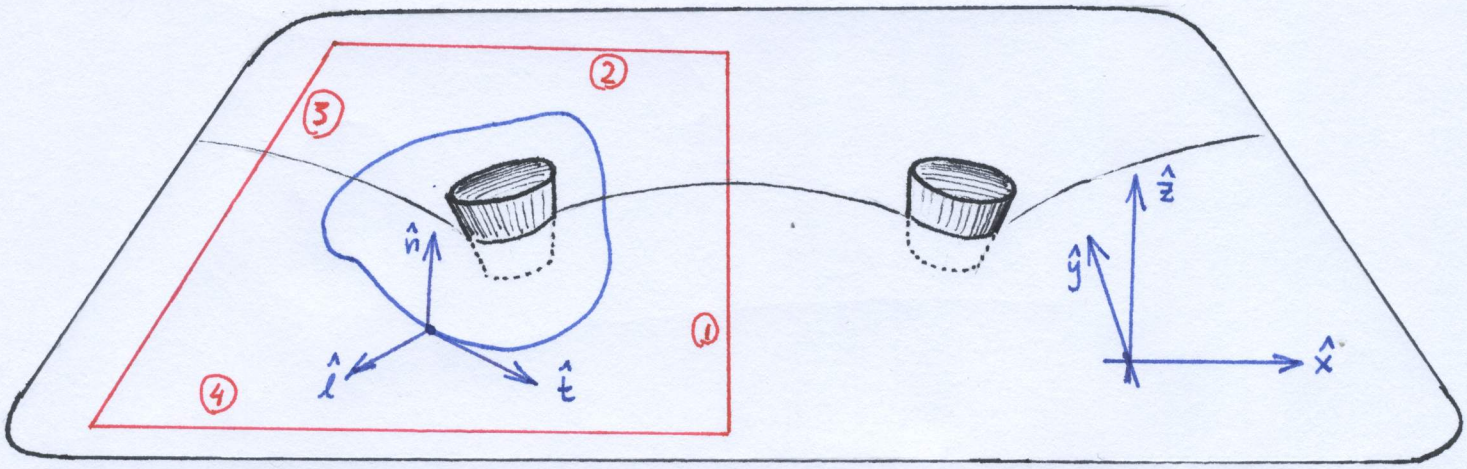
If the particles interact and exert forces on each other, then they will begin to move on the membrane. But we do not want to solve a dynamical problem! So we instead create a "bridge" between the particles which fixes their distance. This requires a force equal to the negative of the interaction force, and we could in principle measure it by putting a calibrated spring into the bridge.

The bridge force is transmitted to the particle, and from there into the membrane, where we can pick it up with a stress tensor integral.

Consider now the following symmetric situation:

We will assume that both the yz-plane and the xz-plane passing exactly through the midpoint between the particles are symmetry planes!

[Müller, Deserno, Guven, Europhys. Lett. 69, 482 (2005);
" " " Phys. Rev. E 72, 061407 (2006)]



The force picked up around the left particle is:

$$\vec{F} = \oint ds \left\{ \left[\tfrac{1}{2}\varkappa\left(K_\perp^2 - K_\parallel^2\right) - \sigma \right] \hat{l} + \varkappa K K_{\perp\parallel} - \varkappa \nabla_\perp K \, \hat{n} \right\}$$

But now imagine deforming the contour such that branch ① lies on the symmetry-cut-curve between the two particles and the three other branches are placed as shown in the figure. Now pull ②, ③, and ④ to infinity, where the membrane is flat and hence $\vec{f}_\perp = -\sigma \hat{l}$. (We assume $K_0 = 0$)

The resulting increase in symmetry permits us to drastically simplify the force expression!

a) branch ② and ④ are parallel, and $\hat{l}_2 = -\hat{l}_4$. Since on these branches the stress tensor becomes $-\sigma\hat{l}$, the integrals over ② and ④ cancel one another.

b) Branch ③ has a constant traction $-\sigma\hat{l}_3 = +\sigma\hat{x}$. However, it does <u>not</u> cancel the contribution from branch ①, for two reasons:

- on branch ① we also have a curvature contribution, because the membrane is not flat there.
- branch ① is <u>not</u> a straight line, and so the integral over ① is longer than that over ③.

c) Mirror symmetry w.r.t. the yz-plane implies $\hat{l}_1 = \hat{x}$

d) Hence, the σ-contribution to the force is:

$$\vec{F}_\sigma = -\int_1 ds\,\sigma\underbrace{\hat{l}_1}_{=\hat{x}} - \int_3 ds\,\sigma\underbrace{\hat{l}_3}_{=-\hat{x}} = -\sigma\left[\int_1 ds - \int_3 ds\right]\hat{x} = -\sigma\,\Delta L\,\hat{x},$$

where ΔL is the amount by which branch ① is longer than its projection into the xy-plane.

e) On branch ① we have $K_{\perp\parallel} = 0$ because of symmetry, and for the same reason we also have $\nabla_\perp K = 0$. Hence, the $\varkappa$-contribution to the total force is:

$$\vec{F}_\varkappa = \int_1 ds\,\tfrac{1}{2}\varkappa\left(K_\perp^2 - K_\parallel^2\right)\hat{l}_1 = \tfrac{1}{2}\varkappa\int_1 ds\left(K_\perp^2 - K_\parallel^2\right)\hat{x}$$

Hence, the total force is along the x-direction, and its magnitude is given by:

$$F = \left(\vec{F}_\sigma + \vec{F}_\kappa\right)\cdot\hat{x} = \underbrace{-\sigma\,\Delta L}_{\text{attraction}} + \underbrace{\tfrac{1}{2}\kappa\int_l ds\left(\underbrace{K_\perp^2}_{\text{repulsion}} - \underbrace{K_\parallel^2}_{\text{attraction}}\right)}$$

The direction of force of the σ-term is easy to deduce, and this leads to the assignment of the forces due to the curvature term.

Observe that this calculation made absolutely no smallness assumptions! This equation is correct even for very strongly deformed membranes (way past linear Monge).

While the sign of the force of the tension part is obvious, the total sign of the curvature part is not. What is bigger? The integral over $K_\perp^2$ or over $K_\parallel^2$? It turns out there is no obvious sign: In fact, the result can be attractive or repulsive depending on the magnitude of the deformation!

See e.g. [Reguwar & Descruo, Soft Matter 7, 8567 (2011)]

However, for very weak deformations, the sign *is* fixed.
To lowest order, one finds for the interaction energy

$$U(r) = 2\pi\kappa\,\alpha_1\alpha_2(\xi a)^2 K_0(\xi r) + \pi\kappa(\alpha_1^2+\alpha_2^2)(\xi a)^4 K_2^2(\xi r) + \cdots$$

modified Bessel functions

α_1, α_2: detachment angles of the two axisymmetric inclusions

a: radius of the two inclusions.

$\xi = \sqrt{\sigma/\kappa}$

[Weikl, Kozlov, Helfrich, Phys. Rev. E 57, 6988 (1988)]

In the tensionless limit this expression simplifies to a power law:

$$U(r) = 4\pi\kappa\,(\alpha_1^2+\alpha_2^2)\,\frac{a^4}{r^4} + \cdots$$

But attention: the prefactor is not right, not even in the erratum. There should be no $\bar{\kappa}$ in the answer.

[Goulian, Bruinsma, Pincus, Europhys. Lett. 22, 145 (1993)
erratum: Europhys. Lett. 23, 155 (1993)]

Calculating higher order corrections can become very tedious. One systematic and fairly insightful way of doing it proceeds via "Effective Field Theory". For a review, see:

[Yolcu, Haussman, Deserno, Adv. Coll. Interface Sci. 208, 89 (2014)]

There is one more limit in which we get an exact result. Imagine that the two particles are very long and lie parallel on the membrane. The deformation field then becomes largely one-dimensional, with small corrections at the tips of the particles. Ignoring these corrections, the following simplifications happen in our interaction formula:

1) $K_{\parallel} = 0$

2) $\Delta L = 0$ (because the midcurve is now a line!)

If the particle has length L, the force per unit length is:

$$f = \frac{\vec{F} \cdot \hat{X}}{L} = +\frac{1}{2} \times K_{\perp}^2 + \mathcal{O}\left(\frac{1}{L}\right)$$

repulsion

Hence, two sufficiently long parallel cylinders on a membrane repel!

This is another exact result that does not rely on any smallness assumptions.

for quantitative nonlinear results in this case, see:

[Müller, Deserno, Guven, Phys. Rev. E 76, 011921 (2007)]