DIFFERENTIATED PRODUCTS
SOLUTIONS TO EQUILIBRIUM MODELS

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ABSTRACT

Differentiated products can be described by a vector of objectively measured physical characteristics. Consumers and producers are assumed to perceive these products as variations of a basic commodity. The vector of physical characteristics is translated into a quality index (a scalar). I present a method for obtaining closed form solutions to an equilibrium model of a market of these products. In this approach, consumer and producer choices, as well as the meaning and nature of market equilibrium, are analyzed for a variety of economic structures. Moreover, the model specifies how the demand and supply for the differentiated product and for the input services, as well as the equilibrium price and wage equations, depend on the income distribution, the distribution of consumer characteristics, and the distribution of the available technologies. The model is suggestive of empirical work to understand the nature of the market for a differentiated product; for example, the market for personal computers, automobiles, rental housing services, stereo equipment, soap.

In principle, the previous theoretical work on hedonic models can estimate the parameters of demand functions for product characteristics and the equilibrium price equation, but it cannot analyze the effects of non – marginal changes in exogenous parameters. Moreover, most of the previous empirical work on hedonic models is unsatisfying because the estimation methods do not yield consistent estimators. For the class of economies that I examine, I derive closed form solutions and the cross – equation restrictions. That enables me to identify and get better parameter estimates of models than I would get if I do not have a closed – form solution. This framework can also analyze the effects of non – marginal changes in exogenous parameters.

The model assumes that consumers have different utility functions and income. Each consumer is characterized by a vector of utility parameters. The vector of the utility parameters and the consumer income follow an exogenously given multi – normal distribution. The model does not require that this vector of utility parameters and the

consumer income are uncorrelated (an assumption that is needed in other work).

The model assumes that competitive firms use different technologies that can be described by a vector of technological parameters. This vector of technological parameters follows an exogenously given multi - normal distribution with a variance - covariance matrix that does not have to be diagonal (an assumption that is needed in other work).

The competitive consumers and producers base their decisions on optimizing behavior and the differentiated product price and the hourly wage rate distributions are determined so that buyers and sellers of all differentiated products and of all labor services are perfectly matched. The price of a differentiated product is a function of the product quality, and the hourly wage rate is a function of a skill index of the labor unit. The equilibrium hourly wage rate is linear in the labor skill index, and, depending on the assumed economic structure, the equilibrium price of the differentiated product is either linear or quadratic in the product quality index.

The model provides demand and supply equations for the differentiated product and the labor services, and specifies the complete set of restrictions among the endogenous and exogenous parameters of the economy. The model introduces a quality index technology; in other respects it has weaker a priori restrictions than have been imposed in past models. The results indicate a simultaneous equation estimation of the economic model in which we impose the specified cross - equation restrictions.

The model has the flexibility of being legitimately usable with several types of data, such as aggregate data, micro data, data from a specific area (city or country), and data from several areas (cities or countries). Moreover, the model can specify the underlying structure of an economy. This is very important since the answer to many interesting questions requires structural analysis that cannot be accomodated by previous work in the area. Therefore, for many applications the closed - form approach dominates the alternative. For example, the computation of the willingnes to pay for a non - marginal change in the air quality distribution requires structural analysis; numerous studies have computed this willingness to pay in a wrong way, since they (implicitly or explicitly) assume that the structure of the economy will not change after the non – marginal change in the air quality distribution.

Using census – tract housing data and SAROAD based data on air quality, I apply my theory to estimate the structural parameters of the economy and to test the model. Given these estimates, I can compute indirect utility functions and the consumers willingness to pay for non – marginal changes in the air quality distribution of a specific city.

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CHAPTER 1

OVERVIEW OF THE HEDONIC APPROACH

1.1. Introduction, and Summary.

In recent years, some economists have adopted a new approach to the theory of individual choices that helps to explain a number of phenomena that are difficult to understand within the confines of the traditional economic theory. This new approach argues that the characteristics of commodities provide (directly or indirectly) utility to individuals and/or services to production processes. Houthakker (1952) pioneered this approach to the problem of quality variation, and to the theory of consumer behavior.

Conventional economic theory treats varieties of a specific commodity as different commodities. This implies a myriad of corner solutions. However, the most interesting conclusions from the theory hold only for positive quantities. Therefore, the attitude of the traditional economic theory amounts to ignoring the problem of quality variations in commodities altogether, inasmuch as varieties of a specific commodity are treated as different commodities.

Conventional economic theory has to assume that there is a list of commodities that are traded in the market. This list of commodities is given exogenously and the complete replacement of a variety of a commodity by another cannot happen. However, such replacements occur frequently and fairly systematically. Houthakker (1952) proposed to study this problem by introducing the quality of a commodity as a separate choice variable. His formulation takes account of the fact that consumers purchase truly negligible fractions of all commodities available to them and does not have to deal with a myriad of corner solutions required by conventional theory.

Becker (1965), Lancaster (1966), and Muth (1966) extended Houthakker's analysis to study consumer behavior but they did not work out the properties of market equilibrium. They assumed that commodities traded in the market do not possess final consumption attributes, and that consumers are also producers. The consumers use the commodities purchased in the market as inputs into a self – production function for ultimate characteristics. Rosen (1976) studies both consumer and producer behavior, and the properties of market equilibrium. Rosen assumes that consumers are not producers, and that all the commodities with their ultimate characteristics are readily available, and traded in the market.

The extension of the conventional economic theory that I study here assumes that commodities are valued for their attributes or characteristics. Hedonic prices are the implicit prices of those attributes. They show how particular attributes may be traded for one another by making substitutions in consumption and/or production. These prices can be computed from information on the physical characteristics of various commodities, and those commodities' market prices. Hedonic prices constitute the only empirical magnitude explained by the models discussed so far. With few exceptions, structural analysis of the hedonic approach is not available, and complete hedonic equilibrium models have not been estimated. The reason is that closed form solutions to hedonic equilibrium models are not available for any class of economies that could serve as the foundation for empirical applications, a gap I hope to fill, in part here.

Tinbergen (1959) supplied the earliest contribution to the formulation and solution of hedonic equilibrium models. However, the model that he studied had several restrictive features; for example, his structure implies that the elasticity of demand with respect to income is zero. Epple (1984) generalizes Tinbergen's model to treat a commodity with an arbitrary number of attributes, and introduces both endogenous demand and supplly.

In my work, I assume that commodities can be accurately described by a vector of objectively measured physical characteristics. Given a vector of objectively measured physical characteristics, a commodity can be assigned a quality index. That is, I make the strong assumption that there is a mapping that transforms physical characteristics into a scalar quality index. Moreover, I assume that commodities are traded in competitive markets, and that the buyers of the commodities care only about the quality index of the commodity that they purchase. For example, a consumer cares about the quality index of a tape recorder that he buys, and a producer of a product cares about the quality index of a labor unit that he hires, where the quality index in both cases is defined by a quality mapping.

This analysis is designed to take account of the fact that a buyer can be indifferent between several varieties of a specific good (which will be the case if those varieties are of the same quality). Unlike all the other studies that I am aware of, the models that I introduce can analyse competitive markets not only for durable but also for non - durable differentiated commodities[1]. I work out the case of exogenous and endogenous supply for differentiated commodities. The inputs to the production can be either traded or not traded in the market (in which second case the inputs are owned by the firms that use them). If the inputs to the production are traded in the market, the short run and the long run decision problem of a competitive firm can be analyzed. Each firm is characterized by a vector of technological parameters that follows a multi - normal distribution.

On the other side of the economy, there are consumers. Consumers have utility functions that depend on the quantity of the numeraire good, on leisure, and on the quantity and quality of the differentiated good(s) that they consume. Consumers can be of different type. Their type is specified by a vector of utility parameters. I assume that the consumer's type vector of parameters follows a multi - normal distribution.

If the consumer uses one unit of the differentiated good, I can work out the case that the consumer's income is exogenously given. The consumer's type - income vector is assumed to follow a multi - normal distribution. An interesting feature of my formulation is that the income elasticity does not have to be zero.

Common characteristics of the class of economies that I study are : 1) cost functions that are quadratic on the quality index of the labor inputs, 2) quadratic utility functions with a second partial derivative with respect to the numeraire good equal zero, and 3) a quality index mapping that is linear in the vector of physical characteristics that describes the

[1] Durable commodities last many periods, and consumers use the services of only one of them each period.

differentiated commodity. In all cases, this structure implies demand functions for the differentiated commodities that are linear in the consumer's type – income vector, and supply functions that are linear in the vector of technological parameters that characterizes a firm. However, using a variable transformation that is similar to the one that I present in the Appendix, Appendix E, I can introduce nonlinearities into the model.

This formulation assumes that a wide variety of alternative differentiated goods are available. The assumption that markets are competitive requires that all the economic agents take the prices of all those varieties as given.

The competitive consumers and producers base their decisions on optimizing behavior, and in equilibrium the prices of all varieties are determined so that the buyers and sellers of all commodities are perfectly matched. Moreover, in equilibrium, none of the economic agents can improve his position, and all of their optimum choices are feasible. The distribution of the equilibrium depends on the assumed structure of the economies that I consider, namely : 1) the distribution of the consumers taste parameters and income, 2) the distribution of the firms technological parameters, 3) the functional form of the utility function, 4) the functional form of the cost function, and 5) the functional form of the quality index mapping.

In principle, the previous theoretical work on hedonic models can estimate the parameters of demand functions for product characteristics and the equilibrium price equation, but it cannot analyze the effects of non – marginal changes in exogenous parameters. However, most of the previous empirical work on hedonic models is unsatisfying because the estimation models do not yield consistent estimators; Epple (1987), Bartik (1987), and Palmquist (1984) are the only exceptions that I am aware of. In my dissertation, I specify a class of models that assume a structure that is different than that of the previous work on structural hedonic models, namely Epple (1984) and Tinbergen (1959). My structure assumes a quality index mapping that is not present in the other work. That quality index mapping allows me to work with a more general utility function (the cross partial derivative of the utility function does not have to be zero and the marginal utility with respect to the numeraire good can have an intercept). Moreover, I do not have to impose restrictions on any one of the exogenously given variance – covariance matrices, I end up with demand functions that do not need to have a zero income elasticity, and the equilibrium price equation can have an intercept. I can also introduce a variable transformation that can make the equilibrium price equation, as well as the demand and supply equations, non – linear.

For the class of models that I consider, I derive closed form solutions and the cross – equation restrictions. That enables me to identify and get better parameter estimates of models than I would get if I did not have a closed form solution. Given the estimates that I obtain for the parameters of the reduced form equations, I can use the theory about the structure of the economy to obtain estimates for the structural parameters that I need to analyze the effects of non – marginal changes in exogenous parameters. The alternative (non – structural) approach cannot analyze the effects of non – marginal changes in exogenous parameters. Therefore, for the class of economies that I consider, the closed form approach dominates the alternative.

As I mentioned earlier, the theoretical model assumes that the quality index for a differentiated commodity is a scalar. If the quality of a differentiated commodity is a vector, analytical solutions could still be obtained using the method that I develope in my dissertation. However, such solutions would require that several constraints among the exogenously given structural parameters of the model are satisfied. For example, a solution would require that the variance – covariance matrices of the multi – normal distributions of the consumer's type and of the firm's vector of technological parameters are diagonal matrices. That condition is unlikely to be satisfied in many cases of interest. Hence, the assumption of a single quality index is an important one that is not easily relaxed.

The thesis is organized as follows. In Chapter 2, I assume that the income distribution is exogenously given and I derive closed form solutions to a model of quality variation in commodities. The nature of the solution depends on the assumptions made about the supply for the differentiated product. The supply for the differentiated product can be either exogenous or endogenous. In the second case the labor can be either homogeneous or heterogeneous. In Appendix A, I introduce leisure into the model and derive closed form solutions to a model of quality variations in commodities under several assumptions about the supply for the differentiated product (exogenous and endogenous supply, homogeneous and heterogeneous labor). In Appendix B, I derive closed form solutions to a model of quantity[2] and quality variations in commodities under several assumptions about the supply for the differentiated product (exogenous and endogenous supply, homogeneous and heterogeneous labor), and under the assumption that leisure is a choice variable for the consumers. In chapter 3, I overview the theoritical model, and in Chapter 4, I present applications.

The applications investigate how far one can go with closed form solutions and how well the resulting model fits the data for cases where a more general functional form might be preferred but cannot solve the equations. This application shows that it is actually feasible not only to estimate a closed - form model but also to test it.

In the next part of chapter 1, I overview the hedonic approach as a tool for studying economic problems.

1.2. Welfare Economics and The Hedonic Approach.

The Hedonic Approach assumes that differentiated commodities can be accurately described by a vector of characteristics, and that consumers can choose their effective consumption of these characteristics through their choice of a differentiated commodity. Observed product prices and the specific amounts of characteristics can provide estimates for implicit or hedonic prices. This information can be used for deriving the demand functions for the characteristics of a differentiated product. These demand functions can be used to compute the willingness to pay for marginal or non - marginal changes in the product characteristics.

The technique just mentioned has many applications in the area of environmental

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[2] Unlike the cases studied in chapter 2 and Appendix A, consumers can now choose not only the quality of the differentiated product but also the quantity.

economics and has been extensively used for estimating the willingness to pay for different distributions of local public goods; for example, Harrison and Rubinfeld (1978), and Ridker and Henning (1967). Moreover, it is worth mentioning that Tiebout's (1956) approach to the local public public goods problem could incorporate the hedonic price framework if we assume that consumers are mobile and that they indicate their preferences for the local public goods by choosing to live in communities that offer the utility maximizing local public goods – taxation package.

Most of the work that uses this approach is unsatisfying because the estimation methods do not yield consistent estimates, Epple (1987); Bartik (1987) and Palmquist (1984) are two exceptions. Many of the previous applications estimate hedonic prices using single equation estimation techniques (Witte et al (1979) is an exception), and none of them makes a structural analysis. However, structural analysis is needed to compute the effects of non – marginal changes in exogenous parameters. Structural analysis would also specify the restrictions among the parameters of demand functions and price equation.

The standard approach is not suitable for structural analysis because it does not provide closed – form solutions. The closed – form approach that I present can specify the restrictions among the parameters and suggests a joint estimation of the demand function and of the price equation in which we impose the specified restrictions among the parameters. These restrictions can be tested and could yield more efficient estimates for the parameters of interest.

Depending on the structure of the economy, and on the question that we are interested in, we do not always need to compute closed form solutions and make a structural analysis; even though that could increase the efficiency of our estimates. For example, the standard approach can estimate the equilibrium price equation and the parameters of the demand functions for product characteristics. However, in many instances estimation of the structural parameters of the model is required for answering interesting questions. These are cases that computation of the willingness to pay requires knowledge of the indirect utility function. For example, this is true for problems that are related to non – marginal changes in exogenously given parameters that are common to all consumers. In these cases, the closed form approach dominates the alternative.

CHAPTER 2

QUALITY VARIATIONS IN COMMODITIES, AND EXOGENOUS INCOME DISTRIBUTION

2.1. Assumptions and Notation.

In this chapter, I consider an economy in which individuals consume a differentiated good, and the numeraire good, x. I assume that consumers use one unit of the differentiated good. For example, this differentiated good could be a house, an automobile, or a computer.

The differentiated good can be accurately described by an (1 x m) vector, v , of objectively measured characteristics. I assume that the consumers care only about the quality index, h , of the differentiated product. The quality, h , is a scalar and a function of the vector of physical characteristics, v . I assume that h is a linear function of v, namely, that

$$h = \epsilon_0 + \epsilon_1 v' \tag{1}$$

where :

ϵ_0 is a parameter,
ϵ_1 is a (1 x m) vector of parameters, and
v' is the transpose of v.

(In the next, a prime " ' " will always denote the transpose of a vector or matrix).

In some applications, the quality index interpretation of the mapping (1) might not make sense. In those cases, (1) can be interpreted as the consumer's self production function; a self production function that is the same for all consumers[3]. That is, it can be argued that

[3] That consumer's self production function can be made be different among consumers or among groups of consumers (for example, see Chapter 4, Section 4.8.).

consumers are also producers and that the differentiated good (as it is offered in the market with characteristics v) does not possess the ultimate consumption attribute that consumers are looking for. A differentiated good with v – physical characteristics is purchased as input into a self – production function for a ultimate characteristic h, equation (1).

The model lets consumers have different utility functions and income.

Each consumer can be described by a [1 x (n + 1)] vector, z . The definition of z follows.

$$z = [\, a \quad I \,]$$

where :

I is the income of a consumer, and

a is a (1 x n) vector of utility parameters that specifies the type of the consumer.

z is assumed to follow a multi – normal distribution with a known mean, $\bar{z}$, and variance, Σ_z . Let it be :

$$N(\bar{z}, \Sigma_z) \tag{2}$$

$U(h, x, a)$ is the utility that an a – type consumer obtains from x and from the services of a differentiated good of h – quality.

The utility function is assumed to be a quadratic of the following form.

$$U(h, x, a) = \delta + (\zeta_0 + \zeta_1 a') h + \theta x + 0.5\, \xi h^2 + \omega x h \tag{3}$$

where :

$\delta, \zeta_0, \xi, \omega$, and θ are utility parameters (scalars), and

ζ_1 is a (1 x n) vector of utility parameters.

An a – type consumer with income I solves the following optimization problem.

$$\max \quad U(h, x, a)$$
$$\text{with respect to } h, x$$
$$\text{subject to } I = P(h) + x$$

where :

$P(h)$ is the equilibrium price equation; it gives the price of the differentiated good as a function of the quality index, h .

Eliminating x , the consumer's optimization problem is equivalent to :

$$\max \quad U(h, I - P(h), a)$$
$$\text{with respect to } h.$$

The first order condition for the last problem is :

$$\frac{\partial P(h)}{\partial h} = \frac{\partial U(h, I - P(h), a) \; / \; \partial h}{\partial U(h, I - P(h), a) \; / \; \partial x} . \tag{4}$$

The optimum decisions of consumers and producers (if any) depend on the equilibrium price equation $P(h)$. The price equation is determined so that buyers and sellers are perflectly matched. In equilibrium, no one of the economic agents can improve his position, all of their optimum decisions are feasible, and the price equation $P(h)$ is determined by the distribution of consumer tastes and income, and by the distribution of the supply for the differentiated product. Next, I specify the equilibrium price equation $P(h)$ for different configurations of the supply side.

2.2. Exogenous Supply for The Differentiated Product.

The supply for the differentiated product is exogenously given. The vector of physical characteristics, v, follows an exogenously given normal distribution with a known mean, $\bar{v}$, and variance, Σ_v. Let it be :

$$N(\bar{v}, \Sigma_v) \tag{5}$$

PROPOSITION.1. The price equation that equilibrates the market described above is :

$$P(h) = \pi_0 + \pi_1 h \tag{6}$$

where :

$$\pi_1 = \frac{1}{2\omega}\left(\xi + \frac{\sqrt{t \Sigma_z t'}}{\sigma_s}\right) \tag{7}$$

$$\pi_0 = \frac{1}{\omega}(\zeta_0 - \theta \pi_1 + t\bar{z}' - (2\omega\pi_1 - \xi)\bar{h}_s) \tag{8}$$

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}'$$

$$\sigma_s = \sqrt{\epsilon_1 \Sigma_v \epsilon_1'} \quad , \ and$$

$$t = [\, \zeta_1 \ \ \omega \,]. \tag{9}$$

t is a [1 x (n + 1)] vector of utility parameters.

Proof[4]

[4] The general strategy of the proof of Proposition 1 was introduced by Tinbergen (1959) and extended by Epple (1984).

Substitute equations (3) and (6) into equation (4), and solve for h to obtain the demand for h, h_d. The demand for h is given by the following equation.

$$h_d = \frac{\zeta_0 - \omega \pi_0 - \theta \pi_1 + t z'}{2 \omega \pi_1 - \xi} \tag{10}$$

where :

t is given in (9).

The second order condition implies :

$2 \omega \pi_1 - \xi > 0.$

The solution that is given in Proposition 1 satisfies the second order condition.

We can now see that the demand for h is linear in z. Therefore, (2) and (10) imply that the aggregate demand for h follows a normal distribution. Let it be :

$f(h) = N(\bar{h}_d, \sigma_d^2)$,

where :

$\bar{h}_d$ is the mean,

σ_d^2 is the variance,

$$\bar{h}_d = \frac{\zeta_0 - \omega \pi_0 - \theta \pi_1 + t \bar{z}'}{2 \omega \pi_1 - \xi} \quad , \quad \textit{and} \tag{11}$$

$$\sigma_d^2 = \frac{t \Sigma_z t'}{(2 \omega \pi_1 - \xi)^2} \tag{12}$$

Equation (1) and the assumption that the supply for v is an exogenously given distribution, that is given in (5), imply that the aggregate supply for h follows a normal distribution. Let it be :

$$g(h) = N(\bar{h}_s, \sigma_s^2) \tag{13}$$

where :

$\bar{h}_s$ is the mean,

σ_s^2 is the variance,

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}', \quad and \tag{14}$$

$$\sigma_s^2 = \epsilon_1 \Sigma_v \epsilon_1' \ . \tag{15}$$

For an equilibrium, we need that :

aggregate demand = aggregate supply .

That is,

$$f(h)dh = g(h)dh \ .$$

Since both aggregate demand and supply follow a normal distribution, the above equilibrium condition is equivalent to:

$$\bar{h}_d = \bar{h}_s \ , \text{ and} \tag{16}$$

$$\sigma_d^2 = \sigma_s^2 \ , \tag{17}$$

where :

$\bar{h}_d$, σ_d^2, $\bar{h}_s$, σ_s^2 are respectively given in (11), (12), (14), and (15).

I can now show that our solution for π_1 and π_0 is equivalent to (16) and (17). The proof follows.

$$\pi_1 = \frac{1}{2\omega}\left(\xi + \frac{\sqrt{t\Sigma_z t'}}{\sigma_s} \right) \Leftrightarrow$$

$$2\omega\pi_1 - \xi = \frac{\sqrt{t\Sigma_z t'}}{\sigma_s} \Leftrightarrow$$

$$\sigma_s = \frac{\sqrt{t\Sigma_z t'}}{2\omega\pi_1 - \xi} \Leftrightarrow$$

$$\sigma_d = \sigma_s \Leftrightarrow \quad (2.17) \quad .$$

Moreover,

$$\pi_0 = \frac{1}{\omega}\left[\zeta_0 - \theta\pi_1 + t\bar{z}' - (2\omega\pi_1 - \xi)\bar{h}_s \right] \Leftrightarrow$$

$$(2\omega\pi_1 - \xi)\bar{h}_s = \zeta_0 - \omega\pi_0 - \theta\pi_1 + t\bar{z}' \Leftrightarrow$$

$$\bar{h}_s = \frac{\zeta_0 - \omega\pi_0 - \theta\pi_1 + t\bar{z}'}{2\omega\pi_1 - \xi} \Leftrightarrow$$

$$\bar{h}_s = \bar{h}_d \Leftrightarrow \quad (2.16)$$

It can be verified that the π_1 and π_0 solutions that follow next

$$\pi_1 = \frac{1}{2\omega}\left(\xi - \frac{\sqrt{t\Sigma_z t'}}{\sigma_s} \right) \quad \text{, and}$$

$$\pi_0 = \frac{1}{\omega}\left(\zeta_0 - \theta\pi_1 + t\bar{z}' - (2\omega\pi_1 - \xi)\bar{h}_s \right) \quad \text{, and}$$

the ones given in (7), and (8) are the only ones that satisfy the equilibrium equations (16), and (17). However, the above solution is ruled out because it does not satisfy the second order condition.

QED

The economy that I examined in this second section of Chapter 2 does not impose any restrictions on the variance – covariance matrix of the vector of product characteristics, and on the variance – covariance matrix of the vector of consumer income and characteristics. The non – existence of restrictions on those variance covariance matrices and the assumed quality index mapping are part of what differentiates this work from the Tinbergen (1959) – Epple (1984) formulation. In addition to that, my formulation does not require that the demand for the differentiated product has a zero income elasticity.

The Tinbergen (1959) – Epple (1984) formulation assumes a different (but still quadratic) functional form for the utility function and they require that the number of consumer characteristics equals the number of product characteristics; to be more specific, their utility function has a cross partial derivative that is equal to zero and a marginal utility with respect to the numeraire good without an intercept. Moreover, for many of the cases that they examine their equilibrium is not unique. In their structure, the equilibrium price equation is quadratic without a constant term.

Unlike the demand and price equations, the restrictions among the price equation parameters and the exogenous parameters of the model are non – linear (see Proposition 1). To illustrate that the price equation is an equilibrium relationship that incorporates features of tastes, supply, and the distributions of income and parameters of taste and supply, I present the following example.

Consider an economy in which consumers have identical preferences that can be described by the following utility function :

$$U = 100 + h + 2x + 0.5h^2 + xh \quad ,$$

where :

x is the numeraire good, and

h is the quality of the differentiated good.

Consumers are assumed to use the services of one differentiated good.

Let the consumer income follow an exogenously given normal distribution with a mean that is equal to 550, and a variance that is equal to 400.

The supply for the differentiated good is assumed to follow an exogenously given normal distribution. This distribution and the quality mapping (1) imply that the supply for the product quality is also a normal distribution; let this distribution have a mean equal to 2 and a variance that is equal to 1.

Proposition 1 implies that the hedonic price equation is :

$$P(h) = 490 + 10.5\,h \quad .$$

Suppose now that the mean of the distribution of the supply for the differentiated product decreases by one unit. Proposition 1 implies that the price equation becomes :

$$P(h) = 510 + 10.5\,h \quad .$$

If, in addition to the previous one unit change in the mean, the variance of the distribution of the supply for the differentiated product decreases by 0.75 of a unit, the price equation becomes :

$$P(h) = 470 + 20.5\,h \quad .$$

2.3. An Example - Application.

For a differentiated product, such as rental residential housing, the model presented in the previous section can be used to characterize the market equilibrium. To be more specific :

The differentiated product rental residential housing can be described by a vector of characteristics v ,

where :

$$v = [\; v_1 \;\; v_2 \;\; v_3 \;]$$

v_1 is the size of the housing unit (number of rooms),

v_2 is an index of how clean is the air, and

v_3 is the travel time to work.

v follows an exogenously given multi – normal distribution.

The quality of housing, h (a scalar), is a linear function of the vector of housing characteristics v , that is,

$$h = \epsilon v' \quad ,$$

where :

ϵ is a vector of parameters.

Consumer preferences are described by utility functions of the type introduced in Chapter 2, Section 2.1. The utility function depend on the quality of the house, h , and on the parameter a ,

where :

a is the number of persons in the family.

The vector $[\ a \ \ I\ ]$ follows an exogenously given multi – normal distribution,

where :

I is the consumer income.

The theoretical model presented in the previous section provides the framework for an analysis of the market described above.

Next, I assume that the supply is endogenous and I study several configurations of the supply side.

2.4. Endogenous Supply, and Exogenously Fixed Inputs That Are Owned by

Competitive Firms.

In this section, I assume that the inputs to the production are exogenously fixed and owned by competitive firms each of which produces the product with the inputs that it owns. The cost of the numeraire input for producing one unit of a differentiated good with characteristics v is $C(v,b)$, and it depends on b ,

where :

b is a (1 x k) vector of technological parameters.

We can think of b as the type of the firm.

I assume that the vector of technological parameters b follows an exogenously given multi − normal distribution with a known mean, $\bar{b}$, and variance, Σ_b . Let it be :

$N(\bar{b}, \Sigma_b)$.

$C(v,b)$ is a quadratic in v of the following form :

$$C(v,b) = \gamma_0 + (\gamma_1 + b\gamma_2)v' + 0.5\, v\gamma_3 v' \tag{18}$$

where:

γ_0 is a parameter (a scalar),

γ_1 is a (1 X m) vector of parameters,

γ_2 is a (k x m) matrix of parameters, and

γ_3 is an (m x m) symmetric and positive definite matrix of parameters.

A b − type profit maximizing competitive firm solves the following problem :

$$\max \quad P(h) - C(v, b)$$

$$\text{with respect to } \; h, v$$

$$\text{subject to } \; h = \epsilon_0 + \epsilon_1 v'$$

where :

$P(h)$ is the equilibrium price equation,

$C(v, b)$ is given in (18).

The above optimization problem is equivalent to :

$$\max \quad P(\epsilon_0 + \epsilon_1 v') - C(v, b)$$

$$\text{with respect to } \; v$$

The first order conditions for the last optimization problem are :

$$\frac{\partial P(\epsilon_0 + \epsilon_1 v')}{\partial v'} = \frac{\partial C(v, b)}{\partial v'} \tag{19}$$

PROPOSITION 2. The price equation that equilibrates the market described above is given in (6) with parameters specified in (7) and (8),

where :

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}' \quad , \tag{20}$$

$$\sigma_s = \sqrt{\epsilon_1 \Sigma_v \epsilon_1'} \quad , \tag{21}$$

$$t = [\, \zeta_1 \;\; \omega \,]$$

$$\bar{v}' = (\gamma_3)^{-1} (\pi_1 \epsilon_1' - \gamma_1' - \gamma_2' \bar{b}') \quad , \text{ and} \tag{22}$$

$$\Sigma_v = (\gamma_3)^{-1} \gamma_2' \Sigma_b \gamma_2 (\gamma_3)^{-1} \tag{23}$$

Proof: see Appendix C1.

The formulation of this section does not impose any restriction on the variance – covariance matrix of the vector of technological parameters and on the variance – covariance matrix of the vector of consumer income and characteristics. The demand, the supply, and the price equations are linear and the restrictions among the price equation parameters and the exogenous parameters are non – linear and unique.

The price equation is an equilibrium relationship that incorporates features of tastes, supply, production technology, and the distributions of income and parameters of taste and technology.

2.5. Endogenous Supply, Exogenous Capital Distribution, Heterogeneous Labor.

In this section, I assume that each competitive firm is exogenously given some capital that is good for producing a specific kind of differentiated product. Thus, we can think of v as a variable that can describe the capital type of a firm. Let the normal distribution

$$N(\bar{v}, \Sigma_v)$$

describe the exogenously given capital distribution,

where :

$\bar{v}$ is the mean, and

Σ_v is the variance.

The labor that is employed in this industry is traded in the market, is heterogeneous, and can be described by an (1 x l) vector, c, of objectively measured physical characteristics. c follows an exogenously given multi – normal distribution. Let it be :

$$N(\bar{c}, \Sigma_c) \quad , \tag{24}$$

where :

$\bar{c}$ is the mean, and

Σ_c is the variance.

I assume that competitive firms care about the quality (skill) of the labor that they hire. The market wage rate is a function of the labor skill index. The labor skill is assumed to be a linear function of the vector that describes the labor's physical characteristics. Namely,

$$s = \eta_0 + \eta_1 c' \quad , \tag{25}$$

where :

η_0 is a parameter,

η_1 is a (1 x I) vector of parameters,

c is the (1 x I) vector of the physical characteristics of a labor unit, and

s is the quality index of a labor unit (a scalar).

Each competitive firm needs $(N_0 + N_1 s)$ units of s – skill type of labor to produce one unit of a differentiated product,

where :

N_0 , and N_1 are two parameters.

However, there is some cost associated with different combinations of a given type of capital and a chosen type of labor.

$C(s, b)$ is the cost (per unit of differentiated product) of adjusting any type of capital to work with s – skill units of labor,

where :

b is a (1 x k) vector of technological parameters.

Each firm is exogenously assigned a b vector of parameters. This vector of parameters follows a multi – normal distribution. Let it be :

$$N(\bar{b}, \Sigma_b) \quad , \tag{26}$$

where :

$\bar{b}$ is the mean, and

Σ_b is the variance.

In addition to the above mentioned adjustment costs, $C(s, b)$ can also iclude other administrative costs. $C(s, b)$ is a quadratic in s of the following form.

$$C(s, b) = \gamma_0 + (\gamma_1 + \gamma_2 b') s + 0.5 \gamma_3 s^2 \quad , \tag{27}$$

where :

γ_0, γ_1, and γ_3 are parameters (scalars), and

γ_2 is a (1 x k) vector of parameters.

A competitive firm has to decide what kind of labor to hire. It solves the following problem.

$$\min \; (N_0 + N_1 s) w(s) + C(s, b)$$
$$\textit{with respect to} \;\; s$$

where :

$w(s)$ is the market wage rate equation.

The first order condition for the last optimization problem is :

$$N_1 w(s) + (N_0 + N_1 s)\frac{\partial w(s)}{\partial s} + \frac{\partial C(s,b)}{\partial s} = 0 \quad . \tag{28}$$

PROPOSITION 3. The price equation that equilibrates the product market described above is given in (6), with parameters given in (7) and (8),

where :

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}' \quad ,$$

$$\sigma_s = \sqrt{\epsilon_1 \Sigma_v \epsilon_1'} \quad , and$$

$$t = [\ \zeta_1 \quad \omega\ ] \quad .$$

The wage equation that equilibrates the market is :

$$w(s) = \rho_0 + \rho_1 s \tag{29}$$

where :

$$\rho_1 = \frac{1}{2N_1}\left(\frac{\sqrt{\gamma_2 \Sigma_b \gamma_2'}}{\sqrt{\eta_1 \Sigma_c \eta_1'}} - \gamma_3\right) \quad , and \tag{30}$$

$$\rho_0 = -\frac{N_0 \rho_1 + \gamma_1 + \gamma_2 \bar{b}' + (\eta_0 + \eta_1 \bar{c}')(2N_1 \rho_1 + \gamma_3)}{N_1} \tag{31}$$

Proof: see Appendix C2.

The model of this section does not impose any restriction on the variance – covariance matrix of the vector of technological parameters and on the variance – covariance matrix of the vector of the consumer income and characteristics. The demand, the supply, the price, and the wage equations are linear and the restrictions among the exogenous parameters and the parameters of the price and wage equations are non – linear and unique.

The price and wage equations are equilibrium relationships that incorporate features of tastes, supply, production technology, and the distributions of income and parameters of taste and technology. To see that, consider the following example.

Suppose that b is a technological parameter such that the lower the b the lower the administration cost, i.e., γ_2 is greater than zero. Proposition 3 implies that a unit decrease in the mean of the exogenously given distribution of b would increase the intercept of the (linear) wage equation by γ_2 / N_1 .

2.6. Endogenous Supply, Endogenous Capital Distribution, Heterogeneous Labor.

In this section, I assume that a competitive firm can choose the type of capital that it buys. However, each type of capital sold in the market can produce only one specific v – type product.

The labor employed in this industry is traded in the market, is heterogeneous, and can be described by an (1 x l) vector of objectively measured physical characteristics, c . c follows an exogenously given normal distribution that is given in (24).

Competitive firms are assumed to care about the skill index of the labor that they hire. The market wage rate is a function of the labor skill index, s . I assume that the labor skill index is a function of the physical characteristics of the labor, c . This function is given in (25).

Each competitive firm needs $(N_0 + N_1 s)$ units of labor to produce one unit of a differentiated product, where N_0 and N_1 are two parameters.

$C(v,s,b)$ is the non – labor cost for producing one unit of a differentiated product with a v – vector of physical characteristics using s – quality labor units, where b is a (1 x k) vector of technological parameters.

b follows an exogenously given multi – normal distribution that is given in (26).

$C(v,s,b)$ includes 1) all the adjustment and other administrative costs that are associated with the production of a v – type differentiated product using s – quality labor, and 2) the cost of a v – type capital, i.e., capital that is good for producing a differentated product with v – physical characteristics.

$C(v,s,b)$ is assumed to be a quadratic of the following form.

$$C(v,s,b) = \gamma_0 + (\gamma_1 + \gamma_2 b') s + 0.5\,\gamma_3 s^2 + (\gamma_4 + b\,\gamma_5) v' + 0.5\, v\, \gamma_6 v' \quad , \tag{32}$$

where :

γ_0, γ_1, γ_3 are parameters (scalars),
γ_2 is a (1 x k) vector of parameters,
γ_4 is a (1 x m) vector of parameters,
γ_5 is a (k x m) matrix of parameters,
γ_6 is a (m x m) matrix of parameters (symmetric).

A profit maximizing competitive firm solves the following problem.

$$\max \quad P(h) - (N_0 + N_1 s)\, w(s) - C(v, s, b)$$

$$\text{with respect to } h, v, s$$

$$\text{subject to } \quad h = \epsilon_0 + \epsilon_1 v'$$

where :

$P(h)$ is the market price equation, and

$w(s)$ is the market wage rate equation.

Eliminating h , I obtain that the above optimization problem is equivalent to :

$$\max \quad P(\epsilon_0 + \epsilon_1 v') - (N_0 + N_1 s)\, w(s) - C(v, s, b)$$

$$\text{with respect to } v, s$$

The first order conditions for the latter optimization problem are given next.

$$\frac{\partial P(\epsilon_0 + \epsilon_1 v')}{\partial v'} = \frac{C(v, s, b)}{v'} \tag{33}$$

$$N_1 w(s) + (N_0 + N_1 s) \frac{\partial w(s)}{\partial s} + \frac{\partial C(v, s, b)}{\partial s} = 0 \tag{34}$$

PROPOSITION 4. The price equation that equilibrates the product market is given in (6), with parameters given in (7) and (8),

where :

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}' \tag{35}$$

$$\sigma_s = \sqrt{\epsilon_1 \Sigma_v \epsilon_1^{\perp}} \tag{36}$$

$$t = [\ \zeta_1 \quad \omega\ ]$$

$$\Sigma_v = (\gamma_6)^{-1} \gamma_5' \Sigma_b \gamma_5 (\gamma_6)^{-1} \quad , \quad and \tag{37}$$

$$\bar{v}' = (\gamma_6)^{-1} (\pi_1 \epsilon_1' - \gamma_4' - \gamma_5' \bar{b}') \ . \tag{38}$$

The wage equation that equilibrates the input market is given in (29), with parameters specified in (30) and (31).

Proof: see Appendix C3.

Like the rest of the models presented in this chapter, the model of this section does not impose any restriction on the variance – covariance matrix of the vector of technological parameters and on the variance – covariance matrix of the vector of the consumer income and characteristics. The demand, the supply, the price, and the wage equations are linear, and the restrictions among the exogenous parameters and the parameters of the price equation and of the wage equation are non – linear and unique.

The price and wage equations are equilibrium relationships that incorporate features of tastes, supply, production technology, and the distributions of income and parameters of taste and technology. To see that, consider the following example.

Suppose that b is a technological parameter such that the lower the b the lower the cost, i.e., γ_2 and γ_5 are greater than zero. Moreover, assume that the product quality is positively related to the characteristics of the product, i.e., ϵ_1 is greater than zero. Suppose now that the mean of the exogenously given distribution for b decreases by one unit. Given that $\omega > 0$, Proposition 4 implies that wages increase by γ_2 / N_1 and that prices decrease by $(2 \omega \pi_1 - \xi) \epsilon_1 (\gamma_6)^{-1} \gamma_5 / \omega$.

2.7. Conclusions.

I anticipate that the class of economies outlined above can have a variety of applications to many practical problems involving equilibrium and structural analysis in cross – section. The assumption that I make throughout this chapter that individuals consume one unit of the differentiated product make me believe that most of those applications are going to be applications on durable goods. This seems to be the case since individuals consume the services of one differentiated good in each period; for example, houses, automobiles, VCR's, computers, etc. However, non – durable goods, e.g., vacation packages, might be analysed using this chapter's framework.

Any analysis of a market for a differentiated product that is based on this chapter's theoretical model would require data on prices, on the physical characteristics of the good, on its quality, and on the consumer's income and type[5]. The rest of the data that is needed is going to depend on the assumptions that we make about the supply side of the economy.

In many applications, it is a common practice to use data on the means of groups of consumers, instead of data on individuals; for example, census tract data. However, this is not always a legitimate step.

In my work, the use of census tract data type of data is legitimate because my structure implies that the demand and the supply functions for the differentiated commodities, as well as the price and the wage equations are linear, i.e., the price equation is linear in quality, the wage equation is linear in labor skill, the demand for the differentiated good is linear in consumer's income and type, and the supply for the differentiated product is linear in the vector of the firm's technological parameters.

As it concerns the estimation technique, the theoretical model suggests a simultaneous estimation of the price equation, of the wage equation (if any), of the demand equation(s), and of the supply (if any) equations in which we impose the constraints across the parameters that are implied by the relevant proposition.

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[5] In many instances, knowledge of the brand name (maker) and of the model could yield the information that we need about physical characteristics, quality and price.

Further elaborations of the basic model are presented in the Appendix. In Appendix A, I introduce leisure into the utility function. In Appendix B, consumers do not have to use one unit of the differentiated product, that is, they can choose not only the quality but also the quantity of the differentiated product.

CHAPTER 3

OVERVIEW OF THE THEORETICAL MODEL

Differentiated commodities can be described by a vector of objectively measured physical characteristics. Consumers and producers are assumed to perceive these products as variations of a basic commodity. The vector of physical characteristics is translated into a quality index (a scalar). I presented a method for obtaining closed form solutions to an equilibrium model of a market of these products, and I analyzed the consumer and producer choices, as well as the meaning and nature of market equilibrium for a variety of economic structures.

The model assumes that consumers have different preferences and income. Preferences can be described by quadratic utility functions with a second partial derivative with respect to the numeraire good equal zero. Each consumer is characterized by a vector of utility parameters. This vector of utility parameters follows an exogenously given multi – normal distribution with a variance – covariance matrix that need not be diagonal or satisfy other exogenously given restrictions.

The utility function depends on the quantity of the numeraire good, on the quality of the differentiated good, and on leisure. Chapter 2 and Appendix A assume that consumers use one unit of the differentiated good. Therefore, it is anticipated that the class of economies presented in these chapters will have a variety of applications to many problems involving structural analysis of markets for durable differentiated goods. However, non – durable goods, such as vacation packages, might be analyzed using those chapters framework. In Appendix B, a consumer is assumed to choose not only the quality but also the quantity of the differentiated product.

In Chapter 2, the income of a consumer is assumed to be exogenously given, and the utility function does not depend on leisure. The vector of the consumer characteristics and income is assumed to follow a multi – normal distribution; unlike other work[6], consumer characteristics, as well as consumer characteristics and income, can be correlated since the theory does not place any restrictions on the vector of consumer characteristics and income.

In equilibrium, the demand for the differentiated good is linear in the vector of consumer characteristics and income. Therefore, the aggregate demand for the differentiated good follows a normal distribution.

On the other side of the economy, there are firms that produce the differentiated good and trade it in competitive markets. In Chapter 2 and Appendices A and B, I characterized the market equilibrium for several configurations of the supply side. The first of them assumes that the supply for the differentiated good is exogenously given. In addition to that, I worked out the case of endogenous supply under the following assumptions :

1) The inputs to the production are exogenously fixed and owned by competitive firms, each of which produces the product with the inputs that it owns,

2) The capital distribution is exogenously given, and

3) None of the inputs to the production is exogenously given.

In all the cases where the supply for the differentiated product is endogenously determined, I assume that the cost function is quadratic in the quality of the labor input, and that the intercept of the marginal cost curve is a linear function of a vector of parameters that characterizes the technology of a firm. This vector of technological parameters is exogenously given to each firm and follows a multi – normal distribution.

Appendices A and B, that introduce leisure as an argument in the utility function, allow the labor inputs to the production to be either homogeneous or heterogeneous. If the labor is heterogeneous, the wage rate is a function of the labor quality index.

In equilibrium, the supply for the differentiated product and the demand for labor of a

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[6] For example, Epple (1984) assumes that the variance covariance matrix of the vector of consumer characteristics is diagonal or that it satisfies other restrictions.

competitive firm are linear in the vector of the technological parameters of the firm. Therefore, the aggregate supply for the differentiated product and the demand for labor inputs follow normal distributions.

For an equilibrium in the product and input markets, Aggregate Demand must equal Aggregate Supply in both markets. These equilibrium equations specify the restrictions among the exogenous and the endogenous (price and wage equations) parameters of the model.

The wage equation that equilibrates the input market is linear in the quality of the labor input, and the equilibrium product price equation is linear in the quality index of the differentiated product for the economies that are specified in Chapter 2 and Appendix B, and quadratic for the ones specified in Appendix A. The equilibrium product price equations that are specified in Chapter 2 and Appendix B have an intercept and they can become non - linear using a variable transformation similar to the one discussed in Appendix E.

The model assumed that the quality of the differentiated product is a linear function of the physical characteristics of the product, and that the quality of labor is linear in the vector of characteristics of the labor unit.

With few exceptions, structural analysis of the hedonic approach is not available, and complete hedonic equilibrium models have not been estimated. The reason is that closed form solutions to hedonic equilibrium models are not available for any class of economies that could serve as the foundation for empirical applications, a gap I hope I filled, in part here.

The models that I present in my thesis can characterize the nature of an equilibrium for a wide variety of markets for differentiated products. These models assume that there is a function that maps physical characteristics into a scalar quality index. While this is a strong restriction, this technology allows me to impose weaker a priori restrictions in other respects. Unlike the previous work on closed form solutions to hedonic equilibrium models[7], the cross partial derivative of the utility function does not have to be zero, the marginal utility with respect to the numeraire good can have an intercept, no restrictions need to be imposed on the exogenously given variance – coveriance matrices, the number of consumer characteristics does not have to equal the number of product characteristics, and the demand function does not have a zero income elasticity.

The nature of the solution for the models that I examined depend on the preference, cost, and production technology structures that I assumed. However, the method employed here is not applicable only to those cases. For example, if the quality index for a differentiated product is a vector, analytical solutions could still be obtained using the method that is developopped in the previous chapters. However, such solutions would require that several constraints among the exogenously given structural parameters of the model are satisfied. One of them would require that the variance – covariance matrices of the multi – normal distributions of the consumer vector of utility parameters and of the firm vector of technological parameters are diagonal matrices. That condition is unlikely to be satisfied in many cases of interest.

The formulation of the hedonic approach that I present here is suggestive of empirical work to understand the nature of the market for a differentiated product like personal computers, automobiles, rental housing services, stereo equipment, and soap. This formulation can help us understand a market for a differentiated product since it can analyze a consumer's decision to replace a variety of a commodity by another; it can analyze a firm's decision to produce one type of a differentiated product, to hire a kind of labor or to invest in a type of capital; it can specify the relationship among the equilibrium price distribution, the income distribution, the distribution of consumer characteristics, and the distribution of the available technologies; it can specify the relationship among the equilibrium hourly wage rate distribution, the distribution of consumer characteristics, and the available technologies.

The model that I present is offered for structural analysis, and can specify the effects of

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[7] I am referring to Tinbergen (1959) and Epple (1984).

a change in any of the exogenously given parameters of the model on the equilibrium of the economy. This is very important for answering many questions of interest. For example, how a technological improvement will affect the supply for the differentiated product, and the price distribution. The ability to compute those effects is very important for applied research in Marketing and in Cost – Benefit Analysis. To illustrate further this important aspect of my work, I present the following example.

A Numerical Example :

Consider an economy in which consumers have identical preferences that can be described by the following utility function :

$$U = 100 + h + 2x + 0.5h^2 + xh \quad ,$$

where :

x is the numeraire good, and
h is the quality of the differentiated good.

Consumers are assumed to use the services of one differentiated good.

Let the consumer income follow an exogenously given normal distribution with a mean that is equal to 550, and a variance that is equal to 400.

The supply for the differentiated good is assumed to follow an exogenously given normal distribution. This distribution and the quality mapping (1) imply that the supply for the product quality is also a normal distribution; let this distribution have a mean equal to 2 and a variance that is equal to 1.

Proposition 1 implies that the equilibrium hedonic price equation is :

$$P(h) = 490 + 10.5h \quad .$$

Equation (10) implies that a consumer with a $500 income demands a differentiated good of quality −0.5; this implies that the demand for x is 15.25. The previous method employed for empirical work[8] in this area would predict that this consumer willingness to pay for a one unit decrease in the quality of the differentiated good is −10.5 since the price equation, Figure 1, implies that the marginal willingness to pay curve is the one given in Figure 2. I note that this consumer willingness to pay for a one unit decrease in quality is −10.5, given that none of the exogenous parameters of the model that can shift the marginal willingness to pay curve changes. However, the ability to compute the willingness to pay for changes in h only in that case is of limited importance to applied research.

A question of the following kind could be of great importance to applied research :

What is the gain (or loss) of a consumer from a unit decrease in the mean, and 0.75 of a unit decrease in the variance of the distribution of the supply for the differentiated product ?

For example, suppose that some policy implies the above specified changes in the distribution of the supply for the differentiated good and that we want to answer the above question to evaluate this policy and compare it to others.

To be able to answer a question like the above, someone must know the closed form solutions to the hedonic model that he considers.

The above changes in the distribution of the supply for the differentiated product imply a new equilibrium. In the new equilibrium, the consumer demands a new combination of the two goods and the previous method cannot provide a measure of the consumer's utility

[8] To derive the willingness to pay for a change in h , the previous work derives a marginal willingness to pay schedule and integrates over the appropriate interval; for example, Ridker and Henning (1967), Harrison and Rubinfeld (1978). To estimate a marginal willingness to pay curve, Harrison and Rubinfeld (1978) propose a four - step estimation procedure. The first step in their procedural model is to specify and estimate a hedonic price equation. The second procedural step, calculating each household's willingness to pay for a marginal change in an attribute, follows from the first order condition for utility maximization. That is, the second step consists of calculating the derivatives of the hedonic price equation with respect to a specific attribute for each household. This derivative is an estimate of the household's willingness to pay for a marginal change in that attribute. The third procedural step provides an estimate of the demand curve for an attribute. It consists of regressing the households' marginal valuations (the derivatives calculated in the second step for each household) on the attribute and other variables (e.g., household income) which may cause the demand for the attribute to shift. In step 4, they integrate the demand curve for that attribute (obtained in step 3) over the appropriate interval to determine each household's willingness to pay for non - marginal changes in an attribute. If the hedonic price equation is linear in attributes, Harrison and Rubinfeld (1978) do not follow the above four step procedural model. Instead, they use their information about the price equation functional form to compute willingness to pay for a non - marginal change in an attribute. In the latter case, they compute consumer benefit in the same way that I do for the linear price equation of the numerical example that I present in this section.

gains or losses from the above changes. Knowledge of the structure of the economy predicts that the \$500 – income utility maximizing consumer gains 38.75 utiles[9]. To find out the consumer's utility before and after the changes in the supply distribution (so that I can compute his gains or losses), I compute 1) his demand for h , 2) the price that he pays for the differentiated good that he consumes, 3) his demand for x (that is equal to his income minus his expenditure on the differentiated good), and 4) I substitute in the utility function his demand for h and x .

Note that the above changes in the supply for the differentiated good imply that the equilibrium price equation is :

$$P(\,h\,) = 470 + 20.5\,h \quad , \text{ and}$$

that the demand for x is 35.125.

The point of the previous example (which point does not depend on the linearity of the marginal willingness to pay schedule of Figure 2) is that the method that integrates a marginal willingness to pay curve over the appropriate interval is unable to answer questions like the above.

If we want to map the previous numerical example to the specific case of the rental housing market, 1) we can think of h as the quality of housing that is a function of air quality, and 2) we can think that the changes in the supply distribution that I considered above are implied by policies that affect the air quality distribution.

If a change in some of the exogenous parameters of the model implies that the consumer's marginal willingness to pay curve does not change significantly, an estimate of the consumer's willingness to pay for an m – unit increase in quality can be obtained by integrating the marginal willingness to pay curve over the appropriate interval. However, for a non – marginal change in one or more of the exogenous parameters of the model, the

[9] Knowledge of the structure of the economy will also let me compute the consumer's willingness to pay for the given changes in the distribution of the supply for the differentiated product. The steps involved in computing a willingness to pay for such changes are illustrated in Chapter 4 with the help of an application that I study there.

latter method would miscalculate the consumer's willingness to pay because it cannot predict the shift of the consumer's marginal willingness to pay curve. For example, suppose that some changes in the exogenous parameters of the model imply that the price equation shifts from position 1 to position 2 (see Figure 3), where h is the quality of the differentiated good. Let D1 and D2 (see Figure 4) be the marginal willingness to pay curves the associated respectively with the price equation curves 1 and 2 of Figure 3. Numerous studies that deal with similar kinds of questions predict that the consumer's willingness to pay for the m – unit increase in h equals the area (efcb). However, knowledge of the structure of the economy can predict that the consumer's willingness to pay equals the area (efda). Therefore, the previous work would underestimate the consumer's willingness to pay by an area that is equal to (abcd). In addition to that, the non – structural approach cannot compute the change in the demand for quality that is implied by a change in one of the exogenous parameters.

The type of errors that I specified in the previous examples are inherent to a non – structural approach and present in every study that addresses similar questions[10]; for example, Harrison and Rubinfeld (1978). They introduce a four step estimation procedure to estimate the marginal willingness to pay for h , where h is an air quality measure; mapping that into the previous example, they estimate a curve like D1, Figure 4. Then they integrate this marginal willingness to pay curve, D1, over the appropriate interval to compute the consumer benefit the associated with, say, an m – unit improvement[11] in h , given a change in the Federal Automobile Emission Control Policy (a change that mandated a roughly 90% reduction in nitrogen oxides, hydrocarbons, and carbon monoxide from[12] 1970 to 1978); for the consumer that Figure 4 refers to, they would predict that his willingness to pay for an m – unit improvement in h equals the area (efcb). However, the measure that they obtained is not the willingness to pay for the environmental policy change that they consider. It is that consumer's willingness to pay for moving into an area

[10] I am referring to studies that I am aware of.

[11] The specific m - unit improvement in h for each consumer was calculated using the Transportation and Air Shed Simulation Model, see Ingram and Fauth (1974).

[12] For the period 1970 to 1990 that they consider, that change would be even larger.

with an m − unit increase in air quality, given that there is no change in any environmental policy. An environmental policy change would shift the price equation curve, suppose from position 1 to position 2, Figure 3. As a result, the marginal willingness to pay curve would shift, suppose from D1 to D2, Figure 4. The consumer of our example would be willing to pay the area (efda) for the environmental policy change that implies for him an m − unit improvement in h .

If the method that integrates a marginal willingness to pay curve over the appropriate interval is not accompanied by a theory about the structure of the economy, it gives an incorrect figure for the consumer willingness to pay for a non − marginal change in an exogenous parameter; for example, the environmental policy change that is discussed in the previous paragraph. Therefore, I believe that structural analysis is necessary to avoid the kind of miscalculations that I mentioned above. The model that I present in my thesis is offered for structural analysis and indicates a simultaneous equation estimation of the economic model in which we impose the cross − equation restrictions that are specified by the relevant proposition. The model has also the nice feature that it can be used legitimately with several types of data. For example, aggregate data, micro data, data from a specific area (city or country), data from several different areas (cities or countries).

CHAPTER 4
APPLICATION

4.1. Introduction.

The model that I presented in Chapter 2 can be used for a study of the differentiated good rental residential housing. The empirical example that follows shows that it is actually feasible to estimate and test a closed – form model.

The results of the empirical example will be used to investigate the willingness to pay for clean air. The theoretical model is written in terms of rental prices and, unlike all the other studies that I am aware of, the empirical work uses rental prices instead of housing values. That builds a consistency between the theory and the empirical work that is not present in other work (for example Harrison and Rubinfeld (1978)). In addition to that, if the stock of housing is exogenously given, I believe that rental prices are more appropriate because they reflect the market's current valuation of housing attributes (the clean air variable is one of them). Housing purchase prices incorporate the present housing conditions, as well as the market's expectations about future.

4.2. The Economic Model.

The differentiated product rental residential housing can be described by a vector of characteristics v ,

where :

$$v = [\ v_1 \ \ v_2 \ \ v_3 \ ]$$

v_1 is the size of the housing unit (number of rooms),

v_2 is an air quality index,

v_3 is the travel time to work.

v follows an exogenously given multi – normal distribution.

The quality of housing, h (a scalar), is a linear function of the vector of housing characteristics v, that is,

$$h = \epsilon v' \quad , \tag{39}$$

where :

$\epsilon = [\ \epsilon_0 \quad \epsilon_1 \quad \epsilon_2\ ]$ is a vector of parameters.

Consumer preferences are described by utility functions. A utility function, $U(h, x, a)$, depends on the quality of the house, h, on the numeraire good, x, and on the parameter a,

where :

a is the number of persons in the family.

A consumer solves the following optimization problem :

$$\begin{aligned} & max \quad U(h, x, a) \\ & with \ respect \ to \quad h, x \\ & subject \ to \quad I = 12\, P(h) + 365\, x \end{aligned}$$

where :

I is the annual income of a consumer

$P(h)$ is the (monthly) rental price equation.

The utility function is given next :

$$\begin{aligned} U(h,x,a) \;=\; & \delta + (\zeta_0 + \zeta_1 a) h + \\ & 0.5\, \xi\, h^2 + x\, h \end{aligned} \tag{40}$$

where :

$\delta, \zeta_0, \xi, \zeta_1$ are utility parameters.

The vector $[\ a\ \ I\ ]$ follows an exogenously given multi – normal distribution.

Proposition 1 implies that the price equation is :

$$P = (365/12)[\ \zeta_0 + \zeta_1 \bar{a} + (\bar{I}/365) - A\bar{h} + 0.5(\ \xi + A\ )h\ ] \tag{41}$$

where :

$\bar{a}$ is the mean size of a family,

$\bar{I}$ is the mean consumer income,

$\bar{h} = \epsilon\, \bar{v}'$ is the mean quality of residential housing,

$\bar{v}$ is the mean of the vector of housing characteristics v ,

$$A = \frac{\sqrt{t\, \Sigma_z\, t'}}{\sigma_s}$$

$$\sigma_s = \sqrt{\epsilon\, \Sigma_v\, \epsilon'} \quad ,$$

$$t = [\ \zeta_1 \quad 1\ ], \ \textit{and}$$

Σ_v is the variance coveriance matrix of the exogenously given distribution of housing characteristics.

Proposition 1 and equation (10) imply that the demand for h is :

$$h = \bar{h} + \frac{\zeta_1}{A}(\ a - \bar{a}\ ) + \frac{1}{365\,A}(\ I - \bar{I}\ ) \tag{42}$$

4.3. The Econometric Model.

The previous section of Chapter 4 implies that the complete model consists of the equations (39), (41), and (42).

For the residential housing market, I assume that the quality of housing is a latent variable. Without loss of generality, the quality of housing can be normalized by setting the parameter ϵ_0 equal to 1.

Substituting equation (39) in (41) and (42), I eliminate the quality of housing[13] and I obtain that the price equation and the first order condition for the consumer's optimization problem are respectively equivalent to :

$$P = (365/12)[\zeta_0 + \zeta_1 \bar{a} + (\bar{l}/365) - A\epsilon \bar{v}' + 0.5(\xi + A)\epsilon v'] \text{ , and}$$

$$\epsilon(v' - \bar{v}') - \frac{\zeta_1}{A}(a - \bar{a}) - \frac{1}{365A}(l - \bar{l}) = 0 \ .$$

I assume an additive error term on the above two equations. To be more specific, I assume that the equations that I will estimate are the following :

$$P = c + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + u_1 \quad \text{, and} \tag{43}$$

$$(v_1 - \bar{v}_1) + \epsilon_1(v_2 - \bar{v}_2) + \epsilon_2(v_3 - \bar{v}_3) + \epsilon_3(a - \bar{a}) + \epsilon_4(l - \bar{l}) + u_2 = 0 \ , \tag{44}$$

where :

$$c = (365/12)[\zeta_0 + \zeta_1 \bar{a} + \bar{l}/365 - A\epsilon \bar{v}'] \ , \tag{45}$$

$$\beta_{i+1} = (365/24)(\xi + A)\epsilon_i \ , \quad \text{for } i = 0, 1, 2 \ , \tag{46}$$

$$\epsilon_3 = -\frac{\zeta_1}{A} \ , \tag{47}$$

$$\epsilon_4 = -\frac{1}{365A} \quad \text{, and} \tag{48}$$

---

[13] I do this because the quality of housing is unobservable by econometricians.

u_1, and u_2 are the econometric errors of the first and second equations respectively. They are assumed to satisfy the following :

A1. u_1 and u_2 are uncorrelated.

A2. a and l are uncorrelated to u_1. They are also uncorrelated to u_2.

A3. v_2 and v_3 are uncorrelated to u_1.

A4. v_2 and v_3 are uncorrelated to u_2.

4.4. Estimation of The Reduced Form Equations.

I estimate the last two equations, (43) and (44), simultaneously via Maximum Likelihood. I also impose the restrictions that are implied by the structure of the model, namely,

$$\epsilon_1 = \frac{\beta_2}{\beta_1} \text{, and} \tag{49}$$

$$\epsilon_2 = \frac{\beta_3}{\beta_1}. \tag{50}$$

I estimate the model using (1980) census tract data on rental prices, number of rooms, travel time to work, size of the family, and consumer income, and (1979) SAROAD based data on air quality[14]. Given that the air quality and travel time to work are census tract variables, assumptions A3 and A4 require that the consumer census tract locational choice is exogenous and uncorrelated to the econometric errors of the equations that I estimate[15]. Unlike other work, e.g., Harrison and Rubinfeld (1978), the model states that it is legitimate

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[14] I define the air quality variable, v_2, to be equal to the inverse of the air pollution variable (Particulate matter).

[15] The dissertation was meant to develop a quality theory for differentiated products. According to the theory, a utility maximizing consumer chooses the quality of a differentiated good and there may exist many goods that can provide that quality. The theory was not meant to specify how the consumer chooses among those equal quality differentiated goods. In terms of the housing market application, the theory does not specify how the consumer makes a locational choice. The consumer is indifferent to all the (housing - location) combinations that can provide the quality of housing that maximizes his happiness. Since a consumer cares only about the quality, I assume that he moves randomly to any census tract and picks a house of the quality that he is looking for. However, if in a census tract demand does not equal supply, the theory suggests that consumers do not bid prices up (that would make the price of a specific house to be different in different census tracts) but they move into another census tract; since there are no moving costs, a consumer will move into another area where he can find a house of the quality that he is looking for. A3 and A4 assume that this random locational choice is uncorrelated to the econometric error terms of the equations that I estimate. Section 4.8. develops an estimation procedure that can give consistent parameter estimates when the assumption A4 is relaxed; I develop that estimation procedure because I thought that A4 was the reason that a fixed effect assumption did not perform well when I pooled the data for all five cities. The assumption A4 may not be satisfied if there is a theory that can make the locational choice endogenous and correlated to the econometric error of the demand for housing quality equation.

to use census tract data. To estimate the model, I use data on Houston, Texas. The results are given on Table[16] 1.

To see if the model is of any value at all, I tested the hypothesis that all the parameters of the equation (43) equal zero, that is,

$$\beta_1 = \beta_2 = \beta_3 = 0 \quad .$$

An F – test implies that this hypothesis is rejected at the 1% significance level.

A similar F – test implies that I cannot accept the hypothesis that the parameters of the second equation, equation (44), equal zero, that is, I cannot accept that

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0 \qquad \text{at the 1\% significance level.}$$

The t – statistics (see Table 1) show that all the parameters are significant at the 10% significance level. Moreover, the size of a house (which is expected to be the main determinant of the rent), as well as the income and the size of the family (that are expected to be the main determinants of the demand for housing quality) are significant at the 1% significance level.

For the residential housing market, I expect the following :

$$\epsilon_1 > 0$$

$$\epsilon_2 < 0 \quad , \quad \textit{and}$$

$$\zeta_1 > 0 \quad .$$

Therefore, the parameter estimates for ϵ_1 , ϵ_2 , and ζ_1 must satisfy the above inequalities. The structural analysis that I make in the next section allows me to check whether my expectations are correct. This will be another test of the model. The sections that follows the structural analysis section investigate other tests of the model.

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[16] All the tables are given in a separate section. The definitions and the notation used in these tables are given at the beginning of that section.

4.5. Structural Analysis.

The parameters estimates that I obtained in the previous section allow me to analyze the structure of the housing market of Houston, Texas. I can also specify how that structure depends on the mean of the air quality distribution. The latter enables me to address interesting questions that a non – structural approach cannot.

4.5.1. The Houston Housing Market.

The parameter estimates that are given on Table 1 and my theoretical model enable me to compute the rental price equation, the demand for housing quality, the quality index equation, the utility function, and the demand for the numeraire good.

Given the parameter estimates on Table 1, given that they satisfy (49) and (50), and given the relationships among the structural parameters and the reduced form equation parameters of the model, equations (45) – (48), I can solve for: A , ζ_1 , ϵ_1 , ϵ_2 , ξ , and ζ_0 . The solutions for these parameters follow:

$$A = 25.10$$
$$\zeta_1 = 12.66$$
$$\epsilon_1 = 33.93$$
$$\epsilon_2 = -0.068$$
$$\xi = -21.91$$
$$\zeta_0 = -1.13$$

The model's predictions are consistent with my a priori expectations that I mentioned at the end of the previous section.

(Note that in order to solve for ζ_0 from (45), I used the statistics that are given on Table 2).

Next, I use the parameter estimates that I have obtained so far to compute the rental price equation, the demand for housing quality, the housing quality index equation, and the utility function.

The rental price equation is:

$$P = 93.51 + 48.59\,v_1 + 1648.67\,v_2 - 3.32\,v_3.$$

The demand for housing quality equation is:

$$h = -0.167 + 0.505\,a + 0.000109\,I \quad .$$

The housing quality index equation is:

$$h = v_1 + 33.93\,v_2 - 0.068\,v_3 \quad . \tag{51}$$

The utility function is:

$$U(\,h\,,\,x\,,\,a\,) = \delta + (\,-1.13 + 12.66\,a\,)\,h + x\,h - 10.95\,h^2 \quad .$$

The price equation can also be written in the following way:

$$P = 93.51 + 48.59\,h \quad .$$

I substitute the last equation and the demand for h into the budget constraint and solve for the demand for x. The demand for x is given next.

$$x = -2.81 - 0.81\,a + 0.0026\,I \quad .$$

We can now see that:

- The rent is positively related to the quality of a house.

- The demand for housing quality is positively related to the size of the family and income.

- The housing quality is positively related to air quality and negatively to travel time to work.

– The marginal utility with respect to housing quality is positively related to the size of a family.

The qualitative properties listed above are consistent with my a priori expectations.

4.5.2. The Houston Housing Market as A Function of The Mean Air Quality.

In this subsection I want to compute how the structure of the Houston housing market depends on the mean air quality. To do that, I repeat the calculations of the previous subsection with the only difference that now I do not substitute 0.0141 imcm[17] for the mean air quality, $\bar{v}_2$, (see Table 2). The results follow.

The parameters A , ζ_0 , ζ_1 , ϵ_1 , ϵ_2 , and ξ do not change because they do not depend on the mean air quality.

The housing quality index equation is given in (51).

The rental price equation is:

$$P = 459.36 - 25904.50\,\bar{v}_2 + 48.59\,v_1 + 1648.7\,v_2 - 3.32\,v_3 \quad ,$$

or equivalently:

$$P = 459.36 - 25904.50\,\bar{v}_2 + 48.59\,h \quad .$$

The equation demand for housing quality is:

$$h = -0.647 + 33.93\,\bar{v}_2 + 0.505\,a + 0.000109\,I \tag{52}$$

The utility function is:

$$U(\,h\,,\,x\,,\,a\,) = \delta + (\,-1.13 + 12.66\,a\,)\,h + x\,h - 10.95\,h^2 \tag{53}$$

The demand for x follows.

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[17] imcm = 1 / (micrograms per cubic meter).

$$x = -14.07 + 797.45\,\bar{v}_2 - 0.81\,a + 0.0026\,I \qquad (54)$$

I can now use the above results to illustrate the kind of questions that my analysis can address.

4.5.3. Illustration: The willingness to pay for an improvement in air quality.

The purpose of this illustration is not to determine the precise dollar figure of the consumer benefit from an improvement in air quality. Rather, it is to illustrate how to perform a general equilibrium analysis that is accomodated by the model and to show that the previous (partial equilibrium) common practice for computing the consumer benefit from a non – marginal change in one of the characteristics of a differentiated good can yield a very different benefit figure. Much greater care would be necessary to estimate with confidence the precise dollar value of the consumer benefit from a change in the air quality distribution. Moreover, for a long run analysis the supply for houses should be made endogenous.

A consumer's willingness to pay for a y% improvement in air quality, W, is defined to be the solution to the following equation :

$$V(a, I, t) = V(a, I + W, t + ty/100) \qquad (55)$$

where :

t is the mean air quality in Houston, and

$V(a, I, t)$ is the indirect utility function of an (a, I) – type consumer given that the mean air quality of the city of Houston equals t .

That is, the consumer's benefit from a y% change in the mean air quality is the part of his income that he is willing to give up so that the utility after the y% change equals the utility before the y% change.

I will compute the benefit of the mean household in Houston[18] from a 1%, 5%, 10%, and 70% improvement in the mean air quality of the city. That is, I will compute W for $y = 1, 5, 10, 70$. The steps involved in the computation are explained next.

To obtain the indirect utility function, I substitute the demand for housing quality and the demand for the numeraire good, equations (52) and (54) respectively, into the utility function, equation (53). Given the indirect utility function, I can specify the functional form of equation (55). I substitute into (55) the indirect utility function, the mean income, the mean number of persons in a household, and the mean air quality of Houston (see Table 27). The latter implies that equation (55) is equivalent to :

$$A W^2 + B(y) W + G(y) = 0$$

where :

W is the willingness to pay (in thousands of dollars),

A , $B(y)$, *and* $G(y)$ are the parameters of the equation that I solve for W,

$A = 0.1533031$, and

the parameter values of $B(y)$, *and* $G(y)$ depend on y (the percentage improvement in the mean air quality).

For $y = 1, 5, 10, 70$ the parameters $B(y)$ *and* $G(y)$ are given next :

B(1) = 7.9356881

B(5) = 7.9887863

B(10) = 8.0551591

B(70) = 8.851632

G(1) = 0.28995

G(5) = 1.66875

G(10) = 3.40517

[18] The mean household of Houston, as well as that of four other cities, is described on Table 27.

G(70) = 25.36271 .

Solving the last equation with respect ot W, I obtained the following solutions[19] :

- W = −0.0365631 thousands of $ for y = 1.
- W = −0.2097305 thousands of $ for y = 5.
- W = −0.4261883 thousands of $ for y = 10.
- W = −3.0236550 thousands of $ for y = 70.

The above solutions imply the consumer benefit figures that are given on Table 3. Table 3 shows that the benefit − percentage change in air quality ratio is slightly increasing[20] in y. That is, if I multiply the percentage change, y, by k, the consumer benefit increases by a factor greater than k. From a technical point of view, this result can be true within my framework because 1) the indirect utility function is quadratic in consumer income and mean air quality, and 2) the coefficients of these variables can be either positive or negative. In other words, 1) the equation that I solve for the willingness to pay , W, is non − linear in the mean air quality, and 2) the solution for W has a first derivative with respect to mean air quality that can be either increasing or decreasing in air quality. In the above application, it turned out that the relationship between mean consumer benefit and mean air quality improvements is increasing.

Ceteris paribus, changes in the mean air quality make a consumer happier because the quality of his house improves (this change in utility is decreasing in air quality improvements because the marginal utility with respect to housing quality is decreasing).

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[19] Moreover, -51.728128, -51.901321, 52.117817, and -54.715762 are also solutions to the same equation for y = 1, 5, 10, and 70 respectively. However, these solutions are rejected because they indicate a willingness to pay that is greater than the mean consumer's income (15.954 thousands of $)

[20] This property is consistent with the following predictions of the model :
- The higher the current mean air quality, the greater the utility change that is implied by a marginal improvement in the mean air quality, and
- The higher the current mean air quality, the greater the part of his income that the mean household is willing to give up for a marginal improvement in air quality.

To prove the first prediction of the model, I computed the second derivative of the indirect utility function with respect to the mean air quality and I found that it was positive (and equal to 41508.825).

To prove the second prediction of the model, I took the indirect utility function and set it equal to a constant. Then I differentiated the last equation (twice), and I computed the first derivative of the willingness to pay for a marginal improvement in the mean air quality with respect to the mean air quality. Given that utility stays at a fixed level, the mean household's willingness to pay for a marginal improvement in the mean air quality is a function of the mean air quality and its first derivative with respect to the mean air quality is greate than zero.

However, increases in the mean air quality shift the price function for housing quality downward. This implies a redistribution of rents (from housing suppliers to consumers of housing) which lets consumers increase their utility even more. In the above application, the change in the distribution of rental prices and the assumed utility function imply that (after the mean air quality improvement) the mean consumer is able to buy a combination of goods that increases his utility at a rate that is greater than the percentage improvement in air quality.

Next, I compute the benefit from the same air quality improvements using the alternative approach[21], and I compare the results.

To compute benefits using the non - structural approach, I first estimate a marginal willingness to pay schedule, and then I integrate the marginal willingness to pay from $\bar{v}_2$ to $\bar{v}_2 + \bar{v}_2 y / 100$ to obtain a measure of the willingness to pay[22] for a y% change in the mean air quality of Houston. To illustrate this method, I experimented with a price equation that is linear in air quality and another that is nonlinear (quadratic). The parameter estimates[23] are given in Tables 4 and 5. I chose to make my calculations using the linear specification for the price equation because the nonlinear specification is not consistent with my apriori expectations[24].

Given a rental price equation that is linear in air quality, the non - structural approach would define the willingness to pay in the following way[25]:

$$W = 12(AQC)(DV)$$

---

[21] Harrison and Rubinfeld (1978) is an example of the non - structural approach that I am referring to in the next paragraph. Here, I am not referring to the four step estimation procedure which is the main contribution of that paper. The four step estimation procedure is the way that they apply that method (namely, the way that they estimate the marginal willingness to pay), given a quadratic " price " equation that they consider. If the price equation is linear in attributes, they use another method to compute benefit (see also footnote 8).

[22] That would assume a uniform improvement in air quality. That is, an improvement in each census tract that equals the mean air quality improvement.

[23] The parameter estimates have been obtained using a single equation estimation technique.

[24] The quadratic model predicts that rent decreases as air quality increases.

[25] For example, Harrison and Rubinfeld (1978), page 92, footnote 28. For a price equation that is linear in air quality, they define willingness to pay, as well as average benefit, in exactly the same way. I recall that if the price equation is linear in attributes, they do not use their four step procedural model to compute benefit (see also footnote 8).

where:

DV is the change in the mean air quality of Houston,

AQC is the coefficient of the air quality variable in the rental price equation, and

$AQC = 6701.2$ (see Table 4).

Calculating the benefit of the mean household using the latter definition for the willingness to pay, I obtained the estimates[26] given on Table 6.

From Tables 3 and 6, we can now see that the two methods imply very different benefit figures. The reason is that if there is a change in one of the exogenous parameters of the model, the latter method (by its nature) is not appropriate for computing the willingness to pay for a change in one of the characteristics of a differentiated good.

To compute benefits, the non – structural approach uses a different method and different estimates of the price equation (the ones given on Table 4). To separate those two issues and to show the difference that arises because of differences in methods of calculation, I compute benefits using AQC=1648.7 (the estimate of the coefficient of the air quality variable in the price equation that is given on Table 1). I obtained the benefit estimates that are given on Table 30.

The benefit figures of Table 6 are approximately 71% below the benefit figure based on the structural model (given on Table 3); this difference arises because of differences in method of calculation as well as differences in coefficients. The benefit figures of Table 30 are approximately 93% below the benefit figures of Table 3; this difference arises because of differences in method of calculation (using the same estimated coefficients). The results show that the non – structural approach can give very different benefit figures

[26] To obtain these estimate, I assumed a uniform improvement in air quality. That is, the improvement in each census tract equals the improvement in the mean air quality of the whole city.

even for marginal changes in the mean air quality[27] (e.g., a 1% change).

Harrison and Rubinfeld (1978) compute the benefit per household from the federal automobile emission control strategy, in which the federal goovernment established tailpipe emission standards[28] for new cars beginning in model year 1971. I believe their procedural model underestimates the benefit from the federal emission policy because 1) their method is not appropriate for estimating the benefit from a policy that implies a non – marginal change in the air quality distribution, 2) the benefit estimate in which they place the greatest confidence is approximately 30% below the figure based on a linear housing value equation, and 3) my application shows that the benefit figure based on a linear price equation is far below (approximately 71%) the willingness to pay that is derived from utility theory.

4.6. Test of The Model[29].

The F – and t – (significance) tests that I performed in Section 4.4, as well as the predictions and the qualitative properties of the model, indicate that the model can be helpful in analysing the housing market of Houston, Texas.

To investigate how general my general equilibrium model can be, I estimated the model for 4 more cities and I also pooled the cross – section data for all 5 cities (see next section).

The model implies specific functional forms for the rental price equation, equation (41), and for the demand for housing quality equation, equation (42), as well as the cross

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[27] To investigate further this interesting result, I took the indirect utility function and set it equal to a fixed utility level. Then I differentiated this equation and I solved for the mean household's willingness to pay for a marginal improvement in the mean air quality. This first order approximation predicts that the mean household is willing to pay $43.39 for a 1% improvement in the mean air quality. The non - structural approach predicts a willingness to pay for the 1% air quality improvement (see Table 6) that is 73.86% below the figure that I obtained from the above first order approximation. (Table 3 shows that the mean household's willingness to pay for the 1% mean air quality improvement is 15.5% below the figure predicted by the first order approximation).

[28] These emission standards became stringent up to 1978 year model, when a roughly 90% reduction from the 1970 levels was mandated for nitrogen oxides, hydrocarbons, and carbon monoxide. Using a linear price equation they obtained that the average annual benefit per household in Boston was $118. This benefit figure is comparable to the willingness to pay for a 10% air quality improvement that is given on Table 6.

[29] For the development of the final version of Section 4.6., I would like to mention helpful discussions with D. Epple and his written comments on an earlier draft.

equation restrictions (49) and (50). A test of all the restrictions requires time series data[30] and could tell something about the internal consistency of the model.

Even though I cannot perform the complete test of the internal consistency of the model, I can test the two cross equation restrictions, (49) and (50). A likelihood ratio test rejects the hypothesis that the restrictions are satisfied. (I took the unrestricted model to consist of equations (43) and (44) without any cross equation restrictions; the parameter estimates of this model are given on Table 7). When I impose the two constraints, the log likelihood function decreases from −65.05 to −76.48; the chi − square statistic becomes equal to 22.86 and the rejection region at the 5% significance level is all values greater than 6. This test provides evidence about the restrictiveness of the model.

The cross equation restrictions (49) and (50) emerge from the particular structure employed to model hedonic equilibrium in this work. Namely, the normality assumptions, the unobserved quality index, and the choice of functional forms for the utility function and the quality index mapping. Having found evidence against these two restrictions, one would like to turn to estimation of a more general structure. However, no such structure is available, and there is no assurance that there exists a more general structure for which closed − form solutions can be derived. Concequently, even though the latter test provided evidence about the restrictiveness of the model, I chose to work and make my analysis using this structural model because of the properties of the model that I discussed in the previous sections. These properties make me believe that the assumed structure is a good approximation to the true structure of the economy.

Should the finding of evidence against the cross − equation restrictions in the model developed here lead to abandonment of the structural approach in favor of the non − structural approach? I believe not, for the following reasons.

As explained in Chapter 3, use of non − structural approaches for evaluating non − marginal changes leads to systematic errors. Hence, even if the marginal willingness to pay

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[30] Appendix D elaborates further this point.

curve were correctly estimated in the non - structural approach, the use of that schedule would give incorrect results.

An advantage of the non - structural approach is that it permits an arbitrary choice of functional form for the hedonic price equation and greater flexibility in choice of functional form for the willingness to pay schedule. However, arbitrarily chosen functional forms for the hedonic price equation and the willingness to pay schedule do not have to be consistent and the non - structural approach cannot provide an econometric test of consistency. The non - structural approach simply avoids the issue that the empirical estimated marginal willingness to pay schedule may not be consistent with the assumed price equation functional form and other equilibrium conditions.

For applications, the tradeoff is as follows. One can employ the structure developed in this thesis. It provides a rigorously derived method for evaluating non - marginal changes in quality, as well as an econometric test, in the form of cross - equation restrictions, that provides evidence about the restrictiveness of the structure. Alternatively, one can adopt the non - structural approach. It affords more flexibility in the choice of functional form than the structural approach, but it gives systematic errors when effects of non - marginal quality changes are calculated, and it provides no econometric test to determine whether the price equation and willingness to pay schedule are internally consistent in equilibrium.

While there are limitations to either approach, I believe that the structural approach will be preferable in many applications because it provides a rigorous procedure for calculating the effects of non - marginal changes in quality. Tests of cross - equation restrictions provide some evidence about the degree of confidence one can have in the results. The non - structural approach evades the econometric consistency check provided by the cross - equation restrictions, but that is a liability rather than an asset of the non - structural approach.

4.7. Applications of the model in other cities.

To obtain an idea about how general the model can be, I repeat the Houston experiment for four more cities, namely, Chicago, Cleveland, Indianapolis, and Dallas. To be more specific, for each of those cities I estimate the equations (43) and (44) imposing the cross equation restrictions (49) and (50) and assuming that the assumptions A1 – A4 are satisfied. The results are given in Tables 8, 9, 10, and 11.

A common characteristic of all those regressions is that all the parameter estimates of interest are insignificant; that was expected because of the small sample size. Moreover, some of the parameter estimates have the wrong sign (see Tables 8, 9, 10, and 11). Wrong signs and insignificant parameter estimates indicate that the assumed structure may not be suitable for Chicago, Cleveland, Indianapolis, and Dallas. To investigate whether this is the case, I have to increase the sample size. To do that, I pooled the cross – section data for all five cities, allowing for fixed effects.

Pooling the cross – section data for all five cities allowed me to increase the sample size to 152. Many different combinations of fixed effect assumptions were tried. These experiments are presented in the appendix, Appendix E. Given the structure that is specified in the previous sections of Chapter 4, the experiments that I performed indicate that a fixed effect assumption is not a good one for studying the problem of the willingness to pay for an improvement in air quality[31].

To be more specific about the fixed effect combinations that I tried, I should mention that, in general, the model predicted the wrong effects of air quality and of travel time to work on rents for all cities other than Houston. The solution to this problem seems to be either more data on the already existing and on other important variables for all five cities (especially, for Chicago, Cleveland, Indianapolis, and Dallas) or an alternative stochastic structure and estimation method that I introduce next.

4.8. An Alternative Estimation Procedure.

In this section, I present an approach for estimation of the reduced form equations that

[31] However, a fixed effect assumption might make sense in the context of other applications; these are applications in which the air quality is not the variable of interest. One of those applications is presented in the Section 4.9 and Appendix J.

does not need satisfaction of the assumption A4. The assumption A4 requires that the census tract locational decision is uncorrelated to the econometric error of the demand for housing quality equation. Violation of that assumption would make the simultaneous equation estimation technique yield inconsistent estimates. I believe that that was the reason that the model predicted the wrong effects of air quality and of travel time to work on rents for Chicago, Cleveland, Indianapolis, and Dallas. Therefore, I introduce a four step estimation procedure that will let me use the structure of the theoretical model and obtain consistent parameter estimates. To illustrate that four step estimation procedure, I poole the cross section data for all five cities (Houston, Chicago, Cleveland, Indianapolis, and Dallas) allowing for fixed effects. The theoretical model follows.

The rental price equation is :

$$P = c + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \quad . \tag{56}$$

The price equation parameters, as well as all the exogenous parameters of the model, are not the same across cities and they should be indexed by j, where j = Houston, Chicago, Cleveland, Indianapolis, Dallas. However, the subscript j has been dropped to simplify the notation. In each city, the price equation parameters satisfy (45) and (46).

The demand for housing quality equation is :

$$h = \gamma - \epsilon_3 a - \epsilon_4 l \tag{57}$$

where :

$\gamma = \bar{v}_1 + \epsilon_1 \bar{v}_2 + \epsilon_2 \bar{v}_3 + \epsilon_3 \bar{a} + \epsilon_4 \bar{l}$, and

ϵ_3 and ϵ_4 satisfy (47) and (48).

Recall that a " – " over a variable means the mean of the variable.

The model requires that the restrictions (49) and (50) are satisfied in each city.

The housing quality index equation is :

$$h = v_1 + \epsilon_1 v_2 + \epsilon_2 v_3 \quad . \tag{58}$$

The parameters of the housing quality index equation can be different across cities. Therefore, they should be indexed by j, where j is defined above. However, as I stated earlier, the subscript j has been dropped. The same is true for the parameters of the demand for housing quality equation.

Since I want to concentrate on the effect of the air quality on the housing market[32], I assume that the model consists of the equations (56), (57), and (58), and that the parameters β_2 , ϵ_1 , and γ are the only ones that can be different across cities. Moreover I make the fixed effect assumption that these three parameters satisfy the following :

$$\beta_2 = \beta_{20} + \sum_{i=0}^{4} \beta_{2i} d_i \quad , \quad and \tag{59}$$

$$\epsilon_1 = \epsilon_{10} + \sum_{i=0}^{4} \epsilon_{1i} d_i \quad , \quad and \tag{60}$$

$$\gamma = \gamma_0 + \sum_{i=0}^{4} \gamma_{1i} d_i \tag{61}$$

where :

β_{2i} , ϵ_{1i} , , and γ_{1i} are exogenously given parameters (they are some of the parameters that I want to estimate), for i = 0, 1, 2, 3, 4, and

$d_1 = 1$ for Chicago,

$d_1 = 0$ else.

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[32] .... and since I cannot estimate the full - model (lack of data on the existing and other important variables).

$d_2 = 1$ for Cleveland,

$d_2 = 0$ else,

$d_3 = 1$ for Indianapolis,

$d_3 = 0$ else,

$d_4 = 1$ for Dallas,

$d_4 = 1$ else,

$d_0 = 0$.

Differences in exogenous factors, for example, humidity, temperature, and rainfall, can make the parameter of the quality index equation ϵ_1 be different across cities.

To estimate the above described model, I follow the next four steps.

STEP 1 :

I assume that there is an additive error term on the price equation, u_1 , that is,

$$P = c + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 + u_1 \quad . \tag{62}$$

v_1 , v_2 , and v_3 are uncorrelated to u_1 , and β_2 satisfies (59).

I think of u_1 as a measurement error in prices.

I estimate the price equation. The parameter estimates are given on Table 23. They imply that the rental price equations are the following :

In Houston : $P = 85.06 + 19.49 v_1 + 12785.74 v_2 - 4.56 v_3$,

In Chicago : $P = 85.06 + 19.49 v_1 + 13826.88 v_2 - 4.56 v_3$,

In Cleveland : $P = 85.06 + 19.49 v_1 + 5888.09 v_2 - 4.56 v_3$, In Indianapolis : $P = 85.06 + 19.49\,v_1 + 4436.75\,v_2 - 4.56\,v_3$, and

In Dallas : $P = 85.06 + 19.49\,v_1 + 9115.34\,v_2 - 4.56\,v_3$.

STEP 2 :

Given (59) and (60), and given that (49) and (50) must be satisfied in all cities, I can use (59), (60), (49), and (50), and the results of Table 23 to obtain estimates for ϵ_2 and ϵ_{1i}, i= 0, 1, 2, 3, 4. Doing that, I obtain that the housing quality index equations are the following :

In Houston : $h = v_1 + 656.05\,v_2 - 0.23\,v_3$,

In Chicago : $h = v_1 + 709.47\,v_2 - 0.23\,v_3$,

In Cleveland : $h = v_1 + 302.12\,v_2 - 0.23\,v_3$,

In Indianapolis : $h = v_1 + 227.65\,v_2 - 0.23\,v_3$, and

In Dallas : $h = v_1 + 467.71\,v_2 - 0.23\,v_3$.

STEP 3 :

I use the above specified housing quality equations for each city to compute the housing quality index for each house of my data set.

STEP 4 :

I use the quality indices that I obtained in step 3 to estimate the following equation via Maximum Likelihood.

$$H = \gamma - \epsilon_3 a - \epsilon_4 I - u_2 \quad , \tag{63}$$

where :

H is the variable estimate for the quality of a house that I obtained in step 3.

u_2 is the econometric error term.

The variables a and I are assumed to be uncorrelated to the error term, and the parameter γ satisfies (61).

The parameter estimates are given on Table 24. They imply that the demand for housing quality equations are the following :

In Houston : $h = -4.51 + 0.138a + 0.000139I$,

In Chicago : $h = -5.53 + 0.138a + 0.000139I$,

In Cleveland : $h = -3.11 + 0.138a + 0.000139I$,

In Indianapolis : $h = -2.41 + 0.138a + 0.000139I$, and

In Dallas : $h = -3.95 + 0.138a + 0.000139I$.

To see if the above model is of any value at all, I tested the hypothesis that all the parameters of the equation (56) equal zero, that is,

$$\beta_1 = \beta_{20} = \beta_{21} = \beta_{22} = \beta_{23} = \beta_{24} = \beta_3 = 0 \quad .$$

An F – test rejected that hypothesis at the 1% significance level.

A similar F – test implies that I cannot accept the hypothesis that the parameters of the second equation, equation (57), equal zero, that is, I cannot accept that

$\epsilon_3 = \epsilon_4 = \gamma_{11} = \gamma_{12} = \gamma_{13} = \gamma_{14} = 0$ at the 1% significance level.

Moreover, the predictions and the qualitative properties of the model, as well the t – statistics that are given on Tables 23 and 24, are consistent with my a priori expectations. To be more specific, the model predicts the following.

– In all five cities, the housing quality is positively related to air quality and negatively to travel time to work.

– In all five cities, the rent is positively related to the size of a house and to the air quality index, and negatively to the travel time to work variable.

– In all five cities, the rent is positively related to housing quality.

– In all five cities, the demand for housing quality is positively related to the size and the income of a household.

– In all five cities, the marginal utility with respect to housing quality is positively related to the size of the family.

In addition to the above, the estimation results imply the following :

– " Indianapolis, Cleveland, Dallas, Houston, Chicago " is the order in which a household (given a family size and income) would order the five cities (from most to least desirable) according to the housing quality that he can enjoy (buy in equilibrium) in each city.

– The housing quality is more sensitive to changes in air quality in Chicago and less in Houston, Dallas, Cleveland, and Indianapolis (the order is from most to least sensitive); more sensitive in the sense that a unit change in air quality changes more the housing quality in Chicago than in other cities.

– Identical house (house described by the same travel time to work, air quality, and number of rooms variables) have different rental prices in each city. Rents are more expensive in Chicago and less expensive in Houston, Dallas, Cleveland, and Indianapolis (the order is from most to least expensive).

Table 28 gives the housing quality and the rent that the mean household enjoys and pays respectively in each city. The mean household for each city is described on Table 27.

Table 29 gives the elasticities of housing qualities and rents with respect to several variables. These results imply that :

- In all five cities, the housing quality is elastic with respect to air quality.

- The elasticity of housing quality with respect to the travel time to work is greater than one (in absolute value) for Chicago, Cleveland, and Indianapolis, and less than one (in absolute value) for Houston and Dallas.

- The elasticity of housing quality with respect to the size of a house is greater than one in Clevaland and Indianapolis, and less than one in Houston, Chicago, and Dallas.

- Rental prices are inelastic with respect to the size of a house, the air quality, and the travel time to work variables.

- The demand for housing quality is inelastic with respect to the size of a family and income.

In addition to the above, I applied the four step procedure to estimate the price equation, the demand for housing quality equation, and the housing index equation for Houston. These estimates are discussed in the Appendix, Appendix K. The qualitative and other properties of this one city estimation are basically the same with the ones of the first experiment of this section that are described above.

4.9. Another Illustration: A Forecasting - Investment Theory Application of The Model.

In this section, I study an investment decision problem to illustrate another application of the model. To be more specific, I consider a competitive firm that wants to compute the present value of the future stream of rents for the next fifteen years. That firm is assumed to supply the services of ten identical houses each year in Houston. The characteristics of those houses are given by the following vector :

$$v = (5 , 0.017 , 30) ,$$

where : v is defined in Section 4.2.

The regression results for the model that is described in Sections 4.2 and 4.3 (see Table 1) imply that, for a 10% annual discount rate, that present value[33] equals $ 241,390. Without specifying a theoritical model, the non – structural approach could compute the present value using a rental price equation that fits the data well. For a linear price equation, the regression results (see Table 4) imply that for a 10% annual discount rate the present value is $ 230,317.

Both of the above present value figures assumed that none of the exogenous parameters of the model will change in the next fifteen years. If one or more of the exogenous parameters of the model changes, the rental price equation shifts, and consequently the present value figure changes. Unlike the alternative non – structural approach, the structural model that I study can compute the present value change that is implied by a change in one of the exogenous parameters of the model.

For example, it is expected that, starting next year, the mean income in Houston will increase 10% every five years. The structural model predicts that[34] the present value increases by $ 216,470 and it becomes equal to[35] $ 457,860. The alternative approach is unable to compute the level of that change; which change turns out to be quite significant since the present value is almost doubled.

In the Appendix, Appendix J, I pool the data for all five cities and I assume that the mean income changes in the above specified way in all cities. The conclusion remains the same, namely, the alternative method would seriously underestimate the present value since the

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[33] To compute the present value, I used the formula

$$PV = 10 \frac{\sum_{t=1}^{15} 12\,P(h)}{(1+i)^t}$$

where :

PV is the present value
$i = 10\%$
$h = (5, 0.017, 30)$
$P(h)$ is the rental price of an h - quality house.

[34] For a 10% annual discount rate.

[35] To obtain that figure, I have to compute how the present value depends on the mean income, that is, I have to compute how the rental price depends on the mean income. The method to do that is analogous to the one that I used to compute the dependence of the price equation on the mean air quality (see Section 4.5.2, and Table 2)

present value is almost doubled in all cities.

4.9. Conclusions.

Most empirical studies that attempt to measure the willingness to pay for an improvement in air quality follow a single equation estimation procedure and estimate a hedonic equation in which housing values are regressed against pollution levels and other housing attributes, e.g., Ridker and Henning (1967); Witte et al (1979) is an exception. As it has been argued by Freeman (1974) and Small (1975), that common practice for estimating benefit[36] is correct for valuing marginal improvements in air quality. However, many researchers have (implicitly) assumed that the value placed on a marginal improvement in air quality is independent of the air quality level, as well as independent of household income and tastes, and have used that method to estimate the total willingness to pay for a non – marginal improvement in air quality. Harrison and Rubinfeld (1978) criticized that method. To estimate the willingness to pay for a non – marginal improvement in air quality, they proposed (and applied) a four – step estimation procedure that does not need to make the above mentioned restrictive assumptions. Their method is appropriate for measuring the willingness to pay for non – marginal changes in one of the characteristics of a differentiated product, given that the marginal willingness to pay curve does not shift (or equivalently, given that none of the exogenous parameters changes). However, for the reasons that I explained at the end of Chapter 3, that method is (by its nature) unable to give a " correct " benefit figure for the application that they consider.

All of the previous work that I am aware of follows a non – structural approach for estimating benefits. However, that method is not appropriate for investigating environmental policy issues, e.g., willingness to pay for an environmental[37] policy change. Next, I elaborate further this point.

When a non – structural approach estimates the willingness to pay for a non – marginal

[36] As it is illustrated by Ridker and Henning (1967).

[37] I am referring to environmental policies because I develop my discussion in the context of the air quality improvement problem. In general, those methods cannot be used to estimate the benefits associated with a policy that changes some of the exogenous parameters of the model.

change in air quality, it takes the marginal willingness to pay schedule as given. That (implicitly) assumes that the air quality distribution does not change; a change in the air quality distribution would shift the price equation and the marginal willingness to pay curve. But if the previous method has to assume that the air quality distribution does not change, then it cannot be used to estimate the benefit associated with a change in the air quality distribution; that is, it cannot be used to evaluate different environmental policies[38]. To make this point more clear, I consider the following example.

A non − structural approach can compute the consumer willingness to pay for moving into a house that is identical to the one that he used to live in and located in a 5% less polluted neighorhood, given an air pollution distribution. It cannot compute the willingness to pay for a policy change that implies a 5% decrease in the mean air pollution.

In principle, the structural approach can provide an estimate for the consumer utility function and can compute the changes in the demand for housing and numeraire good that are implied by an environmental policy change (e.g., a change in the air quality distribution). Consequently, it can compute the change in utility that is associated with a policy change, as well as the willingness to pay for a policy change. The model that I present in my thesis is offered for this kind of structural analysis. It is also an important result that we do not need data for more than one city or time series data in order to estimate the structure of the market for a differentiated product.

The first application shows that it is actually feasible to estimate and test the model. The regression results suggest that the model can be used to investigate environmental policy issues relevant to Houston, Texas[39]. That application also shows that the structural approach and the alternative can give very different benefit figures. To be more specific, the estimation results indicate that the non − structural approach underestimates significantly the consumer benefit from a marginal, as well as non − marginal, air quality improvement. Finally, a fixed effect assumption seems not to be inappropriate for the housing market of

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[38] Presumably, different environmental policies imply changes in the air quality distribution.

[39] However, much greater care would be necessary to estimate with confidence the precise dollar value of the willingness to pay for a change in the air quality distribution.

the five cities, namely, Houston, Chicago, Cleveland, Indianapolis, and Dallas.

In principle and for the class of economies that I consider, the structural approach is superior to the alternative not only for policy evaluation or cost – benefit analysis purposes but also for analysing other problems that can be of interest to the private sector. The second application illustrates one of them. It shows that the structural approach can estimate the change in the present value of a future stream of revenues that is implied by a change in one of the exogenous parameters of the model. It is an important result that for this purpose the analysis does not require time series data or data for more than one cities.

APPENDIX

APPENDIX A

QUALITY VARIATIONS IN COMMODITIES AND ENDOGENOUS SUPPLY FOR LABOR.

A.1. Assumptions and Notation.

Consider an economy in which individuals consume one unit of a differentiated product, the numeraire good, x, and leisure, l.

The differentiated good can be accurately described by an (1 x m) vector v of objectively measured physical characteristics. Consumers are assumed to care about the quality index h of the differentiated product that they consume. The quality h is a function of the physical characteristics v. This function is given in (1).

a is a (1 x n) vector of utility parameters that specifies the type of the consumer. a is assumed to follow an exogenously given multi – normal distribution. Let it be :

$$N(\bar{a}, \Sigma_a) \quad ,$$

where :

$\bar{a}$ is the mean, and

Σ_a is the variance.

$U(h, x, l, a)$ is the utility that an a – type consumer obtains from x, l units of leisure, and the services of an h – quality differentiated product.

The utility function is given by the following equation.

$$U(h, x, l, a) = \delta + (\zeta_0 + \zeta_1 a') h + (\lambda_0 + \lambda_1 a') l + 0.5\, \xi_1 h^2 + 0.5\, \xi_2 l^2 + \omega h l + \theta \cdot x \quad , \tag{64}$$

where:

δ , ζ_0 , λ_0 , ξ_1 ,ξ_2 , ω , and θ are utility parameters (scalars), and ζ_1 , and λ_1 are (1 x n) vectors of utility parameters.

An a – type consumer solves the following problem :

$$\max \; U(h, x, l, a)$$

$$\text{with respect to} \quad h, x, l$$

$$\text{subject to} \quad w()(T - l) = P(h) + x \quad ,$$

where :

$w()$ is the market hourly wage rate equation (to be specified next),
$P(h)$ is the equilibrium market price equation of the differentiated product, and
T is the total amount of time to be allocated between leisure and work.

Eliminating x , the consumer optimization problem becomes equivalent to :

$$\max \quad U(h, w()(T - l) - P(h), h, a)$$

$$\text{with respect to} \quad h , l \quad .$$

The first order conditions for the latter optimization problem are :

$$\frac{\partial P(h)}{\partial h} = \frac{\partial U(h, w()(T - l) - P(h), l, a) / \partial h}{\partial U(h, w()(T - l) - P(h), l, a) / \partial x} \quad ,\text{and} \tag{65}$$

$$w() = \frac{\partial U(h, w()(T - l) - P(h), l, a) / \partial l}{\partial U(h, w()(T - l) - P(h), l, a) / \partial x} \quad . \tag{66}$$

The optimum decisions of consumers and producers (if any) depend on the equilibrium price equation, $P(h)$, and on the equilibrium market hourly wage rate equation $w()$. If labor is heterogeneous, the hourly wage rate depends on a skill (quality) index of the labor unit. $P(h)$ and $w()$ are determined so that the buyers and sellers of all products and inputs are perfectly matched. Next, I study the equilibrium of the above class of economies for different configurations of the supply side.

A.2. Exogenous Supply for v, and Homogeneous Labor.

In section 3.2, the supply for the differentiated product is exogenously given. The vector of the physical characteristics of a differentiated product, v, follows an exogenously given multi – normal distribution with a known mean, $\bar{v}$, and a known variance, Σ_v. Let it be :

$$N(\bar{v}, \Sigma_v) \quad .$$

The labor is assumed to be homogeneous. Therefore, the hourly market wage rate is the same for all individuals. Let this hourly market wage rate be w.

PROPOSITION 5. The price equation that equilibrates the market described above is :

$$P(h) = \pi_1 h + 0.5\, \pi_2 h^2 \tag{67}$$

where :

$$\pi_2 = \frac{1}{\theta}\left(\xi_1 + \frac{\omega^2}{\xi_2} + \frac{\sqrt{t \Sigma_a t'}}{\sigma_s} \right) \quad , \tag{68}$$

$$\pi_1 = \frac{1}{\theta}\left(\zeta_0 - \frac{\omega \lambda_0}{\xi_2} + \frac{\omega\theta}{\xi_2} w + t\bar{a}' - \left(\frac{\omega^2}{\xi_2} - \xi_1 - \theta\, \pi_2\right) \bar{h}_s \right) \tag{69}$$

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}' \quad , \tag{70}$$

$$\sigma_s = \sqrt{\epsilon_1 \Sigma_v \epsilon_1'} \quad , \text{ and} \tag{71}$$

$$t = \zeta_1 - \frac{\omega \lambda_1}{\xi_2} \quad . \tag{72}$$

Proof: see Appendix C4.

The economy that I examine in this second section of Chapter 3 does not impose any restrictions on the variance – covariance matrices of the vector of product and consumer characteristics. That differentiates this work from the Tinbergen (1959) – Epple (1982) formulation.

Tinbergen (1959) – Epple (1982) formulation assumes a different (but still quadratic) functional form for the utility function and they require that the number of consumer characteristics equals the number of product characteristics. Moreover, for many of the cases that they examine their equilibrium is not unique. Their structure assumes the same (quadratic) functional form for the equilibrium price equation.

Unlike the demand and the supply equations, the restrictions among the price equation parameters and the exogenous parameters of the model are non – linear. The price equation is an equilibrium relationship that incorporates features of tastes, supply, and the distributions of taste and supply parameters.

A.3. Exogenous Supply for v and Heterogeneous Labor.

The case that I study in this section is identical to the one of the section 3.2, with the only difference that now labor is heterogeneous.

Competitive firms are assumed to care about the quality (skill index) of the labor that they hire. The market hourly wage rate, $w(s)$, is not the same for everybody and it depends on the skill index of the labor, s . $w(s)$ is assumed to be linear in s , that is,

$$w(s) = \rho_0 + \rho_1 s \tag{73}$$

where :

ρ_0 and ρ_1 are two parameters (scalars).

The labor skill index is a linear function of the consumer type a . That is,

$$s = \eta_0 + \eta_1 a' \quad , \tag{74}$$

where :

η_0 is a parameter (a scalar) ,and
η_1 is a (1 x n) vector of parameters.

PROPOSITION 6. The price equation that equilibrates the market is given in (67) with parameters given in (68) and (69),

where :

$$\sigma_s = \sqrt{\epsilon_1 \Sigma_v \epsilon_1'} \quad ,$$

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}' \quad ,$$

$$t = \zeta_1 - \frac{\omega\lambda_1}{\xi_2} + \frac{\omega\theta\eta_1\rho_1}{\xi_2} \quad , \textit{and} \tag{75}$$

$$w = \rho_0 + \rho_1 \eta_0 \quad . \tag{76}$$

Proof: see Appendix C5.

Next, we assume that the supply for the differentiated product is endogenous and we examine several configurations of the supply side.

A.4. Endogenous Supply for v, Exogenously Fixed Inputs That Are Owned by Competitive Firms, and Homogeneous Labor.

In this section, I assume that the amount of the inputs to the production are fixed exogenously and owned by competitive firms each of which produces differentiated products with the inputs that it owns. The cost of the numeraire input for producing one unit of the good v is $C(v, b)$ and it depends on b,

where :

b is a (1 x k) vector of technological parameters.

We can think of b as the type of the firm.

b is assumed to follow an exogenously given multi – normal distribution with a known mean, $\bar{b}$, and variance, Σ_b. Let it be :

$$N(\bar{b}, \Sigma_b) \quad . \tag{77}$$

$C(v, b)$ is assumed to be a function of the following form.

$$C(v, b) = \gamma_0 + (\gamma_1 + b\,\gamma_2)\, v' \quad , \tag{78}$$

where :

γ_0 is a parameter (a scalar),

γ_1 is a (1 x m) vector of parameters, and

γ_2 is a (k x m) matrix of parameters.

A profit maximizing competitive b – type firm solves the following optimization problem.

$$\begin{aligned}&max \quad P(h) - C(v, b)\\&with\ respect\ to \quad h, v\\&subject\ to \quad h = \epsilon_0 + \epsilon_1 v'\end{aligned}$$

where :

$P(h)$ is the equilibrium market price equation for the differentiated product.

$\epsilon_1' \epsilon_1$ is assumed to be a positive definite matrix.

Eliminating h, the consumer optimization problem becomes equivalent to the following problem.

$$\begin{aligned}&max \quad P(\epsilon_0 + \epsilon_1 v') - C(v, b) \quad .\\&with\ respect\ to \quad v\end{aligned}$$

The first order conditions for the latter optimization problem follow.

$$\frac{\partial P(\epsilon_0 + \epsilon_1 v')}{\partial v'} = \frac{\partial C(v, b)}{\partial v'} \quad . \tag{79}$$

PROPOSITION 7 . The market price equation that equilibrates the market is :

$$P(h) = \pi_1 h + 0.5\, \pi_2 h^2 \quad , \tag{80}$$

where :

$$\pi_2 = (\xi_1 - \frac{\omega^2}{\xi_2}) \frac{\Psi}{\sqrt{t \Sigma_a t' + \theta \Psi}} \tag{81}$$

$$\pi_1 = \Big(\frac{\xi_2 \theta}{\omega^2 - \xi_2(\xi_1 - \theta \pi_2)} - \epsilon_1 (\pi_2 \epsilon_1' \epsilon_1)^{-1} \epsilon_1' \Big)^{-1} [- \epsilon_0 +$$

$$\epsilon_1 (\pi_2 \epsilon_1' \epsilon_1)^{-1} (\epsilon_0 \epsilon_1' \pi_2 - \gamma_1' - \gamma_2' \bar{b}') +$$

$$\frac{\xi_2}{\omega^2 - \xi_2(\xi_1 - \theta \pi_2)} \Big(\zeta_0 - \frac{\omega \lambda_0}{\xi_2} + \frac{\omega \theta}{\xi_2} w \Big)] \quad , \ \textit{and} \tag{82}$$

$$\Psi = \sqrt{\epsilon_1 (\epsilon_1' \epsilon_1)^{-1} \gamma_2' \Sigma_b \gamma_2 (\epsilon_1' \epsilon_1)^{-1} \epsilon_1'} \quad . \tag{83}$$

Proof: see Appendix C5.

A.5. Endogenous Supply for v **, Exogenously Fixed Inputs That Are Owned by Competitive Firms, and Heterogeneous Labor.**

The case that I study in this section is identical to the one of the previous section, with the only difference that now labor is heterogeneous.

The labor skill index, s , is assumed to be a function of the type of the consumer, a . This function is given in (74)

The hourly wage rate depends on the labor skill index, and this function is specified in (73).

PROPOSITION 8. The price equation that equilibrates the market is given in (80) with parameters specified by (81) and (82),

where :

Ψ is given in (83),

$t = \zeta_1 - \dfrac{\omega \lambda_1}{\xi_2} + \dfrac{\omega \theta \rho_1 \eta_1}{\xi_2}$, and

$w = \rho_0 + \rho_1 \eta_0$.

Proof: see Appendix C7.

A.6. Endogenous Supply for v, Exogenous Capital Distribution, and Heterogeneous Labor.

In this section I assume that each competitive firm is exogenously given some capital that is good for producing a specific kind of differentiated product. Thus, we can think of v as a variable that can describe the capital type of a firm. Let the normal distribution

$N(\bar{v}, \Sigma_v)$

describe the exogenously given capital distribution,

where :

$\bar{v}$ is the mean, and

Σ_v is the variance.

The labor is traded in a competitive market and is assumed to be heterogeneous. As in the previous sections of this chapter that were referring to heterogeneous labor, I assume that the labor skill index is a linear function of the consumer type a. This function is given in (74).

Each competitive firm needs $N_0 + N_1 s$ units of s – skill labor units to produce one unit of a differentiated product,

where :

N_0, and N_1 are two parameters (scalars).

It is assumed that there is some cost associated with different combinations of a given kind of capital and a chosen kind of labor. $C(s,b)$ is the cost (per unit of differentiated product) of adjusting any type of capital to work with s – skill labor units,

where :

b is an exogenously given (1 x k) vector of technological parameters.

b follows a multi – normal distribution with a mean $\bar{b}$, and a variance Σ_b . Let this distribution be :

$N(\bar{b}, \Sigma_b)$.

$C(s,b)$ can also include other administrative costs. $C(s,b)$ is given next.

$$C(s,b) = \gamma_0 + (\gamma_1 + \gamma_2 b')s + 0.5\,\gamma_3 s^2 \tag{84}$$

A competitive firm solves the following optimization problem.

$$\begin{aligned}&min\ (N_0 + N_1 s)\,w(s) + C(s,b)\\&with\ respect\ to\ s\end{aligned}$$

where :

$w(s)$ is the equilibrium market hourly wage rate equation.

The first order conditions for this problem are given in (28).

PROPOSITION 9. The price equation that equilibrates the product market, and the wage equation that equilibrates the input market are specified in Propositions 6 and 3 respectively.

Proof: see Appendix C8.

A.7. Endogenous Supply for v, Endogenous Capital Distribution, Heterogeneous Labor.

As in Section 2.4, I assume : 1) that a competitive firm can choose the type of capital that it buys, and 2) that each type of capital sold in the market can produce only one specific v – type differentiated product.

The labor is assumed to be heterogeneous, and the labor skill is a function of the consumer type. Their relationship is given in (74).

$N_0 + N_1 s$ is the number of s – skill labor that is needed to produce one unit of a differentiated product.

$C(v, s, b)$ is the non – labor cost for producing one unit of a v – type differentiated product using s – skill labor and given a vector of technological parameters, b .

b is an (1 x k) vector of technological parameters that is exogenously given to each firm. b is assumed to follow a multi – normal distribution that is given in (77).

The function $C(v, s, b)$ is a quadratic function of the following form.

$$C(v, s, b) = \gamma_0 + (\gamma_1 + \gamma_2 b') s + 0.5\, \gamma_3 s^2 + (\gamma_4 + b\, \gamma_5) v' \quad , \tag{85}$$

where :

γ_0, γ_1, and γ_3 are parameters (scalars),
γ_2 is a (1 x k) vector of parameters,
γ_4 is a (1 x m) vector of parameters, and
γ_5 is a (k x m) matrix of parameters.

A profit maximizing competitive firm solves the following optimization problem.

$$\max \quad P(h) - (N_0 + N_1 s)\, w(s) - C(v, s, b)$$

$$\text{subject to} \quad h = \epsilon_0 + \epsilon_1 v'$$

where :

$P(h)$ is the equilibrium market price equation, and

$w(s)$ is the equilibrium market hourly wage rate equation.

Eliminating h, we obtain that the first order conditions for the firm's optimization problem are given in (33) and (34).

PROPOSITION 10. The price equation that equilibrates the product market, and the wage equation that equilibrates the input market are specified in Propositions 8 and 9 respectively.

Proof: see Appendix C9.

A.8. Conclusions.

In chapter 3, I kept the assumption of the previous chapter that individuals consume one unit of the differentiated product. Therefore, the theoretical models of chapters 2 and 3 can be applied to the same class of empirical applications. The only difference is that now data on wage rates would be needed instead of data on income. In most cases, it will be more difficult to acquire information on consumer income than on consumer employment and/or his hourly wage rate.

Unlike that of the previous chapter, the theoretical model of chapter 3 suggests that the use of census tract data is not legitimate. This is so, because the equilibrium price equation is quadratic.

For empirical applications, the theoretical model of this chapter suggests a simultaneous estimation of the price equation, of the wage equation (if any), of the demand equation(s), and of the supply equations (if any) in which we impose the constraints across the parameters that are specified in the relevant proposition. If we want to estimate the complete model, we must add to our estimation model the equations and the restrictions across the parameters that refer to the labor market.

Note that our theoretical model implies demand and supply functions that are linear.

APPENDIX B

QUANTITY AND QUALITY VARIATIONS IN COMMODITIES AND ENDOGENOUS SUPPLY FOR LABOR

B.1. Assumptions and Notation.

In this chapter, I consider an economy in which there are N differentiated goods, and in which individuals consume x , y , and l ,

where :

x is the numeraire good,

$y = (y_1\ y_2\ ....\ y_N)$ is a (1 x N) vector of differentiated commodities,

y_i is the quantity of the i – differentiated product, where $i = 1, 2, 3, ..., N$, and

$l = (l_1\ l_2\ ....\ l_N)$ is a (1 x N) vector of leisure activities.

The choice variables of a consumer are assumed to be x , y , h , , and l ,

where :

$h = (h_1\ h_2\ ....\ h_N)$, and

h_i is the quality of the ith differentiated good, where $i = 1, 2, 3, ..., N$.

Each of the N differentiated products can be accurately described by a vector of objectively measured characteristics. Let v_i be a (1 x m) vector of objectively measured characteristics that can describe the ith differentiated product.

Consumers are assumed to care about the quantity and the quality of the differentiated commodities that they consume. The quality index of the ith differentiated good, h_i , depends on the physical characteristics, v_i , of that good. The quality index h_i is assumed to be linear in the vector of physical characteristics v_i . That is,

$$h_i = \epsilon_{0i} + \epsilon_{1i} v_i' \quad , \text{ for all } \ i \tag{86}$$

where:

$i = 1, 2, 3, \ldots\ldots, N$,

ϵ_{0i} is a parameter, and

ϵ_{1i} is a (1 x m) vector of parameters.

I interpret equation (86) in the same way that I interpreted equation (1).

a is a (1 x n) vector of utility parameters that specifies the type of the consumer. a is assumed to follow an exogenously given multi – normal distribution. Let it be :

$$N(\bar{a}, \Sigma_a) \quad , \tag{87}$$

where :

$\bar{a}$ is the mean, and

Σ_a is the variance.

$U(h, y, x, l, a)$ is the utility that an a – type consumer obtains from x , an l – vector of leisure activities, and the consumption of N differentiated goods whose quality – quantity combinations are given by (h, y) .

Tha utility function is given by the following equation.

$$\begin{aligned} U(h, y, x, l, a) = \delta + \sum_{i=1}^{N} [(\zeta_{0i} + \zeta_{1i} a') h_i + \tau_i y_i + \\ (\lambda_{0i} + \lambda_{01} a') l_i + 0.5\, \xi_{1i} h_i^2 + 0.5\, \xi_{2i} l_i^2 + 0.5\, \xi_{3i} y_i^2 + \\ \omega_{1i} y_i h_i + \omega_{2i} h_i l_i] + \theta x \end{aligned} \tag{88}$$

where :

$\delta, \zeta_{0i}, \tau_i, \lambda_{0i}, \xi_{1i}, \xi_{2i}, \xi_{3i}, \omega_{1i}, \omega_{2i}$, and θ are parameters (scalars), and ζ_{1i}, and λ_{1i} are (1 x n) vectors of utility parameters.

An a −type consumer solves the following optimization problem.

$$\max\ U(h, y, x, l, a)$$

$$\text{with respect to } h, y, x, l$$

$$\text{subject to} \quad w()(T - \sum_{i=1}^{N} l_i) = \sum_{i=1}^{N} [\, P(h_i) y_i \,] + x$$

where :

$w()$ is the market hourly wage rate equation (to be specified next),
$P(h_i)$ is the price of an ith differentiated product of h_i − quality, and
T is the total amount of time to be allocated between leisure and work.

Eliminating x , the consumer optimization problem becomes equivalent to the following optimization problem.

$$\max\ U(\, h, y, w()(T - \sum_{i=1}^{N} l_i) - \sum_{i=1}^{N} (P(h_i) y_i), l, a\,) \quad . \tag{89}$$

$$\text{with respect to } h, y, l$$

The first order conditions for the latter optimization problem follow.

$$\frac{\partial U(\psi)}{\partial h_i} = \frac{\partial U(\psi)}{\partial x} \frac{\partial P(h_i)}{\partial h_i} \tag{90}$$

$$\frac{\partial U(\psi)}{\partial y_i} = \frac{\partial U(\psi)}{\partial x} P(h_i) \quad , \text{and} \tag{91}$$

$$\frac{\partial U(\psi)}{\partial l_i} = \frac{\partial U(\psi)}{\partial x} w() \quad , \tag{92}$$

where :

$$\psi = w()(T - \sum_{i=1}^{N} l_i) - \sum_{i=1}^{N} (P(h_i) y_i) \quad .$$

The optimum decisions of consumers and producers (if any) depend on the market price equation, $P(h_i)$, and on the market hourly wage rate equation, $w()$. If labor is heterogeneous, the hourly wage rate is going to depend on the quality (skill) of the labor. $P(h_i)$ and $w()$ are determined so that buyers and sellers of all products and inputs are perfectly matched. Next, I study the equilibrium of the above class of economies for several configurations of the supply side.

B.2. Exogenous Supply for v_i, Homogeneous Labor.

In this section, the supply for the differentiated products is exogenously given. The vector of the ojectively measured physical characteristics of the ith differentiated product, v_i, follows an exogenously given normal distribution with a known mean, $\bar{v}_i$, and variance, Σ_{vi}. Let it be :

$$N(\bar{v}_i, \Sigma_{vi}) \quad . \tag{93}$$

Labor is assumed to be homogeneous, and w is the market hourly wage rate that is assumed to be the same for all individuals.

PROPOSITION 11. The price equation that equilibrates the market described above is :

$$P(h_i) = \pi_{0i} + \pi_{1i} h_i \tag{94}$$

where :

$$\pi_{1i} = \frac{1}{\theta}(\omega_{1i} + A^{0.5}) \tag{95}$$

$$\pi_{0i} = \frac{1}{\theta\xi_{2i}A^{0.5}}\{((\zeta_{0i} - \frac{\omega_{2i}\lambda_{0i}}{\xi_{2i}} + \frac{\tau_i}{\xi_{3i}}A^{0.5} + \frac{\theta\omega_{2i}}{\xi_{2i}}w + t_i\bar{a}')\xi_{2i}\xi_{3i} - [(\omega_{2i}^2 - \xi_{1i}\xi_{2i})\xi_{3i} + \xi_{2i}A]\bar{h}_{si}\} \tag{96}$$

$$A = \frac{(\xi_{1i}\xi_{2i} - w_{2i}^2\xi_{3i}}{\xi_{2i}} + \frac{\xi_{3i}}{\sigma_{si}}\sqrt{t_i\Sigma_a t_i'} \quad , \tag{97}$$

$$\sigma_{si}^2 = \epsilon_{1i}\Sigma_{vi}\epsilon_{1i}' \quad , \tag{98}$$

$$\bar{h}_{si} = \epsilon_{0i} + \epsilon_{1i}\bar{v}_i' \quad , \quad \textit{and} \tag{99}$$

$$t_i = \zeta_{1i} - \frac{\omega_{2i}\lambda_{1i}}{\xi_{2i}} \quad . \tag{100}$$

Proof: see Appendix C10.

The economy that I examine in this second section of Chapter 4 does not impose any restrictions on the variance – covariance matrices of the vector of the product and consumer characteristics. Moreover, consumers are assumed to be able to choose not only the quality but also the quantity of the differentiated good; this differentiates this model, as well as all the other models that I present in Chapter 4, from all past work in this area.

Unlike the demand, supply, and price equations the restrictions among the price equation parameters and the exogenous parameters of the model are non – linear. The price equation is an equilibrium relationship that incorporates features of tastes, supply, and the distributions of taste and supply parameters.

B.3. Exogenous Supply for v_i, Heterogeneous Labor.

The case that I study in this section is identical to the one of the previous section, with the only difference that now labor is heterogeneous.

Competitive firms are assumed to care about the quality (skill) index of the labor that they hire. The market hourly wage rate is not the same for everybody; it depends on the quality of the labor, s . The market hourly wage rate equation, $w(s)$, is linear in the quality of the labor; this equation is given in (73).

The labor skill index is assumed to be a linear function of the consumer type, a ; this function is given in (74).

PROPOSITION 12. The price equation that equilibrates the market is given in (94) with parameters given in (95) and (96),

where :

σ^2_{si} and $\bar{h}_{si}$ are given in (4.13) and (4.14) respectively,

$$t_i = \zeta_{1i} - \frac{\omega_{2i}\lambda_{1i}}{\xi_{2i}} + \frac{\theta\omega_{2i}\rho_1\eta_1}{\xi_{2i}} \text{, and} \tag{101}$$

$$w = \rho_0 + \rho_1\eta_0 . \tag{102}$$

Proof: see Appendix C11.

Next, I assume that the supply for the differentiated good is endogenous and I study several configurations of the supply side.

B.4. Endogenous Supply for v_i , Exogenously Fixed Inputs That Are Owned by Competitive Firms, Homogeneous Labor.

In this section, labor is homogeneous and the hourly wage rate w is the same for everybody.

The amount of the inputs to the production are assumed to be fixed exogenously and owned by competitive firms each of which produces differentiated products with the inputs that it owns. The cost of the numeraire input for producing one unit of an ith differentiated good with physical characteristics v_i is $C(v_i, b)$, and it depends on b, where b is an (1 x k) vector of technological parameters. We can think of it as the type of the firm.

b is assumed to follow an exogenously given normal distribution with a known mean, $\bar{b}$, and variance Σ_b. Let it be :

$$N(\bar{b}, \Sigma_b) \quad . \tag{103}$$

$C(v_i, b)$ is assumed to be quadratic in the vector of physical characteristics of the differentiated product, v_i. This function is given next.

$$C(v_i, b) = \gamma_0 + (\gamma_1 + b\,\gamma_2)\,v_i' + 0.5\,v_i\,\gamma_3\,v_i' \ , \tag{104}$$

where :

γ_0 is a parameter (a scalar),

γ_1 is a (1 x m) vector of parameters,

γ_2 is a (k x m) matrix of parameters, and

γ_3 is a (m x m) matrix of parameters.

The matrix γ_3 is assumed to be symmetric and positive definite.

A profit maximizing b – type firm solves the following optimization problem.

$$\begin{aligned} &\textit{max} \quad P(h_i) - C(v_i, b) \\ &\textit{with respect to} \quad h_i\ ,\ v_i \\ &\textit{subject to} \quad h_i = \epsilon_{0i} + \epsilon_{1i}\,v_i' \end{aligned}$$

where :

$P(h_i)$ is the market price of an ith differentiated product of h_i quality.

Eliminating h_i, the above optimization problem becomes equivalent to the following :

$$\max \quad P(\epsilon_{0i} + \epsilon_{1i} v_i^{\prime}) - C(v_i, b) \quad .$$

$$\text{with respect to } v_i$$

The first order conditions for the latter optimization problem are given next, equation (105).

$$\frac{\partial P(\epsilon_{0i} + \epsilon_{1i} v_i^{\prime})}{\partial v_i^{\prime}} = \frac{\partial C(v_i, b)}{\partial v_i^{\prime}} \quad . \tag{105}$$

PROPOSITION 13. The price equation that equilibrates the market is given in (94) with parameters specified in (95), and (96),

where :

A is given in (97),

t_i is given in (100),

$$\sigma_{si}^2 = \epsilon_{1i} \gamma_3^{-1} \gamma_2^{\prime} \Sigma_b \gamma_2 \gamma_3^{-1} \epsilon_{1i}^{\prime} \quad , \quad and \tag{106}$$

$$\bar{h}_{si} = \epsilon_{0i} + \epsilon_{1i} \gamma_3^{-1} (\pi_{1i} \epsilon_{1i}^{\prime} - \gamma_1^{\prime} - \gamma_2^{\prime} b^{\prime}) \quad . \tag{107}$$

Proof: see Appendix C12.

B.5. Endogenous Supply for v_i, Exogenously Fixed Inputs That Are Owned by Competitive Firms, and Heterogeneous Labor.

The case that I study in this section is identical to the one of the previous section, with the only difference that now labor is heterogeneous.

The labor skill index, s, is assumed to be a linear function of the vector of utility parameters that specifies the type of the consumer, a. This function is given in (73).

The labor skill index is assumed to be a linear function of the consumer type, a. This function is given in (74).

PROPOSITION 14. The price equation that equilibrates the product market is given in (94) with parameters specified in (95) and (96),

where :

σ^2_{si}, $\bar{h}_{si}$, t_i, and w are given in (106), (107), (101), and (102) respectively.

Proof: see Appendix C13.

B.6. Endogenous Supply for v_i, Exogenous Capital Distribution, Heterogeneous Labor.

In this section, I assume that each competitive firm is exogenously given some capital that is good for producing a specific kind of a differentiated product. Thus, we can think of v_i as a variable that can describe the capital type of a firm. Let the normal distribution

$$N(\bar{v}_i, \Sigma_{vi})$$

describe the exogenously given capital distribution,

where :

$\bar{v}_i$ is the mean, and

Σ_{vi} is the variance.

The labor is traded in the market and is assumed to be heterogeneous. As in the previous sections that were referring to heterogeneous labor, the quality of labor is assumed to be a linear function of the vector of utility parameters that specifies the consumer type, a. This function is given in (74).

Each competitive firm needs $(N_0 + N_1 s)$ units of s – quality labor to produce one unit of a differentiated product, where N_0 and N_1 are two parameters. However, there is some cost associated with different combinations of a given type of capital and a given type of labor.

$C(s, b)$ is the cost (per unit of differentiated product) of adjusting any type of capital to work with s – quality labor,

where :

b is an exogenously given (1 x k) vector of technological parameters.

b is assumed to follow a multi – normal distribution that is specified next :

$$N(\bar{b}, \Sigma_b) \quad , \tag{108}$$

where :

$\bar{b}$ is the mean, and

Σ_b is the variance.

The function $C(s, b)$ can also include other administrative costs. This function is assumed to be of the following functional form.

$$C(s, b) = \gamma_0 + (\gamma_1 + \gamma_2 b) s + 0.5\, \gamma_3 s^2 \quad , \tag{109}$$

where :

γ_0, γ_1, and γ_3 are parameters (scalars), and
γ_2 is a (1 x k) vector of parameters.

A competitive firm solves the following optimization problem.

$$min \ (N_0 + N_1 s) w(s) + C(s, b) \quad ,$$
$$with \ respect \ to \ s$$

where :

$w(s)$ is the equilibrium hourly market wage rate equation; a function of the quality of labor s .

The first order conditions for the firm's optimization problem are given in (28).

PROPOSITION 15. The price equation that equilibrates the product market and the wage equation that equilibrates the input market are given in Propositions 12 and 3 respectively.

Proof: see Appendix C14.

B.7. Endogenous Supply for v_i , Endogenous Capital Distribution, Heterogeneous Labor.

In this section, I assume 1) that a competitive firm can choose the type of capital that it buys, and 2) that each type of capital can produce only one differentiated product of a specific type.

The labor is heterogeneous, and the quality of labor is a function of the vector of utility parameters that specifies the consumer type, a . This function is given in (74).

$(N_0 + N_1 s)$ is the number of s – quality labor units needed to produce one unit of a differentiated product.

$C(v_i, s, b)$ is the non – labor cost for producing one unit of a v_i – type differentiated product, using s – quality labor, and given a vector of technological parameters b.

b is assumed to follow a normal distribution; this distribution is given in (108).

The equation $C(v_i, s, b)$ is assumed to be of the following functional form.

$$C(v_i, s, b) = \gamma_0 + (\gamma_1 + \gamma_2 b') s + 0.5\, \gamma_3 s^2 + (\gamma_4 + b\, \gamma_5) v' + 0.5\, v_i \gamma_6 v_i' \quad , \tag{110}$$

where :

γ_0 , γ_1 , and γ_3 are parameters (scalars),
γ_2 is a (1 x k) vector of parameters,
γ_4 is a (1 x m) vector of parameters,
γ_5 is a (k x m) matrix of parameters, and
γ_6 is a (m x m) matrix of parameters.

γ_6 is assumed to be a positive definite and symmetric matrix.

A profit maximizing competitive firm solves the following optimization problem.

$$\max \; P(h_i) - (N_0 + N_1 s)\, w(s) - C(v_i, s, b) \quad ,$$

$$\text{with respect to } h_i \, , \, v_i \, , \, s$$

$$\text{subject to } h_i = \epsilon_{0i} + \epsilon_{1i} v'$$

where :

$P(h_i)$ is the equilibrium market price equation, and

$w(s)$ is the equilibrium hourly market wage rate equation.

Eliminating h_i, we obtain that the first order conditions for the firm's optimization problem are given in (33) and (34).

PROPOSITION 16. The price equation that equilibrates the product market, and the hourly wage rate equation that equilibrates the input market are given in Propositions 13 and 15 respectively.

Proof: see Appendix C15.

B.8. Conclusions.

In many instances, consumers choose not only the quality but also the quantity of the differentiated product. The theoretical models presented in chapter 4 can be used for differentiated goods of this kind.

The structure of the class of economies that are considered here imply demand and supply equations, as well as price and wage equations, that are linear. This property of our theoretical model suggests that the use of census tract type of data is legitimate.

As in the previous chapters, our theory suggests a simultaneous equation estimation technique that would utilize the constraints across the parameters that are implied by the relevant proposition.

APPENDIX C

PROOFS OF PROPOSITIONS

C1. PROOF OF PROPOSITION 2.

Substitute (6) and (18) into (19), and solve for v to obtain the supply for v. The supply for v is given by the following equation.

$$v' = (\gamma_3)^{-1}(\pi_1 \epsilon_1' - \gamma_1' - \gamma_2' b') \quad . \tag{111}$$

We can now see that the supply for v is linear in the vector of technological parameters b. This, and the normality of b imply that the supply for v follows a multi – normal distribution with a mean $\bar{v}$, and variance Σ_v. Let it be :

$$N(\bar{v}, \Sigma_v) \quad , \tag{112}$$

where:

the mean is given in (22),

the variance is given in (23).

$h = \epsilon_0 + \epsilon_1 v'$, and the specified normal distribution of the supply for v imply that the supply for h is a normal distribution. Let it be :

$$g(h) = N(\bar{h}_s, \sigma_s^2) \quad ,$$

where :

the mean $\bar{h}_s$ is specified in (20), and the variance σ_s^2 in (21), with $\bar{v}'$ and Σ_v given in (22) and (23) respectively.

Substitute (3) and (6) into (4), and solve for h to obtain the demand for h, h_d. It is given in (10).

Following the same argument as in the proof of Proposition 1, we obtain that the aggregate demand for h, $f(h_d)$, is a normal distribution :

$$f(h) = N(\bar{h}_d, \sigma_d^2) ,$$

where :

the mean $\bar{h}_d$ and variance σ_d^2 are given in (11) and (12) respectively.

Given the normality of the aggregate demand and supply for h, for an equilibrium, the following two equations must be satisfied :

$$\bar{h}_s = \bar{h}_d \tag{113}$$

$$\sigma_s^2 = \sigma_d^2 \tag{114}$$

where :

$\bar{h}_s$, σ_s^2, $\bar{h}_d$, σ_d^2 are given in (20), (21), (11), and (12) respectively, with $\bar{v}'$ specified in (22), and Σ_v in (23).

We can show that the solution for π_1 and π_0 that is given in Proposition 2 is the only one that satisfies the second order condition, and the equilibrium equations (113) and (114). The proof is along the lines of the proof of Proposition 1.

QED

C2. PROOF OF PROPOSITION 3.

Substitute (27) and (29) in (28), and solve for the demand for s. It is given by the following equation.

$$s = -\frac{N_0\rho_1 + N_1\rho_0 + \gamma_1 + \gamma_2 b'}{2N_1\rho_1 + \gamma_3} \tag{115}$$

The second order condition implies :

$$2N_1\rho_1 + \gamma_3 > 0 \quad .$$

We can now see that the demand for s is linear in b. Therefore, the aggregate demand for s follows a normal distribution. Let it be :

$$N(\bar{s}, \sigma_I^2) \quad , \tag{116}$$

where :

$\bar{s}$ is the mean,

σ_I^2 is the variance,

$$\bar{s} = -\frac{N_0\rho_1 + N_1\rho_0 + \gamma_1 + \gamma_2 \bar{b}'}{2N_1\rho_1 + \gamma_3} \quad , \quad \textit{and} \tag{117}$$

$$\sigma_I^2 = \frac{\gamma_2 \Sigma_b \gamma_2'}{(2N_1\rho_1 + \gamma_3)^2} \tag{118}$$

$s = \eta_0 + \eta_1 c'$ and the exogenously given distribution of the supply for c imply that the supply for s follows a normal distribution, the following :

$$N(\eta_0 + \eta_1 \bar{c}', \eta_1 \Sigma_c \eta_1') \tag{119}$$

where :

$\eta_0 + \eta_1 \bar{c}'$ is the mean of the aggregete supply for s, and (120)

$\eta_1 \Sigma_c \eta_1'$ is the variance of the aggregate supply for s. (121)

Since both aggregate demand and supply for s are normal distributions, an equilibrium in the market for inputs requires that the following two equations are satisfied :

$$\sigma_I^2 = \eta_1 \Sigma_c \eta_1' \quad \text{, and} \tag{122}$$

$$\bar{s} = \eta_0 + \eta_1 \bar{c}' \tag{123}$$

where :

the right hand sides of the equations (122) and (123) are given in (120) and (121) respectively, and

$\bar{s}$ is specified in (117), and σ_I^2 in (118).

It can be verified that the only ρ_1 and ρ_0 solutions that satisfy the conditions for equilibrium, (122) and (123), are the ones given in (30) and (31), and the ones that follow :

$$\rho_1 = \frac{1}{2N_1}\left(\frac{\sqrt{\gamma_2 \Sigma_b \gamma_2'}}{\sqrt{\eta_1 \Sigma_c \eta_1'}} + \gamma_3\right) \quad \text{, and} \tag{124}$$

$$\rho_0 = -\frac{N_0 \rho_1 + \gamma_1 + \gamma_2 \bar{b} + (\eta_0 + \eta_1 \bar{c}')(2N_1 \rho_1 + \gamma_3)}{N_1}. \tag{125}$$

However, we rule out the solution specified in (124) and (125) because it does not satisfy the second order condition.

Substitute (3) and (6) into (4) and solve for the demand for h . It is given in (10).

Equation (10) and the assumed multi − normal distribution of z imply that the demand for h follows a normal distribution with a mean given in (11), and a variance given in (12).

Equation (1) and our assumptions about the exogenously distribution of the vector of characteristics v imply that the aggregate supply for h follows a normal distribution that is specified by equations (13), (14), and (15).

Following the same argument as in Proposition 1, we obtain that for an equilibrium :

(16) and (17) must be satisfied,

where :

$\bar{h}_d$, σ_d^2 , $\bar{h}_s$, and σ_s^2 are given in (11), (12), (14), and (15) respectively.

We can show that the π_0 and π_1 solution that is specified in Proposition 3 is the only one that satisfies the second order condition, and the equilibrium equations (16) and (17). For a proof of this see the proof of Proposition 1.

QED

C3. PROOF OF PROPOSITION 4.

Substitute (6), (29), and (32) into (33) and (34), and solve for the demand for s and the supply for v . They are given by the following equations.

$$s = -\frac{N_0\rho_1 + N_1\rho_0 + \gamma_1 + \gamma_2 b'}{\gamma_3 + 2N_1\rho_1} \tag{126}$$

$$v' = (\gamma_6)^{-1}(\pi_1\epsilon_1' - \gamma_4' - \gamma_5' b') \ . \tag{127}$$

(126) and the distributional assumptions about the vector of technological parameters b imply that the demand for s follows a normal distribution, given in (116), with a mean and variance specified in (117) and (118) respectively.

The exogenously given supply for c , and $s = \eta_0 + \eta_1 c'$ imply that the supply for s follows a normal distribution, given in (119), with a mean and variance specified in (120) and (121) respectively.

Given the normality of the aggregate demand and supply for s , an equilibrium must satisfy (122) and (123), where $\bar{h}_s$ and σ_l^2 are given in (117) and (118) respectively. We can show that our solution for ρ_0 and ρ_1 given in (31) and (30) are the only ones that satisfy the equilibrium equations and the second order condition. The proof is along the lines of the proof of Proposition 3.

The supply for v is given in (126). We can see that is linear in b. This and the normality of the vector of technological pararameters b imply that the aggregate supply for v follows a normal distribution with a mean and variance given in (38) and (37) respectively.

The normality of the aggregate supply for v and $h = \epsilon_0 + \epsilon_1 v'$ imply that the aggregate supply for h is a normal distribution with a mean and variance given in (35) and (36) respectively, where $\bar{v}'$ is given in (38), and Σ_v is given in (37).

Substitute (3) and (6) into (4), and solve for h to obtain the demand for h. It is given in (10).

(10) is linear in z. This and our assumption about the normality of z imply that the aggregate demand for h is a normal distribution with a mean given in (11), and a variance given in (12).

Given the normality of the aggregate demand and supply for h, an equilibrium must satisfy the following two equations.

$$\bar{h}_s = \bar{h}_d \quad , \textit{ and} \tag{128}$$

$$\sigma_s^2 = \sigma_d^2 \quad , \tag{129}$$

where :

$\bar{h}_s$, σ_s^2 , $\bar{h}_d$, and σ_d^2 are given in (35), (36), (11), and (12) respectively, with $\bar{v}'$ given in (38), and Σ_v given in (37).

We can show that the solution for π_0 and π_1, that is specified in Proposition 4, is the only one that satisfies the second order condition, and the equilibrium equations (128) and (129). The proof of this is along the lines of the proof of Proposition 1.

QED.

C4. PROOF OF PROPOSITION 5.

Substitute (67) and (64) into (65) and (66), and solve the system for h to obtain the demand for h, h_d. The demand for h is given by the following equation.

$$h_d = \frac{\xi_2}{\omega^2 - \xi_2(\xi_1 - \theta\pi_2)}(\zeta_0 - \theta\pi_1 - \frac{\omega\lambda_0}{\xi_2} - \frac{\omega\theta}{\xi_2} w + t a') \ , \tag{130}$$

where :

t is given in (72).

The second order condition of the consumer optimization problem implies :

$$\frac{\xi_2}{\omega^2 - \xi_2(\xi_1 - \theta\pi_2)} > 0 \quad .$$

We can now see that the demand fo h is linear in a. Therefore, the demand for h follows a normal distribution. Let it be :

$$f(h) = N(\bar{h}_d, \sigma_d^2) \quad , \tag{131}$$

where :

$\bar{h}_d$ is the mean,

σ_d^2 is the variance,

$$\bar{h}_d = \frac{\xi_2}{\omega^2 - \xi_2(\xi_1 - \theta\pi_2)}(\zeta_0 - \theta\pi_1 - \frac{\omega\lambda_0}{\xi_2} +$$

$$\frac{\omega\theta}{\xi_2} w + (\zeta_1 - \frac{\omega\lambda_1}{\xi_2})\bar{a}') , \quad \text{and} \tag{132}$$

$$\sigma_d = \frac{\xi_2}{\omega^2 - \xi_2(\xi_1 - \theta\pi_2)}\sqrt{t\Sigma_a t'} \quad . \tag{133}$$

$h = \epsilon_0 + \epsilon_1 v'$ and the assumption that the supply for v follows an exogenously given multi – normal distribution imply that the supply for h follows a normal distribution. Let it be :

$$g(h) = N(\bar{h}_s, \sigma_s^2) \quad , \tag{134}$$

where :

$\bar{h}_s$ is the mean, given in (70), and

σ_s^2 is the variance, specified in (71).

For an equilibrium, we need that :

aggregate demand = aggregate supply .

That is,

$$f(h)dh = g(h)dh \quad .$$

Since both aggregate demand and supply follow normal distributions, the above equilibrium condition is equivalent to the following two equations.

$$\bar{h}_d = \bar{h}_s \quad , \quad \text{and} \tag{135}$$

$$\sigma_d^2 = \sigma_s^2 \quad , \tag{136}$$

where:

$\bar{h}_d$, σ_d^2, $\bar{h}_s$, σ_s^2 are respectively specified in (132), (133), (70), and (71).

The only π_1 and π_2 solutions that satisfy the equilibrium equations (135) and (136) are the one given in Proposition 5, equations (69) and (68), and the one that follows :

$$\pi_2 = \frac{1}{\theta}\left(\xi_1 - \frac{\omega^2}{\xi_2} - \frac{\sqrt{t\Sigma_a t'}}{\sigma_s}\right), \quad and$$

$$\pi_1 = \frac{1}{\theta}\left(\zeta_0 - \frac{\omega\lambda_0}{\xi_2} + \frac{\omega\theta}{\xi_2} w + t\bar{a}' - \left(\frac{\omega^2}{\xi_2} - \xi_1 - \theta\pi_2\right)\bar{h}\right) .$$

However, we rule out the latter solution because it does not satisfy the second order condition.

QED

C5. PROOF OF PROPOSITION 6.

Substitute (67), (73), and (64) into (65) and (66), and solve the system for h to obtain the demand for h, h_d. h_d is given in (130),

where :

t and w are now given in (75) and (76) respectively.

The rest of the proof is similar to the proof of Proposition 5, with the only difference that now t and w are as specified in (75) and (76).

QED

C6. PROOF OF PROPOSITION 7.

Substitute (80) and (78) into (79), and solve for v to obtain the supply for v. It is given by the following equation, (137).

$$v' = -(\pi_2 \epsilon_1' \epsilon_1)^{-1} (\epsilon_1' \pi_1 + \epsilon_0 \epsilon_1' \pi_2 - \gamma_1' - \gamma_2' b') \ . \tag{137}$$

The second order condition requires that :

$\pi_2 \epsilon_1' \epsilon_1$ is negative definite.

We can now see that the supply for v is linear in the vector of technological parameters b. Therefore, our distributional assumptions about the vector of technological paramaters b, and the equation (137) imply that the aggregate supply for v follows a normal distribution. This distribution is specified next.

$$N(\bar{v}', \Sigma_v) \quad , \tag{138}$$

where :

$\bar{v}'$ is the mean,

Σ_v is the variance,

$$\bar{v}' = -(\pi_2 \epsilon_1' \epsilon_1)^{-1} (\epsilon_1' \pi_1 + \epsilon_0 \epsilon_1' \pi_2 - \gamma_1' - \gamma_2' \bar{b}') \ , \ and \tag{139}$$

$$\Sigma_v = \frac{(\epsilon_1' \epsilon_1)^{-1} \gamma_2' \Sigma_b \gamma_2 (\epsilon_1' \epsilon_1)^{-1}}{\pi_2^2} \ . \tag{140}$$

$h = \epsilon_0 + \epsilon_1 v'$ and the above specified normal distribution for b imply that the aggregate supply for h follows a normal distribution. This distribution is specified next.

$$g(h) = N(\bar{h}_s, \sigma_s^2) \quad , \tag{141}$$

where :

$$\bar{h}_s = \epsilon_0 + \epsilon_1 \bar{v}' \ , \tag{142}$$

$$\sigma_s^2 = \epsilon_1 \Sigma_v \epsilon_1' \quad , \tag{143}$$

$\bar{v}'$ is given in (139), and

Σ_v is given in (140).

Substitute (67) and (64) into (65) and (66), and solve the system for h to obtain the demand for h. It is given in (130).

Following the same argument as in section 3.2., we obtain that the aggregate demand for h, $f(h)$, is a normal distribution with a mean given in (132), and a variance given in (133).

Given the normality of the aggregate demand and supply for h, for an equilibrium the following equations must be satisfied.

$$\bar{h}_s = \bar{h}_d \quad \text{, and} \tag{144}$$

$$\sigma_s^2 = \sigma_d^2 \quad , \tag{145}$$

where :

$\bar{h}_s$, σ_s^2, $\bar{h}_d$, σ_d^2 are given in (142), (143), (132), and (133) respectively, with $\bar{v}'$ given in (139), and Σ_v given in (140).

We can show that our solution for π_1 and π_2 is the only one that satisfies the equilibrium equations (144) and (145).

QED

C7. PROOF OF PROPOSITION 8.

Substitute (80), (73), and (64) into (65) and (66), and solve the system for h to obtain the demand for h, h_d. It is given in (130),

where :

t and w are given in (75) and (2ref(3.20)) respectively.

Following the same argument as in section 3.2., we obtain that the aggregate demand for h , $f(h)$, is a normal distribution with a mean given in (132), and a variance given in (133),

where :

t and w are given in (75) and (76) respectively.

Substitute (80) and (78) into (79), and solve for v to obtain the supply for v . It is given in (137).

Following the same argument as in section 3.4, we can show that the supply for h follows a normal distribution with a mean given in (142) and a variance given in (143), with $\bar{v}'$ specified in (139), and Σ_v in (140).

Given the normality of both aggregate demand and supply, for an equilibrium the following equations must be satisfied.

$$\bar{h}_s = \bar{h}_d \quad \text{, and} \tag{146}$$

$$\sigma_s^2 = \sigma_d^2 \tag{147}$$

where :

$\bar{h}_d$, σ_d^2 , $\bar{h}_s$, σ_s^2 are given in (132), (133), (142), and (143), with t , w , $\bar{v}'$, and Σ_v given in (75), (76), (139), and (140) respectively.

We can show that our solution for π_1 and π_2 is the only one that satisfies the equilibrium equations (146) and (147).

QED

C8. PROOF OF PROPOSITION 9.

Substitute (67), (64), and (29) into (65) and (66), and solve the system for h to obtain the demand for h . It is given in (130), where t is given in (75), and w in (76).

Following the same argument as in section 3.2, we obtain that the aggregate demand for h , $f(h)$, is a normal distribution given in (131)), with a mean given in (132), and a variance given in (133), where t and w are specified in (75) and (76) respectively.

The exogenously given distribution for v and (1) imply that the supply for h is a normal distribution given in (134), with a mean given in (70), and a variance given in (71).

Given the normality of the aggregate demand and supply for h , an equilibrium must satisfy the following two equations.

$$\bar{h}_d = \bar{h}_s \quad \text{, and} \tag{148}$$

$$\sigma_d^2 = \sigma_s^2 \quad , \tag{149}$$

where :

$\bar{h}_d$, σ_d^2 , $\bar{h}_s$, and σ_s^2 are given in (132), (133), (70), and (71) respectively, with t given in (75), and w in (76).

Our solution for π_2 and π_1 is the only one that satisfies the second order condition, and the equilibrium equations (148) and (149).

Substitute (84) and (29) into (28), and solve for the demand for s . It is given in (115). Following the same argument as in the proof of Proposition 3, we obtain that the demand for s follows a normal distribution that is given in (116), with a mean given in (117), and a variance given in (118).

(74) and our distributional assumptions about the vector of utility parameters a imply that the supply fo s follows a normal distribution. This disribution is specified next.

$$N(\eta_0 + \eta_1 \bar{a}', \eta_1 \Sigma_a \eta_1') \quad ,$$

where :

$$\eta_0 + \eta_1 \bar{a}' \quad \text{is the mean, and} \tag{150}$$

$$\eta_1 \Sigma_a \eta_1' \quad \text{is the variance.} \tag{151}$$

Since both aggregate demand and supply for s follow normal distributions, for an equilibrium the following two equations must be satisfied.

$$\sigma_l^2 = \eta_1 \Sigma_a \eta_1' \quad \text{, and} \tag{152}$$

$$\bar{s} = \eta_0 + \eta_1 \bar{a}' \quad , \tag{153}$$

where :

σ_l^2 and $\bar{s}$ are given in (118) and (117) respectively.

It can be verified that our solution for ρ_0 and ρ_1 is the only one that satisfies the second order condition, and the equilibrium equations (152), and (153).

QED

C9 .PROOF OF PROPOSITION 10.

Substitute (80), (85), and (29) into (33) and (34), and solve for the supply for v and the demand for s. They are given by the two equations that follow.

$$v' = (\pi_2 \epsilon_1' \epsilon_1)^{-1} (\pi_1 \epsilon_1' + \epsilon_0 \epsilon_1' \pi_2 - \gamma_1' - \gamma_2' b') \quad , \textit{ and} \tag{154}$$

$$s = -\frac{1}{2 N_1 \rho_1 + \gamma_3} (N_0 \rho_1 + N_1 \rho_0 + \gamma_1 + \gamma_2 b') \quad . \tag{155}$$

Given (154), and following the same argument as in the proof of Proposition 7, we can show that the supply for h is a normal distribution given in (141), with a mean given in (142), and a variance given in (143), where $\bar{v}$ and Σ_v are specified in (139) and (140) respectively.

Substitute (67) and (64) into (65) and (66), and solve the system for h to obtain the demand for h. It is given in (130), with t and w specified in (75) and (76) respectively.

Following the same argument as in Section 3.2, we obtain that the aggregate demand for

h , $f(h)$, is a normal distribution with a mean given in (132), a variance given in (133), and t and w given in (75) and (76) respectively.

Given the normality of the aggregate demand and supply for h , the following two equations must be satisfied for an equilibrium in the product market.

$$\bar{h}_s = \bar{h}_d \quad \text{, and} \tag{156}$$

$$\sigma_s^2 = \sigma_d^2 \quad , \tag{157}$$

where :

$\bar{h}_s$, σ_s^2 , $\bar{h}_d$, and σ_d^2 are given in (142), (143), (132), and (133), with $\bar{v}$, Σ_v , t , and w given in (139), (140), (75), and (76) respectively.

It can be verified that our solution for π_1 and π_2 is the only one that satisfies the equilibrium equations for the product market (156), and (157).

Given (155), and following the same argument as in the proof of Proposition 3, we obtain that the demand fo s follows a normal distribution, given in (116), with a mean given in (117), and a variance given in (118).

(74) and our distributional assumptions about the vector of the utility parameters a imply that the aggregate supply for s is a normal distribution with a mean and variance given in (150) and (151) respectively.

Since both aggregate demand and supply for s follow normal distributions, the following equations must be satisfied for an equilibrium in the input market.

$$\sigma_I^2 = \eta_1 \Sigma_a \eta_1^{\prime} \quad \text{, and} \tag{158}$$

$$\bar{s} = \eta_0 + \eta_1 \bar{a}^{\prime} \quad , \tag{159}$$

where :

$\bar{s}$, and σ_I^2 are given in (117) and (118) respectively.

It can be verified that our solution for ρ_0 and ρ_1 is the only one that satisfies the second order condition, and the equations for an equilibrium in the input market, (158) and (159).

QED

C10. PROOF OF PROPOSITION 11.

Substitute (94), and (88) into (90), (91), and (92), and solve the system for h_i to obtain the demand for h_i, h_{di}. It is given in the next equation.

$$h_{di} = \frac{\xi_{2i}\xi_{3i}}{\xi_{3i}(\omega_{2i}^2 - \xi_{1i}\xi_{2i}) + \xi_{2i}(\omega_{1i} - \theta\pi_{1i})^2}\left[\zeta_{0i} - \frac{\omega_{2i}\lambda_{0i}}{\xi_{2i}} + \frac{(\omega_{1i} - \theta\pi_{1i})(\theta\pi_{0i} - \tau_i)}{\xi_{3i}} + \frac{\theta\omega_{2i}}{\xi_{2i}}w + t_i a^j\right], \tag{160}$$

where :

t_i is given in (100).

The second order condition for the consumer optimization problem implies :

$$\frac{\xi_{2i}\xi_{3i}}{\xi_{3i}(\omega_{2i}^2 - \xi_{1i}\xi_{2i}) + \xi_{2i}(\omega_{1i} - \theta\pi_{1i})^2} > 0 \quad .$$

We can now see that the demand for h is linear in a. Therefore, (160) and (87) imply that the aggregate demand for h_i follows a normal distribution. let it be :

$$f(h_i) = N(\bar{h}_{di}, \sigma_{di}^2) \quad , \tag{161}$$

where :

$\bar{h}_{di}$ is the mean,

σ^2_{di} is the variance,

$$\sigma^2_{di} = \frac{\xi_{2i}\,\xi_{3i}\sqrt{t_i' \Sigma_a t_i}}{\xi_{3i}(\omega^2_{2i} - \xi_{1i}\,\xi_{2i}) + \xi_{2i}(\omega_{1i} - \theta\pi_{1i})^2} \quad , \quad and \tag{162}$$

$$\bar{h}_{di} = \frac{\xi_{2i}\,\xi_{3i}}{\xi_{3i}(\omega^2_{2i} - \xi_{1i}\,\xi_{2i}) + \xi_{2i}(\omega_{1i} - \theta\pi_{1i})^2}\,[\,\zeta_{0i} - \frac{\omega_{2i}\,\lambda_{0i}}{\xi_{2i}} +$$

$$\frac{(\omega_{1i} - \theta\pi_{1i})(\theta\pi_{0i} - \tau_i)}{\xi_{3i}} + \frac{\theta\omega_{2i}}{\xi_{2i}} w + t_i \bar{a}'\,] \quad . \tag{163}$$

(86) and the assumption that the supply for v_i follows an exogenously given distribution, given in (93), imply that the supply for h_i follows a normal distribution. Let it be :

$$g(h_i) = N(\bar{h}_{si}, \sigma^2_{si}) \quad , \tag{164}$$

where :

$\bar{h}_{si}$ is the mean, given in (99), and

σ^2_{si} is the variance, given in (98).

For an equilibrium the following must be satisfied :

aggregate demand = aggregate supply , for all h_i .

That is,

$$f(h_i)\,dh_i = g(h_i)\,dh_i \quad .$$

Since both aggregate demand and supply for h_i follow normal distributions, for all h_i , the above condition for equilibrium is equivalent to :

$$\bar{h}_{di} = \bar{h}_{si} \quad \text{, and} \tag{165}$$

$$\sigma^2_{di} = \sigma^2_{si} \quad , \tag{166}$$

where :

$\bar{h}_{di}$, σ^2_{di} , $\bar{h}_{si}$, and σ^2_{si} are given in (163), (162), (99), and (98) respectively, where t_i is given in (100).

The price equation parameters specified in (95) and (96) satisfy the equilibrium equations (165) and (166). Moreover, they are valid for all admissible set of parameters.

It can be verified that the π_{1i} and π_{0i} solutions, given in (95) and (96), and the three set of solutions that follow are the only ones that satisfy the equilibrium equations (165) and (166).

Set 1 :

$$\pi_{1i} = \frac{1}{\theta}(\omega_{1i} - A^{0.5}) \quad , \tag{167}$$

$$\pi_{0i} = -\frac{1}{\theta \xi_{2i} A^{0.5}} \{ (\zeta_{0i} - \frac{\omega_{2i} \lambda_{0i}}{\xi_{2i}} - A^{0.5} \frac{\tau_i}{\xi_{3i}} + \frac{\theta \omega_{2i}}{\xi_{2i}} w + t_i \bar{a}') \xi_{2i} \xi_{3i} - [(\omega^2_{2i} - \xi_{1i} \xi_{2i}) \xi_{3i} + \xi_{2i} A] \bar{h}_{si} \} \quad , \tag{168}$$

where :

t_i is given in (100),

h_{si} is given in (99),

$$A = \frac{\xi_{3i} (\xi_{1i} \xi_{2i} - \omega^2_{2i})}{\xi_{2i}} - \frac{\xi_{3i}}{\sigma_{si}} \sqrt{t_i \Sigma_a t_i'} \quad \text{, and} \tag{169}$$

σ_{si} is given in (98).

Set 2 :

π_{1i} is given in (167),

π_{0i} is given in (168),

where :

A is given in (97).

Set 3 :

π_{1i} is given in (95),

π_{0i} is given in (96),

where :

A is given in (169).

We rule out the solutions given :

i) in sets 1 and 3 because they do not satisfy the second order conditions for the consumer's utility maximization problem, and

ii) in set 2 because for $\omega_{1i} = 0$ they imply $\dfrac{\partial P(h_i)}{\partial h_i} < 0$.

QED

C11. PROOF OF PROPOSITION 12.

Substitute (94), (73), and (88) into (90), (91), and (92), and solve the system for h_i , to obtain the demand for h_i , h_{di} . It is given in (160), where now t_i is given in (101), and w is given in (102).

The rest of the proof now follows from the proof of Proposition 11, with the only difference that now t_i and w are given in (101) and (102) respectively.

QED

C12. PROOF OF PROPOSITION 13.

Subsitute (94) and (104) into (105), and solve for v_i to obtain the supply for v_i. It is given in the next equation, (170).

$$v_i' = \gamma_3^{-1}(\pi_{1i}\epsilon_{1i}' - \gamma_1' - \gamma_2' b') \quad . \tag{170}$$

We can see now that the supply for v_i is linear in b. Therefore, (103) and (170) imply that the aggregate supply for v_i follows a normal distribution. Let it be :

$$N(\bar{v}_i, \Sigma_{vi}) \quad , \tag{171}$$

where :

$\bar{v}_i$ is the mean,

Σ_{vi} is the variance,

$$\bar{v}_i' = \gamma_3^{-1}(\pi_{1i}\epsilon_{1i}' - \gamma_1' - \gamma_2' \bar{b}') \quad \text{, and} \tag{172}$$

$$\Sigma_{vi} = \gamma_3^{-1}\gamma_2' \Sigma_b \gamma_2 \gamma_3^{-1} \quad . \tag{173}$$

(86) and the specified normal distribution for the supply for v_i, given in (171), imply that the aggregate supply for v_i is a normal distribution; this distribution is specified next, equation (174).

$$g(h_i) = N(\bar{h}_{si}, \sigma_{si}^2) \quad , \tag{174}$$

where :

$\bar{h}_{si}$ is the mean, given in (107), and

σ_{si}^2 is the variance, given in (106).

Substitute (94) and (88) into (90), (91), and (92), and solve the system for h_i, to obtain the demand for h_i, h_{di}. It is given in (160).

Following the same argument as in the proof of Proposition 11, we obtain that the aggregate demand for h_i, $f(h_i)$, is a normal distribution given in (161).

Given the normality of the aggregate demand and supply for h_i , for an equilibrium the following two equation must be satisfied.

$$\bar{h}_{si} = \bar{h}_{di} \quad , \text{ and} \tag{175}$$

$$\sigma^2_{si} = \sigma^2_{di} \quad , \tag{176}$$

where :

$\bar{h}_{si}$, σ^2_{si} , $\bar{h}_{di}$, and σ^2_{di} are given in (107), (106), (163), and (162) respectively.

The parameters π_{0i} and π_{1i} that are specified in Proposition 13 are the only ones that are valid for all admissible set of parameters, and satisfy the second order condition and the equilibrium equation (175) and (176).

QED

C13. PROOF OF PROPOSITION 14.

Substitute (94), (73), and (88) in (90), (91), and (92), and solve the system for h_i , to obtain the demand for h_i . It is given in (160), where t_i is given in (101), and w in (102).

Following the same argument as in the proof of Proposition 11, we obtain that the aggregate demand for h_i , $f(h_i)$, is a normal distribution given in (161), with a mean and variance given in (163) and (162) respectively, where t_i is given in (101) and w in (102).

Substitute (94) and (104) into (105), and solve for v_i , to obtain the supply for v_i . It is given in (170).

Following the same argument as in the proof of Proposition 13, we can show that the aggregate supply for h_i follows a normal distribution that is specified in (172).

Given the normality of both aggregate demand and supply for h_i , for an equilibrium the following equations, (177) and (178) must be satisfied.

$$\bar{h}_{si} = \bar{h}_{di} \quad , \text{ and} \tag{177}$$

$$\sigma^2_{si} = \sigma^2_{di} \quad , \tag{178}$$

where :

$\bar{h}_{si}$, σ^2_{si} , $\bar{h}_{di}$, and σ^2_{di} are given in (163), (162), (107), and (106) respectively.

It can be verified that the parameters specified in Proposition 14 are the only ones that are valid for all admissible set of parameters, and satisfy the second order condition and the equilibrium equations (177) and (178).

QED

C14. PROOF OF PROPOSITION 15.

Substitute (94), (29), and (88) into (90), (91), and (92), and solve the system for h_i , to obtain the demand for h_i , h_{di} . It is given in (160), where t_i is given in (101), and w in (102).

Following the same argument as in the proof of Proposition 11, we obtain that the aggregate demand for h_i , $f(h_i)$, is a normal distribution, given in (161), with a mean given in (163), and a variance in (162), where t_i and w are given in (101) and (102) respectively.

The exogenously given distribution for v_i and (86) imply that the aggregate supply for h_i is a normal distribution, given in (165), with a mean given in (99), and a variance given in (98).

Given the normality of the aggregate demand and supply for h_i , an equilibrium must satisfy the following equations, (179) and (180).

$$\bar{h}_{si} = \bar{h}_{di} \quad , \text{ and} \tag{179}$$

$$\sigma^2_{si} = \sigma^2_{di} \quad , \tag{180}$$

where :

$\overline{h}_{si}$, σ^2_{si} , $\overline{h}_{di}$, and σ^2_{di} are given in (99), (98), (163), and(162) respectively, where t_i is specified in (101), and w in (102).

The solution specified in Proposition 15 is the only one that is valid for all admissible set of parameters, and satisfies the second order condition and the equilibrium equations (179) and (180).

Substitute (109) and (29) into (28), and solve for the demand for s . It is given in (115).

Following the same argument as in the proof of Proposition 3, we obtain that the aggregate demand for s is a normal distribution, given in (116), with a mean given in (117), and a variance given in (118).

Equation (74) and our distributional assumptions about the vector of utility parameters a imply that the aggregate supply for s follows a normal distribution with a mean given in (153), and a variance given in (152).

Since both aggregate demand and supply for s follow normal distributions, an equilibrium must satisfy the following equations :

$$\sigma^2_l = \eta_1 \Sigma_a \eta_1' \quad , \tag{181}$$

$$\overline{s} = \eta_0 + \eta_1 \overline{a}' \quad , \tag{182}$$

where :

σ^2_l and $\overline{s}$ are given in (117) and (118) respectively.

It can be verified that our solution for ρ_0 and ρ_1 , specified in Proposition 15, is the only one that satisfies the second order condition and the equations for an equilibrium in the input market, (181) and (182).

QED

C15. PROOF OF PROPOSITION 16.

Substitute (94), (29), and (110) into (33) and (34), and solve for the supply for v_i and the demand for s. They are given by the following equations :

$$v_i' = \gamma_6^{-1}(\pi_{1i}\epsilon_1' - \gamma_4' - \gamma_5 b') \text{ , and} \tag{183}$$

$$s = -\frac{N_0\rho_1 + N_1\rho_0 + \gamma_1 + \gamma_2 b'}{2N_1\rho_1 + \gamma_3} . \tag{184}$$

Given (183), and following the same argument as in the proof of Proposition 13, we can show that the supply for h_i is a normal distribution, given in (174), with a mean given in (107), and a variance given in (106).

Substitute (94) and (88) into (90), (91), and (92), and solve the system for h_i, to obtain the demand for h_i. It is given in (160), where now t_i and w are given in (101) and (102) respectively.

Following the same argument as in the proof of Proposition 11, we obtain that the aggregate demand for h_i, $f(h_i)$, is a normal distribution, given in (161), with a mean given in (163), a variance given in (162), and t_i and w given in (101) and (102) respectively.

Given the normality of the aggregate demand and supply for h_i, an equilibrium in the product market must satisfy the following equations, (185) and (186).

$$\bar{h}_{si} = \bar{h}_{di} \text{ , and} \tag{185}$$

$$\sigma_{si}^2 = \sigma_{di}^2 \text{ ,} \tag{186}$$

where :

$\bar{h}_{si}$, σ_{si}^2, $\bar{h}_{di}$, and σ_{di}^2 are given in (106), (107), (163), and (162) respectively, with t_i specified in (101), and w in (102).

The solution for the parameters of the price equation that are given in Proposition 16 is the only one that is valid for all admissible set of parameters, and satisfies the second order condition and the conditions for an equilibrium in the product market, namely, equations (185) and (186).

Given (184), and following the same argument as in the proof of Proposition 3, we obtain that the demand for s follows a normal distribution that is given in (116), with a mean and variance specified in (117) and (118) respectively.

(74) and our assumptions about the distribution of the vector of utility parameters, a, imply that the supply for s follows a normal distribution that is given in (150).

Since both the aggregate demand and supply for s are normal distributions, an equilibrium in the input market requires that the following equations be satisfied :

$$\sigma_l^2 = \eta_1 \Sigma_a \eta_1' \quad \text{, and} \qquad (187)$$

$$\bar{s} = \eta_0 + \eta_1 \bar{a}' \quad , \qquad (188)$$

where :

$\bar{s}$ is given in (118), and σ_l^2 in (117).

It can be verified that the parameters of the wage equation that are given in Proposition 16 are the only ones that satisfy the second order condition and the conditions for an equilibrium in the input market, namely, the equations (187) and (188).

QED

APPENDIX D

THE UNRESTRICTED MODEL

To test the internal consistency of the model, I should estimate the unrestricted model and then test the restricted against the unrestricted model using a likelihood ratio test.

The unrestricted model consists of the following two equations:

$$P = b_0 + b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 \bar{a} + b_5 \bar{l} + b_6 \bar{v}_1 + b_7 \bar{v}_2 + b_8 \bar{v}_3 \quad ,and$$

$$d_0 + d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 a + d_5 l + d_6 \bar{v}_1 + d_7 \bar{v}_2 + d_8 \bar{v}_3 + d_9 \bar{a} + d_{10} \bar{l} = 0$$

The theoritical model that I study here imposes the following constraints on the above equations:

$$d_1 = -d_6$$

$$d_2 = -d_7$$

$$d_3 = -d_8$$

$$d_4 = -d_9$$

$$d_5 = -d_{10}$$

$$d_0 = 0$$

$$d_3 = b_3 / b_1$$

$$d_2 = b_2 / b_1$$

$$12 b_4 = d_4 / d_5$$

$$b_5 = 1 / 12$$

$$b_6 = 1 / 12 d_5$$

$$12\,b_7 = d_2 / d_5$$

$$12\,b_8 = d_3 / d_5$$

The restricted model can be estimated in the way that I presented in Chapter 3.

To estimate the unrestricted model, time series data on Houston is needed. An alternative could be pooling the cross – section data for several cities, allowing for fixed effects. However, the validity of the assumed fixed effects could not be tested[40] without time series data on the individual cities.

---

[40] I have in mind likelihood ratio tests that are analogous to the F - tests that are discussed in Maddala (1977), Chapter 14.

APPENDIX E

POOLING THE CROSS SECTION DATA

I pooled the data for all five cities and I tried several fixed effect combinations. Some of those combinations are described next.

Combination 1 : No dummy variables.

Combination 2 : A dummy variable for each city on the intercept of equation (43). No dummy variables on equation (44).

Combination 3 : A dummy variable for each city on the intercept of equation (43). A dummy variable for each city on the variable $(a - \bar{a})$ of equation (44).

Combination 4 : A dummy variable for each city on the variable v_2 of equation (43). A dummy variable for each city on the variable $(v_2 - \bar{v}_2)$ of equation (44).

Combination 5 : A dummy variable for each city on the variable v_1 of equation (43). A dummy variable for each city on the variables $(v_2 - \bar{v}_2)$ and $(v_3 - \bar{v}_3)$ of equation (44).

Combination 6 : A dummy variable for each city on the intercept, and a dummy for Houston on the variables v_1, v_2, and v_3 of equation (43). A dummy variable for Houston on the variables $(v_2 - \bar{v}_2)$ and $(v_3 - \bar{v}_3)$ of equation (44).

Combination 7 : A dummy variable for each city on the intercept of euation (43). A dummy variable for each city on the variable $(a - \bar{a})$ of equation (44).

Combination 6 also assumed that the equation (43), as well as equation (44), is quadratic in air quality. This is justified by assuming that the housing quality mapping is:

$$h = v_1 + \epsilon_1 v_2 + \epsilon_2 v_3 + \epsilon_3 v_4$$

where:

$v_4 = v_2^2$, and

v_1 , v_2 , and v_3 are defined as before.

In that way, I can make the rental price equation be nonlinear (quadratic or of higher degree) in any of the v_i – characteristics of a differentiated good; as a matter of fact, as we will see later, I did that for several other variables. Following a similar variable transformation, I can make my theory able to introduce nonlinearities in the variable that describes the type of the consumer; namely, nonlinearities in the elements of the vector of utility parameters a , e.g., demand for quality and price equations that are nonlinear in the variable " size of a household " and the parameters of the distribution of that variable.

I estimated each of the above combinations with and without imposing the cross – equation restrictions; the cross – equation restrictions require that the equations (49) and (50) are satisfied for each city. Table[41] 12 reports the loglikelihood values for both the restricted and the unrestricted models, for all combinations[42].

Next, I review the qualitative and other properties of the parameter estimates that I obtained from the above experiments (the ones that incorporate the cross – equation restriction). To be more specific, I summarize the WRONG predictions and the problems of insignificant parameter estimates for both equations and for each combination.

Combination 1 : It predicts that i) the relationship between rent and travel time to work is positive and ii) that the relationship between rent and air quality is negative. All the

---

[41] From Table 12 we can see that when I impose the cross - equation restrictions the likelihood ratio decreases significantly. Given the functional forms for the rental price equation and the demand for housing quality, a likelihood ratio test rejects the hypothesis that the cross - equation restrictions are satisfied. However, this is not the appropriate test for testing the internal consistency of the theory (for the reasons discussed in Appendix D). This would be the test if a fixed effect assumption were verified by the data.

[42] Estimating the system of equations (43) and (44) separately for each one of the five cities, I obtain the loglikelihood values that are given on Table 13. Given these values and the ones of Table 12, I can perform a likelihood ratio test to see if any of the assumed fixed effect combinations can be accepted. These tests suggest i) that the parameters of equations (43) and (44) are not the same across cities, and ii) that I cannot accept any of the fixed effect combinations that I tried. However, these likelihood ratio tests take as given the functional form of the rental price equation and of the first order condition for consumer utility maximization problem, given in (43) and (44) respectively. Therefore, I cannot really accept or reject any of the assumed fixed effect assumptions. To do that I should be able to test each one of the assumed fixed effect combinations against the unrestricted model that is specified in the Appendix D; the tests that could be performed are along the lines discussed in Maddala (1977), Chapter 14.

parameter estimates of the rental price equation are insignificant (even the number of rooms).

Combination 2 : It predicts i) that rent and travel time to work are positively related, and ii) that rent and air quality are negatively related.

Combination 3 : It predicts that the relationship between rent and air quality is negative.

Combination 4 : It predicts that the relationship between rent and air quality is positive only for Houston.

Combination 5 : It predicts i) that the relationship between rent and travel time to work is positive and ii) that the relationship between air quality and rent is negative.

Combination 6 : It predicts i) that the relationship between rent and travel time to work is negative only for Houston, and ii) that the relationship between rent and air quality is positive only for Houston.

Combination 7 : It predicts that rent decreases as air quality improves.

From the above list, it seems that the model (when it incorporates a fixed effect assumption) predicts the wrong effects of air quality and of travel time to work on rent; Table 17 presents the parameter estimates for one of those combinations, combination 4. However, when I isolate Houston from the other cities, the model gives the expected predictions for Houston (e.g., see the above comments on combinations 4 and 6). This indicates either that there is need for more data on the already existing and on other important variables for all five cities in order to isolate the effects of air quality and travel time to work on rents, or that the given structure and fixed effect assumptions are not appropriate for Chicago, Cleveland, Indianapolis, and Dallas.

Appendix F presents some other structure that cannot be implied by my theoritical model and which seems to behave well for Chicago, Cleveland, Indianapolis, and Dallas.

In addition to the above, I tried the combinations listed on Table 14. The results were not significantly different.

Since a fixed effect assumption and the assumed structure of the model did not perform that good for Chicago, Cleveland, Indianapolis, and Dallas, I tried a version of the model that yields a price equation that is nonlinear (quadratic) in the number of rooms. To be more specific, I assumed that the housing quality mapping includes a term that is quadratic in v_1 . Then I applied a variable transformation (similar to the one that I used earlier to introduce a quadratic term in air quality) to obtain a version of the model that is linear in the transformed variable. I run separate regressions for each city and I also pooled the cross – section data for Chicago, Cleveland, Indianapolis, and Dallas. The fixed effect combinations that I tried are listed on Table 15. However, the introduction of a quadratic term in the number of rooms did not improve the results. Similar were the results that I obtained when I introduced into the model a quadratic term in air quality and pooled the data for the four cities.

APPENDIX F

AN EMPIRICAL MODEL FOR THE HOUSING MARKET OF CHICAGO, CLEVELAND, INDIANAPOLIS, AND DALLAS.

The rental price equation is:

$$P = d_1 + d_2 + d_3 + d_4 + e v_1 v_2 + g v_3 \ .$$

The demand for number of rooms is:

$$v_1 = l + m I + n a$$

where:

c , e , g , l , m and n are parameters,

I , a , and v_i are defined earlier, i = 1, 2 ,3 , and

d_i is a dummy variable, i = 1, 2, 3, 4.

The dummy variables are defined next.

$d_1 = 1$ for Chicago,

$d_1 = 0$ else.

$d_2 = 1$ for Cleveland,

$d_2 = 0$ else,

$d_3 = 1$ for Indianapolis,

$d_3 = 0$ else,

$d_4 = 1$ for Dallas,

$d_4 = 1$ else.

$$d_0 = 0 \quad .$$

The results are given on Table 16.

APPENDIX J

A FORECASTING - INVESTMENT THEORY APPLICATION OF THE MODEL POOLING THE CROSS SETION DATA FOR ALL FIVE CITIES

A competitive firm supplies the services of a specific type of house in Houston, Chicago, Cleveland, Indianapolis, and Dallas. It supplies the services of ten houses in each city each year. The type of house is given next.

$$h = (5 , 0.017 , 30)$$

To make an investment decision the competitive firm wants to compute the present value of the future stream of rents for the next fifteen years. To compute this present value, I consider the model that I introduced in Sections 4.2 and 4.3, and I assume that now β_2 , *and* ϵ_1 satisfy the following equations:

$$\beta_2 = \beta_{20} + \sum_{i=0}^{4} \beta_{2i} d_i \quad , \quad and$$

$$\epsilon_1 = \epsilon_{10} + \sum_{i=0}^{4} \epsilon_{1i} d_i \quad ,$$

where :

$d_0 = 0$,

β_{20} , β_{2i} , ϵ_{10} , *and* ϵ_{1i} are exogenously given parameters, and

d_i is a dummy variable that is defined in Appendix F, for i = 1 , 2 , 3 , 4 .

I estimate equations (43) and (44) via Maximum Lielihood. I also impose the cross equation restrictions that are implied by the structure of the model, namely,

$$\beta_2 = \beta_3 / \beta_1 \quad ,$$

$$\epsilon_{10} = \beta_{20} / \beta_1 \quad , \quad and$$

$$\epsilon_{10} + \epsilon_{1i} = (\beta_{20} + \beta_{2i}) / \beta_1 \quad , \quad for \quad i = 1 , 2 , 3 , 4 \quad .$$

The regression results are given in Table 17. They indicate that for a 10% annual discount rate the total present value is $ 907,790. The present value for each individual city is given in Table 18.

Estimating the above model without imposing the cross equation restrictions, I obtain the parameter estimates that are given in Table 19. These estimates imply that for a 10% annual discount rate the total present value is $ 1,139,250. The present value for each individual city is given in Table 20.

Both of the above present value figures assumed that none of the exogenous parameters of the model will change in the next fifteen years. If one or more of the exogenous parameters of the model changes, the rental price equation shifts, and consequently the present value figure changes. Unlike the alternative non – structural approach, the structural model that I study can compute the present value change that is implied by a change in one of the exogenous parameters of the model. To illustrate this point, I consider a situation that is similar to the one of Section 4.8, namely, the following.

It is expected that, starting next year (1981), the mean income increases 10% every five years, see Table 21. The structural model predicts that[43] the total present value[44] increases by $ 963,350 and it becomes equal to $ 1,871,140; the present values for each individual city are given in Table 22. We can now see that the alternative method would seriously underestimate the present value since the present values are (roughly speaking) doubled for all cities.

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[43] For a 10% annual discount rate.

[44] To compute those present values, I have to compute how the present value depends on the mean income, that is, I have to compute how the rental price equation depends on the mean income. I do that in the same way that I computed the dependence of the price equation on the mean air quality (see Section 4.5.2 and Table 2).

APPENDIX K

AN ALTERNATIVE ESTIMATION PROCEDURE FOR HOUSTON

The econometric model for the housing market of Houston consists of equations (62) and (63). I estimate the model using the four step procedure that I introduced in Chapter 4, Section 4.8. The parameter estimates that I obtaine are given on Tables 25 and 26. They imply that the rental price equation, the housing quality index equation, and the demand for housing quality equation are respectively given by the following equations :

$$P = 172.2 + 45.77\,v_1 + 6701.21\,v_2 - 8.65\,v_3 \quad ,$$

$$h = v_1 + 146.41\,v_2 - 0.189\,v_3 \quad ,, \text{ and}$$

$$h = -1.52 + 0.026\,a + 0.000172\,I \quad .$$

To see if the model is of any value at all, I tested the hypothesis that all the parameters of the rental price equation equal zero, that is,

$$\beta_1 = \beta_2 = \beta_3 = 0 \quad .$$

An F – test implies that this hypothesis is rejected at the 1% significance level.

A similar F – test implies that I cannot accept the hypothesis that the parameters of the demand for housing quality equation equal zero, that is I cannot accept that

$\epsilon_3 = \epsilon_4 = 0$ at the 1% significance level.

The predictions and the qualitive properties of the model, as well as the t – statistics that are given on Tables 25 and 26, are consistent with my a priori expectations and comparable to the ones of the parameter estimates that I obtained when I applied the simultaneous equation technique.

Moreover, the parameter estimates of interest that I obtained from the two different procedures are not very different in magnitute; only the three of them are different in magnitute by a factor greater than three. The parameters that I expect to be the main determinants of the rental price equation (namely, the coefficients of the number of rooms and of the travel time to work variables) and of the demand for housing quality equation (namely, the coefficient of income) are different in magnitute by a factor less than three[45]. That is, the two estimation techniques yielded parameter estimates that were different by a factor greater than three only for the coefficients that I did not expect to be the main determinants of rental prices and demand for quality.

I believe that the just mentioned difference in the magnitudes of the estimates that I obtained from the two estimation techniques would disappear if in the first step the four step estimation procedure yielded a better estimate for the coefficient of v_2 in the price equation. For that reason more data on the existing and on other important variables might be important.

[45] .... with the two of them being pretty close; namely the coefficient of v_1 in the price equation, and the coefficient of income in the demand for housing quality equation.

FIGURES

FIGURE 1

The Price Equation of The Starting Situation of The First Example of Chapter 3.

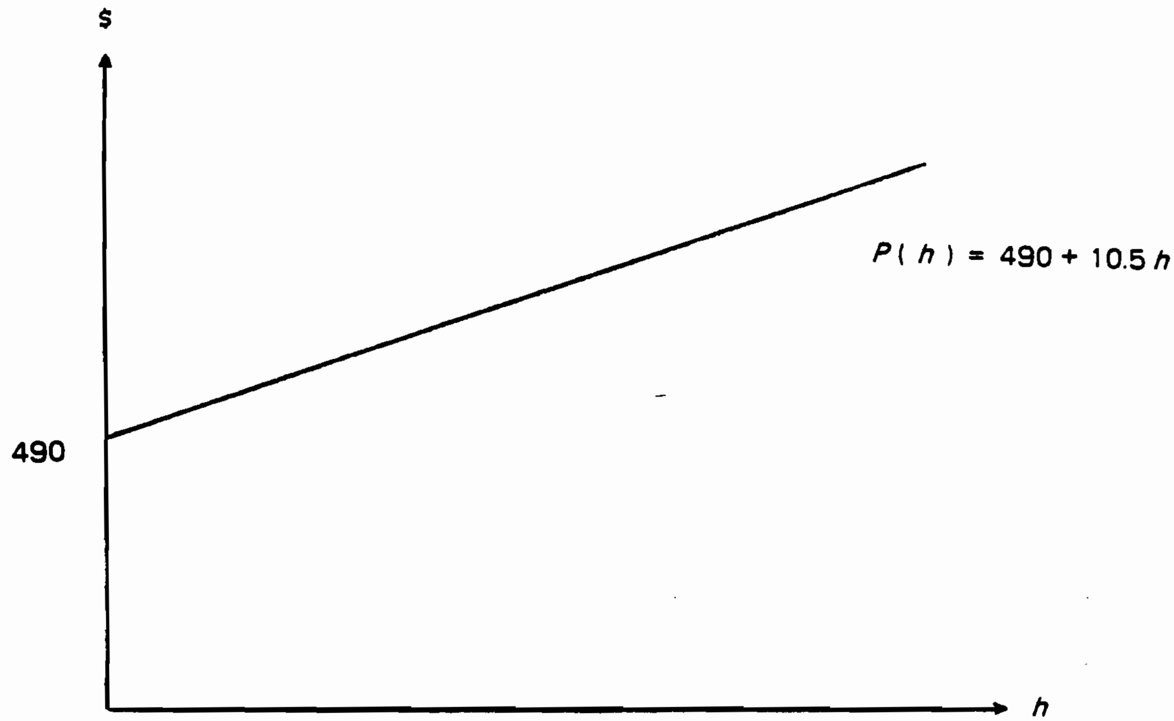


FIGURE 2

The Marginal Willingness to Pay Curve The Associated with The Price Equation of Figure 1.

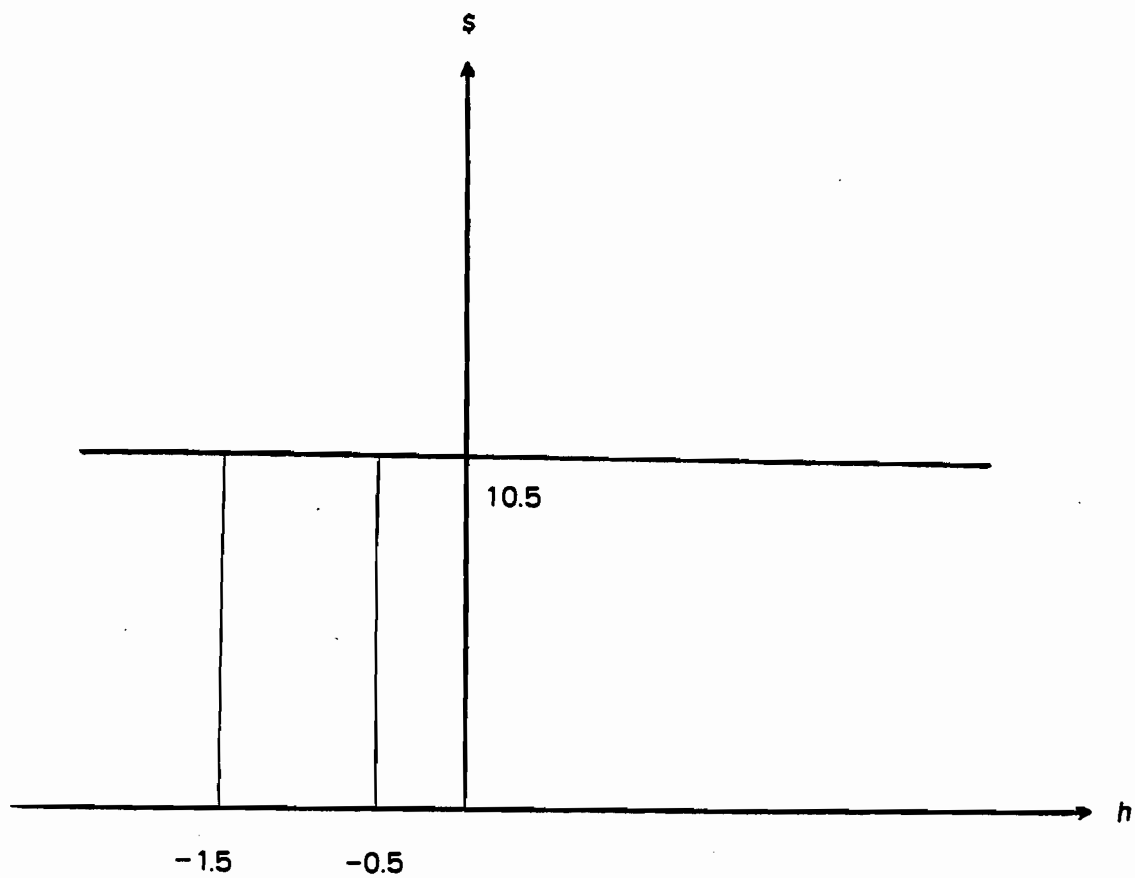


FIGURE 3

The Price Equations of The Second Example of Chapter 3.

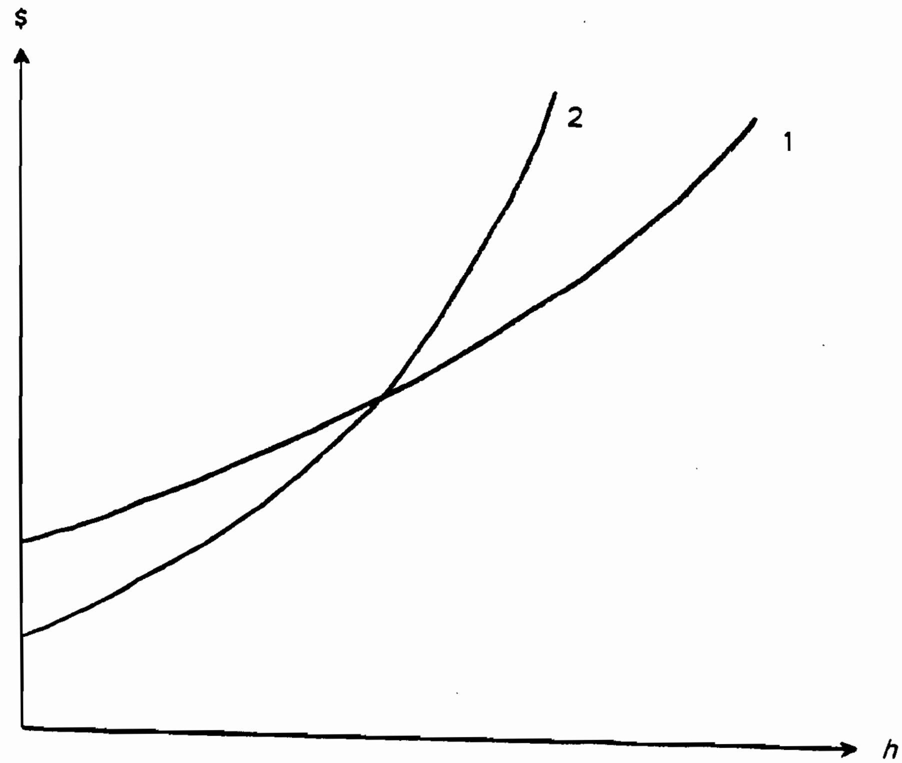
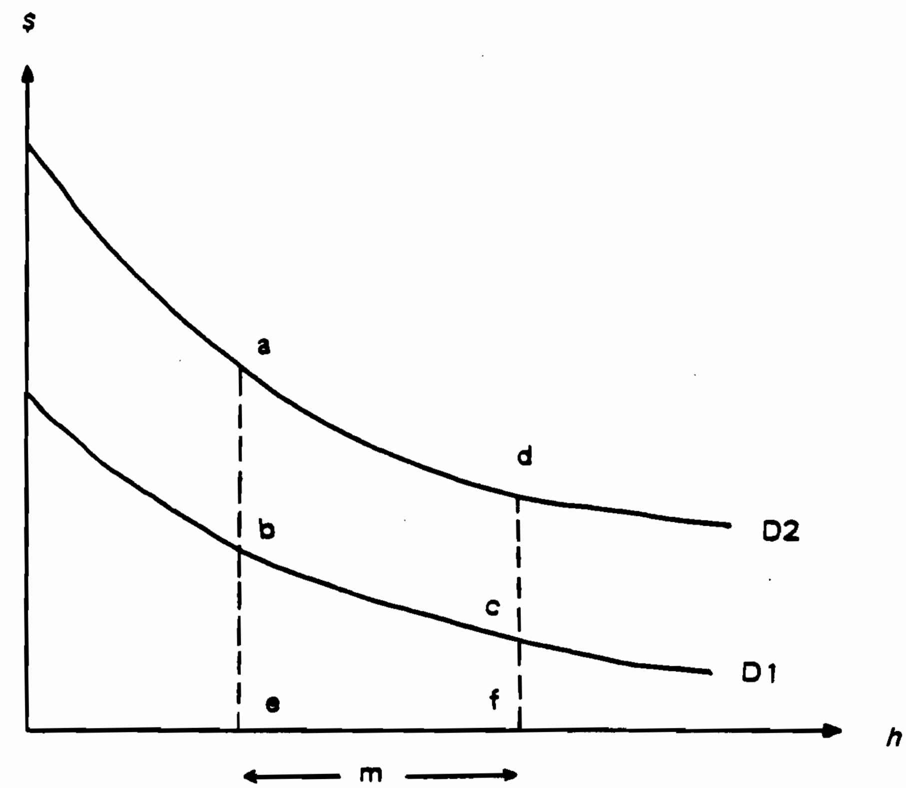


FIGURE 4

The Marginal Willingness to Pay Curves of The Second Example of Chapter 3.



TABLES

NOTATION AND DEFINITIONS FOR TABLES

N: is the number of observations.

h: is the housing quality.

P: is the rent.

V1: is the number of rooms of a house.

V2: is the air quality.

V3: is the travel time to work.

a: is the size of a family.

I: is the consumer income.

MV1, MV2, MV3, Ma, MI are the means of V1, V2, V3, a, and I respectively.

dX = X − MX, for X = V1, V2, V3, a, I.

C: is the intercept of the price equation.

D: in front of a variable D indicates a dummy variable.

RHO: is the estimate of the inter − equation error correlation.

FUNCTION: is the negative of the loglikelihood function.

EQUATION 1: is the equation(43).

EQUATION 2: is the equation (44).

SIGMA: is the standard deviation of the model error.

E(Y,X): is the elasticity of Y with respect to X.

TABLE 1

ESTIMATION RESULTS - HOUSTON, TEXAS - THE RESTRICTED MODEL.

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-3.318238	0.9485320	-3.498288
	V1	48.59252	14.452640	3.362189
	V2	1648.671	949.02110	1.737233
	C	93.51385	49.934550	1.872728
	SIGMA	59.95461	5.9348540	10.10212
EQUATION 2:	dV2	33.92849	19.899270	1.705012
	dV3	-0.06828701	0.01574382	-4.337385
	da	-0.5044743	0.09248937	-5.454403
	dI	-0.0001091461	0.0000153241	-7.122514
	SIGMA	0.3736063	0.03698296	10.10212
	RHO	-0.2432288	12.19337	-0.01994762

N = 57

FUNCTION = 76.48

TABLE 2

HOUSTON STATISTICS.

Mean number of rooms:	4.1281
Mean air quality:	0.014123
Mean travel time to work:	25.956
Mean number of persons in a family:	2.4998
Mean income:	15954

TABLE 3

THE BENEFIT OF THE MEAN HOUSEHOLD
ESTIMATES IMPLIED BY THE STRUCTURAL ANALYSIS.

AIR QUALITY IMPROVEMENT	ANNUAL BENEFIT
1 %	$ 36.56
5 %	$ 209.73
10 %	$ 426.19
70 %	$ 3023.65

TABLE 4

PRICE EQUATION LINEAR IN AIR QUALITY.

VARIABLE	COEFFICIENT	STANDARD DEVIATION	T-STATISTIC
V3	-8.650030	1.574976	-5.492168
V1	45.76947	11.22621	4.077019
V2	6701.209	2578.897	2.598479
C	172.2020	58.04411	2.966743
SIGMA	52.08527	4.878232	10.67708

N = 57

FUNCTION = 43.70

TABLE 5

PRICE EQUATION QUADRATIC IN AIR QUALITY.

VARIABLE	COEFFICIENT	STANDARD DEVIATION	T-STATISTIC
V3	-8.656569	1.572942	-5.503424
V1	45.96413	11.22070	4.095871
V2	12983.47	16221.11	0.8004059
V2 square	-254478.6	648741.6	-0.3922650
C	135.4400	110.1952	1.2290920
SIGMA	52.01511	4.871661	10.67708

N = 57

FUNCTION = 43.62

TABLE 6

THE BENEFIT OF THE MEAN HOUSEHOLD
ESTIMATES IMPLIED BY THE PREVIOUS METHOD

AIR QUALITY IMPROVEMENT	ANNUAL BENEFIT
1 %	$ 11.34
5 %	$ 56.69
10 %	$ 113.38
70 %	$ 793.69

TABLE 7

ESTIMATION RESULTS - HOUSTON - THE UNRESTRICTED MODEL

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-8.650030	1.995053	-4.335740
	V1	45.76947	11.67806	3.919271
	V2	6701.209	3570.176	1.876997
	C	172.2020	76.31932	2.256336
	SIGMA	52.08527	7.137568	7.297341
EQUATION 2:	dV2	16.80116	22.69788	0.7402084
	dV3	-0.04591578	0.01752044	-2.620697
	da	0.5919534	0.09511213	6.223743
	dI	0.0009786965	0.0001627315	6.014180
	SIGMA	0.3647809	0.04745587	7.686739
	RHO	-0.2631797	17.27334	-0.0153618

N = 57

FUNCTION = 65.05

TABLE 8

ESTIMATION RESULTS - CHICAGO - THE RESTRICTED MODEL.

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-0.006470678	0.1481955	-0.04366312
	V1	5.884917	53.457280	0.1100864
	V2	-482.7077	41224.141	-0.1170444
	C	180.3079	175.3906	1.028036
	SIGMA	46.34383	12.37562	
3.744769				
EQUATION 2:	dV2	-82.02455		
	dV3	0.001099536		
	da	-0.8331512	0.3662429	2.274860
	dI	0.00006860878	0.00005111136	1.342339
	SIGMA	0.4643174	0.1644645	2.823208
	RHO	0.006609137	0.03923263	0.1684602

N = 22

FUNCTION = 28.64

NOTE: t - statistics and standard errors have not been computed for the restricted parameters.

TABLE 9

ESTIMATION RESULTS - CLEVELAND - THE RESTRICTED MODEL.

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	0.3290233	0.5170494	0.6363479
	V1	11.53795	15.68396	0.7356527
	V2	362.6105	648.0004	0.5595837
	C	60.07079	81.54489	0.7366592
	SIGMA	33.80079	5.560607	6.078616
EQUATION 2:	dV2	31.42764		
	dV3	0.02851662		
	da	-1.578467	0.2195937	-0.0000007832915
	dI	-0.0003156647	0.0002905941	1.086274
	SIGMA	0.3600843	0.07500576	4.800755
	RHO	0.0005528120	0.05958675	0.009277432

N = 28

FUNCTION = 20.49

NOTE: t - statistics and standard errors have not been computed for the restricted parameters.

TABLE 10

ESTIMATION RESULTS - INDIANAPOLIS - THE RESTRICTED MODEL.

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-0.07216579	0.6014478	-0.1199868
	V1	-3.859933	31.49128	-0.1225715
	V2	164.9637	1262.578	0.1306563
	C	143.0363	127.1001	1.125383
	SIGMA	44.57563	10.26277	4.343428
EQUATION 2:	dV2	-42.73745		
	dV3	0.01868613		
	da	-0.8149931	0.3319744	-2.454988
	dI	-0.0006890227	0.00028471	-2.420086
	SIGMA	0.4089882	0.1094133	3.738014
	RHO	0.04403254	0.1690214	0.2605145

N = 19

FUNCTION = 21.06

NOTE: t - statistics and standard errors have not been computed for the restricted parameters.

TABLE 11

ESTIMATION RESULTS - DALLAS - THE RESTRICTED MODEL.

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-0.4758249	1.816221	-0.2619862
	V1	-11.00349	35.50741	-0.3098928
	V2	1719.227	5917.013	0.2905566
	C	227.2570	87.40706	2.599985
	SIGMA	104.7485	29.44304	3.557665
EQUATION 2:	dV2	-156.2438		
	dV3	0.04324310		
	da	-0.9669829	0.222302	-4.349861
	dI	-0.0002630585	0.0003083035	-0.8532452
	SIGMA	0.6037945	0.1997041	3.023446
	RHO	0.3678123	2.121229	0.173443

N = 26

FUNCTION = 61.87

NOTE: t -statistics and standard errors have not been computed for the restricted parameters.

TABLE 12

POOLING OF CROSS SECTION DATA FOR ALL 5 CITIES

INTRODUCTION OF SEVERAL DUMMY VARIABLES COMBINATIONS IN EQUATIONS (43) AND (44)

COMBINATION	CROSS EQUATION RESTRICTIONS	FUNCTION	
		RESTRICTED	UNRESTRICTED
1	2	290.8	267.1
2	2	260.5	231.0
3	2	255.9	227.0
4	6	280.4	251.1
5	10	260.8	220.4
6	4	256.9	219.6
7	3	255.8	224.1

TABLE 13

ESTIMATION OF EQUATIONS (43) AND (44) WITHOUT ANY CROSS EQUATION RESTRICTIONS.

CITY	FUNCTION	N
Houston	67.10	57
Chicago	27.51	22
Cleveland	8.939	28
Indianapolis	16.43	19
Dallas	44.26	26
TOTAL	164.239	152

TABLE 14

POOLING THE CROSS SECTION DATA FOR ALL 5 CITIES

SOME OTHER FIXED EFFECT ASSUMPTIONS THAT I TRIED.

COMBINATION	DUMMY VARIABLES ON THE COEFFICIENTS OF THE FOLLOWING VARIABLES	
	EQUATION 43	EQUATION 44
8	V1 ,V2	dV3
9	V1, V3	dV2
10	V1, V2, V3	none
11	V3	dV3
12	intercept	dI
13	intercept	da, dI

TABLE 15

POOLING OF CROSS SECTION DATA FOR 4 CITIES (HOUSTON NOT INCLUDED)

THE FIXED EFFECT ASSUMPTIONS THAT I TRIED

VERSION OF THE MODEL THAT IS QUADRATIC IN THE NUMBER OF ROOMS.

COMBINATION	DUMMY VARIABLES ON THE COEFFICIENTS OF THE FOLLOWING VARIABLES	
	EQUATION 43	EQUATION 44
a	intercept	none
b	intercept, V1 square	dI, V1 square
c	intercept	intercept
d	V1 square	d[V1 square]
e	V1 square	da, V1 square
f	V1 square	V1 square

NOTE: the variable d[V1 square] equals V1 square minus its mean.

TABLE 16

POOLING OF CROSS SECTION DATA FOR THE 4 CITIES (HOUSTON NOT INCLUDED)

ESTIMATION OF A MODEL THAT IS NOT IMPLIED BY MY THEORY.

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-3.888989	1.085596	-3.582354
	(V1)(V2)	2086.851	319.5622	6.530343
	d1	211.0204	37.62419	5.608636
	d2	116.8344	30.72123	3.803051
	d3	94.71353	29.44482	3.216645
	d4	156.3945	32.46285	4.817647
	SIGMA	52.58693	3.815057	13.78405
EQUATION 2:	I	0.0002682911	0.000125149	2.143774
	a	0.74330240	0.0991034	7.497462
	C	2.180308	0.2850502	7.648856
	SIGMA	0.6598734	0.04762226	13.85641
	RHO	0.0143644	0.1981039	0.07250946

N = 96

FUNCTION = 170.1

TABLE 17

POOLING THE CROSS SECTION DATA FOR ALL FIVE CITIES

THE RESTRICTED MODEL

	VARIABLE	COEFFICIENT	STANDARD ERROR	T - STATISTIC
EQUATION 1:	V3	-0.1283015	0.3175333	-0.4040568
	V1	32.56683	7.4686649	4.360471
	V2	1331.408	898.9843	1.481014
	DV2 CHICAGO	-2152.440	1824.745	-0.1.179584
	DV2 CLEVELAND	-5132.951	1325.874	-3.871372
	DV2 INDIANAPOLIS	-6168.475	1613.227	-3.823686
	DV2 DALLAS	-2067.615	873.7504	-2.366368
	C	70.28733	30.33801	2.316807
	SIGMA	70.63500	4.716114	14.97737
EQUATION 2:	dV2	40.88233		
	DdV2 CHICAGO	-66.09302		
	DdV2 CLEVELAND	-157.6129		
	DdV2 INDIANAPOLIS	-189.4097		
	DdV2 DALLAS	-63.48837		
	dV3	-0.003939637		
	da	-0.8094912	0.07088658	-11.41953
	dI	-0.00006090387	0.000005492406	-11.08874
	SIGMA	0.5270966	0.03620903	14.55705
	RHO	0.03928353	0.8944206	0.04392065

N = 152

FUNCTION = 280.4

Note: t - statistics and standard errors have not been computed for the restricted parameters.

TABLE 18

THE PRESENT VALUES FOR ALL FIVE CITIES THAT ARE IMPLIED BY THE STRUCTURAL MODEL - GIVEN THE ASSUMPTION THAT NONE OF THE EXOGENOUS PARAMETERS OF THE MODEL WILL CHANGE IN THE NEXT FIFTEEN YEARS

	PRESENT VALUE
HOUSTON	$ 229,900
CHICAGO	$ 196,520
CLEVELAND	$ 149,370
INDIANAPOLIS	$ 134,210
DALLAS	$ 197,790
TOTAL	$ 907,790

TABLE 19

POOLING THE CROSS SECTION DATA FOR ALL FIVE CITIES

AN EMPIRICAL MODEL

	VARIABLE	COEFFICIENT	STANDARD ERROR	T-STATISTIC
EQUATION 1:	V3	-2.456470	1.006203	-2.441327
	V1	15.89273	6.901100	2.302927
	V2	12629.29	2082.937	6.063212
	DV2 CHICAGO	2124.967	1340.099	1.585680
	DV2 CLEVELAND	-4466.044	1326.619	-3.366486
	DV2 INDIANAPOLIS	-1899.818	996.6019	-1.906295
	DV2 DALLAS	5718.168	9136.521	0.6258584
	C	24.29229	41.24524	0.5889720
	SIGMA	64.17864	3.680897	17.43560
EQUATION 2:	dV2	-11.00449	23.88285	-0.4607694
	DdV2 CHICAGO	-71.16296	60.10029	-1.184070
	DdV2 CLEVELAND	4.669519	42.45915	0.1099767
	DdV2 INDIANAPOLIS	-24.39432	109.7109	-0.2223510
	DdV2 DALLAS	-123.5883	37.66427	-3.281314
	dV3	0.006899758	0.009387393	0.7350025
	da	-0.8801964	0.06375797	-13.80527
	dI	-0.00005603019	0.000009169505	-6.110492
	SIGMA	0.4780662	0.02750962	17.37815
	RHO	0.06291262	7.383495	0.00852071

N = 152

FUNCTION = 251.1

TABLE 20

THE PRESENT VALUES FOR ALL FIVE CITIES

THAT ARE IMPLIED BY AN EMPIRICAL MODEL

	PRESENT VALUES
HOUSTON	$ 223,280
CHICAGO	$ 256,190
CLEVELAND	$ 154,030
INDIANAPOLIS	$ 193,780
DALLAS	$ 311,970
TOTAL	$ 1,139,250

Note: the empirical model implicitly assumes that none of the exogenous parameters of the model will change in the next fifteen years.

TABLE 21

MEAN INCOME IN THE FIVE CITIES

INCOME IN $

YEAR	HOUSTON	CHICAGO	CLEVELAND	INDIANAPOLIS	DALLAS
1980	15954	16928	10784	13889	14567
1981-85	17549	18621	11862	15278	16024
1986-90	19304	20483	13049	16806	17626
1991-95	21235	22531	14354	18486	19389

TABLE 22

PRESENT VALUE FOR EACH CITY GIVEN A 10%

INCREASE IN THE MEAN INCOME EVERY 5 YEARS

	PRESENT VALUE
HOUSTON	$ 446,350
CHICAGO	$ 426,050
CLEVELAND	$ 280,540
INDIANAPOLIS	$ 322,700
DALLAS	$ 395,500
TOTAL	$ 1,871,140

TABLE 23

AN ALTERNATIVE ESTIMATION PROCEDURE

FOR ALL FIVE CITIES

THE FIRST EQUATION

VARIABLE	COEFFICIENT	STANDARD ERROR	T - STATISTIC
V3	-4.557341	9.261684	-4.920640
V1	19.48912	6.027161	3.233549
V2	12785.74	1767.226	7.234924
DV2 CHICAGO	1041.140	1177.005	0.8845673
DV2 CLEVELAND	-6897.647	1205.995	-5.719468
DV2 INDIANAPOLIS	-8348.988	1192.448	-7.001555
DV2 DALLAS	-3670.402	896.7612	-4.092954
C	85.05893	36.63616	2.321720
SIGMA	55.87912	3.204888	17.43560

N = 152

FUNCTION = 127.2

TABLE 24

AN ALTERNATIVE ESTIMATION PROCEDURE

FOR ALL FIVE CITIES

THE SECOND EQUATION

VARIABLE	COEFFICIENT	STANDARD ERROR	T - STATISTIC
I	0.0001393595	0.0000202607	6.878315
a	0.1379081	0.133066	1.036389
C	-4.509668	0.5332514	-8.456927
DC CHICAGO	-0.8252417	0.2822666	-2.923625
DC CLEVELAND	1.401357	0.2914920	4.807532
DC INDIANAPOLIS	2.102350	0.3154746	6.664085
DC DALLAS	0.5582720	0.2640562	2.114217
SIGMA	1.101503	0.06317553	17.43560

N = 152

FUNCTION = 230.4

TABLE 25

AN ALTERNATIVE ESTIMATION PROCEDURE

FOR HOUSTON

THE FIRST EQUATION

VARIABLE	COEFFICIENT	STANDARD ERROR	T - STATISTIC
V3	-8.65003	1.74207	-4.965374
V1	-45.76947	10.76549	4.251498
V2	6701.209	3365.305	1.991263
C	172.2020	66.23065	2.600035
SIGMA	52.08527	6.221286	8.372107

N = 57

FUNCTION = 43.70

TABLE 26

AN ALTERNATIVE ESTIMATION PROCEDURE

FOR HOUSTON

THE SECOND EQUATION

VARIABLE	COEFFICIENT	STANDARD ERROR	T - STATISTIC
I	0.000172258	0.00002530637	6.806901
a	0.02581847	0.1270296	0.2032476
C	-1.522414	0.5712655	-2.664985
SIGMA	0.6462210	0.06402520	10.09323

N = 57

FUNCTION = 55.99

TABLE 27

THE MEAN HOUSEHOLD IN EACH ONE OF THE FIVE CITIES

	CHICAGO	HOUSTON	CLEVELAND	INDIANAPOLIS	DALLAS
V1	4.0	4.1	4.5	4.1	3.8
V2	0.0121	0.0141	0.0106	0.0129	0.0167
V3	31.6	26.0	24.0	20.5	22.4
a	1.95	2.50	1.84	1.85	2.19
I	16928	15954	10784	13889	15795

TABLE 28

RENT AND HOUSING QUALITY OF THE MEAN HOUSEHOLD IN EACH CITY

	RENT	HOUSING QUALITY
HOUSTON	227.73	7.43
CHICAGO	186.58	5.33
CLEVELAND	125.93	2.19
INDIANAPOLIS	128.91	2.34
DALLAS	209.03	6.45

TABLE 29

MEAN HOUSEHOLD'S ELASTICITIES

	HOUSTON	CHICAGO	CLEVELAND	INDIANAPOLIS	DALLAS
E(P,V1)	0.35	0.42	0.70	0.62	0.35
E(P,V2)	0.79	0.74	0.50	0.45	0.73
E(P,V3)	-0.52	-0.77	-0.87	-0.73	-0.49
E(h,I)	0.30	0.44	0.68	0.83	0.34
E(h,a)	0.046	0.050	0.116	0.109	0.047
E(h,V1)	0.77	0.54	2.05	1.75	0.59
E(h,V2)	1.25	1.61	1.47	1.26	1.21
E(h.V3)	-0.803	-1.36	-2.52	-2.01	-0.80

TABLE 30

THE BENEFIT OF THE MEAN HOUSEHOLD ESTIMATES IMPLIED BY THE PREVIOUS METHOD AND THE PARAMETER ESTIMATES OF TABLE 1

AIR QUALITY IMPROVEMENT	ANNUAL BENEFIT
1%	$ 2.79
5%	$ 13.95
10%	$ 27.90
70%	$ 195.27

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