Essays on Market/Mechanism Design

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All errors are my own.

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Chapter 1

Accommodating Cardinal, Ordinal and Mixed Preferences: An Extended Preference Domain for the Assignment Problem

Abstract

We extend the preference domain of the assignment problem to accommodate ordinal, cardinal and mixed preferences and thereby allow the mechanism designer to elicit different levels of information about individuals' preferences. Given a fixed preference relation over a finite set of alternatives, our domain contains preferences over lotteries that are monotonic, continuous and satisfy an independence axiom. Under a natural coarseness relation, the stochastic dominance relation is the coarsest element of the domain and represents *fully* ordinal preferences. Any von Neumann-Morgenstern expected utility preference is a finest element and represents *fully* cardinal preferences. The extended domain can be characterized by an expected multi-utility representation. Although it is possible to construct a mechanism in the extended domain where the agents with ordinal preferences don't have an incentive to deviate from truth telling, agents with cardinal preferences may deviate even if the deviations are restricted to ordinal preference reports.
Key Words: market design, cardinal preferences, ordinal preferences, preference reporting language, the assignment problem
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1.1 Introduction

A prevalent assumption of the market design literature is that the agents are equally able to report their preferences (or their types) regardless how complex the underlying allocation problem is. However, reporting preferences over very large finite sets as in the case of combinatorial allocation problems, or over infinitely many lotteries as in the case of probabilistic mechanisms are potentially challenging tasks for agents. In particular, consider a probabilistic mechanism which requires agents to report preferences over lotteries. One particular method is to ask for cardinal preferences over the finite set of objects. Agents would be asked to provide a vector of scores or ratings for each object which essentially determines the intensity of their preferences. Specifying the intensity of preferences is often not a straightforward task in market design environments, such as resident matching or school choice, as opposed to auctions where one can use willingness to pay as a measure of intensity of preferences. Essentially, the problem is that the agents preferences over lotteries may not be complete due to their lack of sophistication or purely normative reasons.[1]

The difficulty that agents may face when asked to report cardinal preferences is a well-known issue among practitioners and experimenters (see Kagel and Roth (1995)). Most recently, in a course allocation setup, Budish and Kessler (2016) show that the inconsistencies between the reported cardinal preferences and the revealed preferences through comparisons of outcomes of alternative mechanisms are not rare. According to their findings, 15.6% of subjects made mistakes in reporting their cardinal preferences.

One possible remedy would be to ask only ordinal preferences—that is rankings over objects. Unfortunately, abandoning cardinal preferences and focusing only on ordinal preferences potentially lead to an information loss which can be detrimental in terms of social welfare. Carroll (2018) shows that focusing only on ordinal mechanisms is not without loss of generality in the sense that a mechanism designer eliciting only ordinal preferences has to give up either efficiency or strategy-proofness if some agents have cardinal preferences.

The purpose of this chapter is to extend the preference domain of the assignment problem to allow agents to report their preferences in a varying degree of sophistication in terms of the intensity of their preferences. The domain we introduce includes usual cardinal preferences, ordinal preferences in the form of simple rankings and some *mixed* preferences that are neither fully cardinal nor fully ordinal. These mixed preferences essentially enable a richer preference reporting language where agents can choose to be selective about the objects for which they want to report

[1] A vast literature starting with Aumann (1962) and Bewley et al. (1986) discusses positive and normative reasons as to why the axiom of completeness may fail to hold in common economic environments.

intensities. To fix this idea, consider a resident who is asked to report preference over 3 hospitals, $\{h_1, h_2, h_3\}$. Assume that her rank order list is such that $h_1 \succ h_2 \succ h_3$, and additionally, she values her first choice much more than the other two, but she is unsure about how much she values her second choice relative to the third one. Now consider a preference reporting language requiring her to submit cardinal preferences via reporting scores between 0 and 1 for each object. Since she is not certain about the intensity of her preferences for the bottom two objects, she is forced to randomly choose scores for them. For example, she may end up reporting $(1, 0.2, 0)$ or $(1, 0.2, 0.1)$ since her true preference is coarser than a full cardinal report. Our mixed type allows the resident in this scenario to report an incomplete scoring vector such as $(1, 0.2, -)$. In the extended domain of this chapter, such a report may be mapped to a preference relation which would rank a lottery better than another if all fully cardinal utility vectors consistent with this incomplete vector agree on this ranking. An empty vector in this preference reporting language, therefore, would correspond to an ordinal preference—a pure rank order. Such a preference language enables the mechanism designer to improve upon an ordinal mechanism if there are agents with cardinal or mixed preferences (see Example 1).

The extended domain of the preferences is constructed using a complete and transitive relation over the finite set of objects and extending this relation to the lotteries over this set. Given the preferences over the set of objects, which we shall refer as ordinal preferences, we define inducements which are pre-orders over lotteries consistent with the ordinal preferences in the following sense: An inducement ranks a degenerate lottery better than another degenerate lottery if and only if the corresponding objects are ranked in the same order under the ordinal preferences. Then, we introduce a *monotonicity* notion for inducements requiring that given a compound lottery, if we construct another compound lottery by replacing some of the degenerate lotteries with better ranked degenerate lotteries, then the latter should be ranked better than the former. The extended domain includes all monotonic inducements for some ordinal preferences which also satisfy continuity and independence axioms. This domain includes the stochastic dominance order, which we refer to as the fully ordinal preference, as well as any von Neumann-Morgenstern (vNM) expected utility preferences which we refer to as the fully cardinal preferences. Under a natural coarseness relation defined over inducements, the stochastic dominance order is the unique minimal element of the extended domain, while any fully cardinal preference is a maximal element. In line with the expected multi-utility theorem of Dubra et al. (2004), we show that the set of inducements of an ordinal preference can be characterized as the meet of a set of fully cardinal preferences consistent with that ordinal preference. That is, each inducement of the extended domain can be represented via a set of utility vectors. This, in turn, allows us to use a preference reporting language allowing agents to submit multiple utility vectors, or alternatively, incomplete or set valued vectors.

We study the extended domain for the assignment problem of n objects to n agents. We say that a mechanism which only depends on ordinal preferences is an ordinal mechanism, while any other mechanism is called a cardinal mechanism. Ordinal mechanisms such as the Probabilistic Serial (PS) mechanism (Bogomolnaia and Moulin, 2001) or the Random Priority (RP) mechanism may not produce efficient allocations when some agents have cardinal preferences. One can produce an efficient cardinal mechanism from an ordinal mechanism which is efficient in the ordinal domain by modifying what the mechanism produces for the preference profiles outside the ordinal domain. The Pseudomarket Mechanism (PM) of Hylland and Zeckhauser (1979) can also be used in the extended domain whenever a pseudo-equilibrium exists. Given an inducement from the extended domain for each agent, a Pseudomarket equilibrium produces a probabilistic allocation which is individually optimal for each agent and clears the market. Since the inducements are weakly coarser than fully cardinal preferences, there may be a multiplicity of equilibria.

Allowing both ordinal and cardinal mechanisms, the extended domain enables us to compare the Probabilistic Serial mechanism and the Random Priority mechanism with the Pseudomarket mechanism. It turns out that neither PS nor RP outcome can always be supported by a Pseudomarket equilibrium where all agents have ordinal preferences and equal endowments [2]. With a Pseudomarket equilibrium from equal endowments, the demand for each object is the main determinant of the resulting allocation. Therefore, Pseudomarket mechanisms punish agents whose preferences are similar to each other more in comparison with the PS mechanism, as they need to compete over same objects. Incidentally, Pseudomarket mechanisms with equal endowments are not *ordinally fair* while the PS is ordinally fair (Hashimoto et al., 2014).[3] On the other hand, according to social welfare domination notion of Doğan et al. (2018), the PS solution is dominated by a Pseudomarket mechanism.[4]

The incentive problems of the mechanisms defined on the extended domain are not surprising. The impossibility result by (Zhou, 1990) applies to the extended domain as well: There is no probabilistic mechanism on the extended domain which is efficient, weakly strategy-proof and weakly envy-free for $n > 2$. To get a better sense of the tension between efficiency and incentives, we study what kind of deviations are the main culprit for efficient mechanisms to be incompatible

[2]By contrast, Kesten (2006) obtains PS solutions as a market mechanism based on a Top Trading Cycles procedure from equal endowments.

[3]A probabilistic mechanism is ordinally fair if an agent's surplus at an object can not exceed other agents' surplus at the same object if that agent may receive that object with positive probability.

[4]A probabilistic assignment sw-dominates another probabilistic assignment if the former is ex-ante optimal for a larger set of vNM utility profiles.

with truth-telling incentives. It turns out that one can construct an efficient and weakly envy-free mechanism which is weakly strategy-proof for the subset of agents with ordinal preferences. This means that even when there are other agents with cardinal preferences, there is an efficient and weakly envy-free mechanism for which agents with ordinal preferences have no incentive to misreport their preferences. This mechanism is an extension of the PS mechanism of Bogomolnaia and Moulin (2001). Therefore, this possibility result is an extension of the possibility result of Bogomolnaia and Moulin (2001) showing that the PS mechanism is weakly strategy-proof and efficient. This result enables us to conclude that using the extended domain is innocuous for incentive compatibility considerations as it preserves the sole possibility result of the ordinal domain.

Using the extended domain, we are able to ask certain questions that were not possible to be asked in the pure cardinal or pure ordinal domain. One such question is that whether cardinal deviations are essential for the stronger impossibility result in the cardinal domain. For example, if we only allow ordinal deviations, that is the deviations that require a change in rank order, would agents with cardinal preferences still be better off by deviating from their true preferences? Unfortunately, the answer is affirmative. For agents with cardinal preferences, even if we only allow ordinal deviations, there may be a deviation which is strictly better than truth-telling. This shows that for any mechanism, agents with sophisticated fully cardinal preferences may have incentive to appear less sophisticated and report only ordinal preferences. Indeed, for the Pseudomarket mechanism with equal endowments, the agents with fully cardinal preferences may be better of by reporting ordinal preferences.

Related Literature: This is the first paper studying mechanisms eliciting ordinal, mixed and cardinal preferences in the assignment problem.[5] While the difference among agents in terms of the type of strategies that they can come up with has been studied especially in the matching literature extensively (see Pathak and Sönmez (2008)), the differences in terms of the type of preferences has been relatively understudied. Recently, Carroll (2018) studies the shortcomings of the ordinal mechanisms when the agents have cardinal preferences. Ehlers et al. (2020) shows that in a voting setting cardinal preference information can be ignored. On the other hand, Kim (2017) reiterates Carroll (2018) by showing that cardinal mechanisms can improve upon ordinal mechanisms in a voting setting where agents' preferences are private information. Lastly, this paper also relates to the burgeoning literature on market mechanisms in the assignment problem (see Gul et al. (2019), Le (2017) and Miralles and Pycia (2020)).

[5]The closest paper is due to Fisher (2018) where the true preference of every agent is cardinal, but only a subset of agents know their true cardinal preferences while the rest is endowed with a prior over the set of consistent utility vectors.

1.2 Extended Preference Domain

In this section, we introduce the extended domain and provide a characterization for this domain which is similar to the expected multi-utility theorem of Dubra et al. (2004). Using this characterization, we describe a preference reporting language that allows agents to report their preferences from this domain. Fixing a single agent, we start with a preference relation over the finite set of objects and then extend this preference relation, which is referred as ordinal preferences, to a preference relation over lotteries.[6]

Let $\succsim$ be a complete and transitive preference relation over a set of n objects, A. Let $\succ$ denote the asymmetric part of $\succsim$ and $\sim$ denote the symmetric part. A random allocation P is a probability distribution over the set A that is $P \in \triangle(A)$ and $P_a \in [0,1]$ denotes the probability for receiving object $a \in A$. We want to define a pre-order over the set of probability distributions $\triangle(A)$ which is consistent with the preference relation $\succsim$. Any reasonable consistency concept between a pre-order over $\triangle(A)$ and the ordinal preference $\succsim$ requires that the degenerate distributions yielding a certain object with probability one to be ranked same way as those objects are ranked under $\succsim$. We define the notion of inducement following this idea: Let I be a mapping between preference relations over the non-empty set A and pre-orders over the set $\triangle(A)$. I is called an **inducement** of the ordinal preference $\succsim$ if $a \succ b$ implies $l_a I(\succ) l_b$ and $a \sim b$ implies $l_a I(\sim) l_b$ where l_a and l_b denote the degenerate lotteries that surely awards objects a and b respectively. We shorthand $I(\succsim)$ as $\succsim^I$.

Definition of inducement says nothing about the non-degenerate lotteries. We employ a monotonicity notion very similar to Anscombe et al. (1963) so that we can use degenerate lotteries to compare at least some of the non degenerate lotteries. We say that the inducement $\succsim^I$ is a **monotonic** inducement of the ordinal preference $\succsim$, if for every $\{a_1, \cdots, a_k\}, \{a'_1, \cdots, a'_k\} \subseteq A$ with $a_m \succsim a'_m$ for each $m \in \{1, ..., k\}$ and for every $(\lambda_1, \cdots, \lambda_k) \in \triangle^{k-1}$, we have $\sum_m \lambda_m l_{a_m} \succsim^I \sum_m \lambda_m l_{a'_m}$ and if, in addition, there exists $m' \in \{1, \cdots, k\}$ with $a_{m'} \succ a'_{m'}$ and $\lambda_{m'} > 0$, we have $\sum_m \lambda_m l_{a_m} \succ^I \sum_m \lambda_m l_{a'_m}$.

Monotonicity basically says that a compound lottery of degenerate lotteries is preferred to another compound lottery of degenerate lotteries which is constructed by replacing some of the degenerate lotteries with less preferred ones. It corresponds to monotonicity of preferences with respect to the probability of the preferred outcome when $n = 2$. Furthermore it is equivalent to the idea that given any lottery, one should get a weakly better lottery by increasing the probability of a

[6] One may, of course, introduce this domain without relying onto underlying ordinal preferences. We build upon the ordinal preferences to highlight the fact that cardinal preferences are merely additional information accompanying ordinal preferences.

desired outcome while decreasing the probability of a less desired one by the same amount:

Proposition 1. *Let $\succsim^I$ be an inducement of the ordinal preference $\succsim$. $\succsim^I$ is a monotonic inducement if and only if for each $P, P' \in \triangle(A)$ with $P_a > P'_a$, $P_b < P'_b$ for some $a, b \in A$ and $P_c = P'_c$ for every $c \in A\backslash\{a, b\}$, $a \succ b$ implies $P \succ^I P'$ and $a \sim b$ implies $P \sim^{\mathcal{I}} P'$.*

Proof. Suppose that for each $P, P' \in \triangle(A)$ with $P_a > P'_a$, $P_b < P'_b$ for some $a, b \in A$ and $P_c = P'_c$ for every $c \in A\backslash\{a, b\}$, $a \succ b$ implies $P \succ^I P'$ and $a \sim b$ implies $P \sim^{\mathcal{I}} P'$. Let $\{a_1, \cdots, a_k\}, \{a'_1, \cdots, a'_k\} \subseteq A$ with $a_m \succsim a'_m$ for each $m \in \{1, \cdots, k\}$ and let $\lambda \in \triangle^{n-1}$. Let $P_0 := \sum_i \lambda_i l_{a_i}$ and $P_k := \sum_i \lambda_i l_{a'_i}$. Furthermore, for $m \in \{1, \cdots, k-1\}$, define $P_m \in \triangle(A)$ recursively such that $P_{ma_m} = P_{m-1a_m} - \lambda_m$, $P_{ma'_i} = P_{m-1a'_m} + \lambda_m$ and $P_{mc} = P_{m-1c}$ for each $c \in A\backslash\{a_m, a'_m\}$. By construction, we have $P_{m-1} \succsim^I P_i$. By transitivity, therefore, $P_0 \succsim^I P_k$. Furthermore if $a_j \succ a'_j$ and $\lambda_j > 0$ for some $j \in \{1, ..., k\}$, then $P_{j-1} \succ^I P_j$, thus $P_0 \succ^I P_k$. Proof of the converse of the statement is straightforward. $\square$

The property above is a necessary and sufficient condition for the first order stochastic dominance. Therefore, the weakest monotonic inducement of a preference relation defines the first order stochastic dominance relation. To make this statement precise, we first define a partial order over the set of pre-orders over the set $\triangle(A)$. Given two pre-orders $\succsim^I$ and $\succsim^{I'}$ defined over the set $\triangle(A)$, we say that $\succsim^I$ is **coarser** than $\succsim^{I'}$ if for each $P, P' \in \triangle(A)$, $P \succsim^I P'$ implies $P \succsim^{I'} P'$. Alternatively we can say that $\succsim^I$ is coarser than $\succsim^{I'}$ if the upper (lower) contour sets with respect to $\succsim^I$ are subsets of the upper (lower) contour sets with respect to $\succsim^{I'}$ respectively. That is let $L_{\succsim^I}(P) := \{P' \in \triangle(A) | P' \succsim^I P\}$ and $U_{\succsim^I}(P) := \{P' \in \triangle(A) | P \succsim^I P'\}$, we say that $\succsim^I$ is coarser than $\succsim^{I'}$ if:

$$\forall P \in \triangle(A), \quad L_{\succsim^I}(P) \subseteq L_{\succsim^{I'}}(P) \text{ or equivalently } U_{\succsim^I}(P) \subseteq U_{\succsim^{I'}}(P)$$

According to above defined coarseness relation, an inducement is coarser than another inducement if the latter obeys the former whenever two lotteries are comparable according to the former inducement. Naturally, if $\succsim^I$ is coarser than $\succsim^{I'}$ we say that $\succsim^I$ is **finer** than $\succsim^{I'}$. Obviously if an inducement is finer than a complete inducement, than that inducement is equal to that complete inducement. That is, there can be no inducement which is strictly finer than a complete inducement. Thus, we may say a coarser inducement is less capable of comparing lotteries than a finer inducement, or simply *less complete*. An agent endowed with a coarser inducement than another agent's inducement either agrees with the latter's choice between two lotteries or is unable to compare the two. Therefore one may interpret having coarser preference relation as being less sophisticated or less precise about how to compare lotteries.

Now we define the stochastic dominance order as an inducement. Given a preference relation $\succsim$ over A, **the stochastic dominance order** $\succsim^{sd}$ over $\triangle(A)$ is an inducement of the ordinal preference $\succsim$ and is defined such that for each $P, P' \in \triangle(A)$, $P \succsim^{sd} P'$ if and only if for every $a \in A$, we have $\sum_{b \succsim a} P_b \geqslant \sum_{b \succsim a} P'_b$.

Proposition 2. *$\succsim^{sd}$ is a monotonic inducement of the ordinal preference $\succsim$. Furthermore $\succsim^{sd}$ is the unique coarsest monotonic inducement of the ordinal preference $\succsim$.*

Proof. First statement is trivial. Since for every $P, P' \in \triangle(A)$, $P \sim^{sd} P'$ implies $P = P'$, it suffices to show that for every inducement $\succsim^I$ of $\succsim$, $P \succ^{sd} P'$ implies $P \succ^{\mathcal{I}} P'$ to show that $\succsim^{sd}$ is coarser than every monotonic inducements of the ordinal preference $\succsim$.

Let $P, P' \in \triangle(A)$ such that $P \succ^{sd} P'$. When $n = 2$, we directly have $P \succ^I P'$. Assume $n > 2$. Enumerate the objects in A such that $A = \{a_1, a_2, ..., a_n\}$ where $a_i \succsim a_j$ implies $i \leqslant j$. Let $\{P^m\}_{m=1}^{n-2}$ be a finite sequence of probabilistic allocations such that:

$$P^m_{a_j} := \begin{cases} P'_{a_j} & \text{if } \ j \leqslant m \\ P_{a_{m+1}} + \sum_{k=1}^{m}(P_{a_m} - P'_{a_m}) & \text{if } \ j = m+1 \\ P_{a_j} & \text{if } \ j > m+1 \end{cases}$$

By construction, it follows that for each monotonic inducement $\succsim^I$ of $\succsim$, we have $P \succsim^I P^1$, $P^1 \succsim^I P^2$, $\cdots$, $P^{n-3} \succsim P^{n-2}$ and $P^{n-2} \succsim P'$ Furthermore $P \succ^I P^1$ or $P^{n-2} \succ^I P'$ or $P^i \succ^I P^{i+1}$ for some $i \in \{1, ..., n-3\}$ since otherwise we must have $a_1 \sim a_2 \sim \cdots \sim a_{n-1} \sim a_n$ contradicting the fact that $P \succ^{sd} P'$. Then transitivity implies $P \succ^I P'$.

Uniqeness follows from the fact that if there is another monotonic inducement $\succsim^{I'}$ which is coarser than every other monotonic inducement, then $\succsim^{I'}$ is coarser than $\succsim^{sd}$ and $\succsim^{sd}$ is coarser than $\succsim^{I'}$ which implies $\succsim^{I'} = \succsim^{sd}$. ☐

Above result tells us that the stochastic dominance order is the weakest preference relation among the monotonic inducements. There are of course many stronger monotonic inducements that agrees with the stochastic dominance order whenever the stochastic dominance order can compare two lotteries and be able to compare lotteries that can't be compared by the stochastic dominance order. One important class of such monotonic inducements is the class of von Neumann-Morgenstern orders which we refer as fully cardinal preferences. Given an ordinal preference $\succsim$ over A, a **von Neumann-Morgenstern (vNM) order** $\succsim^u$ over $\triangle(A)$ is defined such that there exists $u \in \mathbb{R}^n_+$ with $u_i \geqslant u_j \ \Leftrightarrow a_i \succsim a_j$ such that $P \succsim^v P' \ \Leftrightarrow \ u \cdot P \geqslant u \cdot P'$. It is clear that, a vNM order is finer than the stochastic dominance order.

Lemma 1. *Fix an ordinal preference $\succsim$ over A. Any vNM order $\succsim^v$ is a monotonic inducement of $\succsim$. Furthermore, if a monotonic inducement $\succsim^I$ is finer than a vNM order $\succsim^v$, then $\succsim^I = \succsim^v$.*

Proposition 2 and lemma 1 shows that the coarseness relation over the domain of monotonic inducements has a unique minimal element which is the stochastic dominance relation and a vNM order is a maximal element. There are of course other maximal inducements which aren't vNM orders such as the lexicographic order. In this chapter, however, we restrict our attention to monotonic inducements which also satisfy the axioms continuity and independence of irrelevant alternatives. These axioms help us to characterize the preference domain as the meet of a set of fully cardinal preferences.

Let $\succsim^I$ be an inducement of the ordinal preference $\succsim$, we say that $\succsim^I$ is/satisfies:

- **Continuity**[7] if $U_{\succsim^I}(P)$ and $L_{\succsim^I}(P)$ are closed for each $P \in \triangle(A)$.
- **Independence of irrelevant alternatives** if for each $P, P', P'' \in \triangle(A)$ with $P \succsim^I P'$ and $\forall \lambda \in (0, 1]$, we have $\lambda P + (1-\lambda)P'' \succsim^I \lambda P' + (1-\lambda)P''$.

Let $\mathcal{M}(\succsim)$, $\mathcal{MC}(\succsim)$ and $\mathcal{MIC}(\succsim)$ denote the set of monotonic inducements of $\succsim$, the set of monotonic and continuous inducements of $\succsim$ and the set of monotonic and continuous inducements of $\succsim$ satisfying the independence axiom respectively. Obviously, $\mathcal{M}(\succsim) \subseteq \mathcal{MC}(\succsim) \subseteq \mathcal{MIC}(\succsim)$. The coarseness relation over the monotonic inducements constitutes a meet semi-lattice:

Proposition 3. *The set of monotonic and continuous inducements of an ordinal preference $\succsim$, $\mathcal{M}(\succsim)$, is a partially ordered set under the coarseness relation. Furthermore $\mathcal{M}(\succsim)$ is a meet-semilattice where the meet of $\succsim^I$ and $\succsim^{I'}$, $\succsim^I \bigwedge \succsim^{I'} := \succsim^{I \wedge I'}$, is defined such that:*

$$\forall P, P' \in \triangle(A) \;\; P \succsim^{I \wedge I'} P' \;\Leftrightarrow\; P \succsim^I P' \text{ and } P \succsim^{I'} P'$$

Proof. Since the coarseness is equivalently described trough an inclusion relation over the upper and lower contour sets, it is a partial order over inducements. Furthermore clearly $L_{\succsim^{I\wedge I'}}(P) = L_{\succsim^I}(P) \cap L_{\succsim^{I'}}(P)$ and $U_{\succsim^{I\wedge I'}}(P) = U_{\succsim^I}(P) \cap U_{\succsim^{I'}}(P)$ for each $P \in \triangle(A)$, therefore $\succsim^{I\wedge I'}$ is coarser than both $\succsim^I$ and $\succsim^{I'}$. Furthermore for each $P \in \triangle(A)$, if $S \subseteq L_{\succsim^I}(P)$ and $S \subseteq L_{\succsim^{I'}}(P)$, then $S \subseteq L_{\succsim^I}(P) \cap L_{\succsim^{I'}}(P)$. Similarly, if $S \subseteq U_{\succsim^I}(P)$ and $S \subseteq U_{\succsim^{I'}}(P)$, then $S \subseteq U_{\succsim^I}(P) \cap U_{\succsim^{I'}}(P)$. Then any inducement coarser than both $\succsim^I$ and $\succsim^{I'}$ is also coarser than $\succsim^{I\wedge I'}$, implying that $\succsim^{I\wedge I'}$ is the meet of $\succsim^I$ and $\succsim^{I'}$. □

[7] Here we use the Herstein Milnor concept of continuity as opposed to the usual continuity axiom of vNM utility theorem. When a preference is complete and satisfies the independence axiom, the two are equivalent (Aumann (1962) and Karni (2007)). Since this chapter introduce a preference domain including some preferences failing completeness, we assume Herstein Milnor continuity.

Since upper and lower contour sets of the meet of a collection of inducements is the intersection of the respective sets of the collection, the meet preserves monotonicity and continuity as well as the independence axiom.

The extended domain that this chapter focuses on is the domain of the monotonic and continuous inducements satisfying the independence axiom. By the expected multi-utility theorem of Dubra et al. (2004), we know that the continuity and independence axioms are necessary and sufficient for expected multi-utility representation. Our characterization translates this results to the domain of inducements and states that $\mathcal{MIC}(\succsim)$ is equivalent to meets of arbitrary collections of vNM orders coarser than $\succsim^{sd}$. Therefore, the preference of an agent in this domain admits an expected multi-utility representation:

Theorem 1. *An inducement of an ordinal preference is monotonic, continuous and satisfies independence if and only if it is the meet of a collection of vNM orders finer than that inducement. That is, let $\succsim^I$ be an inducement of the ordinal preference $\succsim$. $\succsim^I \in \mathcal{MIC}(\succsim)$ if and only if $\succsim^I = \bigwedge_{u \in V} \succsim^u$ where $\{\succsim^u | u \in V\}$ is the set of all vNM orders finer than $\succsim^I$.*

Proof. Since the meet preserves continuity and independence, an inducement which is equal to the meet of every vNM order finer than that inducement satisfies continuity and independence. For the converse of this statement, let $\succsim^I \in \mathcal{MIC}(\succsim)$ for an ordinal preference $\succsim$. First observe that if $\forall a, b \in A$, $a \sim b$ then the statement holds trivially. Assume $\exists a, b \in A$, $a \succ b$ implying that relative interior of $\triangle(A)$ is non-empty. We prove this theorem by three claims:

Claim 1: For each P in the relative interior of $\triangle(A)$, there exists $u \in \triangle^{n-1}$ such that for each $P_1, P_2 \in \triangle(A)$, $P_1 \succsim^I P \succsim^I P_2$ implies $u \cdot P_1 \geqslant u \cdot P_2$.

Let $X := \{x \in \mathbb{R}^n \,|\, \sum_i x_i = 0\}$. $X \subset \mathbb{R}^n$ is an $n-1$ dimensional vector space where $\triangle(A) - \frac{1}{n}e$[8] $\subset X$. Suppose $P - \frac{1}{n}e$ is in the relative interior of $\triangle(A) - \frac{1}{n}e \subset X$. Sets $\tilde{U}_{\succsim^I}(P) := \{P' - \frac{1}{n}e \in \triangle(A) - \frac{1}{n}e | P' \succsim^I P\}$ and $\tilde{L}_{\succsim^I}(P) := \{P' - \frac{1}{n}e \in \triangle(A) - \frac{1}{n}e | P \succsim^I P'\}$ are both convex since $\succsim^I$ satisfies independence. Now assume the intersection of the relative interior of $\tilde{U}_{\succsim^I}(P)$ and the relative interior of $\tilde{L}_{\succsim^I}(P)$ is non-empty, that is relint$(\tilde{U}_{\succsim^I}(P)) \cap$ relint$(\tilde{L}_{\succsim^I}(P)) \neq \varnothing$. Let $P'' - \frac{1}{n}e \in$ relint$(\tilde{U}_{\succsim^I}(P)) \cap$ relint$(\tilde{L}_{\succsim^I}(P))$. Obviously $P'' \sim^I P$ and for each $P_u \in \tilde{U}_{\succsim^I}(P)$ and $P_l \in \tilde{L}_{\succsim^I}(P)$ there exists $\lambda_u, \lambda_l > 1$ such that $\lambda_u P'' + (1-\lambda_u)P_u \in \tilde{U}_{\succsim^I}(P)$ and $\lambda_l P'' + (1-\lambda_l)P_l \in \tilde{L}_{\succsim^I}(P)$. Let $P_u \in \tilde{U}_{\succsim^I}(P)$ and $P_l \in \tilde{L}_{\succsim^I}(P)$ such that $P_u \succ^I P$ and $P \succ^I P_l$ which is possible since $P \in$ relint$(\triangle(A))$ and $\succsim^I$ is monotonic with $a \succ b$ for some $a, b \in A$. Let $P'_u := \lambda_u P'' + (1-\lambda_u)P_u$ and $P'_l := \lambda_l P'' + (1-\lambda_l)P_l$. Since $P'_u \in \tilde{U}_{\succsim^I}(P)$ and $P'_l \in \tilde{L}_{\succsim^I}(P)$ we have $P'_u \succsim^I P \sim^I P'' \succsim^I P'_l$. Let $\delta_u := 1/\lambda_u$ and $\delta_l := 1/\lambda_l$, we have $\delta_u P'_u + (1-\delta_u)P_u =$

[8] $e := (1, \cdots, 1) \in \mathbb{R}^n$

$\delta_l P'_l+(1-\delta_l)P_l = P''$. But $P_u \succ^I P''$ and $P'' \succ^I P_l$ implies $\delta_u P'_u+(1-\delta_u)P_u \succ^I \delta_l P'_l+(1-\delta_l)P_l$, which is a contradiction. Hence relint$(\tilde{U}_{\succsim^I}(P)) \cap$ relint$(\tilde{L}_{\succsim^I}(P)) = \varnothing$.

Then by separating hyperplane theorem, there exists a vector $u' - \frac{1}{n}e \in X$ such that for each $P' \in \triangle(A)$, $P' \succsim P$ implies $(u' - \frac{1}{n}e) \cdot P' \geqslant (u' - \frac{1}{n}e) \cdot P'$ and $P \succsim P'$ implies $(u' - \frac{1}{n}e) \cdot P \geqslant (u' - \frac{1}{n}e) \cdot P$. This implies for some sufficiently small $\lambda \in (0,1)$, letting $u := \lambda u' + (1-\lambda)\frac{1}{n}e$ we have that $P' \succsim^I P$ implies $u \cdot P' \geqslant u \cdot P$ and $P \succsim P'$ implies $u \cdot P \geqslant u \cdot P'$ where $u \in \triangle^{n-1}$. Hence for each $P_1, P_2 \in \triangle(A)$, $P_1 \succsim^I P \succsim^I P_2$ implies $u \cdot P_1 \geqslant u \cdot P_2$.

Claim 2: For any $P_1, P_2 \in \triangle(A)$ with $P_1 \succsim^I P_2$, there exists $u \in \triangle^{n-1}$ such that $u \cdot P_1 \geqslant u \cdot P_2$ and for each $P_1, P_2 \in \triangle(A)$ with $P_1 \succ^I P_2$, $u \cdot P_1 > u \cdot P_2$.

Let $P_1, P_2 \in \triangle(A)$. Take $P \in$ relint$(\triangle(A))$. There exists $\lambda > 1$ such that $P_3 := \lambda P + (1-\lambda)P_1 \in \triangle(A)$. Then for $\delta := 1/\lambda \in [0,1]$, $P = \delta P_3 + (1-\delta)P_1$. By independence, $P = \delta P_3 + (1-\delta)P_1 \succsim^I \delta P_3 + (1-\delta)P_2$. Then by Claim 1 above, we have there exists $u \in \triangle^{n-1}$ such that $u \cdot P \geqslant u \cdot (\delta P_3 + (1-\delta)P_2)$ that is $u \cdot (\delta P_3 + (1-\delta)P_1) \geqslant u \cdot (\delta P_3 + (1-\delta)P_2)$. Then we have $u \cdot P_1 \geqslant u \cdot P_2$. Hence we get that for any $P_1, P_2 \in \triangle(A)$, $P_1 \succsim^I P_2$ implies $u \cdot P_1 \geqslant u \cdot P_2$.

Assume there is $P_1 \succ^I P_2$ with $u \cdot P_1 \leqslant u \cdot P_2$. Since $P_1 \succsim^I P_2$ implies $u \cdot P_1 \geqslant u \cdot P_2$, we have $u \cdot P_1 = u \cdot P_2$. If $P_1 \sim^I l_a$ where $a \in \max_{\succsim} A$, then $u \cdot P_1 = u \cdot l_a = u \cdot P_2$. Since $\succsim^I$ is monotonic, $u \cdot l_a = u \cdot P_2$ implies $P_2 \sim^I l_a$ which contradicts $P_1 \succ^I P_2$. Alternatively suppose $l_a \succ^I P_1$ so that $u \cdot l_a > u \cdot P_1$. Then by continuity of $\succsim^I$, there is $\varepsilon \in (0,1)$ such that $P_1 \succ \varepsilon l_a + (1-\varepsilon)P_2$. But $u \cdot P_1 < u \cdot (\varepsilon l_a + (1-\varepsilon)P_2)$ since $u \cdot P_1 < u \cdot l_a$. Contradiction. Then $u \cdot P_1 > u \cdot P_2$.

Claim 3: $\succsim^I = \bigwedge_{u \in V} \succsim^u$ where V represents the set of all monotonic vNM orders finer than $\succsim^I$.

Observe that monotonicity of $\succsim^I$ implies that $u_a \geqslant u_b \;\Leftrightarrow a \succsim b$. This is because $l_a \succsim^I l_b$ implies $u_a \geqslant u_b$. Then above claims show that a monotonic, continuous inducement satisfying independence is coarser than some vNM order defined by $u \in \triangle^{n-1}$. Now we need to show that $\succsim^I = \bigwedge_{u \in V} \succsim^{vNM(u)}$ where V represents the set of all monotonic vNM orders finer than $\succsim^I$. Assume not true. By definition, $\succsim^I$ is coarser than $\bigwedge_{u \in V} \succsim^{vNM(u)}$ which means that $P \succsim^I P'$ implies $P \bigwedge_{u \in V} \succsim^{vNM(u)} P'$.

Assume for some $P, P' \in \triangle(A)$, $P \bigwedge_{u \in V} \succsim^{vNM(u)} P'$ but $\neg(P \succsim^I P')$. Since $\succsim^I$ is coarser, we have $\neg(P' \succsim^I P)$. Since two lotteries remain incomparable when same convex combination with an interior lottery is taken due to continuity and independence, without loss of generality we may let $P, P' \in$ int$(\triangle(A))$. Since $P' \in \triangle(A) \backslash (U_{\succsim^I}(P) \cup L_{\succsim^I}(P))$, and since by continuity $\triangle(A) \backslash (U_{\succsim^I}(P) \cup L_{\succsim^I}(P))$ is open, there is $\varepsilon > 0$ such that $B_\varepsilon(P') \subset \triangle(A) \backslash (U_{\succsim^I}(P) \cup L_{\succsim^I}(P))$. By monotonicity we can find $P_u, P_d \in B_\varepsilon(P')$ with $P_u \succ^{sd} P' \succ^{sd} P_d$. Let $U_1 := U_{\succsim^I}(P)$ and $L_1 :=$ co$(L_{\succsim^I}(P) \cup \{P_u\})$. Assume relint$(U_1) \cap$ relint$(L_1) \neq \varnothing$, then there is $P'' \in$ relint$(U_1) \cap$

relint(L_1). Since $P'' \in$ relint(U_1), we have $P'' \succsim^I P$ and since $P'' \in$ relint(L_1), there is $P_{dd} \in L_{\succsim^I}(P)$ and $\lambda \in [0,1]$ with $P'' = \lambda P_{dd} + (1-\lambda)P_u$. By independence, we have $P_{dd} \in L_{\succsim^I}(P)$, $\lambda P + (1-\lambda)P_u \succsim^I \lambda P_{dd} + (1-\lambda)P_u = P'' \succsim^I P$. Then $\lambda P + (1-\lambda)P_u \succsim^I P$ and again by independence we have $P_u \succsim^I P$ which contradicts the fact $P_u \in B_\varepsilon(P') \subset \triangle(A) \backslash (U_{\succsim^I}(P) \cup L_{\succsim^I}(P))$. Hence we have relint$(U_1) \cap$ relint$(D_1) = \varnothing$. Then by separating hyperplane theorem we have $u_1 \in \mathbb{R}^n$ such that $u_1 P_0 \geqslant u_1 P_1$ for each $P_0 \in U_1$ and $P_1 \in L_1$. Since u_1 also separates $U_{\succsim^I}(P))$ and $L_{\succsim^I}(P)$, claim 2 implies $\succsim^{u_1}$ is coarser than $\succsim^I$. Furthermore by construction $u_1 \cdot P \geqslant u_1 \cdot P_u > u_1 \cdot P'$. Similarly let $L_2 = L_{\succsim^I}(P)$ and $U_2(P) = $ co$(U_{\succsim^I}(P) \cup \{P_d\})$. Using similar arguments, we can show that there is $u_2 \in \mathbb{R}^n$ such that $u_2 P_0 \geqslant u_2 P_1$ for each $P_0 \in U_2$ and $P_1 \in L_2$. Again, since u_2 also separates $U_{\succsim^I}(P))$ and $L_{\succsim^I}(P)$, claim 2 implies $\succsim^{u_2}$ is coarser than $\succsim^I$. Furthermore, by construction $u_2 \cdot P' > u_2 \cdot P_d \geqslant u_2 \cdot P$. But then, we can't have $P \bigwedge_{u \in V} \succsim^{vNM(u)} P'$. Hence we conclude that $\succsim^I = \bigwedge_{u \in V} \succsim^u$ where V represents the set of all monotonic vNM orders finer than $\succsim^I$. $\square$

Finally, we can identify the minimal element and maximal elements of the domain:

Corollary 1. *Let $\succsim$ be an ordinal preference.*

- *The stochastic dominance order $\succsim^{sd}$ is the minimal element of $\mathcal{MIC}(\succsim)$ with respect to the coarseness relation.*
- *$\succsim^I \in \mathcal{MIC}(\succsim)$ is a maximal element of $\mathcal{MIC}(\succsim)$ with respect to the coarseness relation if and only if $\succsim^I$ is a vNM order whcih is an inducement of $\succsim$.*

The corollary above states that the domain $\mathcal{MIC}(\succsim)$ contains the stochastic dominance order as its unique minimal element and all monotonic vNM orders as its maximal elements. Furthermore the coarseness relation on $\mathcal{MIC}(\succsim)$ can be described in terms of the Bernouilli utility vector sets that characterize each inducement:

Proposition 4. *Let $\succsim^I, \succsim^{I'} \in \mathcal{MIC}(\succsim)$ with $\succsim^I = \bigwedge_{u \in V} \succsim^u$ and $\succsim^{I'} = \bigwedge_{u \in V'} \succsim^u$. $\succsim^I$ is coarser than $\succsim^{I'}$ if and only if co$(V) \supseteq$ co(V').*

Proof. If part is straightforward. For the only if part, firstly observe that $\bigwedge_{u \in \text{co}(V)} \succsim^u$ is the maximal representation of $\bigwedge_{u \in V} \succsim^u$ in the sense that $\bigwedge_{u \in V} \succsim^u = \bigwedge_{u \in \text{co}(V)} \succsim^u$ and if $\bigwedge_{u \in V} \succsim^u = \bigwedge_{u \in S} \succsim^u$ then $S \subseteq$ co(V). Suppose that there is $u \in$ co$(V')\backslash$co(U). Because of the maximality of co(V), $u \notin$ co(U) implies that $\succsim^I$ is not coarser than $\succsim^u$. Then there exists $P, P' \in \triangle(A)$ with $P \succ^I P'$ but $u \cdot P < u \cdot P'$. But since $\succsim^I$ is coarser than $\succsim^{I'}$ we have $P \succ^{I'} P'$ which implies $u \cdot P \geqslant u \cdot P'$ since $u \in$ co(V'). Contradiction. Then co$(V')\backslash$co$(V) = \varnothing$. $\square$

In the remainder, we will abuse notation and let $\succsim^I \in \mathcal{MIC}(\succsim)$ defined such that $\succsim^I = \bigwedge_{u \in I} \succsim^u$ where I is a convex subset of $\triangle^{n-1}$ with the property that $u_i \geqslant u_j \Leftrightarrow a_i \succsim a_j$. Of course if we pick the set of all Bernouilli utility vectors associated with each vNM inducement of $\succsim$ in $\mathcal{MIC}(\succsim)$ that is if we let $I = \{u \in \triangle^{n-1} | u_i \geqslant u_j \Leftrightarrow a_i \succsim a_j\}$, then we get the stochastic order, $\succsim^{sd}$. On the other hand, picking a single Bernouilli utility vector gives us a maximal element - a vNM order defined by that single Bernouilli utility vector. Above proposition also shows that if we take a set whose convex hull is a superset of some singleton Bernouilli utility vector and a proper subset of the set of all Bernouilli utility vectors associated each vNM inducement of $\succsim$, then we get an inducement which is neither maximal nor minimal according to the coarseness relation. To see this in an example, let $A = \{a, b, c\}$ and $a \succ b \succ c \succ$. Also let $I_0 = \{(x, y, z) \in \triangle^{|A|-1} | x > y > z\}$, $I_1 = \text{co}\{(3/4, 1/4, 0), (2/3, 1/3, 0)\}$ and $I_2 = \{(2/3, 1/3, 0)\}$. Obviously, $I_0 \supset I_1 \supset I_2$ implying $\succsim^{I_0}$ is coarser than $\succsim^{I_1}$ and $\succsim^{I_1}$ is coarser than $\succsim^{I_2}$. Furthermore $\succsim^{I_0}$ is the stochastic dominance order while $\succsim^{I_2}$ is a vNM order with the utility vector $(3/4, 1/4, 0)$. Now consider the upper contour sets of $(1/3, 1/3, 1/3)$ for each inducement:

$$U_{\succsim^{I_0}}(1/3,1/3,1/3) = \text{co}\{(1,0,0),(1/3,2/3,0),(2/3,1/3,0),(1/3,1/3,1/3)\}$$
$$U_{\succsim^{I_1}}(1/3,1/3,1/3) = \text{co}\{(1,0,0),(1/6,5/6,0),(1/2,0,1/2),(1/3,1/3,1/3)\}$$
$$U_{\succsim^{I_2}}(1/3,1/3,1/3) = \text{co}\{(1,0,0),(0,1,0),(1/2,0,1/2)\}$$

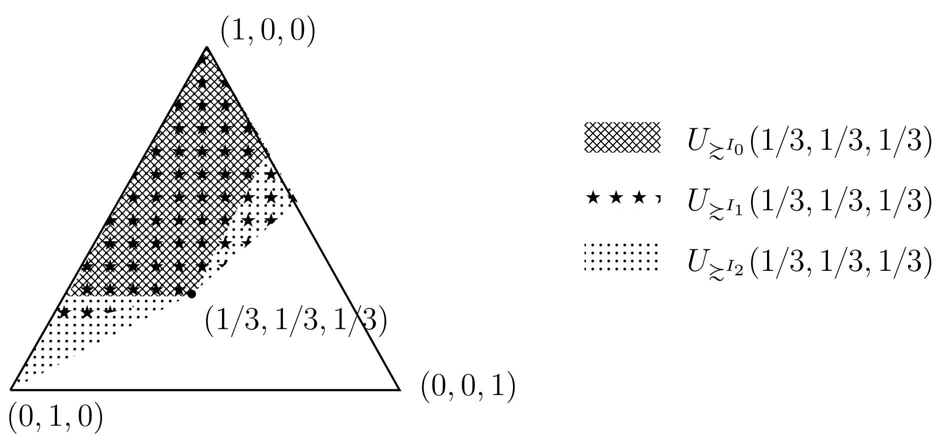


Figure 1.1: Upper Contour Sets for the uniform lottery according to $\succsim^{I_0}$, $\succsim^{I_1}$ and $\succsim^{I_2}$

Clearly, $U_{\succsim^{I_0}}(1/3,1/3,1/3) \subset U_{\succsim^{I_1}}(1/3,1/3,1/3) \subset U_{\succsim^{I_2}}(1/3,1/3,1/3)$. Notice that ,as an implication of coarseness relation, $U_{\succsim^{I_2}}(1/3,1/3,1/3) \backslash U_{\succsim^{I_0}}(1/3,1/3,1/3)$ describes the set of lotteries that are incomparable with $(1/3,1/3,1/3)$ according to $\succsim^{I_0}$ and $U_{\succsim^{I_2}}(1/3,1/3,1/3) \backslash U_{\succsim^{I_1}}(1/3,1/3,1/3)$

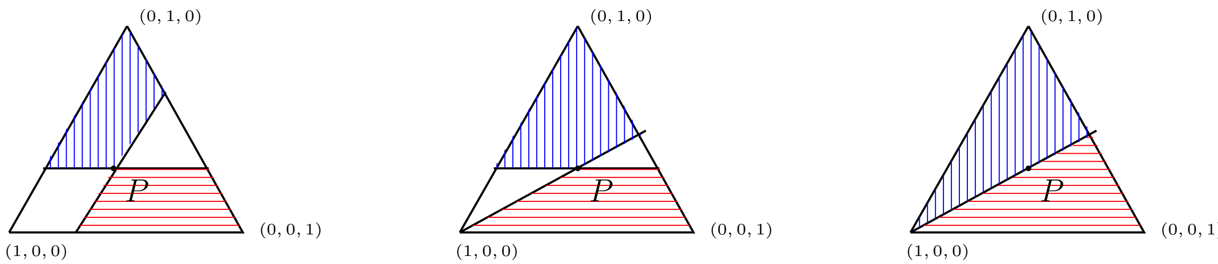
describes the set of lotteries that are incomparable with $(1/3, 1/3, 1/3)$ according to $\succsim^{I_1}$. Therefore, one can interpret the coarseness relation over this extended domain of preference relations over $\triangle(A)$ as a measure of completeness. The stochastic dominance is *the least complete element* of this domain while a monotonic vNM order is *a most complete element*.

The natural relation between the completeness of a preference relation over $\triangle(A)$ and the coarseness relation has a practical implication. We can use this domain with a specific *preference reporting language* where agents can express the ambiguity they have about the preferences over lotteries trough incomplete cardinal preference reports in addition to their ordinal preference reports (rankings). Imagine a preference reporting language for which agents submit a ranking over the alternatives they face and a score for each of them. The scores, of course, have to be consistent with the rankings so that the designer is able to come up with a monotonic inducement. The designer may put arbitrary bounds for the scores or may scale the reports to a desired interval. Without loss of generality, assume that the designer communicates that the scores should be in the interval $[0, 1]$. Furthermore, assume that agents have the choice of not reporting any score for some or all alternatives. The designer would explain the participants that an empty score would be interpreted as the agent is not certain about the intensity of her preference for that alternative relative to other alternatives. Going back to the example of a resident who is certain about the relative intensities of her preferences for the first two hospitals, a score report of $(1, 0.2, -)$ basically means that the agent has a strong preference for the first alternative while she is not positive about her preference for third alternative relative to other two. This report would correspond to the Bernouilli utility vector set $V = \{(1, 0.2, x) \in \mathbb{R}^3 \,|0 \leqslant x < 0.2\}$ in our domain so that, for instance, the agent would strictly prefer $(0.41, 0, 0.59)$ to the uniform lottery while wouldn't be able to compare $(0.39, 0, 0.61)$ with the uniform lottery. Notice that both lotteries are incomparable with the uniform lottery according to the stochastic dominance order. Therefore this incomplete score report leads to a preference relation which is finer than just a ranking.

Consider another example with three objects $A = \{a, b, c\}$ with $a \succ b \succ c$ according to agent's ranking (ordinal) preferences over a, b and c. Consider three possibilities:

	sd	mix	vNM
a	-	1	1
b	-	0.5	0.5
c	-	-	0

First report is the empty report and therefore is interpreted as that the agent only has ordinal preferences. Third report is a full cardinal report and therefore is interpreted as that the agent has a



(a) Stochastic Dominance (b) mix (c) vNM

Figure 1.2: Upper and Lower Contour Sets for three types

vNM order with reported Bernouilli utilities. Second report is an incomplete cardinal report. In figure 1.2, we depict upper and lower contour sets of the uniform lottery for each type.

In figure 1.2, horizontally dashed blue areas represent the upper contour sets of $P = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and red vertically dashed areas represent the lower contour sets of P. Undashed areas represents the lotteries that are incomparable to P. As in figure 1.1, the set of incomparable lotteries shrinks as the inducement gets finer. The mixed preference is capable of comparing more lotteries than stochastic dominance but less lotteries than vNM.

If we additionally allow agents to report disjoint intervals instead of singleton scores, we again end up with similar mix preference types. For example consider the cardinal report for the same ordinal preferences in figure 1.3.

	mix
a	1
b	$[1/3, 2/3]$
c	0

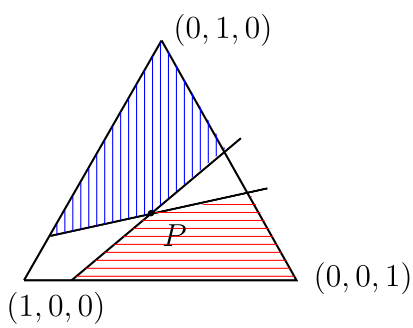


Figure 1.3: Upper and Lower Contour sets with interval scores

Obviously, a preference reporting language allowing disjoint intervals is more expressive than the one where agents can only report at most one score for each alternative. For example, the inducement described in figure 1.3 can not be reported trough reports with at most one score. However disjoint intervals are not expressive enough neither. Letting $n = 4$, $\succsim^I$ where $I = \{x \in [0,1]^4 | x_3/x_4 = 2, x_1 > x_2 > x_3 > x_4\}$ can not be expressed by disjoint intervals. One can allow for arbitrary intervals and than take an intersection with the set $\{x \in \mathbb{R}^n_+ | x_i \geqslant x_j \Leftrightarrow x_i \gtrsim x_j\}$ which takes care of monotonicity requirement. Such a preference reporting language would be able to produce any inducement in $\mathcal{MIC}(\gtrsim)$. However, in practice, such a preference reporting language might be difficult to communicate with participants

We conclude this section by stressing the difference between being indifferent among two alternatives and not specifying scores for two alternatives. An agent reporting $a \succ b \succ c \sim d$ and $(1, 0.5, -, -)$ and another agent reporting $a \succ b \succ c \succ d$ and $(1, 0.5, -, -)$ are not considered to have the same preferences. The lotteries $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are equivalent for the former while the latter strictly prefers $(0, 0, 1, 0)$. This implies that we can't skip ordinal preference report for above described preference reporting language as identical score vectors may lead to different inducements.

1.3 The Assignment Problem

In this section, we introduce the assignment problem on the extended domain. As in the random assignment problem presented in Bogomolnaia and Moulin (2001), N denotes the set of agents and A denotes the set of objects where $|A| = |N| = n$. [9] A *deterministic assignment* is a one-to-one mapping from N onto A which can be represented by a permutation matrix Π entries of which takes the value of either one or zero and has exactly one non-zero entry per row and per column. Given $i \in N$ and $a \in A$, $\Pi_{ia} = 1$ implies that agent i has the object a. A *random allocation* is a probability distribution over A and the set of random allocations is denoted by $\triangle(A)$. A *random assignment* is a probability distribution over deterministic assignments. The doubly stochastic matrix P defines a random assignment where:

$$P = \sum_{\Pi \in \triangle A} \lambda_\Pi \Pi \quad \text{with} \quad \forall \Pi \in \triangle A, \ \lambda_\Pi \geqslant 0, \ \text{ and } \sum_{\Pi \in \triangle A} \lambda_\Pi = 1$$

The fact that P is a doubly stochastic matrix implies:

$$\forall a \in A, \forall i \in N, \ P_{ia} \geqslant 0, \ \sum_{a \in A} P_{ia} = 1, \ \text{ and } \sum_{i \in N} P_{ia} = 1$$

By Birkhoff–von Neumann decomposition theorem, we know that any doubly stochastic matrix can be represented as a lottery over permutation matrices which corresponds to deterministic assignments. Therefore the co-domain of the probabilistic mechanisms is simply the space of doubly

[9] It is possible to consider the case where $|A| \neq |N|$ as long as each agent can have at most one object. One would simply introduce null objects or fictitious agents. Fictitious agents would be indifferent between any outcome so that they end up with the excess objects.

stochastic matrices. Let $\mathcal{L}_n$ be the set of doubly stochastic $n \times n$ matrices - set of random assignements, $\mathcal{P}(A)^n$ denote the set of complete and transitive preference profiles over A for n agents and $\mathcal{I} = \{\mathcal{I}_i\}_{i \in N}$ be the domain of inducements. Furthermore let $\mathcal{D} := \{(\succsim, I) \in \mathcal{I} \times \mathcal{P}(A)^n | \succsim_i^{I_i} \in \mathcal{MIC}(\succsim_i)$ for each $i \in N\}$ denote the space of monotonic continuous inducements satisfying independence axiom. A probabilistic allocation mechanism, $F : \mathcal{D} \to \mathcal{L}_n$, maps a preference and inducement profile to a doubly stochastic matrix. If $F(I, \succsim) = F(I', \succsim)$ for each $\succsim \in \mathcal{P}(A)^n$ and inducement profiles $I, I' \in \mathcal{I}$, then we say that the probabilistic mechanism F is an *ordinal mechanism*. If F is not an ordinal mechanism then it is a *cardinal mechanism*.

The PS mechanism of Bogomolnaia and Moulin (2001) and the random priority (RP) mechanism are examples of ordinal mechanisms. Ordinal mechanisms are robust to errors on cardinal preference reporting and requires less complex preference reporting language. However, especially when some agent have clear cardinal preferences, ordinal mechanisms may produce inefficient outcomes. Consider the following example:

Example 1. Let $n = 3$ and $A = \{a, b, c\}$ with $a \succ_i b \succ_i c$ for each $i \in N$. For such a profile, any ordinal mechanism which satisfies equal treatment of equals, produce the probabilistic allocation where every agent has equal probability of getting an object. Let's denote this allocation with P_0. Assume the first agent actually has a vNM preference for some $u \in \mathbb{R}^3$ and the other two has only the ordinal preference which is represented by the stochastic dominance order. Without loss of generality, let $u_c = 0$. Now consider below two allocations:

P_1	1	2	3
a	0	1/2	1/2
b	1	0	0
c	0	1/2	1/2

P_2	1	2	3
a	1/2	1/4	1/4
b	0	1/2	1/2
c	1/2	1/4	1/4

Now when $u_a/u_b < 2$, agent 1 prefers P_1 over P_0 and P_2, and when $u_a/u_b > 2$, she prefers P_2 over P_0 and P_1. For other agents all three probabilistic allocations are incomparable. Then, we might argue, cardinal mechanisms may provide an additional room to improve agents with cardinal preferences even when there is only one agent with cardinal preferences.

The efficiency loss associated with using ordinal mechanisms become more stark, if more agents have cardinal preferences. Assume that for the above example second agent also has a vNM preference for some $v \in \mathbb{R}^3$ and again without loss of generality, let $v_c = 0$. Consider following probabilistic allocation :

P_3	1	2	3
a	1/2	1/6	1/3
b	0	2/3	1/3
c	1/2	1/6	1/3

Now if $u_a/u_b > 2$ and $v_a/v_b < 2$ then for each agent P_3 is weakly better than P_0. That is, the allocation P_3 is dominated by P_0, implying that P_0 is not efficient for the given preferences over lotteries. This efficiency improvement, however, is in expense of strategy-proofness even when $n = 3$. We will revisit this point in section 1.5.

One important example of cardinal mechanisms is the Pseudomarket mechanism of Hylland and Zeckhauser (1979). The Pseudomarket mechanism determines the final outcome trough a Pseudomarket where each agent has identical income, and trade for probability shares. The domain of the Pseudomarket mechanism is the domain of the vNM orders. We can straightforwardly extend the Pseudomarket mechanism by replacing the domain of the Pseudomarket mechanism with the extended domain. An important property of Pseudomarket mechanisms is that they achieve efficiency with respect to any underlying inducement profile of the extended domain provided that a pseudo equilibrium exists. Furthermore, all efficient assignments can be achieved by some Pseudomarket mechanism.

We define Pseudomarket mechanisms in the extended domain through the Pseudomarket equilibrium outcomes. Naturally, to be able to define Pseudomarket equilibrium, we need to define endowments as well. We can impose identical endowments for all agents, or allow different initial endowments to consider environments where each agent initially own or has claim for a certain object as in Abdulkadiroğlu and Sönmez (2003).

A random assignment $P \in \mathcal{L}_n$ is said to be **achievable under a Pseudomarket mechanism** at induced preferences $(\gtrsim_i^{I_i})_{i \in N}$ with doubly stochastic endowment matrix $E \in \mathcal{L}_n$ if there exists a price vector $p \in \triangle^{n-1}$ satisfying:

- **Individual Optimality:** For each $i \in N$:
  - $P_i \in \triangle(A)$ and $p \cdot P_i \leqslant p \cdot E_i$.
  - If $P_i' \in \triangle(A)$ and $p \cdot P_i' \leqslant p \cdot E_i$, then $\neg(P_i' \succ_i^{I_i} P_i)$.
- **Market Clearing:** $\forall a \in A,\ \sum_{i \in N} P_{ia} = \sum_{i \in N} E_{ia} = 1$.

Then a Pseudomarket mechanism simply produces a probabilistic allocation which is achievable under a pseudomarket mechanism at the given induced preferences. Unfortunately a Pseudomarket equilibrium may not always exists if we restrict the endowment structure ex-ante. Particularly, there might be no Pseudomarket equilibrium with equal endowments:

Example 2. Consider the following ordinal preference profile:

1	2	3
a	a	b,c
b,c	b,c	a

Assume that agents did not report any cardinal preferences meaning that we assume each agent has a stochastic dominance ordering over lotteries based on above ordinal preferences. Let $E \in \mathcal{L}_n$ be an endowment matrix such that $E_i = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for $i \in \{1, 2, 3\}$. We will attempt to find a price vector $p \in \triangle^2$ and a random assignment $P \in \mathcal{L}_n$ such that for each agent individual optimality condition is satisfied and market clears.

Firstly, observe that we can't have $p_a = 0$ since otherwise individual optimality for agent 1 and 2 dictates that $P_{1a} = P_{2a} = 1$ which is a contradiction. Since $p_a > 0$, we have that $P_{3a} = 0$. Similarly, we must have $p_b = p_c$ since otherwise we will either have $P_{1b} = P_{2b} = P_{3b} = 0$ or $P_{1c} = P_{2c} = P_{3c} = 0$ failing the market clearing condition. Now let $p_b = p_c = p$. Since $p \in \triangle^2$, we have $p_a = 1 - 2p$. Then $p \cdot E_i = \frac{1}{3}$ for all $i \in \{1, 2, 3\}$. Now assume $p = 0$ implying $p_a = 1$. Then $P_{1a} \leqslant \frac{1}{3}$ and $P_{2a} \leqslant \frac{1}{3}$. But then $P_{3a} = 0$ implies $P_{1a} + P_{2a} + P_{3a} \leqslant \frac{2}{3}$ contradicting the market clearing condition. Then $p > 0$ and therefore the budget constraint for the agent 3 must bind meaning that $\frac{1}{3} = p(P_{3b} + P_{3c})$. Since $P_{3a} = 0$, this implies $p = \frac{1}{3}$. But then $p_a = \frac{1}{3}$ and therefore $P_{1a} = P_{2a} = 1$ which is again a contradiction. Then we conclude that given the equal endowment matrix, there exists no price vector supporting a Pseudomarket equilibrium that is there is no Pseudomarket equilibrium.

This does not mean, however, there can be no Pseudomarket equilibrium for this preference profile. Consider the following endowment matrix:

E	1	2	3
a	1/2	1/2	0
b	0	0	1
c	1/2	1/2	0

Now consider the equilibrium price vector $p = (1, 0, 0)$ and the random assignment $P = E$. Clearly, $P = E$ with price vector p satisfies the individual optimality condition for each agent and the market clearing condition. Therefore for the above endowment matrix E, there exists a

Pseudomarket equilibrium. In fact, if the endowment matrix is efficient, no trade outcome can always be supported as a Pseudomarket equilibrium. We will illustrate this point in section 1.4.

We will loosely use the term PM mechanism with equal endowments to imply a mechanism that produces the Pseudomarket equilibrium with equal endowments for the preference profiles for which such an equilibrium exists. Now, we present an example consisting 3 agents with ordinal preferences and compare PS, RP and PM outcomes:

Example 3. Consider the following ordinal preference profile:

1	2	3
a	a	b
b	c	c
c	b	a

Assume that all agent have fully ordinal preferences, that is each agent's preference ordering over lotteries is determined by the stochastic dominance order based on above rank orders. Now, solutions for PM, PS and RP are following:

P^{PS}	1	2	3
a	1/2	1/2	0
b	1/4	0	3/4
c	1/4	1/2	1/4

P^{RP}	1	2	3
a	1/2	1/2	0
b	1/6	0	5/6
c	1/3	1/2	1/6

P^{PM}	1	2	3
a	$\frac{\lambda}{1+\lambda}$	$\frac{1}{1+\lambda}$	0
b	$\frac{1-\lambda}{2-\lambda}$	0	$\frac{1}{2-\lambda}$
c	$\frac{1-\lambda+\lambda^2}{2+\lambda-\lambda^2}$	$\frac{\lambda}{1+\lambda}$	$\frac{1-\lambda}{2-\lambda}$

Depending on the equilibrium price selection, Pseudomarket mechanism with equal endowments produces above probabilistic assignment where $\lambda \in (1/2, 1]$. Notice that agent two prefers lower λ while the third one prefers higher λ. For the first agent, they are not comparable. If we choose $\lambda = 3/4$ so that the equilibrium price is equally distanced from preferred prices of both agent two and three we get:

$$P^{PM'} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline a & \frac{3}{7} & \frac{4}{7} & 0 \\ b & \frac{1}{5} & 0 & \frac{4}{5} \\ c & \frac{13}{35} & \frac{3}{7} & \frac{1}{5} \end{array}$$

Observe that, not only above particular Pseudomarket mechanism with equal endowments produces an outcome different than PS and RP, but also regardless of equilibrium price selection (choice of λ), PM is different than PS and RP. This is because RP and PS can not be achievable under equal incomes/endowments. To see this assume P^{PS} is an equilibrium allocation for equal endowment matrix E where $E_{ia} = 1/3$ for each $i \in N$ and each $a \in A$. For P^{PS} to be achievable, the net trades from equal endowments should be affordable in the sense that $(P_i^{PS} - E_i) \cdot p \leqslant 0$. Furthermore if $(P_i^{PS} - E_i) \cdot p < 0$ then agent i can be better off by buying more of her top ranked object which implies P_i^{PS} is not optimal, then $(P_i^{PS} - E_i) \cdot p = 0$ for each i.

$$
\begin{aligned}
P_1^{PS} - E_1 = (1/6, -1/12, -1/12) \cdot p = 0 \quad &\Rightarrow \quad 2p_a = p_b + p_c \\
P_2^{PS} - E_2 = (1/6, 0, -1/6) \cdot p = 0 \quad &\Rightarrow \quad p_a = p_c \\
P_3^{PS} - E_3 = (-1/3, 5/12, -1/12) \cdot p = 0 \quad &\Rightarrow \quad 5p_b = 4p_a + p_c
\end{aligned}
$$

From here, it is straightforward to show that $p_a = p_b = p_c = 1/3$. But then for each agent, the degenerate lottery giving them their best option is affordable. This contradicts the optimality of P_i^{PS} for each $i \in N$.

Similarly for P^{RP}, we have:

$$
\begin{aligned}
P_1^{RP} - E_1 = (1/6, -1/6, 0) \cdot p = 0 \quad &\Rightarrow \quad p_a = p_b \\
P_2^{RP} - E_2 = (1/6, 0, -1/6) \cdot p = 0 \quad &\Rightarrow \quad p_a = p_c \\
P_3^{Rp} - E_3 = (-1/3, 1/2, 1/6) \cdot p = 0 \quad &\Rightarrow \quad 3p_b + p_c = 2p_a
\end{aligned}
$$

Above we directly get $p_a = p_b = p_c$ and $3p_b + p_c = 2p_a$ which is a contradiction. Then neither RP allocation nor PS can be achievable with equal incomes.

This interesting observation about the celebrated PS mechanism, that it is not achievable from equal probability share endowments trough trade, might seem odd at a first glance. However, the reason for this issue is that the prices in this pseudomarket is driven solely by demand. Therefore, the agents whose demand are more aligned with each other need to compete over certain objects and therefore are worse off under equal incomes scenario. In this example, among 3 possible binary comparisons, agent 1 and agent 2 have aligned preferences in two of them and agent 1 and agent 3 have aligned preferences in one of them while agent 2 and agent 3 have opposite preferences in all three of them. This puts agent 1 in disadvantage with the Pseudomarket mechanism compared to the Probabilistic serial. On the other hand, having relatively misaligned preferences, agent 3 is better off with the Pseudomarket mechanism.

We can further analyze the differences between the PS and the Pseudomarket mechanism in terms of the axioms that they satisfy. The PS mechanism employs a cake eating algorithm which awards the objects ranked best by the greatest number of agents are distributed evenly among those who ranks that object as their best. One interesting axiom used in characterization of the PS mechanism is ordinal fairness (Hashimoto et al., 2014). A probabilistic mechanism is said to be ordinally fair if the surplus at an object for an agent who may get the object with positive probability may not exceed the surplus at the same object for other agents. More precisely, if F is ordinally fair and $F((\succsim_i)^{I_i}) = P$, then for each $i, j \in N$ and $a \in A$ with $P_{ia} > 0$, $\sum_{b \succsim_i a} P_{ib} \leqslant \sum_{b \succsim_j a} P_{jb}$. It is clear that P^{PM} is not ordinally fair for the preference profile of Example 3, for any $\lambda \in (1/2, 1]$. This implies that the Pseudomarket mechanism with equal endowments is not ordinally fair.

While Pseudomarket mechanisms with equal endowments fail to be ordinally fair, they are preferred to the PS solution by two agents and therefore we may argue that if the agents are asked to choose between PS and PM solution in the example 3, the majority would pick PM. Furthermore, from a utilitarian perspective, PM performs better compared to PS at least for this example. To show this, we use social welfare domination notion introduced by Doğan et al. (2018). A probabilistic assignment P is said to be ex-ante efficient at vNM utility profile $(u_i)_{i \in N}$ if $P \in \arg\max_{P' \in \mathcal{L}_n} \sum_{i \in N} P_i \cdot u_i$. A probabilistic assignment P sw-dominates another probabilistic allocation P' if P is ex-ante efficient at vNM utility profile $(u_i)_{i \in N}$ whenever P' is also ex-ante efficient at vNM utility profile $(u_i)_{i \in N}$ and there exists a vNM utility profile $(u'_i)_{i \in N}$ such that only P is ex-ante efficient at. Now, consider P^{PM} where $\lambda = 1$ that is:

$$P^{PM''} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline a & \frac{1}{2} & \frac{1}{2} & 0 \\ b & 0 & 0 & 1 \\ c & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

Wlog, let $u_{1c} = u_{2b} = u_{3a} = 0$, $u_{1b} = u_{2c} = u_{3c} = 1$ and $u_{1a} = \gamma_1$, $u_{2a} = \gamma_2$, $u_{3b} = \gamma_3$ where $\gamma_1, \gamma_2, \gamma_3 > 1$. It is straightforward to show that P^{PS} is ex ante efficient whenever $\gamma_2 = 1 + \gamma_1$ and $\gamma_3 = 2$ while $P^{PM''}$ is ex ante efficient whenever $\gamma_2 = 1 + \gamma_1$ and $\gamma_3 \geqslant 2$. This implies that $P^{PM''}$ sw-dominates P^{PS}.

1.4 Efficiency on the Extended Domain

An efficient random assignment is one for which there is no other random assignment that weakly improves all agents and strictly improves one of the agents. That is, a random assignment $P \in \mathcal{L}_n$ is $(\succsim_{\mathbf{i}}^{\mathbf{I_i}})_{\mathbf{i \in N}}$**-efficient** if there is no random assignment $P' \in \mathcal{L}_n$ such that for each $i \in N$, we have $P'_i \succsim_i^{I_i} P_i$ and for some $j \in N$, we have $P'_i \succ_i^{I_i} P_i$. Since the inducements in the extended domain are not complete in general, we can come up with a stronger efficiency notion which allows for some agents not to be able to compare the alternative assignment with the original one. More precisely, a random assignment $P \in \mathcal{L}_n$ is called to be **strongly** $(\succsim_{\mathbf{i}}^{\mathbf{I_i}})_{\mathbf{i\in N}}$**-efficient** if there is no random assignment $P' \in \mathcal{L}_n$ such that for each $i \in N$, we have $\neg(P_i \succ_i^{I_i} P_i')$ and for some $j \in N$, we have $P_j' \succ_j^{I_j} P_j$. Note that the weak version coincide with the strong version if $\succsim_i^{I_i}$ is complete for each agent. However, if the induced preferences are not complete, as in the case of stochastic dominance, there can be two random assignments where one agent prefers one to the other and other agents can't compare the two. In this case both random assignments may be efficient but only one can be strongly efficient. Consider the following example:

Example 4. Let $N = \{1, 2, 3\}$, $A = \{a, b, c\}$ and for each $i \in N$, $\succsim_i^{I_i} = \succsim_i^{sd}$. Table on the right depicts the ordinal preference $\succsim_i$ for each $i \in N$ and matrices on the right are random assignments:

1	2	3
a	a	a
b	c	c
c	b	b

P	1	2	3
a	1/3	1/3	1/3
b	2/3	1/6	1/6
c	0	1/2	1/2

P'	1	2	3
a	1/2	0	1/2
b	1/2	0	1/2
c	0	1	0

Above, agent 1 prefers P' over P while for other agents they are not comparable. This means P is not strongly efficient. One thing to note about this example is that P is the PS solution for this preference profile given ordinal preferences $(\succsim_i)_{i\in N}$. The fact that the PS solution for this preference profile is not strongly $\succsim^{sd}$-efficient is not surprising since strong $\succsim^{sd}$-efficiency is equivalent to ex-ante efficiency and we know that the PS mechanism is not ex-ante efficient in general.

If an allocation is efficient with respect to an inducement profile, then it is also efficient with respect to a coarser inducement profile. This is a direct result of the fact that finer inducements obey coarser inducements so that, if an allocation dominates another with respect to a coarser inducement, it also dominates the other with respect to the finer inducement:

Corollary 2. *Let $(\succsim_i^{I_i})_{i\in N}$ and $(\succsim_i^{I_i'})_{i\in N}$ be profiles of monotonic inducements where $I_i \supseteq I_i'$ for each $i \in N$. If $P \in \mathcal{L}_n$ is $(\succsim_{\mathbf{i}}^{\mathbf{I'}})_{\mathbf{i\in N}}$-efficient then it is $(\succsim_{\mathbf{i}}^{\mathbf{I}})_{\mathbf{i\in N}}$-efficient.*

Now we can prove that any PM mechanism under the inducement profile $\{I_i\}_{i\in N}$ yields $(\succsim_i^{I_i})_{i\in N}$-efficient random assignments if Pseudomarket equilibrium exists:

Theorem 2. *Let F, be a PM mechanism. For any inducement profile $(\succsim_i^{I_i})_{i\in N} \in \mathcal{D}$ such that a Pseudomarket equilibrium exists, $F((\succsim_i^{I_i})_{i\in N})$ is $(\succsim_i^{I_i})_{i\in N}$-efficient.*

Proof. Assume there exists $(\succsim_i^{I_i})_{i\in N} \in \mathcal{D}$ such that $F((\succsim_i^{I_i})_{i\in N}) = P$ is not $(\succsim_i^{I_i})_{i\in N}$-efficient.

Then there is $P' \in \mathcal{L}^n$ such that for each $i \in N$, we have $P'_i \succsim_i^{I_i} P_i$ and for some $j \in N$, we have $P'_j \succ_j^{I_j} P_j$. Individual optimality implies that for each $i \in N$, $p \cdot P'_i \geqslant p \cdot P_i$ and $p \cdot P'_j > p \cdot P_j$ where p is an equilibrium price vector for which P is an equilibrium allocation. Then we have $\sum_i p \cdot P'_i > \sum_i p \cdot P_i$ implying $p \cdot \sum_i P'_i > p \cdot \sum_i P_i$. But since $P, P' \in \mathcal{L}^n$, we end up with $p \cdot e > p \cdot e$. □

The contrapositive statement of this theorem, the fact that if a mechanism is not $(\succsim_i^{I_i})_{i \in N}$-efficient, then it is not a PM mechanism under the inducement profile $\{\mathcal{I}_i\}_{i \in N}$, and corollary 2 tell us that the Random Priority mechanism is not a PM mechanism even if we restrict the preference domain to ordinal preferences. To illustrate this, consider following example from (Bogomolnaia and Moulin, 2001):

Example 5. Let $N = \{1, 2, 3, 4\}$, $A = \{a, b, c, d\}$. Table on the right depicts $\succsim_i$ for $\imath in N$ and the matrices are random assignments:

1	2	3	4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

P	1	2	3	4
a	5/12	5/12	1/12	1/12
b	1/12	1/12	5/12	5/12
c	5/12	5/12	1/12	1/12
d	1/12	1/12	5/12	5/12

P'	1	2	3	4
a	1/2	1/2	0	0
b	0	0	1/2	1/2
c	1/2	1/2	0	0
d	0	0	1/2	1/2

Here P is the RP solution while P' is the PS solution. Since for each $i \in N$, we have $P'_i \succ_i^{sd} P_i$, it follows that P is not $\succsim_i^{sd}$-efficient. By corollary 2, this implies P is not $\succsim_i^{I_i}$-efficient for every monotonic inducement profile and therefore can't be achievable by a PM mechanism due to Theorem 2. We can further illustrate this point by assuming for each $i \in N$, $\succsim_i^{I_i} = \succsim_i^{sd}$. If P is achievable with some endowment matrix E with price vector $p \in \triangle^{n-1}$, then for each $i \in N$, we have $P_i \cdot p < P'_i \cdot p$. But this implies $p_b + p_d < p_a + p_c$ and $p_b + p_d > p_a + p_c$. Then P is not achievable by a PM mechanism.

Above theorem is the analogue of the first welfare theorem for the Pseudomarket mechanisms. The next theorem is the second welfare theorem counterpart and it is a generalization of the second welfare theorem for the assignment problem of Miralles and Pycia (2014):

Theorem 3. *Every efficient random assignment is achievable under a Pseudomarket mechanism. That is, if for each $i \in N$, $\succsim_i^{I_i} \in \mathcal{D}$ and $P \in \mathcal{L}^n$ is $(\succsim_i^{I_i})_{i \in N}$-efficient and , then it is achievable under a Pseudomarket mechanism at induced preferences $(\succsim_i^{I_i})_{i \in N}$ with some doubly stochastic matrix $E \in \mathcal{L}^n$.*

Proof. Let $P \in \mathcal{L}^n$ be a $(\succsim_i^{I_i})_{i\in N}$-efficient random assignment . For each $i \in N$, let $S_i \subseteq \triangle(A)$ be defined by the following:

$$S_i := \begin{cases} \{P_i' \in \triangle(A) | P_i' \succ_i^{I_i} P_i\} & \text{if } \{P_i' \in \triangle(A) | P_i' \succ_i^{I_i} P_i\} \neq \varnothing \\ \{P_i\} & \text{otherwise} \end{cases}$$

Since $U_{\succsim_i^{I_i}}(P_i)$ is convex, for each $P_0, P_1 \in \{P_i' \in \triangle(A)|P_i' \succ_i^{I_i} P_i\} \subseteq U_{\succsim_i^{I_i}}(P_i)$ and each $\lambda \in [0,1]$, we have $\lambda P_0 + (1-\lambda)P_1 \in U_{\succsim_i^{I_i}}(P_i)$. If $\lambda P_0 + (1-\lambda)P_1 \notin \{P_i' \in \triangle(A)|P_i' \succ_i^{I_i} P_i\}$, then $\lambda P_0 + (1-\lambda)P_1 \sim_i^{I_i} P_i$ which implies that for each $u \in I_i$, we have $u \cdot (\lambda P_0 + (1-\lambda)P_1) = u \cdot P_i$. Since for each $u \in I_i$, we have $u \cdot P_0 \geqslant u \cdot P_i$ and $u \cdot P_1 \geqslant u \cdot P_i$, it follows that $u \cdot P_0 = u \cdot P_i$ and $u \cdot P_1 = u \cdot P_i$. But then $P_0 \sim_i^{I_i} P$ and $P_1 \sim_i^{I_i} P$ contradicting $\lambda P_0 + (1-\lambda)P_1 \notin \{P_i' \in \triangle(A)|P_i' \succ_i^{I_i} P_i\}$. Therefore, for each $i \in N$, S_i is convex. Then $S := \sum_i S_i$ is also convex.

Now suppose $\sum_i P_i = e \in S$. Since P is $(\succsim_i^{I_i})_{i\in N}$-efficient, this implies that for each $i \in N$, we have $\{P_i' \in \triangle(A)|P_i' \succ_i^{I_i} P_i\} = \varnothing$. But then, letting $E = P$, P is achievable by any price $p \in \triangle^{n-1}$. Alternatively, suppose that $\sum_i P_i = e \notin S$. Then by Separating Hyperplane Theorem, there exists a linear functional strictly separating S and $\{\sum_i P_i\}$ in the sense that, there is $p \in \mathbb{R}^n$ such that for each $P' \in \mathcal{L}^n$ with $\sum_i P_i' \in S$, we have $p \cdot \sum_i P_i \leqslant p \cdot \sum_i P_i'$ and for some $P'' \in \mathcal{L}^n$ with $\sum_i P_i'' \in S$, we have $p \cdot \sum_i P_i < p \cdot \sum_i P_i''$.

Claim: If $P_j' \succ_j^{I_j} P_j$, then $p \cdot P_j' > p \cdot P_j$.

Let $P_j', P_j \in \mathcal{L}^n$ with $P_j' \succ_j^{I_j} P_j$ for some $j \in N$. For each $i \in N$, if there is $a, b \in A$ with $a \succ_i b$, let the sequence $\{P_{i_k}\}_{k=1}^{\infty}$ be defined by the following:

$$P_{id_k} = \begin{cases} P_{ia} + \varepsilon/n & \text{if } d = a \\ P_{ib} - \varepsilon/n & \text{if } d = b \\ P_{ic} & \text{otherwise} \end{cases}$$

And if for each $a, b \in A$, $a \sim_i b$,let $P_{i_k} = P_i$ for every $k \in N$. Since $\succsim_i^{I_i}$ is a monotonic inducement, for each $k \in \mathbb{N}$, we have $P_j' + \sum_{i\neq j} P_{i_k} \in S$ and $P_j' + \sum_{i\neq j} P_{i_k} \to P_j' + \sum_{i\neq j} P_i$. But then since $p \cdot (P_j' + \sum_{i\neq j} P_{i_n}) \geqslant p \cdot \sum_i P_i$, we have $p \cdot (P_j' + \sum_{i\neq j} P_i) \geqslant p \cdot \sum_i P_i$ which implies $p \cdot P_j' \geqslant p \cdot P_j$.

Then either we have $p \cdot P_j' > p \cdot P_j$ or $p \cdot P_j' = p \cdot P_j$. Suppose $p \cdot P_j' = p \cdot P_j$.

Case 1: Suppose there is no $P_j'' \in \triangle(A)$ with $P_j \succ_j^{I_j} P_j''$. Then for each $a \in A$, we have $l_a \succsim_j^{I_j} P_j$. This implies $p \cdot l_a \geqslant p \cdot P_j''$ and therefore if $P_{ja} > 0$ then $p_a = \min\{p_b | b \in A\}$. Furthermore since P is $(\succsim_i^{I_i})_{i\in N}$-efficient, if $P_{ja} > 0$ then for each $i \in N$, we have $a \in \min_{\succsim_i} A$. Then for some $\varepsilon > 0$, let $p' \in \mathbb{R}^n$ be defined such that for each $a \in A$ with $P_{ja} > 0$, $p_a' = p_a - \varepsilon$ and $p_a' = p_a$ otherwise. Then $p'\,\mathbb{R}^n$ strictly separates S and $\{\sum_i P_i\}$ and $p' \cdot P_j' > p' \cdot P_j$.

Case 2: Suppose there is $P''_j \in \triangle(A)$ with $P_j \succ_j^{I_j} P''_j$. Since $\succsim_j^{I_j}$ is continuous, there is $\lambda \in (0,1)$ with $\lambda P''_j + (1-\lambda)P'_j \succ_i^{I_j} P_j$. Again, $p \cdot \lambda P''_j + p \cdot (1-\lambda)P'_j \geqslant p \cdot P_j$. Then $p \cdot P'_j = p \cdot P_j$ implies $p \cdot P''_j \geqslant p \cdot P_j$. Then for each $P''_j \in \triangle(A)$ with $P_j \succ_i^{I_j} P''_j$, we have $p \cdot P''_j \geqslant p \cdot P_j$ and each $P'_j \in \triangle(A)$ with $P'_j \succ_i^{I_j} P_j$, we have $p \cdot P'_j \geqslant p \cdot P_j$. Then we get $p_a = p_b$ for each $a, b \in A$. But if for each $a, b \in A$, we have $p_a = p_b$, then for each $P'_i \in S_i$ and each $i \in N$, we have $p \cdot P_i = p \cdot P'_i$. Then for each $P' \in \mathcal{L}^n$ with $\sum_i P'_i \in S$, we have $p \cdot \sum_i P_i = p \cdot \sum_i P'_i$ contradicting the fact that p strictly separates S and $\{\sum_i P_i\}$.

Then we conclude that for each $i \in N$, $P'_i \succ_i^{I_i} P_i$ implies $p \cdot P'_i > p \cdot P_i$.

Finally, consider $p' = \frac{1}{p \cdot e + kn}(p + ke) \in \triangle^{n-1}$ for sufficiently large $k > 0$. Let $x, y \in \triangle(A)$ with $x \cdot p' \leqslant y \cdot p'$ for some $p \in \mathbb{R}^n$. Then $x \cdot (p+ke) \leqslant y \cdot (p+ke)$ implying $x \cdot p + kx \cdot e \leqslant y \cdot p + ky \cdot e$. Since $x \cdot e = y \cdot e = 1$, this implies $x \cdot p \leqslant y \cdot p$. Hence we have that for each $i \in$N, $P'_i \succ_i^{I_j} P_i$ implies $p' \cdot P'_i > p' \cdot P_i$. Then P is achievable under a Pseudomarket mechanism at induced preferences $(\succsim_i^{I_i})_{i \in N}$ with doubly stochastic endowment matrix $E = P$ and price vector $p' \in \triangle^{n-1}$. □

The fact that every efficient random assignment is achievable under a Pseudomarket mechanism also suggests that for any preference profile there exists a Pseudomarket equilibrium where the initial endowment is the resulting random assignment. Unfortunately, this work around for the existence problem of the Pseudomarket equilibrium is not very practical since there is no straightforward way to find out the suitable endowment matrix for an arbitrary preference profile.

On the other hand, since the PS solution is $(\succsim_i^{sd})_{i \in N}$-efficient, we can say that there exists a PM mechanism in the extended domain which agrees with the PS in the ordinal domain:

Corollary 3. *There exists a PM mechanism which extends the PS mechanism into the extended domain. That is, there exists a PM mechanism, F, such that for each $(\succsim_i^{I_i})_{i \in N} \in \mathcal{D}$ with for each $i \in N$, $\succsim_i^{I_i} = \succsim_i^{sd}$, $F((\succsim_i^{I_i})_{i \in N})$ is the PS solution for $(\succsim_i)_{i \in N}$.*

Indeed, the random assignments P and P' in example 4 are achievable trough a pseudomarket mechanism. Both random assignments can be supported by any price vector with $p_a > p_c > p_b$ and endowment matrices P and P' respectively.

We say that a random assignment is envy-free if every agent prefers their individual allocation over what other agents' receive. Similarly ,we say that a random assignment is weakly envy-free if no agent strictly prefers another agents' allocation over her own. More precisely, a random assignment $P \in \mathcal{L}_n$ is $(\succsim_i^{I_i})_{i \in N}$**-envy free** if for each $i, j \in N$, we have $P_i \succsim_i^{I_i} P_j$. Similarly, a random assignment $P \in \mathcal{L}_n$ is **weakly** $(\succsim_i^{I_i})_{i \in N}$**-envy free** if for each $i, j \in N$, we have $\neg(P_j \succ_i^{I_i} P_i)$.

As with the efficiency, two versions of no envy coincides if we use complete inducements. By definition, PM mechanisms are weakly envy-free if the endowments are identical among agents:

Proposition 5. *Let $P \in \mathcal{L}^n$ be achievable under a Pseudomarket mechanism at some induced preferences $(\succsim_i^{I_i})_{i \in N}$ with doubly stochastic endowment matrix E. If $E_i = E_j$ for each $i, j \in N$, then P is weakly $(\succsim_i^{I_i})_{i \in N}$-envy free.*

For asymmetric endowment matrices weak no envy is not guaranteed even if the endowment matrix admits no envy. This is because, an agent may have envy for the equilibrium outcome of another one who owns a highly demanded object even if she ranks that object as the worst alternative.

PM mechanisms in general won't achiave strong efficiency or no envy. Incidentally, one cannot find a mechanism which is strongly efficient and satisfies weak no envy. We end this section by this result:

Proposition 6. *For $n \geqslant 3$ there exists an inducement profile $(\succsim_i)_{i \in N}$ for which there is no random assignment which is both strongly efficient and weakly envy-free.*

Proof. Let $n = 3$, and for each $i \in N$, assume $a \succ_i b \succ_i c$. Assume first agent has a vNM preference with Bernouilli utility vector $u \in R^3_+$ where $u_c = 0$ and second agent has a stochastic dominance inducement Now let $P \in \mathcal{L}^n$. For some $\varepsilon \in \mathbb{R}\backslash\{0\}$ and $\delta \in (0,1)$ define P' such that $P'_1 := P_1 - \varepsilon((1-\delta), -1, \delta)$, $P'_2 := P_2 + \varepsilon((1-\delta), -1, \delta)$ and $P'_3 := P_3$. Since P is a probabilistic assignment, by construction, P' is also a probabilistic assignment, if all of its entries are non-negative. Now observe that whenever P' is a probabilistic assignment P_2 and P'_2 are incomparable for the second agent. For the first agent, for each $\varepsilon \in \mathbb{R}$ sufficiently close to 0, there exists $\delta_1, \delta_2 \in [0,1]$ such that $u \cdot \varepsilon((1-\delta_1), -1, \delta_1) > 0$ and $u \cdot \varepsilon((1-\delta_1), -1, \delta_1) < 0$. This implies if P' is a probabilistic allocation then P is not strongly efficient. But this implies if P is strongly efficient then either $P_1 = l_a$ or $P_1 = l_c$ which implies P is not weakly envy-free. □

1.5 Incentives vs Efficiency on the Extended Domain

We define strategy-proofness in two forms just like efficiency and no envy. In its strong form, strategy-proofness implies that for all agents, truth-telling is weakly better than any deviation. Weak strategy-proofness, however, only requires that deviations can't be strictly better than truth-telling. More precisely, a probabilistic mechanism $F : \mathcal{D} \to \mathcal{L}_n$ is **strategy-proof** if for each inducement profile $(\succsim_i^{I_i})_{i \in N}$ and $((\succsim_i^{I_i})_{i \in N\backslash\{j\}}, \succsim_j^{I_j}) \in \mathcal{D}$, we have $F_i(\succsim_i^{I_i})_{i \in N} \succsim_j^{I_j} F_i((\succsim_i^{I_i})_{i \in N\backslash\{j\}}, \succsim_j^{I_j})$. And similarly, a probabilistic mechanism $F : \mathcal{D} \to \mathcal{L}_n$ is **weakly strategy-proof** if for each inducement profile $(\succsim_i^{I_i})_{i\in N}$ and $((\succsim_i^{I_i})_{i\in N\setminus\{j\}}, \succsim_j^{I_j}) \in \mathcal{D}$, we have $\neg(F_i((\succsim_i^{I_i})_{i\in N\setminus\{j\}}, \succsim_j^{I_j}) \succ_j^{I_j} F_i(\succsim_i^{I_i})_{i\in N})$. Again, weak and strong versions coincide for complete inducements.

By Zhou (1990), we know that there is no probabilistic mechanism satisfying ex-ante efficiency, strategy-proofness and symmetry[10] for the domain of cardinal preferences. Since the extended domain includes cardinal preferences domain, this result applies to the extended domain as well. We can replace strategy-proofness with weak strategy-proofness as two notions coincide in the cardinal domain and replace weak no envy with symmetry since the former implies the latter.

Corollary 4. *For $n > 2$, there is no probabilistic mechanism which is efficient, weakly strategy-proof and weakly envy-free.*

Since the extended domain includes both fully cardinal and fully ordinal preference profiles, it is not surprising that the impossibility results of these domains apply to the extended domain as well. We may ask, however, how severe is the incompatibility between strategy-proofness and efficiency in the extended domain. Firstly, we study the potential deviations that agents with ordinal prefrences may make in the extended domain. Bogomolnaia and Moulin (2001) show that there is no probabilistic mechanism satisfying ordinal efficiency, strategy-proofness and no envy in the ordinal preferences domain. There exists, however, weakly strategy-proof, envy-free and ordinally efficient mechanisms such as the PS in the ordinal preferences domain. One may ask if this result applies to a set of agents with only ordinal preferences in the extended domain as well. The problem in the extended domain is that, agents with ordinal preferences may report cardinal preferences which is not possible in the domain of Bogomolnaia and Moulin (2001). It turns out that, we may use efficient improvements of the PS solution since such an improvement won't yield a stochastically dominating allocation for any agent:

Proposition 7. *Let P^{PS} be the PS solution for the inducement profile $(\succsim_i^{I_i})_{i\in N} \in \mathcal{D}$. If P is a $(\succsim_i^{I_i})_{i\in N}$-efficient improvement of P^{PS} that is P is $(\succsim_i^{I_i})_{i\in N}$-efficient and for every agent $i \in N$, $P_i \succsim_i^{I_i} P_i^{PS}$, then there is no agent $j \in N$ with $P_j \succ_j^{sd} P_j^{PS}$.*

Proof. We start with the following claim:

Claim: Given the inducement profile $(\succsim_i^{I_i})_{i\in N} \in \mathcal{D}$, let $P \in \mathcal{L}_n$ be an efficient probabilistic allocation. If there exists $\{a, b\} \subseteq A$ and $\{i, j\} \subseteq N$ with $a \succsim_i b$, $b \succsim_j a$ and $\neg(a \sim_k b)$ for either $k = i$ or $k = j$, then either $P_{ib} = 0$ or $P_{ja} = 0$.

Assume not true, that is $P_{ib}, P_{ja} > 0$. Consider $P' \in \mathcal{L}_n$ where $P'_i = P_i + \varepsilon(l_a - l_b)$ and $P'_j = P_j + \varepsilon(l_b - l_a)$ where $\varepsilon = \min\{P_{ib}, P_{ja}\}$. Clearly P' improves P, contradicting that P is

[10] Symmetry requires that agents with identical preferences to do not envy envy each others' probabilstic allocations.

efficient.

Now, suppose there exists an efficient improvement of P^{PS} where $P_{i_1} \succ^{sd}_{i_1} P^{PS}_{i_1}$ for some $i_1 \in N$. This implies $P_{i_1 a} > P^{PS}_{i_1 a}$ for some $a \in A$. Let $a_0 \in \max_{\succsim_{i_1}} \{a \in A | P_{i_1 a} > P^{PS}_{i_1 a}\}$. Clearly, $P_a = P^{PS}_a$ for $a \succ_{i_1} a_1$ since $P_{i_1} \succ^{sd}_{i_1} P^{PS}_{i_1}$. Then there exists $i_2 \in N$ with $P_{i_2 a_0} < P^{PS}_{i_2 a_0}$. Then there exists $a_1 \succsim_{i_2} a_0$ with $P_{i_2 a_1} > P^{PS}_{i_2 a_1}$ since otherwise $P^{PS}_{i_2} \succ^{sd}_{i_2} P_{i_2}$ contradicting $P_{i_2} \succsim^{I_{i_2}}_{i_2} P^{PS}_{i_2}$. Now $a_0 \succsim_{i_1} a_1$ by construction. Then above claim implies $P^{PS}_{i_1 a_1} = 0$ since $P^{PS}_{i_2 a_0} > P_{i_2 a_0} \geqslant 0$. Since $P_{i_2 a_1} > P^{PS}_{i_2 a_1}$, there exists $i_3 \in N$ with $P_{i_3 a_1} < P^{PS}_{i_3 a_1}$. Then there exists $a_2 \succsim_{i_3} a_1$ with $P_{i_3 a_2} > P^{PS}_{i_3 a_2}$ since otherwise $P^{PS}_{i_3} \succ^{sd}_{i_3} P_{i_3}$ contradicting $P_{i_3} \succsim^{I_{i_3}}_{i_3} P^{PS}_{i_3}$. Again $a_0 \succsim_{i_1} a_2$ by construction. Then above claim implies $P^{PS}_{i_1 a_2} = 0$ since $P^{PS}_{i_3 a_1} > P_{i_3 a_1} \geqslant 0$. Again, $P_{i_3 a_2} > P^{PS}_{i_3 a_2}$ implies that there exists $i_4 \in N$ with $P_{i_4 a_2} < P^{PS}_{i_4 a_2}$. Then there exists $a_3 \succsim_{i_4} a_2$ with $P_{i_4 a_3} > P^{PS}_{i_4 a_3}$ since otherwise $P^{PS}_{i_4} \succ^{sd}_{i_4} P_{i_4}$ contradicting $P_{i_4} \succsim^{I_{i_4}}_{i_4} P^{PS}_{i_4}$. Again $a_0 \succsim_{i_1} a_3$ by construction. Then above claim implies $P^{PS}_{i_1 a_3} = 0$ since $P^{PS}_{i_4 a_2} > P_{i_4 a_2} \geqslant 0$. Repeating this argument, we get that $P^{PS}_{i_1 a} = 0$ for all $a \in A$ with $a_0 \succsim_{i_1} a$ which is a contradiction.

□

Since efficient improvements can't make agents with ordinal preferences strictly better, a mechanism which uses efficient improvements is weakly strategy-proof for the agents with ordinal preferences:

Proposition 8. *There exists an efficient and weakly envy-free mechanism such that, agents with ordinal preferences can't be strictly better off by misreporting their preferences. That is, there exists an efficient and weakly envy-free probabilistic mechanism $F : \mathcal{D} \to \mathcal{L}_n$ such that for each $(\succsim^{I_i}_i)_{i \in N} \in \mathcal{D}$, we have $\neg \Big(F_j((\succsim^{I_i}_i)_{i \in N \setminus \{j\}}, \succsim^{I'_j}_j) \succ^{sd}_j F_j((\succsim^{I_i}_i)_{i \in N}) \Big)$.*

Proof. Let F be a probabilistic mechanism where $F((\succsim^{I_i}_i)_{i \in N})$ is an efficient improvement of the PS solution for the inducement profile $(\succsim^{I_i}_i)_{i \in N}$ a la proposition 7. Such an improvement always exists since if there is no distinct allocation improving the PS solution, the PS solution itself is an improvement.

Now by construction F is efficient and by proposition 7, $F_i((\succsim^{I_i}_i)_{i \in N}) = P^{PS}_i$ if $I_i = sd$ where P^{PS} is the PS solution for $(\succsim^{I_i}_i)_{i \in N}$. Suppose for some $j \in N$, $F_j((\succsim^{sd}_i)_{i \in N \setminus \{j\}}, \succsim^{I'_j}_j) \succ^{sd}_j F_j((\succsim^{sd}_i)_{i \in N})$. Let $\bar{P}^{PS}$ be the PS solution for the inducement profile $((\succsim^{sd}_i)_{i \in N \setminus \{j\}}, \succsim^{I'_j}_j)$, $F((\succsim^{sd}_i)_{i \in N}) = P$ and $F((\succsim^{sd}_i)_{i \in N \setminus \{j\}}, \succsim^{I'_j}_j) = \bar{P}$. Since the PS is weakly strategy-proof in the ordinal domain, $P^{PS}_j \succsim^{sd}_j \bar{P}^{PS}_j$. Now if $\bar{P}_j \succ^{sd}_j P^{PS}_j$ then $\bar{P}_j \succ^{sd}_j \bar{P}^{PS}_j$ contradicting proposition 7. Then $P_j \succsim^{sd}_j P^{PS}_j$ implies $\neg(\bar{P}_j \succ^{sd}_j P_j)$ that is $\neg \Big(F_j((\succsim^{I_i}_i)_{i \in N \setminus \{j\}}, \succsim^{I'_j}_j) \succ^{sd}_j F_j((\succsim^{I_i}_i)_{i \in N}) \Big)$.

Lastly, assume $P_j \succ^{I_i}_i P_i$. Since $P_i \succsim^{I_i}_i P^{PS}_i \succ^{sd}_i P^{PS}_j$, we have $P_j \succ^{sd}_i P^{PS}_j$. Then by proposition 7, we have $\succsim_i \neq \succsim_j$. But then $P_j \succ_i^{sd} P_j^{PS}$ contradicts $P_j \succsim_j^{I_j} P_j^{PS}$. Hence we get $\neg(P_j \succ_i^{I_i} P_i)$ implying that F satisfies weak no envy. $\square$

Above result simply states that there exists an efficient and weakly envy-free extension of the PS mechanism which is weakly strategy-proof in the ordinal domain. That means, agents with ordinal preferences can't make themselves strictly better off by reporting cardinal preferences. Therefore, the sole positive result of the ordinal domain survives in the extended domain so that cardinal deviations won't be disruptive for incentives for a population of agents with ordinal preferences.

Next we ask whether restricting all agents to only ordinal deviations would recover weak strategy-proofness. By restricting all agents to only ordinal deviations what we mean is that we require agents to report either their true preferences or an inducement with an underlying ordinal preference different than their true ordinal preference. It turns out that it is not possible to find an efficient and weakly envy-free mechanism which is immune to such deviations:

Proposition 9. *There exists an inducement profile such that agents with cardinal preferences may be better off by misreporting their ordinal preferences. That is let $n > 2$ and $F : \mathcal{D} \to \mathcal{L}_n$ be a probabilistic mechanism which is efficient and weakly envy-free. There exists an inducement profile $(\succsim_i^{I_i})_{i \in N} \in \mathcal{D}$ such that for some $\succsim_j^{I'_j}$ we have $F_j((\succsim_i^{I_i})_{i \in N \setminus \{j\}}, \succsim_j^{I'_j}) \succ_j^{I_j} F_j((\succsim_i^{I_i})_{i \in N})$ where there exists $a, b \in A$ with $l_a \succ_j^{I_j} l_b$ and $l_b \succ_j^{I'_j} l_a$.*

Proof. Let $n = 3$. Consider the ordinal preferences below:

1	2	3
a	a	b
c	c	a
b	b	c

(1)

1	2	3
a	a	b
b	c	a
c	b	c

(2)

1	2	3
a	a	b
b	b	a
c	c	c

(3)

1	2	3
a	a	a
b	c	b
c	b	c

(4)

Any efficient random assignment should be also ordinally efficient by corollary 2. The unique ordinally efficient outcome for the profile (1) is:

	1	2	3
a	1/2	1/2	0
b	0	0	1
c	1/2	1/2	0

So regardless of the inducements or the cardinal reports for the profile (1), an efficient probabilistic mechanism must yield above probabilistic allocation. For $x, y \in [0, 1]$ with $x \geqslant y$, ordinally efficient assignments for the profile (2) can be represented such that:

P	1	2	3
a	$1-x$	x	0
b	y	0	$1-y$
c	$x-y$	$1-x$	y

Consider agent 1 in profile (1). For this agent not to deviate from profile (1) to profile (2) regardless of his cardinal preferences we must have $x \geqslant 1/2$ since otherwise for some vNM profile, the random assignment under (2) yields higher expected utility.

Both agent 1 and agent 2 can switch to (2) from (3). Then if $x > 1/2$, one of them surely switch for some vNM profiles since one of them receives less than $1/2$ probability share of a. Then $x = 1/2$ so that the ordinally efficient assignments for the profile (2) becomes:

P	1	2	3
a	$1/2$	$1/2$	0
b	y	0	$1-y$
c	$1/2-y$	$1/2$	y

Now assume agent 1 and agent 3 has vNM preferences represented by u_1 and u_3 respectively in profile (2). Let $u_{1a}/u_{1b} = \gamma_1$, $u_{3b}/u_{3a} = \gamma_3$ and $u_{1c} = u_{3c} = 0$. Weak no envy implies $\gamma_1/2 + y \geqslant (1-y)$ since otherwise agent 1 envies agent 3. Then $y \geqslant 1/2 - \gamma_1/4$. Then agent 3 receives $(0, 1/2 + \gamma_1/4, 1/2 - \gamma_1/4)$ at best at profile (2).

Lastly, assume agent 1 and agent 3 has vNM preferences represented by v_1 and v_3 respectively in profile (4). Assume $v_{1a}/u_{1b} = v_{3a}/u_{3b} = \gamma_1$. By weak no envy, agent 1 and 3 must receive the same allocation which implies they both get $1/2$ of b. Then ordinally efficient and weakly envy-free assignments for profile (4) for these preferences can be represented as:

P'	1	2	3
a	z	$1-2z$	z
b	$1/2$	0	$1/2$
c	$1/2-z$	$2z$	$1/2-z$

We must have $1/2 + z\gamma_1 \geqslant (1-2z)\gamma_1$ since otherwise agents 1 and 3 envies agent 2. Then we have $z \geqslant 1/2 - 1/(6\gamma_1)$. But then the expected payoff of agent 3 at profile (2) is $\gamma_3(1/2 + \gamma_1/4)$

while her expected payoff is $\gamma_3/2 + z \geqslant \gamma_3/2 + 1/2 - 1/(6\gamma_1)$ if she switches to (4) and reports γ_1 . Then we must have $\gamma_3\gamma_1 1/4 + 1/(6\gamma_1) \geqslant 1/2$ otherwise agent 3 at profile (2) with vNM preference γ_3 can switch to profile (4) and report γ_1. However for $\gamma_1, \gamma_3 \in (1, 1+\varepsilon)$ for sufficiently small $\varepsilon > 0$, we have $\gamma_3\gamma_1 1/4 + 1/(6\gamma_1) < 1/2$. Contradiction. □

Above result shows that agents with cardinal preferences may gain by misreporting their ordinal preferences. The proof of Zhou (1990) regarding the impossibility of efficient, symmetric and strategy-proof mechanisms in cardinal domain relies on deviations on cardinal utility profiles that preserves underlying ordinal preferences. This result shows that even if we require misreports to induce different ordinal preferences, agents with cardinal preferences can still benefit from deviation.

We can further illustrate this point on the PM mechanism with equal endowments. Consider a 3×3 assignment problem with the following ordinal preference profile for agents:

1	2	3
a	a	b
c	b	c
b	c	a

Instead of picking a price on the simplex we may equivalently use a normalization where the smallest price is zero. Assume $p_c \geqslant p_a$, then$P_{1c} = P_{2c} = 0$ and therefore $P_{3c} = 1$ implying that $p_b > p_c$. But then $p_b > p_c \geqslant p_a = 0$ implies $P_{1a} = P_{2a} = 1$ which is a contradiction. Then $p_a > p_c$. If $p_b \geqslant p_a > p_c = 0$, then $P_{1b} = P_{2b} = 0$ implying $P_{3b} = 1$. Then $p \cdot E_3 = \frac{p_a + p_b}{3} \geqslant p_b$ implying $p_a \geqslant 2p_b$. Then we have $p_a > p_b > p_c = 0$.

Given this, agent 1 demands $\left(\frac{1}{p_a}, 0, 1 - \frac{1}{p_a}\right)$ and agent 3 demands $\left(0, \frac{1}{p_b}, 1 - \frac{1}{p_b}\right)$. If we assume stochastic dominance order for agent 2, their demand is a convex combination of $\left(\frac{1}{p_a}, 0, 1 - \frac{1}{p_a}\right)$ and $\left(0, \frac{1}{p_b}, 1 - \frac{1}{p_b}\right)$. Therefore the allocations look like:

	1	2	3
a	$\frac{1}{p_a}$	$\lambda\frac{1}{p_a}$	0
b	0	$(1-\lambda)\frac{1}{p_b}$	$\frac{1}{p_b}$
c	$\frac{p_a-1}{p_a}$	$\lambda\frac{p_a-1}{p_a} + (1-\lambda)\frac{p_b-1}{p_b}$	$\frac{p_b-1}{p_b}$

This implies $p_a = 1 + \lambda$ and $p_b = 2 - \lambda$. Then equilibrium allocations look like:

	1	2	3
a	$\frac{1}{1+\lambda}$	$\frac{\lambda}{1+\lambda}$	0
b	0	$\frac{1-\lambda}{2-\lambda}$	$\frac{1}{2-\lambda}$
c	$\frac{\lambda}{1+\lambda}$	$\frac{1-\lambda+\lambda^2}{2+\lambda-\lambda^2}$	$\frac{1-\lambda}{2-\lambda}$

for some $\lambda \in \left(\frac{1}{2}, 1\right]$.

Now assume agent 2 uses a vNM inducement with bernoulli utilities $(u_a, u_b, 0)$ where $u_a > u_b > 0$. Let $\gamma = u_a/u_b$. If $\gamma > \frac{p_a}{p_b} = \frac{1+\lambda}{2-\lambda}$, agent 2 demands no probability shares of b therefore we get $\lambda = 1$. $\gamma < \frac{p_a}{p_b} = \frac{1+\lambda}{2-\lambda}$ implies agent 2 demands no probability shares of a which means $\gamma < 1/2$ contradicting $u_a > u_b$. For $\gamma = \frac{p_a}{p_b} = \frac{1+\lambda}{2-\lambda}$ agent 2 is indifferent between two extreme points and can potentially demand positive probability share of each object. Then we get that $\lambda = \frac{2\gamma-1}{\gamma+1}$ for $1 < \gamma \leqslant 2$. Then in terms of γ, equilibrium allocations look like:

	1	2	3
a	$\frac{1+\gamma}{3\gamma}$	$\frac{2}{3} - \frac{1}{3\gamma}$	0
b	0	$\frac{2-\gamma}{3}$	$\frac{1+\gamma}{3}$
c	$\frac{2}{3} - \frac{1}{3\gamma}$	$\frac{1-\gamma+\gamma^2}{3\gamma}$	$\frac{2-\gamma}{3}$

where $\gamma \in (1, 2]$. Notice that given the true $\hat{\gamma}$, agent 2 should report γ by solving below maximization problem:

$$\max_{\gamma \in (1,2]} \left(\frac{2}{3} - \frac{1}{3\gamma}\right) u_a + \left(\frac{2}{3} - \frac{\gamma}{3}\right) u_b$$

First order condition for this problem yields $\gamma = \sqrt{\hat{\gamma}}$ and the objective function is concave in the relevant domain. Then reporting $\sqrt{\hat{\gamma}}$ instead of $\hat{\gamma}$ yields higher expected utility for agent 2. This is not surprising, considering that pseudomarket mechanisms are in general not strategy-proof. What is surprising is that, if $\hat{\gamma} = 2$, reporting only ordinal preferences and thereby choosing a stochastic dominance inducement is weakly better than reporting true vNM order for a vNM utility maximizer. Below graph shows expected utility of agent 2 as a function of reproted γ when $\hat{\gamma} = 2$. It turns out $\gamma = 2$ is actually a minimum of what the second agent may get from reporting only ordinal preferences.

Proposition 10. *For PM mechanisms, an agent with fully cardinal preferences may be weakly better off by reporting only ordinal preferences. That is, there is a preference inducement profile $(\succsim_i^{I_i})_{i \in N} \in \mathcal{D}$ such that for every PM mechanism F, there is $j \in N$ with $P'_j \succsim_j^{I_j} P_j$ where $F(\succsim_i^{I_i})_{i \in N} = P$ and $F((\succsim_i^{I_i})_{i \in N \setminus \{j\}}), \succsim_j^{sd}) = P'$.*

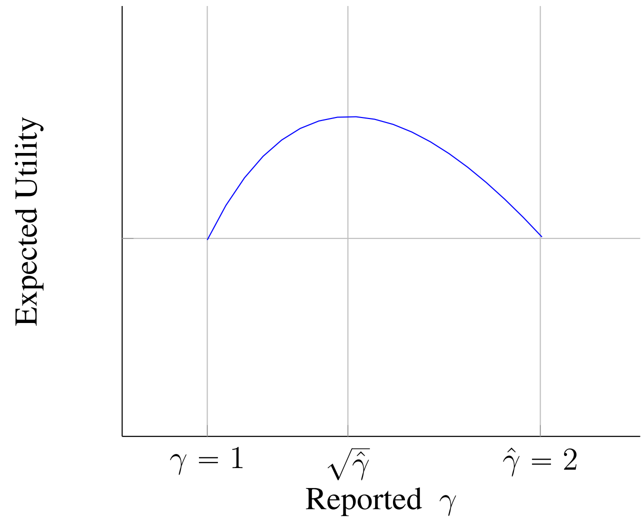


Figure 1.4: True report of cardinal preferences yields the worst outcome when $\hat{\gamma} = 2$

By truthfully reporting their cardinal preference, agent 2 ignores the effect of their preferences on the prices. In this particular example, agent 2 is pivotal in the sense that their preference directly determines the prices and therefore the allocation. A true report of cardinal preferences has two effects on the resulting outcome for agent 2: Firstly, given the prices, agent 2 receives more of what they value more according to their preferences -let's call this the demand effect. Secondly, true report of preferences increases the aggregate demand and therefore the price of more desired objects -and let's call this THE price effect. When the price effect is stronger, as in this example, a rational agent should downplay the cardinal utility differences between different objects, whereas when the demand effect is higher, a rational agent should exaggerate the cardinal utility differences between different objects. The discrepancy between the best report of the cardinal preferences and the true report of the cardinal preferences can be very large that even reporting only ordinal preferences and thereby hiding cardinal preferences may be better than the true report of cardinal preferences.

1.6 Conclusion

This chapter introduces an extension to the preference domain of the assignment problem to include cardinal, ordinal and mixed preferences along with a preference reporting language enabling agents to report preferences from this domain. Practically, this domain allows the mechanism designer to offer agents access to a richer preference reporting language. Doing so improves efficiency of the outcomes and grants flexibility to the participants to report their preferences in varying degrees of detail. Theoretically, we are able to ask new questions about the cardinal domain and the ordinal domain that we were not able to ask before. We ask, in particular, whether the positive results of the ordinal domain survives when agents have the ability to report cardinal preferences. It turns out that, the weak strategy-proofness of the PS mechanism is preserved via efficient improvements of the PS solution for the agents with ordinal preferences. We were also able to gauge the strength of the impossibility result concerning incentive compatibility and efficiency in the cardinal domain. We show that, requiring cardinal agents to make ordinal deviations is not enough to defeat this impossibility result. Furthermore, we explore the Pseudomarket mechanisms in this domain and show that the equal income Pseudomarket equilibrium outcome may be distinct from the PS outcome in the ordinal domain. Although the PS is often cited to be the standard mechanism to be used in the ordinal domain, we showed that the equal income Pseudomarket solution has certain desirable properties that the PS is lacking. On the other hand, the tension between incentive compatibility and efficiency is striking for the equal income Pseudomarket mechanisms. Agents with cardinal preferences may be weakly better by reporting only ordinal preferences.

One obvious avenue for future research is to employ the extended domain for other market design environments, such as the two-sided matching markets or the combinatorial assignment problem. Another avenue for further research may be to find computationally feasible mechanisms in the extended domain. Pseudomarket mechanisms require computation of equilibrium price vectors and there is no straightforward way to select a particular equilibrium. A mechanism, which still belongs to the class of PM mechanims if it is efficient, that would skip the equilibrium price determination can be very helpful for the actual implementation of this domain and associated preference reporting language. Lastly, the extended domain can be used to study the relationship between sophistication of agents and truth-telling in a richer domain. We show that one may find a mechanism for which agents with ordinal preferences may not benefit form appearing more sophisticated than they are. Can we find a mechanism that would generalize this result to mixed types? The same question can be asked in the other direction. Would it be beneficial to appear less sophisticated? The answer is yes for the equal income Pseudomarket mechanism where cardinal agents may benefit from reporting ordinal preferences. Is there a mechanism that would punish agents appearing less sophisticated? We leave these questions for future research.

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Chapter 2

Information Aggregation from Anonymous Sources in Competitive Environments

Abstract

We extend a two-person simultaneous move guessing the state game with a communication step facilitating information transfer between players. There are two types of players, the insider and the trader where the trader does not know who he is playing against. We study two mechanisms in this environment that is prevalent in practice, specifically in opening and closing call auctions. In the optimal deterministic communication mechanism, the principal produces an optimal public signal that aggregates information to be shared in the communication process. In the direct probabilistic communication mechanism, the game reaches the communication step with some probability and in the communication step, a fully revealing -not necessarily optimal- public signal is observed. We show that the former mechanism fails to aggregate information with rational agents while the latter mechanism achieves some information aggregation if the communication step is reached. If the traders are not fully rational, however, the optimal deterministic communication mechanism too achieves some information aggregation. For a Gaussian signal structure, the trader's ex-ante payoff with the optimal deterministic communication mechanism is higher as long as the trader's information is noisy but the likelihood of having an insider is low.
Key Words: financial market design, information aggregation, market micro structure, insider trading, cheap talk
JEL Codes: D82, D83, D84, G14

2.1 Introduction

How can a market maker incentivize information sharing in a competitive environment? Consider a simultaneous game with pre-play communication where players' payoffs are perfectly misaligned. Individually, sharing any valuable information is sub-optimal since agents' payoffs are perfectly misaligned. Yet, sharing information increases overall welfare as it leads agents to make better decisions as a group. One way to stimulate information sharing would be to penalize for wrong messages. This arises naturally in stock markets during day trading. The orders submitted by traders may be considered as signals about the traders' beliefs on the fundamentals, and misleading signals -orders in a different direction or magnitude than what the private information suggests- are costly. For the opening and closing auctions which determine open and close prices, however, there is no direct cost to a misleading or noisy signal/order. This is because it is possible to submit an order beforehand to be executed in open or close with the option to change the order before the execution time. The market maker may try to replicate the cost of misleading/noisy orders in day trading by introducing restrictions on order submission. These restrictions, however, will restrict the information aggregation as well. Particularly, a common method for the opening/closing auctions is to make execution time random, so that traders who attempt to hide information or deceive others would risk the execution of their faulty order. Although this does prevent such behavior to some extent, it also diminishes the time horizon for information aggregation. An alternative approach is to allow traders to submit or cancel orders freely without risk of orders being executed at a random time, but censor the information about submitted orders. This approach involves a more direct restriction to information aggregation to find the optimal information aggregation scheme. Our central question in this chapter is that if directly penalizing wrong messages is not possible, what kind of restrictions would aggregate information more efficiently.

In particular, we analyze two mechanisms, the optimal deterministic communication mechanism, and the direct probabilistic mechanism. We start with a simple guessing the state game with two players and two types. The insider type knows the true state while the trader type receives a noisy signal. With some probability, one of the players is of insider type while with the remainder probability both players are trader types. We, then, introduce a pre-play communication step where we introduce two mechanisms of information aggregation. With both mechanisms, players send a message about their private information to a third party, the principal. The principal, then, creates an optimal public signal in the optimal deterministic communication mechanism. In the direct probabilistic mechanism, on the other hand, the game ends right after the messages are sent to the principal where the messages are considered to be players' final guesses for the state. With the remainder probability, the principal directly reveals the messages she receives.

We show that the optimal deterministic communication mechanism fails to aggregate information, even when we assume the players of the trader type can coordinate on truth-telling strategies to achieve socially optimal outcomes. The possibility that there might exist an insider who can't benefit from information aggregation prevents any player from sending any kind of informative message. The direct probabilistic mechanism, on the other hand, does achieve full information aggregation if the game proceeds to the revelation of the messages. The dominance of the direct probabilistic mechanism does not hold if we relax the rationality assumption, however. If the trader types have the incorrect belief that there is no insider in the game, the optimal deterministic communication mechanism outperforms the direct probabilistic mechanism when the noise of the private information of trader types is high enough. This is because the probability that the game ends without communication in the direct probabilistic mechanism needs to be higher to force the insider to reveal the true state when the private information of trader types is noisier.

The most straightforward application of our theoretical analysis is with the financial markets. Indeed, In financial markets, prices as well as any other information disseminated by the market makers are essentially public signals that aggregate information potentially imperfectly. The effectiveness of the information aggregation directly relates to market efficiency as it determines the informativeness of the prices. The efficient market hypothesis, the supposition that prices fully and immediately reveal all relevant information, can be only achieved when there is perfect information aggregation. In reality, however, the markets are not perfectly efficient. One particular reason for this is that in the presence of a trader with significantly superior information, such as an insider, information aggregation may lead to manipulation since the insider might increase their benefit by misleading the market. Having superior information lets the insider choose the correct position while misleading the market decreases the cost of holding that position. As a matter of fact, in financial markets, traders strive to capture excess gains relative to market performance. This can happen naturally when the insider has an opposite view about the worth of an asset relative to the market. The insider has an incentive to expand the extent of this disagreement through manipulative strategies to get additional benefits. Such manipulations not only lead other traders to act on inaccurate signals, but it also disrupts the information exchange among them.

These kinds of manipulative strategies are difficult to detect in general. However, we can look at some indirect effects in the form of reversals to get a sense of how such manipulative strategies would affect the prices. If there were no manipulation, no spoofing[1], we would expect the prices

[1]Spoofing is a term for this type of manipulation, where an insider submits an order that may not eventually be executed to mislead other players.

following a somewhat steady trend, upward, downward, or flat once the market is open. With the opening prices being determined by call actions that will be described in detail below, whatever information that arrived during the off-market hours would be priced in and the arrival of new information would lead the price during the daytime trading.

However, in reality, trend reversals right after market opening is not a rare occurrence. On the contrary, seasoned traders are not just aware of the possibility of morning reversals, but they devise strategies that help them detecting these reversals in time.[2] We provide a preliminary empirical study of reversals in the section 2.2. In a nutshell, our finding is in line with the literature in the sense that as the market gets thinner, price discovery becomes a more volatile process and price efficiency is harder to achieve. Indeed, many empirical studies including Felixson and Pelli (1999), Hillion and Suominen (2004) and Khwaja and Mian (2005) reiterates this point. This is an important point highlighting the fact that the models with non-atomic players whose decisions directly affect each other may perform better in explaining the information aggregation in financial markets.

Reversals happening right after market opening suggests that the market maker should pay special attention to market openings. In practice, market makers use a trading procedure named call auction. Opening call auctions are the double auctions conducted to determine the opening price in various stock exchanges including NYSE and NASDAQ. Although the use of call auctions are a common practice among stock exchanges, the actual implementation varies greatly among them. All exchanges open the floor for orders a few hours before the market opening. They start to disseminate information about submitted orders within the last 30 minutes to the last 5 minutes. The type of information disseminated varies among exchanges and may include market imbalance, the hypothetical price if the auction was to be executed at that time, total volume, etc. NYSE and NASDAQ execute the auction at a pre-determined time while London Stock Exchange executes the auction at a random time in a given time period thereby ending the communication process probabilistically. Based on these differences in how communication is facilitated in these stock exchanges, we analyze two mechanisms of information aggregation. With the optimal deterministic communication mechanism, the market maker designs an information dissemination procedure that facilitates information transaction among investors while disincentivizing manipulation. The direct probabilistic communication mechanism, however, adopts a simple information aggregation structure and instead has a communication step that is reached probabilistically.

Under the optimal deterministic communication mechanism, the market maker tries to come up

[2] Playing the gap or gap reversal is a common trading strategy that focuses on predicting reversals including early morning reversals.

with an optimal information aggregation structure that would maximize price efficiency. The main problem with this approach is that although the market maker is aware of the dealers and brokers that submit orders, they have no way of knowing who that particular dealer or a broker is trading on behalf of. In essence, the problem lies in the fact that the market maker is unaware of which traders have an informational advantage. Indeed, if the market maker is aware of say who has insider information, the information aggregation problem would be simpler if not trivial[3]. When the market maker is unaware of who has what kind of information, however, the effectiveness of information aggregation procedures is not clear, especially in thin markets. With the direct probabilistic communication mechanism, on the other hand, the market maker uses a simple reveal all information structure, but randomizes the execution time of the auction. This randomization in the execution time essentially makes the orders that are put with the purpose of manipulation costly. A manipulative order that sells the undervalued or buys the overvalued stock might get executed.

This chapter studies the optimal deterministic communication and direct probabilistic communication mechanisms for the information aggregation problem with anonymous agents. We utilize a stylized guessing the true state game with two players. In this game, players get additional benefits from their opponents' mistakes which captures the additional benefits derived from going against the market when the market is wrong. There are two types of players in this game, one is an insider who knows the true state and the other is a trader who has noisy information about the true state. There is always one trader type playing the game, but the other player may be of insider type or another trader. The crucial part of the game is that the trader does not know if he is playing against an insider or a trader. Studying a simple guessing the state game instead of an auction or demand schedule games allow us to study information-side strategic interaction in a tractable way.

After introducing the model in the second section, the third section studies the optimal deterministic communication mechanism by introducing the principal to guessing the true state game to facilitate communication. We show that there can be no information aggregation with the optimal deterministic communication mechanism even when we assume the trader types coordinate on strategies that maximize the ex-ante welfare of all trader type players. The fourth section studies the direct probabilistic communication mechanism where the players' messages are taken as their final action with some probability and their reports are directly revealed with the remainder probability. Information aggregation is possible with this mechanism, and under some mild conditions, the information aggregation is perfect whenever the communication step is reached.

[3]Knowing which trader has what quality of information would allow the market maker to selectively censor order information.

Lastly, we introduce a behavioral variant of the model where the traders hold the incorrect belief that they always play against another trader -that is, there is no trader. With the behavioral model, both mechanisms aggregate information. With the optimal deterministic communication mechanism, the trader types learn from each other while the principal dampens information aggregation to minimize insider manipulation. On the other hand, with the direct probabilistic communication mechanism, the trader types learn from the insider as the principal sets the probability that the game ends without communication high enough. For a Gaussian signal structure, the trader's ex-ante payoff with the optimal deterministic communication mechanism is higher as long as the trader's information is noisy but the likelihood of having an insider is low.

Literature Review: Although relatively an understudied subject, there is a widening strand of theoretical research working on information based market manipulations (as opposed to market based manipulation as in Allen and Gale (1992) and Allen and Gorton (1991)). An early attempt by Vila (1989) studies simple games where noise traders provide cover for manipulators to hide their trades. In Bagnoli and Lipman (1996), a manipulator induce incorrect beliefs on stockholders about a takeover. Van Bommel (2003) utilizes Kyle (1985) framework to model informed investors that manipulate prices by spreading imprecise rumor. Eren and Ozsoylev (2006) uses the same framework to study hype-and-dump manipulation. In Chakraborty and Yılmaz (2004), the market faces uncertainty about the existence of the insider and long-lived informed traders manipulate short lived informed traders.

Another line of literature related to this chapter studies strategic information aggregation in markets with finitely many players. Dubey et al. (1987) extends the usual rational expectation equilibrium setting by introducing a multiperiod price discovery process. In Vives (2011), finitely many sellers with common and private values engage in supply function competition, while Rostek and Weretka (2012) studies information aggregation in a Gaussian model with private values. In Ostrovsky (2012), partially informed strategic traders dynamically trade securities and asymptotic information aggregation is achieved for 'separable' securities. Another dynamic strategic market model, (Rostek and Weretka, 2015), studies a consumption based model of demand function competition.

This chapter also relates to the cheap talk literature, (Crawford and Sobel, 1982). In particular, the costly signal cheap talk model of Kartik (2009) finds that the information aggregation among sender and receiver increases with the size of the cost introduced. In our model, the indirect cost introduced by the direct probabilistic communication mechanism forces the insider to reveal his information fully if the communication introduced. The indirect cost, the probability that the game ends before the communication step is reached, to be introduced for full information aggregation increases with the noisiness of the information of the traders.

Lastly, our player is related to financial market design literature. Budish et al. (2015) suggest using batch auction instead of continuous limit order book mechanism. O'Hara (2015) proposes potential directions to reform current high frequency market microstructure.

2.2 Reversals and S&P 500

As an empirical evidence of the potential manipulation in opening auctions, in this section we briefly analyze the reversals that happen right after market opening. To illustrate what we mean by a trend reversal, we plot the price movement of two stocks right after the market opening.

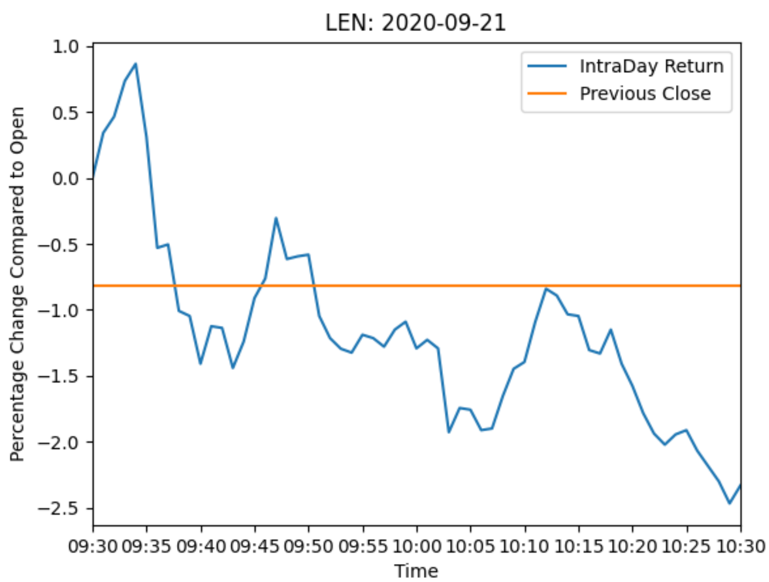

Figure 2.1: Downward Reversal After Market Opening

On figure 2.1, we see that the opening price is larger than the previous close and during the first five minutes, the intraday return compared to the opening price reaches almost 1%. But this gain is reversed within a few minutes and by 9:40 am, the stock price is below the previous close.

On figure 2.2, a similar dynamic plays out in the opposite direction. The opening price is lower than the previous close and in the first 5 minutes of daytime trading, the intraday return almost reaches -0.5%. However, this trend is quickly reversed, and by 10:00 am the stock price passes the previous close level.

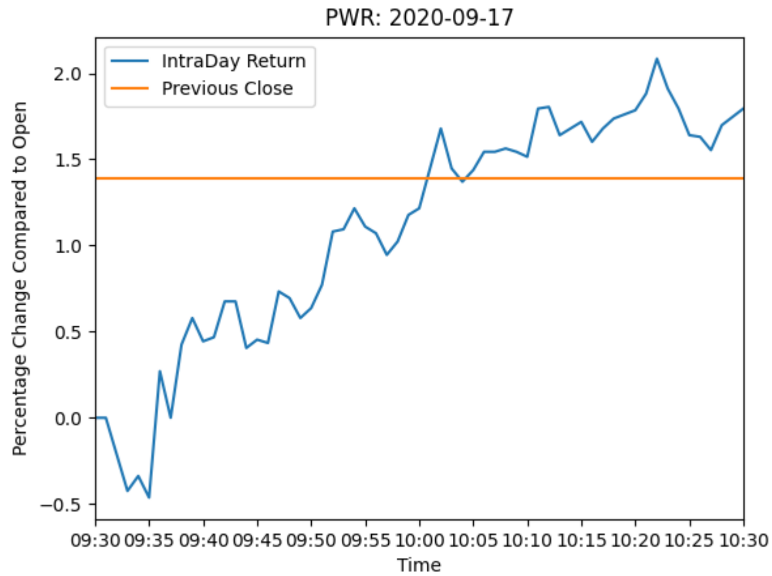

Figure 2.2: Upward Reversal After Market Opening

Of course, these graphs do not definitively demonstrate that there is an insider manipulation of stock prices. The above graphs do provide anecdotal evidence for reversals right after market opening suggesting inefficiency of the market opening process. To get a feeling of how often such reversals occur, we define a reversal parameter for a stock within a given time period by multiplying maximum percent gain with maximum percent loss. More specifically, let P_t denote the mid-price of a stock at time t and P_{t_0} denote the opening price. The reversal parameter for a given time window, $\triangle$, is defined such that:

$$\text{Reversal}_{\triangle} = \left(100 \times \frac{\max\{P_s | t_0 \leqslant s \leqslant t_0 + \triangle\} - P_{t_0}}{P_{t_0}}\right) \times \left(100 \times \frac{P_{t_0} - \min\{P_s | t_0 \leqslant s \leqslant t_0 + \triangle\}}{P_{t_0}}\right)$$

Notice that the reversal parameter is always non-negative. It will take the value of zero if the stock price does not fall below the opening price or does not exceed the opening price during the time window. If the stock price increases a maximum of 1% and decreases a maximum of -1% during the time window, the reversal parameter will take the value of one. The reversal parameter behaves like variance except that it will increase with the variance around the opening price. A stock price that is consistently higher than the opening price during the time window will have a zero reversal value no matter how volatile the price is.

We compute the reversal parameter for each stock listed in S& P500 for every trading day in 2018, 2019, and 2020 until October 10, 2020. Below we plot the kernel density distribution for time windows of 10, 15, and 30 minutes for different quantiles based on market capitalization:

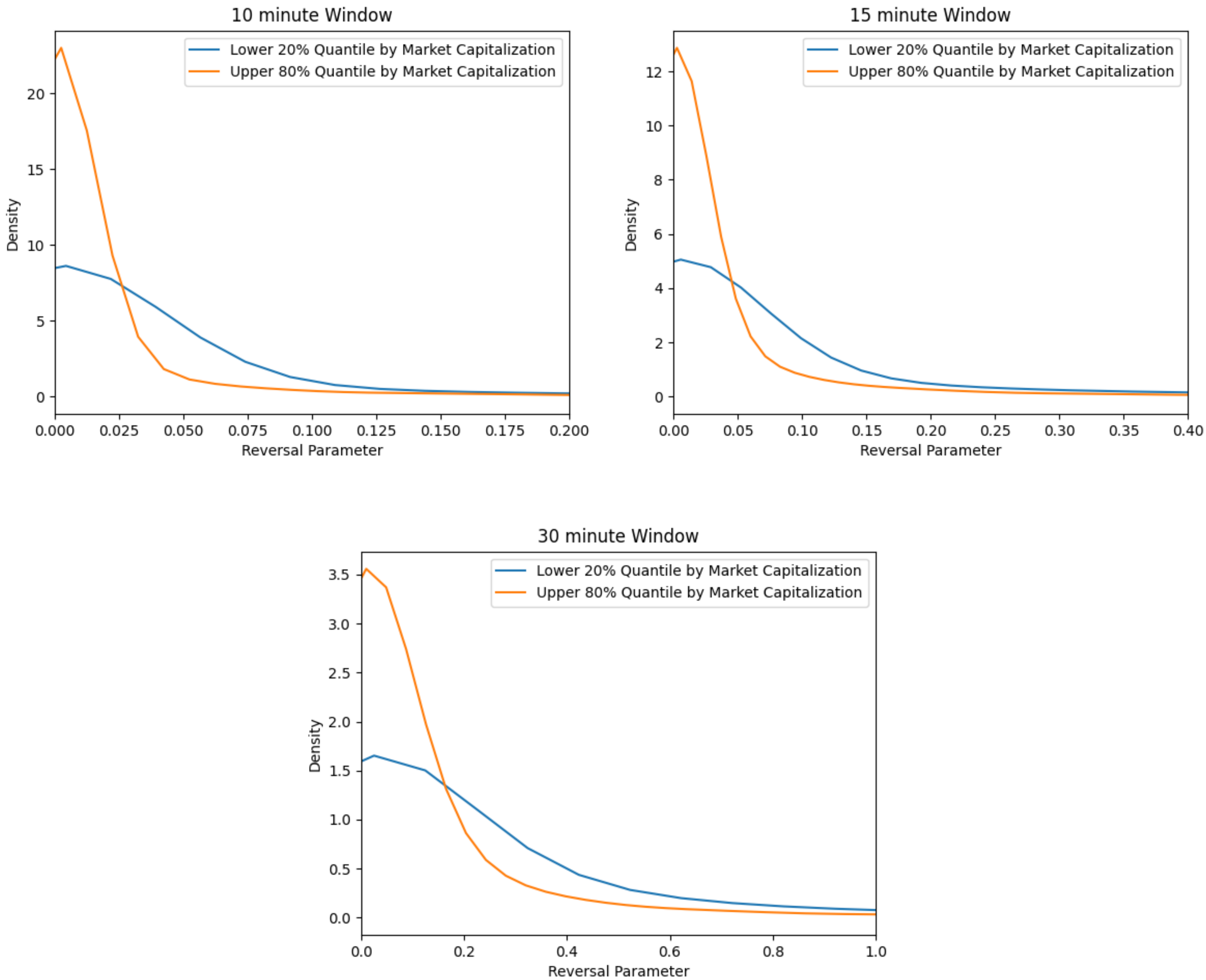

Figure 2.3: Reversal Parameter and Market Capitalization

As expected the mean and variance of the reversal parameter increases with the window size. More strikingly, for the stocks belonging to the lower 20 % quantile of the market capitalization distribution, the reversal parameter has significantly higher mean and variance. This suggests that the reversals are more common with low market capitalization stocks. This is in line with the literature in the sense that as the market gets thinner, price discovery becomes a more volatile process and price efficiency is harder to achieve. This finding highlights the importance of studying manipulation with models having finitely many players as opposed to a continuum players. The finite number of players approach allow for players to be informationally large and affect the beliefs of rest of the players.

2.3 The Model

We consider the following simultaneous move game with two players, $i \in \{0, 1\}$, where the payoff structure is given such that:

$$u_i(a, \theta) = -(a_i - \theta)^2 + \kappa(a_j - \theta)^2$$

Here $\kappa \in (0, 1)$ is a constant and θ is a random variable following a know prior distribution, $F_0(\cdot)$,with full support $\Theta \subseteq \mathbb{R}$. Action space is S, that is $a_i \in S \subseteq \mathbb{R}$ for $i \in \{1, 2\}$ where $S = \text{co}(\Theta)$.Each player receive a private signal that is independent conditional on θ. There are two types of players, the trader type and the insider type. If player i is of insider type, he receives the signal $s_i = \theta$ while if player i is of trader type, he receives a noisy signal $s_i \in S$ where $F(s_i|\theta)$ has a known distribution. The distribution $F(s_i|\theta)$ follows monotone likelihood ratio principle in the sense that for all $\theta, \theta' \in \Theta$ and $s, s' \in S$ with $\theta \geqslant \theta'$ and $s \geqslant s'$, we have that:

$$\frac{f(s|\theta)}{f(s|\theta')} \geqslant \frac{f(s'|\theta)}{f(s'|\theta')}$$

The type space is such that with π probability both players are the trader type, that is $\Pr(0, 1 \in T) = \pi$, while with $1 - \pi$ probability one of the players is of the insider type that is $\Pr((0 \in I \wedge 1 \in T) \vee (0 \in T \wedge 1 \in I)) = 1 - \pi$.

Interpretation of the payoff structure: This game is simply a guessing the state game with additional benefit from the opponents failure. Although the opponent's failure directly improves the payoff of a player, the game is not a zero-sum game since $\kappa < 1$. One can interpret θ as the fundamental value of an asset, and players as two investors trying to decide which position to take. Picking the same position as the other investor increases the cost of the correct position, therefore picking the right position yields better return if the opponent picks the wrong position.

In the unique Bayesian Nash equilibrium of this game, each player should set $a_i = \mathbb{E}[\theta|s_i]$ regardless what strategy their opponent follows. As a result, the players payoff will be proportional to their private information. Now, consider an outsider (the principal) intervening before the game to facilitate communication between players. The principal's goal is to maximize ex-ante welfare of the trader. The assumption that the principal maximizes the trader's welfare as opposed to total welfare does not change our qualitative results. Since the insider has nothing more to learn about θ, his main goal is to make sure that the trader relies on as faulty information as possible. Because of this, we use a notion of optimality that only depends on the trader's ex-ante welfare.

Clearly the best the principal can make is to reveal the true space to the trader type. The principal, however, does not posses private information about θ or the players' types. The principal receives anonymous reports of the private signals of the players and given these reports, she produces a public signal. Finally, upon observing the public signal, the players play above simultaneous move game.

In our baseline model, which we dub as the optimal deterministic communication mechanism, the timing of the game is summarized as:

The Optimal Deterministic Communication Mechanism

Step 0: The nature picks θ and the types of the players. Each player receives their private signal, s_i, based on their type.

Step 1: Each player sends a report $m_i \in S$ to the principal.

Step 2: Given the set of reports, $\{m_i, m_j\}$, the principal picks a public signal s_p from the set $S_p = S^2$. Upon observing s_p, each player picks their action and the outcomes are realized.

For the alternative mechanism, which we call the direct probabilistic communication mechanism, with probability q the game ends after the first step while with probability $1 - q$ the game proceed the second step. Then we can re-write the second step for the direct probabilistic communication mechanism as:

The Direct Probabilistic Communication Mechanism

Step 2' With probability $q \in (0, 1)$, returns are realized where $a_i = m_i \in S$ and with probability $1 - q$, the public signal $\{m_i, m_j\}$ is observed by the players who in turn picks their final action a_i.

Optimal Deterministic Communication vs Direct Probabilistic Communication: The principal's goal, maximizing the trader's payoff, is equivalent to maximizing the information that the trader receives before taking the final action. The two mechanisms we consider here differ in terms of what the principal chooses. With the optimal deterministic communication mechanism, the principal picks an optimal public signal that would update the trader's beliefs to a sharper posterior. On the other hand, with the direct probabilistic communication mechanism, the principal simply picks the probability that the game ends without the communication step.

Before going into our analysis, we note that the revelation principal does not apply to our setting since F_0 and $F(\cdot|\theta)$ are full support and the signal space is equal to the action space. This means that the players cannot communicate their types to the principal without interfering their report for their private signals. As we like to capture the behavior of the players and the related design question for real world competitive environments such as the stock markets, we refrain from enabling the players to report richer messages than their private information about θ, 'the fundamental'. On the other hand, if we consider an extended version of this model where the players can send reports from a richer set which would allow them to report their types without interference, the revelation principal will be applicable. Since it is not possible to improve the trader without making the insider worse off, however, the incentive compatibility constraints dictate any outcome that the principal can implement won't involve information transfer from the insider to the trader. This is in line with our findings in the optimal deterministic communication mechanism section. For the direct probabilistic communication mechanism, we find that the insider does give away some information. The discrepancy comes from the fact that our direct probabilistic communication mechanism does not correspond to a classical mechanism design problem due to the restriction on the domain of mechanisms that the principal can choose.

2.4 The Optimal Deterministic Communication Mechanism

With the optimal deterministic communication mechanism, the principal's problem is simply choosing the public signal that would maximize trader's ex-ante welfare. Since all players are Bayes rational and the principal possesses no private information, it turns out that directly revealing the reported signals is an optimal public signal:

Proposition 11. *Without loss of generality, the principal sets $s_p = (m_i, m_j)$.*

Proof. Assume that given $\{m_i, m_j\}$, the principal sends the signal $s_p \in S_p$ with probability $\sigma_p(s_p|m_i, m_j)$ where $\sigma_p(\cdot|m_i, m_j) \in \triangle(S_p)$. Let $\Pr(i \in T|m_i)$ denote the probability that i is of trader type given his report m_i. The interim social welfare , given $\hat{s}_i, \hat{s}_j$ is then:

$$
\begin{aligned}
W(m_i, m_j) &= \mathbb{E}[-\Pr(i \in T|m_i)(\mathbb{E}[\theta|s_i^T, m_i, s_p] - \theta)^2 - \Pr(j \in T|m_j)(\mathbb{E}[\theta|s_j^T, m_j, s_p] - \theta)^2)|m_i, m_j] \\
&= -\Pr(i \in T|m_i)\,\mathbb{E}[(\mathbb{E}[\theta|s_i^T, m_i, s_p] - \theta)^2|m_i, m_j] \\
&\qquad - \Pr(j \in T|m_j)\,\mathbb{E}[(\mathbb{E}[\theta|s_j^T, m_j, s_p] - \theta)^2)|m_i, m_j]
\end{aligned}
$$

Now we have that:

$$
\Pr(\theta|s_p) = \frac{\sigma_p(s_p|m_i, m_j)\Pr(m_i, m_j|\theta) f_0(\theta)}{\int_{m_i}\int_{m_j}\sigma_p(s_p|m_i, m_j)\Pr(m_i, m_j|\theta) f_0(\theta) dm_i dm_j}
$$

Above implies s_p is a garbling of (m_i, m_j) and therefore for all $s_p \in S^2$:

$$
\mathbb{E}[(\mathbb{E}[\theta|s_i^T, m_i, s_p] - \theta)^2|m_i, m_j] \geqslant (\mathbb{E}[\theta|s_i^T, m_i, s_p = (m_i, m_j)] - \theta)^2|m_i, m_j]
$$

And similarly for all $s_p \in S^2$:

$$
\mathbb{E}[(\mathbb{E}[\theta|s_j^T, m_j, s_p] - \theta)^2|m_i, m_j] \geqslant (\mathbb{E}[\theta|s_j^T, m_j, s_p = (m_i, m_j)] - \theta)^2|m_i, m_j]
$$

Then we have:

$$
\begin{aligned}
W(m_i, m_j) &= -\Pr(i \in T|m_i)\,\mathbb{E}[(\mathbb{E}[\theta|s_i^T, m_i, s_p] - \theta)^2|m_i, m_j] \\
&\qquad - \Pr(j \in T|m_j)\,\mathbb{E}[(\mathbb{E}[\theta|s_j^T, m_j, s_p] - \theta)^2)|m_i, m_j] \\
&\leqslant -\Pr(i \in T|m_i)\,\mathbb{E}[(\mathbb{E}[\theta|s_i^T, m_i, s_p = (m_i, m_j)] - \theta)^2|m_i, m_j] \\
&\qquad - \Pr(j \in T|m_j)\,\mathbb{E}[(\mathbb{E}[\theta|s_j^T, m_j, s_p = (m_i, m_j)] - \theta)^2)|m_i, m_j]
\end{aligned}
$$

Hence, we can conclude that, without loss of generality, the principal sets $s_p = (m_i, m_j)$. □

Above proposition allows us to simplify the game so that we can replace the step 2 with:

Step 2": Principal sets $s_p = (m_i, m_j)$. Upon observing s_p, each player picks their action and the outcomes are realized.

The principal is redundant for the optimal public signal design and what we end up with is a game where players first communicate with each other and then take their final action. The communication step is a cheap talk step in the sense that the messages -the reported signals that they send each other- need not to be correct, there is no penalty for incorrect reports. Nevertheless, this is different than a classical cheap talk game a la Crawford and Sobel (1982) as the players are potentially unsure about the type of their adversary, their payoffs are diametrically opposite of each other and they both take actions determining each other's payoff.

2.4.1 Perfect Bayesian Equilibrium

Since the final action choice will be $a_i = \mathbb{E}[\theta|s_i, m_j]$ for each player, the Perfect Bayesian equilibrium of this game can be described by a function $\sigma_i(\cdot|s_i)$ for each player $i \in \{1, 2\}$ where player i sends the report m_i with probability $\sigma_i(\cdot|s_i)$ conditional on receiving the private signal s_i. If the player i is of trader type, his beliefs upon receiving their adversary j's report is determined by:

$$\Pr_i(j \in T) = \frac{\pi \Pr(m_j|j \in T)}{\pi \Pr(m_j|j \in T) + (1-\pi)Pr(m_j|j \in I)}$$

$$\Pr_i(\theta|s_i, m_j) = \frac{f(s_i|\theta)\Pr(m_j|\theta)f_0(\theta)}{\int_\theta f(s_i|\theta)\Pr(m_j|\theta)f_0(\theta)d\theta}$$

And if player i is of the insider type, then $\Pr_i(j \in T) = 1$ and $\Pr_i(\theta|s_i, m_j) = \mathbb{1}\{\theta = s\}$.

Given $a_i = \mathbb{E}[\theta|s_i, m_j]$, the problem that player i faces in the first step is:

$$\max_{m_i} -\mathbb{E}[(\theta - \mathbb{E}[\theta|s_i, m_j])^2|s_i, m_i] + \kappa\,\mathbb{E}[(\theta - \mathbb{E}[\theta|s_j, m_i])^2|s_i, m_i]$$

Since s_i is a sufficient statistic for m_i, only the second term of player i's objective function depends on m_i. Then above problem is equivalent to:

$$\max_{m_i} \mathbb{E}[\mathrm{var}(\theta|s_j, m_i)|s_i, m_i]$$

Clearly the variance of θ for player j's can not get worse upon observing m_i, therefore:

$$\mathrm{var}(\theta|s_j, m_i) \leqslant \mathrm{var}(\theta|s_j) \quad \Longrightarrow \quad \mathbb{E}[\mathrm{var}(\theta|s_j, m_i)|s_i, m_i] \leqslant \mathbb{E}[\mathrm{var}(\theta|s_j)|s_i, m_i]$$

Then an uninformative message such that $\sigma_i(m_i|s_i) = \sigma_i(m_i'|s_i)$ for all $m_i, m_i' \in S$ and $s_i \in S$ is a weakly dominating strategy for player i. But then in any perfect Bayesian equilibrium, every player will end up with relying on only their private information:

Proposition 12. *In any PBE, each player will end up with relying on only their private information and no information is aggregated.*

It is not surprising that, even when sharing information would increase the payoff of the trading type, no player will share any information. Indeed, any deviation from any level of information sharing strictly increases players' payoff. This leads to a prisoner dilemma type of defecting for all players and results in no information aggregation.

Despite this result, we see a lot of information sharing in competitive environments. We can come up with a few explanations as to why some people end up sharing information instead of defecting, including how socially optimal equilibrium is sustained in repeated versions of the prisoner dilemma game.[4] For the specific setting of stock markets, one can argue that investors with limited private information would benefit from sustained information sharing. We remain agnostic about how such information sharing behavior can arise, but introduce a special equilibrium concept which rules out individual deviations and only considers the deviations applying to every copy of a type, Symmetric Bayesian Equilibrium.

2.4.2 Symmetric Bayesian Equilibrium

Symmetric Bayesian Equilibrium is more restrictive than Bayesian Nash Equilibrium and need not coincide with it as it does not allow players of the same type to consider different off-path deviations. This notion of symmetric equilibrium has been used especially in strategic speculation models, such as Kyle (1989). The main idea behind the symmetric equilibrium is that the players of the same type coordinate in certain strategies and does not deviate individually. In our context, this means that the traders will not choose a strategy that benefits them at the expense of another trader. Since such a strategy will be followed by all players of trader type, allowing for this strategy would yield a prisoner's dilemma type socially sub-optimal outcome within the group of players of trader type. By ruling this out, symmetric equilibrium assumes that the trader type players coordinate on the strategy that maximizes ex-ante payoff of the trader type. One can motivate this assumption by thinking this model in a repeated game setup where traders follow the group-optimal strategy to make sure high payoff in future encounters.

With the symmetric equilibrium, the first term of the ex-ante return will be the same for each player of same type and as a result, the trader type's ex-ante return becomes $-(1-\kappa\pi)\,\mathbb{E}[\mathrm{var}(\theta|s^T, m_j)]$. Similarly, if player i is of insider type, his ex-ante return becomes $\kappa\,\mathbb{E}[\mathrm{var}(\theta|s^T, m_i)]$. To simplify notation and avoid confusion, let $q(s, m) = \mathrm{var}(\theta|s^T = s, m_j = m)$.

A symmetric equilibrium in this game is described by functions $\sigma^T(\cdot|s) \in \triangle(S)$ and $\sigma^I(\cdot|\theta) \in \triangle(S)$ where $\sigma^T(m|s)$ denotes the probability that the trader type reports m given the private signal s and $\sigma^I(m|\theta)$ denotes the probability that the insider type reports m given the private signal θ satisfying:

- $\sigma^T \in \arg\max -(1-\kappa\pi)\,\mathbb{E}[q(s, \hat{s}_j)]$
- $\sigma^I \in \arg\max \kappa\,\mathbb{E}[q(s_j, \hat{s}^I)]$

The function q describes the variance of θ conditioned on the trader type's information after

[4] Indeed there is a plethora of experiments starting with Axelrod (1980) showing that people do cooperate if the game is played repeatedly

the signal reports are revealed.

$$
\begin{aligned}
q(s,m) =& \operatorname{var}(\theta | s_i^T = s, m_j = m) \\
=& \mathbb{E}[(\theta - \mathbb{E}[\theta | s_i^T = s, m_j = m])^2 | s_i^T = s, m_j = m] \\
=& \mathbb{E}[\theta^2 | s_i^T = s, m_j = m] - \mathbb{E}[\theta | s_i^T = s, m_j = m]^2
\end{aligned}
$$

For the rest of the chapter, we will assume that the insider will not pick any signal report that will change the trader type's prior about the type space. Specifically, we assume that $\Pr(m^T = m) = \Pr(m^I = m)$ for any $m \in S$. A direct consequence of this assumption is that $\Pr(j \in T | m_j = m) = \pi$. Then, we have:

$$
\begin{aligned}
\mathbb{E}[q(s,m)] =& \mathbb{E}[\theta^2] - \mathbb{E}[\mathbb{E}[\theta | s_i^T = s, m_j = m]^2] \\
=& \mathbb{E}[\theta^2] - \mathbb{E}[(\pi\, \mathbb{E}[\theta | s_i^T = s, m_j^T = m] + (1-\pi)\, \mathbb{E}[\theta | s_i^T = s, m_j^I = m])^2]
\end{aligned}
$$

This implies that the trader type's problem is equivalent to the following problem:

$$
\max_{\sigma^T} \quad \mathbb{E}[(\pi\, \mathbb{E}[\theta | s_i^T = s, m_j^T = m] + (1-\pi)\, \mathbb{E}[\theta | s_i^T = s, m_j^I = m])^2]
$$

And the insider type's problem is equivalent to the following problem:

$$
\min_{\sigma^I} \quad \mathbb{E}[(\pi\, \mathbb{E}[\theta | s_i^T = s, m_j^T = m] + (1-\pi)\, \mathbb{E}[\theta | s_i^T = s, m_j^I = m])^2]
$$

Given above formulation we can prove the following:

Proposition 13. *There exists no pure strategy symmetric equilibrium.*

Proof. Clearly, $\mathbb{E}[\mathbb{E}[\theta | s_i = s, m_j = m]^2]$ is a convex function of $\mathbb{E}[\theta | s_i = s, m_j^T = m]$. Furthermore since $\int_m \mathbb{E}[\theta | s_i = s, m_j^T = m] \Pr(m_j^T = m) dm = \mathbb{E}[\theta | s_i = s]$, the trader type will set $\mathbb{E}[\theta | s_i = s, m_j^T = m] > \mathbb{E}[\theta | s_i = s]$ if $\mathbb{E}[\theta | s_i = s, m_j^I = m] > \mathbb{E}[\theta | s_i = s]$ and $\mathbb{E}[\theta | s_i = s, m_j^T = m] < \mathbb{E}[\theta | s_i = s]$ if $\mathbb{E}[\theta | s_i = s, m_j^I = m] < \mathbb{E}[\theta | s_i = s]$.

The insider type, however, minimizes $\mathbb{E}[\mathbb{E}[\theta | s_i, m_j]^2]$, implying that $\mathbb{E}[\theta | s_i = s, m_j^I = m] > \mathbb{E}[\theta | s_i = s]$ if $\mathbb{E}[\theta | s_i = s, m_j^T = m] < \mathbb{E}[\theta | s_i = s]$ and $\mathbb{E}[\theta | s_i = s, m_j^I = m] < \mathbb{E}[\theta | s_i = s]$ if $\mathbb{E}[\theta | s_i = s, m_j^T = m] > \mathbb{E}[\theta | s_i = s]$. Then there can be no pure strategy symmetric equilibrium where $\mathbb{E}[\theta | s_i = s, m_j^T = m] \neq \mathbb{E}[\theta | s_i = s]$ and $\mathbb{E}[\theta | s_i = s, m_j^I = m] \neq \mathbb{E}[\theta | s_i = s]$ for all $s, m \in S$.

Lastly if $\mathbb{E}[\theta | s_i = s, m_j^I = m] = \mathbb{E}[\theta | s_i = s]$ for all $s, m \in S$, then the trader type sets $\mathbb{E}[\theta | s_i = s, m_j^T = m] = \mathbb{E}[\theta | s_i = s, m_j^T = s_j^T]$. Then there is no pure strategy symmetric equilibrium.

$\square$

To illustrate why there is no pure strategy symmetric equilibrium, consider the following example with the state space $\Theta = \{0, 1\}$, signal/action space $A = \{0, 1\}$ and the information structure:

$$\begin{bmatrix} f(s=1|\theta=1) & f(s=0|\theta=1) \\ f(s=1|\theta=0) & f(s=0|\theta=0) \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Since the trader's problem is a convex function of $\mathbb{E}[\theta|s_i = s, m_j^T]$, the trader will always pick an atomic distribution for $\sigma^T(m|s)$ where each signal is mapped to a unique signal report. Assume the trader maps signal $s = 0$ to $m = 0$ and $s = 1$ to $m = 1$. Then the best the insider can do would be the exact opposite which is $\sigma^I(m|s) = 1$ if and only if $m \neq s$. But then the trade will switch the reported signals and use the same mapping between the signals and reported signals to mimic the insider. Since the trader is trying to mimic the insider while the insider is going exactly opposite, there won't be a pure strategy equilibrium.

It is possible to construct a mixed strategy symmetric equilibrium, but a mixed strategy symmetric equilibrium means that the players of the same type should be able to collectively mix. That is, there must exist a semi-public randomization device that is only observable by traders, so that even with the mixed strategy, the traders will continue playing the same strategy at every realization of the public randomization device. Of course, this is very unlikely to happen in any real-world application of this model. Therefore, we skip the analysis of the mixed strategy symmetric equilibrium.

Hence, we conclude that the optimal public signal mechanism fails to aggregate any information. We will revisit the optimal public signal mechanism once we introduce the behavioral traders.

2.5 The Direct Probabilistic Communication Mechanism

With the direct probabilistic communication mechanism, the principal makes the signal reports costly instead of trying to come up with the optimal public signal. The cost is introduced by making having a probabilistic communication step. That is, after the players submit their signal reports, with probability $q \in (0, 1)$, the game ends with their signal reports considered to be their final action choice and with probability $1 - q$, the game proceeds to the communication step. If the game proceeds to the communication step, the principal directly reveals the reports she received to the players and they, in turn, pick their final actions.

Since the public signal directly reveals signal reports, we can let $s_p = m_j$ when solving player i's problem. This implies that $a_i = m_i$ with probability q and $a_i = \mathbb{E}[\theta|s_i, m_j]$ with probability

$1-q$. Therefore, player i solves the following problem:

$$\max_{m_i} -q\,\mathbb{E}[((m_i-\theta)^2-\kappa(m_j-\theta)^2))|s_i]-(1-q)\,\mathbb{E}[((\mathbb{E}[\theta|s_i,m_j]-\theta)^2-\kappa(\mathbb{E}[\theta|s_j,m_i]-\theta)^2))|s_i]$$

We focus on Perfect Bayesian Equilibrium (PBE) in pure strategies such that $s_i : S \to S$ for $i \in \{1,2\}$ describe a PBE if s_i solves above maximization problem where the beliefs of a trader type player i after observing m_j become:

$$\Pr_i(j \in T) = \frac{\pi \Pr(m_j|j \in T)}{\pi \Pr(m_j|j \in T) + (1-\pi)Pr(m_j|j \in I)}$$

$$\Pr_i(\theta|s_i = s) = \frac{f_0(\theta)f(\theta|s)}{\int_\theta f_0(\theta)f(\theta|s)d\theta}$$

And if the player i is of insider type, $\Pr_i(j \in T) = 1$ and $\Pr_i(\theta|s_i = s) = \mathbb{1}\{\theta = s\}$. Now define the function $g : S \times S \to \mathbb{R}$ such that:

$$g(s,m) = \mathbb{E}[\theta|s_i^T = s, m_j = m]$$

Function $g(s,m)$ determines the expectation of a trader type player in equilibrium given his private signal s and report of the other player m.

We will focus on PBE where players of same type follows the same strategy, $m_i(\cdot)$ which is monotonic, twice continuously differentiable and $\Pr(m_j^T = m|s_i^T = s) = \Pr(m_j^I = m|s_i^T = s)$ that is the reported signal does not contain any information about the type of the other player to a player of trader type.

Letting $h_i = \pi$ if i is of trader type and $h_i = 1$ otherwise, the maximization problem of player i is equivalent to solving:

$$\max_m -q\,\mathbb{E}[(m-\theta)^2|s_i m_i = m] + (1-q)\kappa h_i\,\mathbb{E}[(g(s,m)-\theta)^2|s_i, m_i = m]$$

The first order condition yields:

$$-2q(m - \mathbb{E}[\theta|s_i]) + 2(1-q)\kappa h_i\,\mathbb{E}\left[(g(s,m)-\theta)\frac{\partial g(s,m)}{\partial m}\middle| s_i = s, m\right] = 0$$

where $h_i = \pi$ if i is of trader type and $h_i = 1$ otherwise. Then an interior solution -if exists- would yield:

$$m_i = \mathbb{E}[\theta|s_i] + \frac{1-q}{q}h_i\kappa\,\mathbb{E}\left[(g(s,m)-\theta)\frac{\partial g(s,m)}{\partial m}\middle| s_i = s, m\right]$$

And the second order condition guaranteeing the interior solution is:

$$\forall s, m \in S, \quad -2q + 2(1-q)\kappa h_i \, \mathbb{E}\left[\left(\frac{\partial g(s,m)}{\partial m} \right)^2 + (g(s,m) - \theta) \frac{\partial^2 g(s,m)}{\partial m} \middle| s_i = s, m \right] \leqslant 0$$

Furthermore, since the maximization problem is unconstrained, there can be no pure strategy PBE when the second order condition fails for any players. Then we have:

Proposition 14. *There exists a pure strategy PBE in the costly signal game if for any $s_i, m \in S$, we have:*

$$\forall s, m \in S, \quad -2q + 2(1-q)\kappa h_i \, \mathbb{E}\left[\left(\frac{\partial g(s,m)}{\partial m} \right)^2 + (g(s,m) - \theta) \frac{\partial^2 g(s,m)}{\partial m} \middle| s_i = s, m \right] \leqslant 0$$

Furthermore, any pure strategy PBE, if exists, satisfies that for all $i \in \{1, 2\}$:

$$m_i = \mathbb{E}[\theta | s_i] + \frac{1-q}{q} h_i \kappa \, \mathbb{E}\left[(g(s,m) - \theta) \frac{\partial g(s,m)}{\partial m} \middle| s_i, m \right]$$

Clearly any pure strategy PBE is symmetric in the sense that all players of trader type send the same report m given a signal $s \in S$. We can make a sharper characterization for PBE if the function $g(s,m)$ is separable in the sense that $\frac{\partial^2 g(s,m)}{\partial s \partial m} = 0$:

Proposition 15. *Assume $\frac{\partial^2 g(s,m)}{\partial s \partial m} = 0$, then a unique PBE exists if:*

$$\forall s_i, m \in S, \quad \kappa \left(\frac{\partial g(s,m)}{\partial m} \right)^2 \leqslant \frac{q}{1-q}$$

And in the unique PBE, we have $m_i = \mathbb{E}[\theta | s_i]$.

Proof. By proposition 14, we know that a pure strategy PBE exists if for any $s_i, m \in S$, we have:

$$\forall s, m \in S, \quad -2q + 2(1-q)\kappa h_i \, \mathbb{E}\left[\left(\frac{\partial g(s,m)}{\partial m} \right)^2 + (g(s,m) - \theta) \frac{\partial^2 g(s,m)}{\partial m} \middle| s_i, m \right] \leqslant 0$$

Since $\frac{\partial^2 g(s,m)}{\partial s \partial m} = 0$, above implies:

$$(1-q)\kappa h_i \, \mathbb{E}\left[\left(\frac{\partial g(s,m)}{\partial m} \right)^2 + (g(s,m) - \theta) \frac{\partial^2 g(s,m)}{\partial m} \middle| s_i, m \right] \leqslant q$$

$$(1-q)\kappa h_i \left[\left(\frac{\partial g(s,m)}{\partial m} \right)^2 + \frac{\partial^2 g(s,m)}{\partial m} \, \mathbb{E}[(g(s,m) - \theta) | s_i, m] \right] \leqslant q$$

By law of iterated expectations, we have that:

$$\begin{aligned}\mathbb{E}[(g(s,m)-\theta)|s_i,m] &= \mathbb{E}[(\mathbb{E}[\theta|s_j^T=s,m_i=m]-\theta)|s_i,m]\\ &= \mathbb{E}[(\mathbb{E}[\theta|s_j^T=s,m_i=m]|s_i,m]-\mathbb{E}[\theta|s_i,m]=\mathbb{E}[\theta]|s_i,m]-\mathbb{E}[\theta|s_i,m]=0\end{aligned}$$

Then the existence condition simplifies to:

$$\forall m\in S,\quad \kappa h_i\left(\frac{\partial g(s,m)}{\partial m}\right)^2 \leqslant \frac{q}{1-q}$$

And since $h_i=1$ for the trader type above implies:

$$\forall m\in S,\quad \kappa\left(\frac{\partial g(s,m)}{\partial m}\right)^2 \leqslant \frac{q}{1-q}$$

Furthermore, again from proposition 14, we have that:

$$\begin{aligned}m_i &= \mathbb{E}[\theta|s_i]+\frac{1-q}{q}h_i\kappa\,\mathbb{E}\left[(g(s,m)-\theta)\frac{\partial g(s,m)}{\partial m}\middle|\, s_i,m\right]\\ &= \mathbb{E}[\theta|s_i]+\frac{1-q}{q}h_i\kappa\frac{\partial g(s,m)}{\partial m}\,\mathbb{E}[(g(s,m)-\theta)|s_i,m]\end{aligned}$$

Now, again since $\mathbb{E}[(g(s,m)-\theta)|s_i,m]=0$ by the law of iterated expectations, above implies $m_i=\mathbb{E}[\theta|s_i]$. □

2.5.1 Normal Linear Signal Structure

As with the optimal deterministic communication mechanism in the previous section, we will use a normal linear signal structure as an example:

$$\theta\sim N(0,1),\qquad s^T=\theta+\varepsilon\quad\text{where}\quad \varepsilon\sim N(0,\sigma^2)$$

If $g(s,m)$ is linear on s and m due to normal-linear signal structure, then by proposition 15, we get $m_i=\mathbb{E}[\theta|s_i]$ which in turn implies $\mathbb{E}[\theta|s_i^T=\hat{s},m_j=m]$ is linear on s and m. Then focusing on such equilibria, we have that a PBE exists with $m_i=\mathbb{E}[\theta|s_i]$ if $\kappa\left(\frac{\partial g(s,m)}{\partial m}\right)^2\leqslant\frac{q}{1-q}$. Furthermore, we have:

$$\begin{aligned}g(s,m)=\mathbb{E}[\theta|s_i^T=\hat{s},m_j=m] &= \frac{\pi}{2+\sigma^2}(s+(1+\sigma^2)m))+(1-\pi)m\\ &= \frac{\pi}{2+\sigma^2}s+\left(\pi\frac{1+\sigma^2}{2+\sigma^2}+(1-\pi)\right)m\end{aligned}$$

Then $\kappa\left(\frac{\partial g(s,m)}{\partial m}\right)^2 \leqslant \frac{q}{1-q}$ implies:

$$\kappa\left(\pi\frac{1+\sigma^2}{2+\sigma^2}+(1-\pi)\right)^2 \leqslant \frac{q}{1-q} \quad \Rightarrow \quad q \geqslant \frac{\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}{1+\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}$$

Assuming above condition is satisfied, the ex-ante error of a trader type conditional on the game proceeds to the second step is:

$$\begin{aligned}
\mathbb{E}[(\theta-g(s,m))^2] &= \mathbb{E}\left[\left(\theta-\frac{\pi}{2+\sigma^2}s-\left(\pi\frac{1+\sigma^2}{2+\sigma^2}+(1-\pi)\right)m\right)^2\right] \\
&= \pi\,\mathbb{E}\left[\left(\theta-\frac{\pi}{2+\sigma^2}s-\left(\pi\frac{1+\sigma^2}{2+\sigma^2}+(1-\pi)\right)\frac{1}{1+\sigma^2}s'\right)^2\right] \\
&+ (1-\pi)\,\mathbb{E}\left[\left(\theta-\frac{\pi}{2+\sigma^2}s-\left(\pi\frac{1+\sigma^2}{2+\sigma^2}+1-\pi\right)\theta\right)^2\right] \\
&= \left(\frac{\pi}{2+\sigma^2}\right)^2\sigma^2+\pi\left(\pi\frac{\sigma^2}{2+\sigma^2}+(1-\pi)\frac{\sigma^2}{1+\sigma^2}\right)^2+\pi\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2\sigma^2
\end{aligned}$$

And the ex-ante error of a trader type conditional on the game does not proceed to the second step is $\frac{\sigma^2}{1+\sigma^2}$. So the ex-ante payoff of the trader type becomes:

$$\begin{aligned}
V_{trader} &= -q(1-\kappa\pi)\frac{\sigma^2}{1+\sigma^2} \\
&\quad -(1-q)(1-\kappa\pi)\left[\left(\frac{\pi}{2+\sigma^2}\right)^2\sigma^2+\pi\left(\pi\frac{\sigma^2}{2+\sigma^2}+(1-\pi)\frac{\sigma^2}{1+\sigma^2}\right)^2+\pi\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2\sigma^2\right]
\end{aligned}$$

Clearly, V_{trader} is a decreasing function of q. This implies that the principal will set $q = \frac{\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}{1+\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}$ and therefore ex-ante payoff of the trader becomes:

$$\begin{aligned}
V^*_{trader} &= -\frac{\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}{1+\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}(1-\kappa\pi)\frac{\sigma^2}{1+\sigma^2} \\
&\quad -\frac{1}{1+\kappa\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2}(1-\kappa\pi)\left[\left(\frac{\pi}{2+\sigma^2}\right)^2\sigma^2+\pi\left(\pi\frac{\sigma^2}{2+\sigma^2}+(1-\pi)\frac{\sigma^2}{1+\sigma^2}\right)^2\right. \\
&\qquad\left.+\pi\left(\frac{2+\sigma^2-\pi}{2+\sigma^2}\right)^2\sigma^2\right]
\end{aligned}$$

Below we plot the probability q and the ex-ante payoff of the trader for different σ levels:

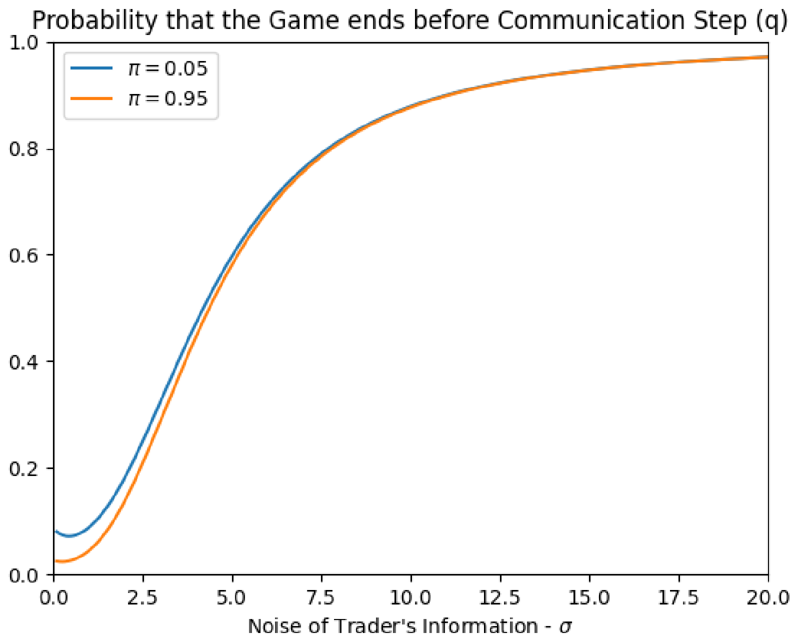

Figure 2.4: Direct Probabilistic Communication Mechanism - q parameter

The optimal q set by the principal determining the probability that the game ends before the communication step increases with σ, which means that the principal will have to make the signals costlier as the variance of the trader's estimate of θ increases. This is because as the information that the trader type receives gets noisier, the traders rely less on their own private signals and more on the public signal. This increases the benefit of manipulation for the insider type. To offset this benefit, the principal needs to introduce higher cost in the form of higher probability for the game to end without a communication period. On the other hand, the optimal q does not change much with respect to π. The fact that the optimal q is insensitive to π suggests that even when the probability that one of the players is insider is very low, if the trader type's information is very noisy, the principal needs to introduce very high q. This is a major drawback for the direct probabilistic communication mechanism, since when the trader's have little to none private information, full information aggregation is achieved at a very high cost of not having the communication at all most of the time.

The ex-ante payoff of the trader decreases with σ as expected. It decreases with π as well. This is because, under the direct probabilistic communication mechanism, increasing likelihood of playing against insider increases the trader's payoff through what trader can learn from the insider.

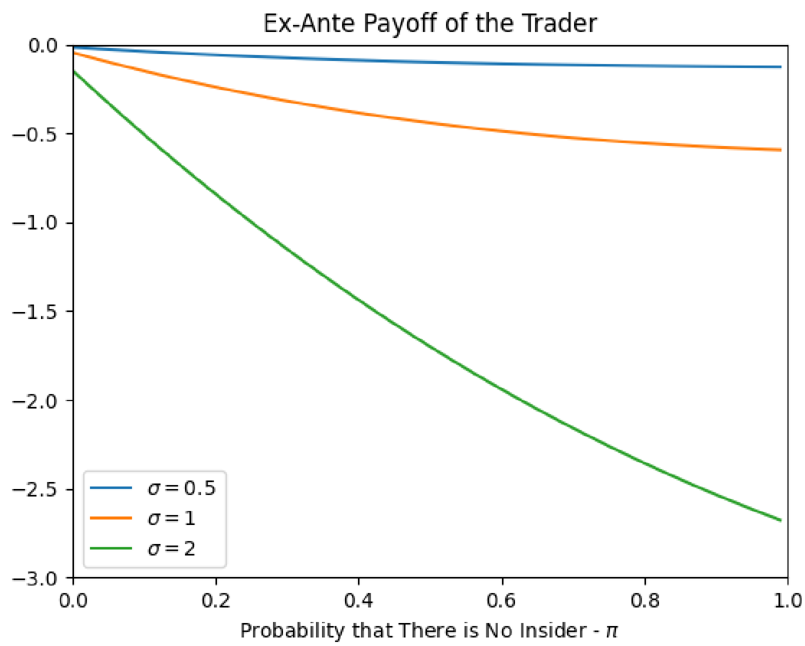

Figure 2.5: Direct Probabilistic Communication Mechanism - Ex-Ante Payoff of the Trader

With this point, we conclude our analysis for rational traders and consider a behavioral variation.

2.6 A Behavioral Variation

So far, we modeled traders to be fully rational which enabled us to analyze the optimal deterministic communication mechanism without the principal. The fully revealing public signal is an optimal public signal, so there is no need for principal to devise a complicated public signal. Being able to process all the information revealed through public information may not be realistic, however. Indeed, relaxing the rationality assumption in competitive learning environments proved to be useful in explaining market inefficiencies in financial markets (see Barberis and Thaler (2003) for a survey of academic work on behavioral finance and Shiller (2015) for more recent empirical evidence).

Although there are many behavioral learning models including evolutionary learning, reinforcement learning and belief learning (see chapter 6 on Camerer (2011)), it is difficult to identify which model would be more suitable for learning in competitive environments, specifically stock markets. In this chapter, instead of specifying a learning mechanism for traders, we assume that they have incorrect beliefs about the likelihood of a trader. More specifically, we assume that the traders believe that $\pi = 1$, that is traders believe that they play against another trader for certain. Moreover, we assume that based on this incorrect belief, traders always report their private signals truthfully and assume that the principal's public signal is an aggregation of private signals from traders.

The principal, on the other hand, is aware that one of the players might be an insider and produces a public signal given the reported signals to maximize trader's ex-ante payoff. The trader type interprets the public signals under the assumption that the principal received true signals of two trader type players. To simplify our analysis, assume that the insider follows non-revealing strategies in the sense that the reported signals does not change the prior of the principal about the types of the players that is $\Pr(m_i = m|i \in T) = \Pr(m_i = m|i \in I)$. Since we assume that traders truthfully report their signal, this implies $\int_\theta f(s|\theta)f_0(\theta)d\theta = \int_\theta \sigma^I(s|\theta)f_0(\theta)d\theta$ where $\sigma^I(s|\theta)$ determines the probability that the principal sends the signal $s \in S$ given θ.

Assume that the principal produces the signal $s_p \in S_p$ with probability $\sigma^P(s_p|\hat{s}_i, \hat{s}_j) \in [0,1]$ where $\int_{s_p} \sigma^P(s_p|\hat{s}_i, \hat{s}_j)ds_p = 1$ and $\sigma^P(s_p|s_i = s, s_j = s') = \sigma^P(s_p|s_i = s, s_j = s')$ for all $s_p \in S$ as the principal can't tell which signal is coming from which type. Given the principal's strategy σ^P, the trader type's expectation of θ conditional on own signal s_i and public signal s_p can be written as:

$$\mathbb{E}^T[\theta|s_i^T = s, s_p] = \frac{\int_{s_j}\int_\theta \theta f_0(\theta)f(s|\theta)f(s_j|\theta)\sigma^P(s_p|s,s_j)ds_j d\theta}{\int_{s_j}\int_\theta f_0(\theta)f(s|\theta)f(s_j|\theta)\sigma^P(s_p|s,s_j)ds_j d\theta}$$

Note that above expectation operator $\mathbb{E}^T$ computes the expectation of θ given own private signal s and public signal s_p under the incorrect belief that player j is surely another trader.

2.6.1 The Optimal Deterministic Communication Mechanism

We start our analysis with the optimal deterministic communication mechanism where the game follows step 0, 1 and 2 as described above except that traders' final action is determined by $\mathbb{E}^T[\theta|s_i^T = s, s_p]$ and they always report their signal truthfully. We assume that given a public signal structure $\sigma^P(s_p|m_i, m_j)$, the insider maximizes his ex-ante payoff by picking a function $\sigma^I(m|\theta)$ determining the probability that the insider reports $m \in S$ given the true state θ.

Our first result for the behavioral model is that the principal will set $\mathbb{E}^T[\theta|s_i^T = s, s_p]$ to the expectation of θ under a condition:

Proposition 16. *Assume that for every $\theta' \in \Theta$ there exists $s \in S$ such that $\mathbb{E}[\theta|s_i^T = s] = \theta'$. Then the principal picks a public signal structure $\sigma^P(s_p|m_i, m_j)$ such that $\mathbb{E}^T[\theta|s_i^T = s, s_p] =$*

$\mathbb{E}^T[\theta|s_j^T = s', s_p] = \mathbb{E}[\theta|m_i = s, m_j = s']$.

Proof. The ex-ante welfare of the trader is:

$$-\pi(1-\kappa)\,\mathbb{E}[(\theta - \mathbb{E}^T[\theta|s_i^T = s, s_p])^2 + (\theta - \mathbb{E}^T[\theta|s_i^T = s', s_p])^2]$$
$$- (1-\pi)\,\mathbb{E}[(\theta - \mathbb{E}^T[\theta|s_i^T = s, s_p])^2/2 + (\theta - \mathbb{E}^T[\theta|s_j^T = s', s_p])^2/2]$$
$$= -\frac{1+\pi(1-2\kappa)}{2}\,\mathbb{E}[(\theta - \mathbb{E}^T[\theta|s_i^T = s, s_p])^2 + (\theta - \mathbb{E}^T[\theta|s_i^T = s', s_p])^2]$$

Then the principal's problem becomes:

$$\max_{\sigma^P} -\frac{1+\pi(1-2\kappa)}{2}\,\mathbb{E}[(\theta - \mathbb{E}^T[\theta|s_i^T = s, s_p])^2 + (\theta - \mathbb{E}^T[\theta|s_i^T = s', s_p])^2]$$

Now given the reports $m_i = s, m_j = s'$, letting $\mathbb{E}^T[\theta|s_i^T = s, s_p] = \mathbb{E}^T[\theta|s_j^T = s', s_p] = \mathbb{E}[\theta|m_i = s, m_j = s']$ achieves the maximum. Then if we can show that for any $s, s' \in S$, there exists $s_p \in S_p$ with $\mathbb{E}^T[\theta|s_i^T = s, s_p] = \mathbb{E}^T[\theta|s_j^T = s', s_p] = \mathbb{E}[\theta|m_i = s, m_j = s']$ we are done. To show this, we rearrange $\mathbb{E}^T[\theta|s_i^T = s, s_p]$ so that:

$$\begin{aligned}
\mathbb{E}^T[\theta|s_i^T = s, s_p] &= \frac{\int_{s_j}\int_\theta \theta f_0(\theta) f(s|\theta) f(s_j|\theta)\sigma^P(s_p|s,s_j)ds_j d\theta}{\int_{s_j}\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)\sigma^P(s_p|s,s_j)ds_j d\theta} \\
&= \int_{s_j} \frac{\frac{\int_\theta \theta f_0(\theta) f(s|\theta) f(s_j|\theta)d\theta}{\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)d\theta}\sigma^P(s_p|s,s_j)\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)d\theta}{\int_{s_j}\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)\sigma^P(s_p|s,s_j)ds_j d\theta} ds_j \\
&= \int_{s_j} \mathbb{E}[\theta|s_i^T = s, s_j^T = s_j]\frac{\sigma^P(s_p|s,s_j)\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)d\theta}{\int_{s_j}\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)\sigma^P(s_p|s,s_j)ds_j d\theta} ds_j
\end{aligned}$$

Then letting $\lambda_{s_p}(s,s_j) = \sigma^P(s_p|s,s_j)\int_\theta f_0(\theta) f(s|\theta) f(s_j|\theta)d\theta$, $\lambda_{s_p}(s) = \int_{s_j}\lambda_{s_p}(s,s_j)ds_j$ and $\lambda_{s_p}(s_j) = \int_s \lambda_{s_p}(s,s_j)ds_i$, we have:

$$\mathbb{E}^T[\theta|s_i^T = s, s_p] = \int_{s_j}\frac{\lambda_{s_p}(s_i,s_j)}{\lambda_{s_p}(s_i)}\,\mathbb{E}[\theta|s_i^T = s_i, s_j^T = s_j]ds_j$$

This implies instead of choosing $\sigma^P(s_p|s,s')$, the principal can choose $\lambda_{s_p}(s,s') \in [0,1]$ for all $s_p \in S_p$ with $\int_{s_p}\lambda_{s_p}(s,s') = \Pr(s_i^T = s, s_j^T = s')$. Now if $\{\mathbb{E}[\theta|s_i^T = s_i, s_j^T = s_j] | s_i, s_j \in S\} = \Theta$, then there exists some $\lambda_{s_p}(s,s') \in [0,1]$ with $\mathbb{E}^T[\theta|s_i^T = s, s_p] = \mathbb{E}^T[\theta|s_j^T = s', s_p] = \mathbb{E}[\theta|m_i = s, m_j = s']$. Then we only need to show $\{\mathbb{E}[\theta|s_i^T = s_i, s_j^T = s_j] | s_i, s_j \in S\} = \Theta$.

Now by assumption we have that $\{\mathbb{E}[\theta|s_i^T = s] | s \in S\} = \Theta$. Furthermore for any $s \in S$, there exists $s' \in S$ with $\mathbb{E}[\theta|s_i^T = s, s_j^T = s'] = \mathbb{E}[\theta|s_i^T = s]$. To see this, first we have that $\mathbb{E}[\theta|s_i^T = s, s_j^T = s']$ is a continuous and increasing function of s' due to MLRP. Now assume

$\mathbb{E}[\theta|s_i^T = s, s_j^T = s'] > \mathbb{E}[\theta|s_i^T = s]$ for all $s' \in S$. But then $\mathbb{E}[\mathbb{E}[\theta|s_i^T = s, s_j^T = s']||s_i^T = s] > \mathbb{E}[\mathbb{E}[\theta|s_i^T = s]|s_i^T = s]$ implying $\mathbb{E}[\theta|s_i^T = s] > \mathbb{E}[\theta|s_i^T = s]$ which is a contradiction. Similarly, we can rule out $\mathbb{E}[\theta|s_i^T = s, s_j^T = s'] < \mathbb{E}[\theta|s_i^T = s]$ for all $s' \in S$. Then there exists $s', s" \in S$ with $\mathbb{E}[\theta|s_i^T = s, s_j^T = s'] > \mathbb{E}[\theta|s_i^T = s] > \mathbb{E}[\theta|s_i^T = s, s_j^T = s"]$. Then since $\mathbb{E}[\theta|s_i^T = s, s_j^T = s']$ is a continuous and increasing function of s', there exists $s' \in S$ with $\mathbb{E}[\theta|s_i^T = s, s_j^T = s'] = \mathbb{E}[\theta|s_i^T = s]$. But then $\{\mathbb{E}[\theta|s_i^T = s_i, s_j^T = s_j]|s_i, s_j \in S\} = \{\mathbb{E}[\theta|s_i^T = s]|s \in S\} = \Theta$. □

Given above proposition, the insider's problem becomes:

$$\max_{\sigma^I(s'|\theta)} \kappa\, \mathbb{E}[(\theta - \mathbb{E}[\theta|m_i = s, m_j = s'])^2] \quad \text{s. to} \quad \forall s \in S \quad \Pr(m_i = s|i \in T) = \Pr(m_i = s|i \in I)$$

Above problem is equivalent to:

$$\min_{\sigma^I(s'|\theta)} \mathbb{E}[\mathbb{E}[\theta|m_i = s, m_j = s']^2] \quad \text{s. to} \quad \forall s \in S \quad \int_\theta f(s|\theta) f_0(\theta) = \int_\theta \sigma^I(s|\theta) f_0(\theta)$$

We will use the linear-normal structure to further analyze the insider's problem. Recall that the linear-normal structure assumes:

$$\theta \sim N(0,1), \quad s^T = \theta + \varepsilon \quad \text{where} \quad \varepsilon \sim N(0, \sigma^2)$$

Further assume that the principal's strategy is such that, $m^I = \alpha\theta + \mu$ where $\alpha \in \mathbb{R}$ and $\mu \sim N(0, \delta^2)$ where θ and μ are independent.Since $\forall s \in S$, $\int_\theta f(s|\theta) f_0(\theta) = \int_\theta \sigma^I(s|\theta) f_0(\theta)$, we have $\phi(\frac{m}{\alpha^2+\delta^2}) = \phi(\frac{m}{1+\sigma^2})$ for all $m \in \mathbb{R}$. This implies that $\alpha^2 + \delta^2 = 1 + \sigma^2$. Then, non-revealing strategies impose a trade-off between the noisiness of the reported signals and the scaling coefficient α. Since $m^I = \alpha\theta + \mu$, $\mathbb{E}[\theta|m_i, m_j]$ is linear on m_i, m_j. Furthermore, since both players have equal probability of being insider, we have $\mathbb{E}[\theta|m_i, m_j] = \beta(m_i + m_j)$ for some $\beta \in \mathbb{R}$. Given $\mathbb{E}[\theta|m_i, m_j]$, the insider's problem is simply to maximize the variance of the trader's estimate conditional on the other player being insider:

$$\max_{\alpha,\delta} \kappa\, \mathbb{E}[(\theta - \beta(m_i + m_j))^2|i \in T, j \in I] \quad \text{subject to} \quad \alpha^2 + \delta^2 = 1 + \sigma^2$$

Observe that:

$$
\begin{aligned}
\mathbb{E}[(\theta-\beta(m_i+m_j))^2|i\in T, j\in I] =& \mathbb{E}[(\theta-\beta(\theta+\varepsilon+\alpha\theta+\mu-\theta)^2]\\
=& \mathbb{E}[(1-\beta-\beta\alpha)\theta-\beta\varepsilon+\beta\mu)^2]\\
=&(1-\beta-\beta\alpha)^2+\beta^2\sigma^2+\beta^2\delta^2\\
=&(1-\beta-\beta\alpha)^2+\beta^2\sigma^2+\beta^2(1+\sigma^2-\alpha^2)\\
=&(1-\beta)^2+2\beta^2\sigma^2+\beta^2-2(1-\beta)\beta\alpha\\
=&(1-2\beta)^2+2\beta^2\sigma^2+2\beta(1-\beta)(1-\alpha)
\end{aligned}
$$

And the trader's ex-ante payoff becomes:

$$
\begin{aligned}
V^T &= -(1-\kappa\pi)(\pi\,\mathbb{E}[(\theta-\beta(m_i+m_j))^2|i\in T, j\in T]+(1-\pi)\,\mathbb{E}[(\theta-\beta(m_i+m_j))^2|i\in T, j\in I])\\
&= -(1-\kappa\pi)(\pi[(1-2\beta)^2+2\beta^2\sigma^2+2\beta(1-\beta)(1-\alpha)]+(1-\pi)[(1-2\beta)^2+2\beta^2\sigma^2])\\
&= (1-\kappa\pi)[(1-2\beta)^2+2\beta^2\sigma^2+2\beta(1-\pi)(1-\beta)(1-\alpha)]
\end{aligned}
$$

If the principal were to directly report back the signal she received, then $\mathbb{E}[\theta|m_i,m_j] = \frac{1}{2+\sigma^2}(m_i+m_j)$ therefore $\beta=\frac{1}{2+\sigma^2}$. Then the insider sets $\alpha=-\sqrt{1+\sigma^2}$. And the trader's ex-ante payoff is:

$$
V_0^T = -(1-\kappa\pi)\left(\frac{\sigma^2}{2+\sigma^2}+2(1-\pi)\frac{1+\sigma^2}{2+\sigma^2}\frac{1+\sqrt{1+\sigma^2}}{2+\sigma^2}\right)
$$

On the other hand with the optimal public signal, given this signal structure, we have that:

$$
\mathbb{E}[\theta|m_i,m_j] = \frac{1+\pi+\alpha(1-\pi)}{2(1+\pi+\alpha(1-\pi)+\sigma^2)}(m_i+m_j)
$$

Then given $\beta=\frac{1+\pi+\alpha(1-\pi)}{2(1+\pi+\alpha(1-\pi)+\sigma^2)}$, the insider solves:

$$
\max_{\alpha}(1-2\beta)^2+2\beta^2\sigma^2+2\beta(1-\beta)(1-\alpha) \quad \text{subject to} \quad \alpha^2\leqslant 1+\sigma^2
$$

It turns out that the insider always set α as small as possible and as a result sets $\alpha=-\sqrt{1+\sigma^2}$. This means that even with the optimal public signal, the insider always sends a signal that suggests opposite of the correct position. As a result, plugging back to β, we get $\beta=\frac{1+\pi-\sqrt{1+\sigma^2}(1-\pi)}{2(1+\pi-\sqrt{1+\sigma^2}(1-\pi)+\sigma^2)}$. As a result β is negative for small π values as can be seen below:

The weight β has a smaller domain as σ increases since as the trader's information gets noisier, the principal pushes the posterior of the trader towards the prior distribution of θ. On the other hand as the trader's information gets bigger, the principal relies more on the messages. Furthermore, β

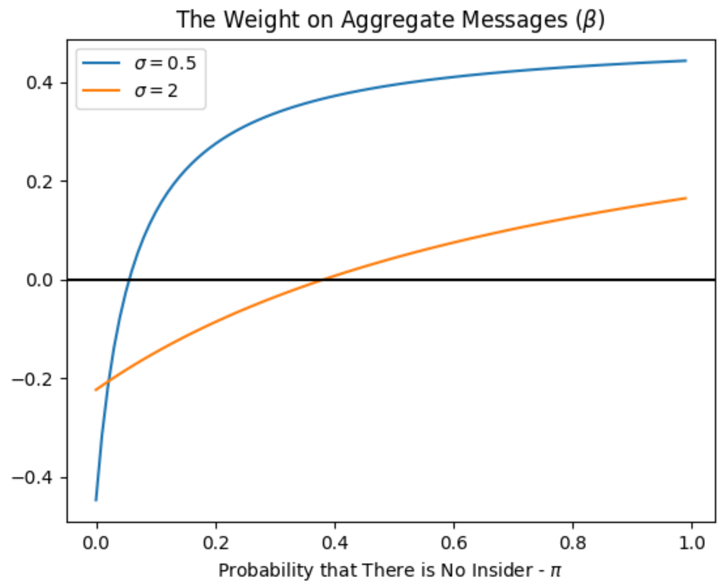

Figure 2.6: The Weight of Aggregate Messages

is negative for low π values. This is because with higher chance that one of the messages belongs to the insider, the principal wants to negate this by making traders pick an action that is opposite of the aggregate messages. Finally, we graph the trader's ex-ante payoff:

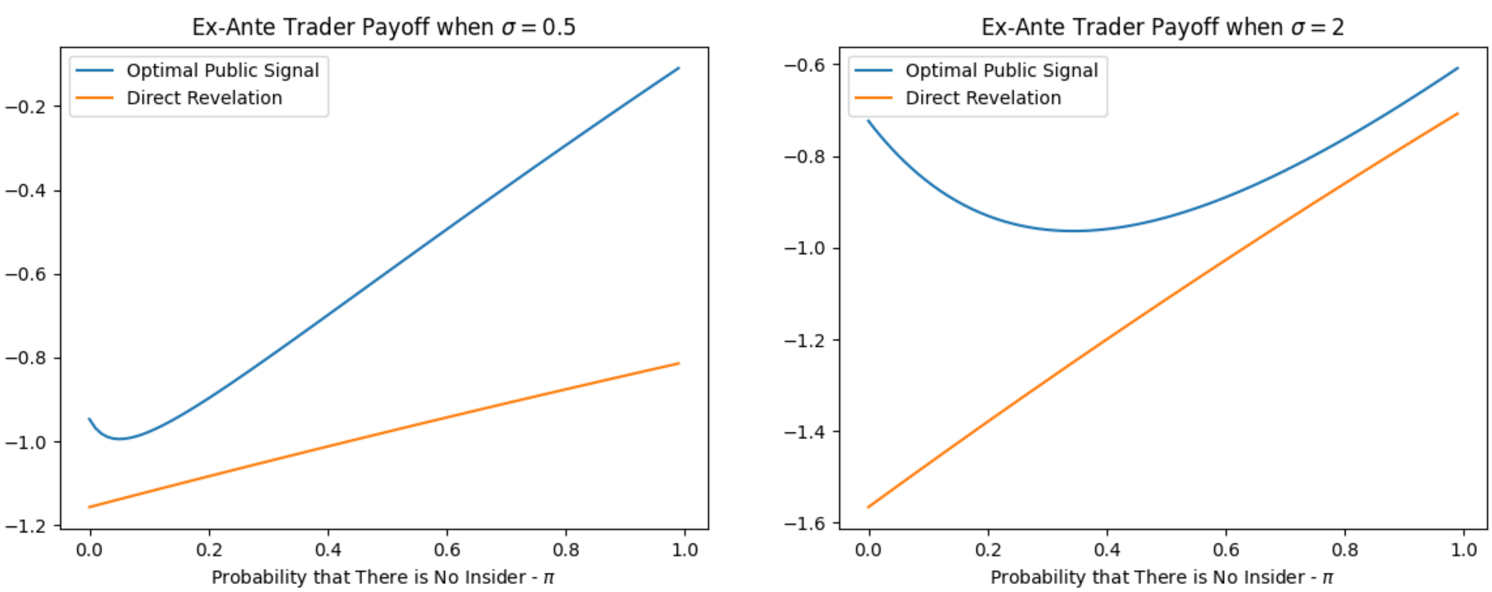

Figure 2.7: Optimal Deterministic Mechanism - Ex-Ante Trader Payoff

Trader's ex-ante payoff decreases with π for small π values. This is a result of β being negative for small π values. When β is negative, the trader is actually worse off when he is playing against another trader. Therefore, as long as β is negative, increasing the likelihood of playing against another trader decreases the welfare of the trader. However, once β becomes positive, higher likelihood of playing against another trader means more information aggregation. As a result, when β is positive, trader's payoff increases with π.

2.6.2 The Direct Probabilistic Communication Mechanism

For the direct probabilistic communication mechanism, we assume that the principal sets $s_p = (m_i, m_j)$ and traders who are unaware that there might exist an insider truthfully report their signal and if the game proceeds to the communication step, they set their action to be $a^T = \mathbb{E}[\theta | s_i^T = m_i, s_j^T = m_j]$. Again, with probability q, the game ends right after the signal reports with $a_i = m_i$. Insider's problem is then:

$$\max_m \mathbb{E}[-q[(m-\theta)^2 + \kappa(s-\theta)^2] + (1-q)\kappa(\mathbb{E}[\theta | s_i^T = s, s_j^T = m] - \theta)^2 | m, \theta]$$

Notice that, letting $r(s,m) = \mathbb{E}[\theta | s_i^T = s, s_j^T = m]$, we can directly utilize proposition 14 so that we have :

Corollary 5. *There exists a pure strategy PBE in the costly signal game with behavioral traders if for any $s_i, m \in S$, we have:*

$$\forall s, m \in S, \quad -2q + 2(1-q)\kappa\, \mathbb{E}\left[\left(\frac{\partial r(s,m)}{\partial m} \right)^2 + (r(s,m) - \theta) \frac{\partial^2 r(s,m)}{\partial m} \middle| \theta, m \right] \leqslant 0$$

Furthermore, in any pure strategy PBE, if exists, the insider sets:

$$m^I = \theta + \frac{1-q}{q} \kappa\, \mathbb{E}\left[(r(s,m) - \theta) \frac{\partial r(s,m)}{\partial m} \middle| \theta, m \right]$$

Since the law of iterated expectations fails in the behavioral model, we can not use proposition 15. However, we can still get a sharper characterization if $r(s,m)$ is linear that is $r(s,m) = \beta(s+m)$ for some $\beta \in \mathbb{R}$:

Proposition 17. *Assume $r(s,m)$ is linear that is $r(s,m) = \beta(s+m)$ for some $\beta \in \mathbb{R}$. If $q \geqslant \frac{\kappa\beta^2}{1+\kappa\beta^2}$, then the principal's reporting strategy is such that:*

$$m^I = \frac{q\theta + (1-q)\kappa\beta(\beta\, \mathbb{E}[s|\theta] - \theta)}{q - \beta^2(1-q)\kappa}$$

Proof. Using corollary 5, we have that if a pure strategy PBE exists, then:

$$
\begin{aligned}
m =& \theta + \frac{1-q}{q}\kappa\,\mathbb{E}\left[(r(s,m)-\theta)\frac{\partial r(s,m)}{\partial m}\middle|\,\theta, m\right] \\
=& \theta + \frac{1-q}{q}\kappa\,\mathbb{E}\left[(\beta(s+m)-\theta)\beta|\,\theta, m\right] \\
=& \theta + \frac{1-q}{q}\kappa(\beta^2\,\mathbb{E}[s|\theta] + \beta^2 m - \beta\theta)
\end{aligned}
$$

Then:

$$
m = \frac{q\theta + (1-q)\kappa\beta(\beta\,\mathbb{E}[s|\theta] - \theta)}{q - \beta^2(1-q)\kappa}
$$

Furthermore, again corollary 5 implies, a pure strategy PBE exists if:

$$
-q + (1-q)\kappa\beta^2 \leqslant 0 \qquad \Rightarrow \qquad q \geqslant \frac{\kappa\beta^2}{1+\kappa\beta^2}
$$

□

Again we will use the linear-normal signal structure, so that $\mathbb{E}[s|\theta] = \theta$ and $\beta = \frac{1}{2+\sigma^2}$. Then if $q \geqslant \frac{\kappa}{\kappa+(2+\sigma^2)^2}$, we have:

$$
m = \frac{q + (1-q)\kappa\beta(\beta-1)}{q - \beta^2(1-q)\kappa}\theta = \left(1 - \frac{(1-q)\kappa\sigma^2}{q(2+\sigma^2)^2 - (1-q)\kappa}\right)\theta
$$

Given the insider's strategy, we can compute the ex-ante welfare of trader. Let $\gamma = \frac{(1-q)\kappa\sigma^2}{q(2+\sigma^2)^2-(1-q)\kappa}$, then:

$$
\begin{aligned}
V^T =& -q\left(\frac{\sigma^2}{1+\sigma^2} - \kappa\pi\frac{\sigma^2}{1+\sigma^2} - \kappa(1-\pi)\gamma^2\right) - (1-q)(1-\kappa\pi)\,\mathbb{E}[(\beta(s+m)-\theta)^2] \\
&= -q\left((1-\kappa\pi)\frac{\sigma^2}{1+\sigma^2} - \kappa(1-\pi)\gamma^2\right) \\
&- (1-q)(1-\kappa\pi)[\pi\,\mathbb{E}[((2\beta-1)\theta + \beta\varepsilon + \beta\varepsilon')^2] + (1-\pi)\,\mathbb{E}[((\beta+\beta(1-\gamma)-1)\theta + \beta\varepsilon)^2]] \\
&= -q\left((1-\kappa\pi)\frac{\sigma^2}{1+\sigma^2} - \kappa(1-\pi)\gamma^2\right) \\
&- (1-q)(1-\kappa\pi)[\pi((1-2\beta)^2 + 2\beta^2\sigma^2) + (1-\pi)((1-2\beta+\beta\gamma)^2 + \beta^2\sigma^2)] \\
&= -q\left((1-\kappa\pi)\frac{\sigma^2}{1+\sigma^2} - \kappa(1-\pi)\gamma^2\right) - (1-q)(1-\kappa\pi)\left[\frac{\sigma^2}{2+\sigma^2} + (1-\pi)\frac{\gamma^2 - \sigma^2 + 2\gamma\sigma^2}{(2+\sigma^2)^2}\right]
\end{aligned}
$$

Numerically solving for optimal q, we plot the trader's ex-ante return:

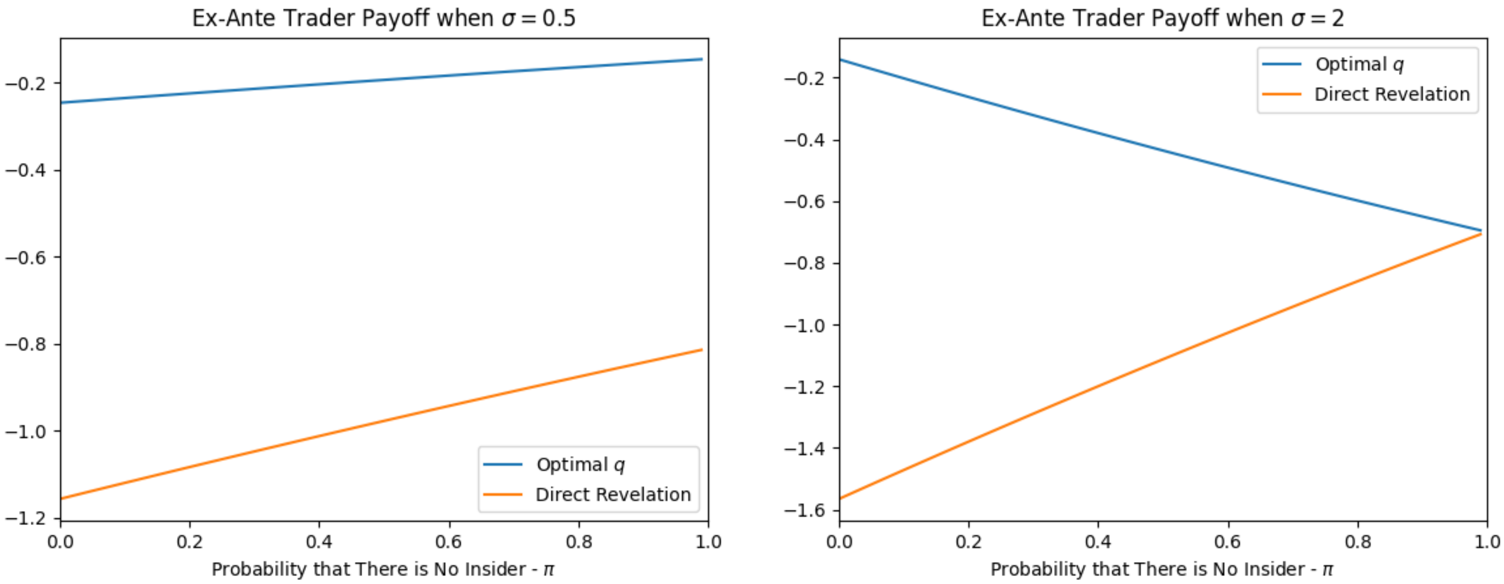

Figure 2.8: Ex-Ante Trader Payoff - Behavioral Variant

It turns out that when σ is low, trader's ex-ante return increases with π. This is because when the quality of trader's information is good enough, trader's welfare is increasing with the probability that the game proceeds to the communication step. And, the probability that the game proceeds to the communication step, $1 - q$, increases with the probability that both players are traders. On the other hand, when σ is high, traders mostly learn from the insider rather than each other. As a result, when the trader's information is noisy enough, increasing likelihood of an insider increases the trader's payoff although the probability that the game ends before the communication step increases with the likelihood of an insider being present.

Lastly, we compare the optimal deterministic communication and direct probabilistic communication mechanisms, by comparing the outcomes for both mechanisms under the same linear-normal signal structure and identifying the mechanism yielding higher ex-ante payoff for the trader:

Above we observe that for both κ values, the optimal deterministic communication mechanism performs better if the trader's information is noisy enough and the likelihood of an insider is low enough. This is because, when the likelihood of an insider is low, the optimal deterministic communication mechanism allows the traders to aggregate information among themselves. The optimal deterministic communication mechanism works relatively poorly when the likelihood of an insider is low, since the principal sends the posterior of the traders to the opposite direction of

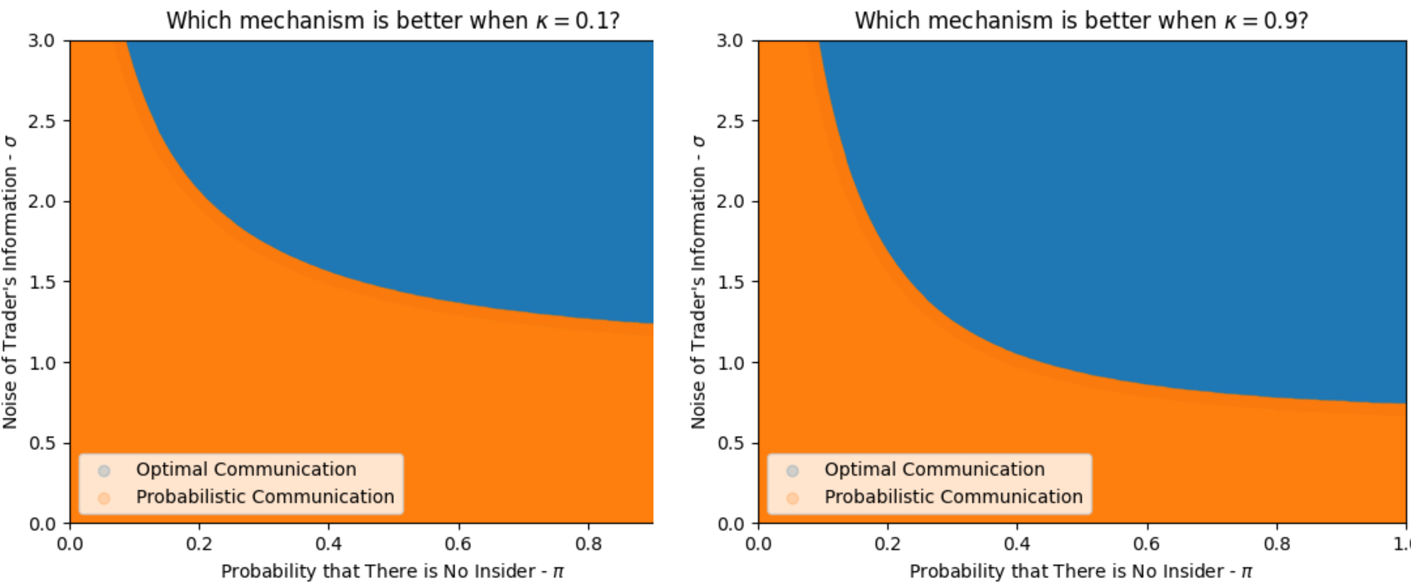

Figure 2.9: Comparison of Two Mechanisms

what their private signals suggest. The direct probabilistic communication mechanism, however, works better if there is a higher chance that there is an insider. When the likelihood of an insider is low, the direct probabilistic communication becomes less desirable as it eliminates communication with some probability.

2.7 Conclusion

This chapter studies two mechanisms of aggregating information from anonymous sources. We have established that with rational players, it is not possible to aggregate information using the optimal deterministic communication mechanism. This is because, with the optimal deterministic communication mechanism, rational players face a prisoner dilemma type situation where the socially optimal information aggregation outcome is never reached due to individual deviations. To avoid this problem, we consider a non-standard equilibrium notion (due to Kyle (1985)) that restricts players from individual deviations and forces them to coordinate on strategies that would benefit all agents of this type. Even with such an equilibrium notion, the optimal deterministic communication mechanism fails to aggregate information. This is because any information-sharing strategy of the trader type can be exploited by the insider and any misleading strategy of the insider can be turned against the insider in equilibrium. As a result, there is no information aggregation under the optimal deterministic communication mechanism even for this specific equilibrium notion.

Then we introduce the direct probabilistic communication mechanism which achieves some information aggregation with rational players. The probabilistic mechanism forces the insider to reveal the true state but this happens at a cost. The cost is that with probability q the game never reaches the communication step. The bigger the likelihood of an insider, the higher the principal should set the probability q.

Lastly, we consider a behavioral variant of this model, where the traders are unaware that they might be playing against an insider. In this case, both mechanisms aggregate some information. It turns out that, the optimal deterministic communication mechanism performs better if the trader's information is noisy but the likelihood of an insider is low. With a low likelihood of an insider, the principal is able to produce an optimal public signal that enables the traders to aggregate information among themselves. On the other hand, when there is a high likelihood of an insider, the optimal deterministic communication mechanism sends the posterior of the traders to the opposite of what their private information suggests and as a result prevents information aggregation. The direct probabilistic communication mechanism, however, benefits from the high likelihood of an insider as it forces the insider to share some information about the true state.

There are three potential avenues for future research following this chapter. Firstly, one can introduce heterogeneity among trader types in terms of the quality of their information. Especially in the model with behavioral traders, if trader types have different levels of information, the optimal deterministic communication mechanism may perform relatively well compared to the direct probabilistic communication mechanism when the likelihood of an insider is low. Secondly, our comparisons rely on a linear normal signal structure for the behavioral case. The extent to which these comparisons can be generalized is an open question. Lastly, incorporation of an explicit trade mechanism with a pricing rule can test how much our result would survive with non-separable price effects.

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Chapter 3

Strategic Trading, Ambiguity and No Trade Theorems

Abstract

In the absence of ambiguity, Walrasian equilibrium prices are generically fully revealing and therefore it is impossible for Bayes rational agents to benefit from their private information as Milgrom and Stokey (1982) among others have shown in a series of results which are referred as no trade theorems. Condie and Ganguli (2011) shows that the rational expectations equilibria may be only partially revealing when some traders have non-smooth ambiguity-averse preferences. In this chapter, we show that even when the prices are partially revealing under ambiguity, there won't be any trade that strictly improves any of the agents if the initial endowments are interim Pareto optimal. Potentially non-revealing prices may increase the trade volume and even with interim Pareto optimal initial endowments, there may be some trade in the equilibrium. However, any such trade won't strictly improve any trader and therefore there is no way to benefit from private information even with ambiguity averse traders.

Key Words: Ambiguity, Strategic Trading, No Trade Theorems

JEL Codes: D50, D82,D83, D84, D90

3.1 Introduction

Theoretical foundations of strategic trading between agents with asymmetric information under common prior requires extensions to usual asset exchange framework for fully informed agents. This is because, in a market with asymmetric information a *fully revealing equilibrium* generically exists (Radner (1979), Allen (1981) and Morris (1994)). A fully revealing equilibrium reveals the private information of individual agents through prices and thereby negates the value of private information. Therefore, agents with common prior won't trade regardless of information dispersion if the current allocation is efficient with respect to the join of the private information of all agents (Milgrom and Stokey, 1982). Then if information acquisition is costly, agents have no incentive to acquire information since there is no way to benefit from private information (Grossman and Stiglitz, 1980).

The so called Grossman Stiglitz Paradox, asserting that agents don't invest in private information since there is no way to benefit from private information, is a theoretical paradox rather than an empirical one. In stock markets, for example, investors are willing to gather private information on stock performances. Stock prices won't perfectly reveal the private information that agents have and learning trough prices takes time. Following this intuition, Hellwig (1980) utilizes aggregate shocks hindering price revelation. Kyle (1985) and Glosten and Milgrom (1985) offers a model with noise traders. Dubey et al. (1987) studies a two period Shapley Shubik Market game where learning takes time. Ausubel (1990) uses a state space with higher dimension than prices. Harris and Raviv (1993) simply drops the co mmon prior, agents agree to disagree and hold different opinions.

In this chapter, we study the trading environment of Condie and Ganguli (2011) with asymmetric information and ambiguity averse agents. Agents are endowed with max-min expected utility with multiple priors a la Gilboa and Schmeidler (1989). Since agents have max-min expected utility instead of a smooth ambiguity aversion preferences, they are not always responsive to private information. When the signal they recieve does not push the posterior sufficiently away from the prior, they act as if they did not receive the signal. This is the reason why Condie and Ganguli (2011) shows that the prices are only partially revealing with ambiguity averse agents. This chapter explores the extent of the strategic trading and the value of private information under the partially revealing equilibrium.

We start with an exposition of the decision making problem under ambiguity when ambiguity is defined using multiple priors. Multiple priors potentially lead to multiple posteriors which in turn implies that the agents will hedge themselves against every possible posterior when making decisions. Therefore, unless the prices are extreme enough, ambiguity averse agents try to follow a certainty equivalent strategy where they demand same consumption level in all states. As a result the Walrasian Equilibrium under ambiguity is not unique generally. Indeed, when the ambiguity is extreme in the sense that the agents' common multiple prior set includes every prior, any price is a Walrasian equilibrium. Furthermore, the prices when there is a multiplicity of equilibria potentially do not reveal the private signals of individual agents.

This is a stark difference from Walrasian equilibrium under no ambiguity. Indeed, it is straightforward to show that there is a one-to-one mapping from non-redundant signals to prices when the underlying Bernoulli utility is strictly concave. Furthermore, introducing a demand schedule game where agents submit demand schedules that maximizes their expected utility conditional on not just their private signal but also prices, one can show that the equilibrium prices always reveal the private signals in the Bayesian Nash equilibrium if it exists. This implies that there can be no trade in the Bayesian Nash equilibrium of this game if initial allocations are interim Pareto optimal. This is in line with the no trade theorem of Milgrom and Stokey (1982).

In the next step, we study the Bayesian Nash equilibrium of the demand schedule game with ambiguity. One may expect that the fact that ambiguity causes non-revealing prices would lead to a violation of no trade result. However, it turns out that non-revealing prices do not benefit agents with informational advantage. This is because that ambiguity yields to non-revealing prices only if some agents follow certainty equivalent strategy. As a result, in the equilibrium of the demand schedule game, given a non-revealing price vector, agents will have same utility level regardless of their private signal. This implies that even though there can be trade given interim Pareto optimal endowments, any such trade won't strictly improve any agent.

The organization of this chapter is as follows: Next section gives an exposition of the decision making problem under ambiguity. We introduce a market for Arrow securities in the third chapter and study the Walrasian equilibrium without any ambiguity. Then, we repeat this exercise with ambiguity averse agent where agents' common prior is a multiple prior set. In the fourth section, we introduce a demand schedule game for both cases. We show that when agents are not ambiguity averse, prices are revealing and the no trade theorems apply. With ambiguity averse agents, however, we show that prices are not fully revealing. Nevertheless, we show that non-revealing prices does not imply that there can be any trade that is beneficial for some agent if the initial endowments are Pareto optimal.

Related Literature: There is a plethora of multiple prior ambiguity and trading models starting with Gilboa and Schmeidler (1989) which introduces multiple prior ambiguity model and Gilboa and Schmeidler (1993) which studies different procedures that incorporates Bayesian updating when there is no unique prior. De Castro and Chateauneuf (2011) shows that ambiguity aversion may lead to lesser trading volume. Strzalecki and Werner (2011) study efficient allocations under ambiguity aversion using the concept of conditional beliefs. Angelopoulos and Koutsougeras (2015) studies incentive compatibility in a pure exchange economy with ambiguity averse agents. De Castro and Yannelis (2018) shows that incentive compatibility agrees with efficiency if agents are ambiguity averse in the form of maximin preferences.Beissner (2017) studies ambiguity about continuous-time volatility in a general equilibrium setup with agents with different preferences for risk and ambiguity. Lastly, Beissner and Riedel (2019) study economies with Knightian uncertainty about state prices.

3.2 Decision Making under Ambiguity

Following Gilboa and Schmeidler (1989), we model ambiguity aversion through an agent with multiple priors over the state space of a decision problem and evaluating each outcome based on the prior that yields the lowest utility for that outcome. That is, given a state space Ω, an ambiguity averse agent endowed with the convex set of priors $\mathcal{P} \subset \triangle(\Omega)$ solves the following maximization problem given a set of alternative state dependent outcomes, $X \subset \mathbb{R}^{|\Omega|}$:

$$\max_{x \in X} \min_{\hat{\pi} \in \mathcal{P}} \mathbb{E}_{\hat{\pi}}[u(x(\omega))]$$

where $u(\cdot)$ is a Bernoulli utility function that is strictly increasing and strictly concave. The fact that the decision maker is evaluating the outcomes based on the prior in $\mathcal{P}$ that minimizes the expected utility forces the decision maker to prefer degenerate lotteries over risky ones. To observe this point, consider a simple example with a binary state space $\Omega = \{H, L\}$ with a prior set such that $\hat{\pi}(H) \in [\underline{\pi}, \bar{\pi}]$ for some $0 \leqslant \underline{\pi} \leqslant \bar{\pi} \leqslant 1$. Further assume that $u(x) = \log(x)$ and consider the following problem:

$$\max \min_{\hat{\pi}(H) \in [\underline{\pi}, \bar{\pi}]} \mathbb{E}_{\hat{\pi}}[\log(x)] \quad \text{subject to} \quad p_L x(L) + p_H x(h) \leqslant I$$

Above p_L and p_H can be interpreted as the prices of Arrow securities for states L and H respectively. Now observe that:

$$
\min_{\hat{\pi}(H)\in[\underline{\pi},\bar{\pi}]} \mathbb{E}_{\hat{\pi}}[\log(x)] = \begin{cases} \underline{\pi}\log(x(H)) + (1-\underline{\pi})\log(x(L)) & \text{if} \quad x(H) > x(L) \\ \log(x(H)) & \text{if} \quad x(H) = x(L) \\ \bar{\pi}\log(x(H)) + (1-\bar{\pi})\log(x(L)) & \text{otherwise} \end{cases}
$$

Letting $\bar{\pi} = \frac{1}{3}$ and $\underline{\pi} = \frac{2}{3}$, below we plot the indifference curve for utility level one:

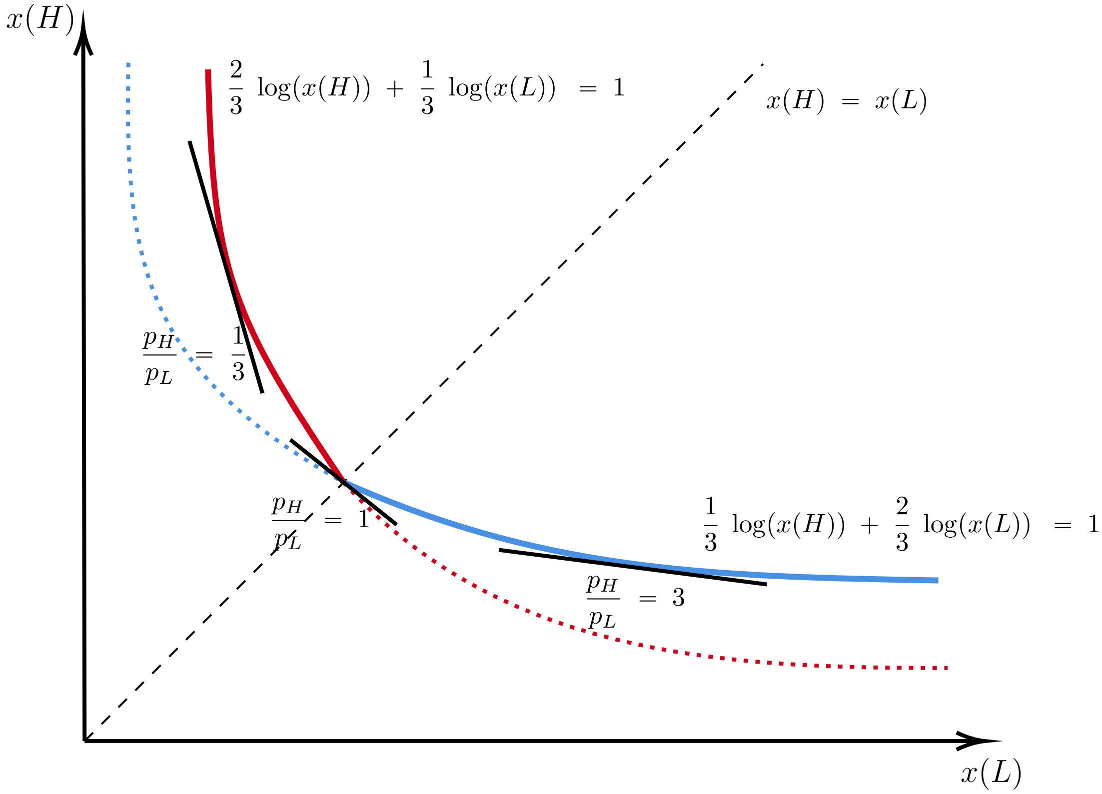

Figure 3.1: Decision Under Ambiguity

As demonstrated in figure 3.1, the ambiguity averse decision maker effectively uses the minimum of two von Neumann Morgenstern expected utility preference under two extreme priors of the set $\mathcal{P}$. As a result, the ambiguity averse decision maker will set $x(H) = x(L)$ as long as $\frac{p_H}{p_L} \in \left[\frac{1}{2}, 2\right]$. This implies that the ambiguity averse decision maker is forced to hedge herself with a certainty equivalent if the Arrow securities' prices. We can generalize this observation by firstly defining the problem of an ambiguity averse decision maker endowed with convex set of prior, $\mathcal{P} \subseteq \triangle(\Omega)$, to buy Arrow securities in a finite state space, Ω, given a monetary endowment, $I \in \mathbb{R}$, and Arrow security price p_ω for each state $\omega \in \Omega$:

$$\max_{x \in \mathbb{R}^{|\Omega|}} \min_{\hat{\pi} \in \mathcal{P}} \mathbb{E}_{\hat{\pi}}[u(x)] \quad \text{subject to} \quad p \cdot x \leqslant I \tag{AA - problem}$$

Given above problem we show that if $\mathcal{P}$ is compact and convex, Ω is finite and $p \in \mathbb{R}^{|\Omega|}_{++}$ then the minimax theorem implies that the ambiguity averse decision maker will set her demand equally if the normalized price is in $\mathcal{P}$:

Theorem 4. *Let $\mathcal{P}$ be compact and convex, $|\Omega| = k \in \mathbb{N}$, $p \in \mathbb{R}^k_{++}$ and u increasing and concave in the AA - problem. Let $\tilde{p}$ be the normalized price such that $\tilde{p} = \frac{p}{\|p\|_1}$. If $\tilde{p} \in \mathcal{P}$ then the unique solution to the AA - problem satisfies $x_\omega = \frac{I}{\|p\|_1}$.*

Proof. Firstly consider the following problem:

$$\max_{x \in X} \min_{\hat{\pi} \in \mathcal{P}} \mathbb{E}_{\hat{\pi}}[u(x)] \tag{AA* - problem}$$

where $X = \{x \in \mathbb{R}^k_+ \,|p \cdot x \leqslant I\}$. Now since for any $x \notin X$ is not feasible for AA - problem, the AA - problem and the AA* - problem are equivalent. Then since $p \in \mathbb{R}^n_{++}$, both X and $\mathcal{P}$ are convex and compact. Furthermore $\mathbb{E}_{\hat{\pi}}[u(x)]$ is continuous on x and $\hat{\pi}$ and it is concave in x since u is concave and convex in $\hat{\pi}$ as it is linear on $\hat{\pi}$. Then the minimax theorem implies:

$$\max_{x \in X} \min_{\hat{\pi} \in \mathcal{P}} \mathbb{E}_{\hat{\pi}}[u(x)] = \min_{\hat{\pi} \in \mathcal{P}} \max_{x \in X} \mathbb{E}_{\hat{\pi}}[u(x)]$$

Then AA - problem is equivalent to:

$$\min_{\hat{\pi} \in \mathcal{P}} \max_{x \in X} \mathbb{E}_{\hat{\pi}}[u(x)] \tag{AA** - problem}$$

Now solving the maximization problem, first order conditions imply that for some $\lambda \in \mathbb{R}$:

$$\frac{\pi_\omega u'(x_\omega)}{p_\omega} = \lambda \quad \Rightarrow \quad x_\omega = u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)$$

Since $u(\cdot)$ is concave, above condition must be satisfied for the optima. Then the minimization problem becomes:

$$\min_{\hat{\pi} \in \mathcal{P}} \sum_{\omega \in \Omega} \hat{\pi}_\omega u\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)$$

Firstly to see that above objective function is convex in $\hat{\pi}$ note that for any $f : \mathbb{R}_+ \to \mathbb{R}$ where f is increasing and convex, $zf(z)$ is convex. Then the objective function is convex if $u\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)$ is increasing and convex in $\hat{\pi}_\omega$. Now since u is increasing and concave, u' is decreasing and therefore

$u\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)$ is increasing in $\hat{\pi}_\omega$. And the first derivative of $u\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)$ with respect to $\hat{\pi}_\omega$ is:

$$-u'\left(u'^{-1}\left(\frac{\lambda p_\omega}{\hat{\pi}_\omega}\right)\right)\frac{1}{u''\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)}\frac{\lambda p_\omega}{\pi_\omega^2} = -\frac{1}{u''\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)}\left(\frac{\lambda p_\omega}{\pi_\omega}\right)^2$$

Now since above expression is increasing in $\hat{\pi}_\omega$, then the objective function is convex. Then we can use the first order condition to find interior solution:

$$\forall\omega\in\Omega\quad u\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right) - \frac{1}{u''\left(u'^{-1}\left(\frac{\lambda p_\omega}{x_\omega}\right)\right)}\left(\frac{\lambda p_\omega}{\pi_\omega}\right)^2 = \mu$$

where $\mu\in\mathbb{R}$. Note that $\frac{p_\omega}{x_\omega} = \frac{p'_\omega}{x'_\omega}$ satisfies above condition. Then if $\hat{\pi}\in\mathcal{P}$ where $\frac{p_\omega}{\hat{\pi}_\omega} = \frac{p'_\omega}{\hat{\pi}_{\omega'}}$ for all $\omega,\omega'\in\Omega$, we have an interior solution with $x_\omega = x_{\omega'}$ for all $\omega,\omega'\in\Omega$. Furthermore $\frac{p_\omega}{\hat{\pi}_\omega} = \frac{p'_\omega}{\hat{\pi}_{\omega'}}$ for all $\omega,\omega'\in\Omega$ if and only if $\tilde{p}\in\mathcal{P}$. Hence we have that if $\tilde{p}\in\mathcal{P}$ then $x_\omega = \frac{I}{\|p\|_1}$. $\square$

Above theorem tells us that the ambiguity averse investors will hedge against every possibility when the normalized prices of Arrow securities are equal to the probabilities of the associated states for some probability measure in their multiple prior set. This makes the ambiguity averse investors less sensitive to the prices than ambiguity neutral investors.

In the next section, we will introduce a market for Arrow securities with finitely many potentially ambiguity averse agents. Ambiguity averse agents will be able to receive signals that will update their multiple prior set and then we will consider the Walrasian equilibrium in this securities market. We will, then, introduce a model of strategic trading and solve for Bayesian Nash Equilibrium.

3.3 A Market for Arrow Securities with Ambiguity Averse Agents

Let N denote the set of finitely many agents and Ω denote the finite set of states, $|\Omega| = k$. Let $e_{i\omega}\in\mathbb{R}_+$ denote the endowment of agent $i\in N$ at state $\omega\in\Omega$. We assume that all agents are ambiguity averse at least ex-ante, before receiving their private signals. Let $\mathcal{P}_0\subseteq\triangle(\Omega)$ be the set of common priors. Now consider a set of signals S with known conditional distribution $f(s|\omega)$ for each $s\in S$ and $\omega\in\Omega$. Receiving the signal $s\in S$, every prior in the common prior set $\mathcal{P}_0$ is updated so that set of posteriors after receiving $s\in S$ becomes:

$$\mathcal{P}_s = \{\hat{\pi}^s|\hat{\pi}\in\mathcal{P}_0\}\quad\text{where}\quad\forall\omega\in\Omega\quad\hat{\pi}^s_\omega = \frac{\hat{\pi}_\omega f(s|\omega)}{\sum_{\omega'\in\Omega}\hat{\pi}_{\omega'}f(s|\omega')}$$

Each agent is endowed with a set of signals S_i and a known distribution $f_i(s_i|\omega)$ for each $s_i \in S_i$. Each agent is expected utility maximizers with a Bernoulli utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$ which is strictly increasing and strictly concave.

3.3.1 Walrasian Equilibrium

Now we can define a Walrasian equilibrium of an Arrow securities market after the private signals are realized:

Definition 1. *Given a private signal realization $s \in \bigtimes_{i \in N} S_i$, a price vector $p \in \triangle^{k-1}$ is a Walrasian equilibirum if:*

- *For all $i \in N$, $x_i = \arg\max_{x \in X_i} \min_{\hat{\pi} \in \mathcal{P}_{s_i}} \mathbb{E}_{\hat{\pi}}[x]$ where $X_i = \{x \in \mathbb{R}^k_+ \, | p \cdot x \leqslant p \cdot e_i\}$.*
- $\sum_{i \in N} x_i = \sum_{i \in N} e_i$

Notice that the Walrasian equilibrium assumes price taking behavior in the sense that prices are exogenous to the agents' problem and the agents doesn't learn anything about what the other agents received as their private signal. This is an important point that we will relax when we consider a strategic version of the securities market.

Now let's consider an example with two investors, $N = \{1, 2\}$ two states: $\{H, L\}$. The endowments are given by the following such that:

	H	L
1	1	0
2	0	1

The common multiple prior set is the following:

$$\mathcal{P}_0 = \left\{ (\hat{\pi}_H, \hat{\pi}_L) | (\hat{\pi}_H \in \left[\tfrac{3}{7}, \tfrac{4}{7}\right] \right\}$$

And the private signal structure is given such that:

$f_1(s_1\|\omega)$	H	L
h	1	$\frac{1}{2}$
l	0	$\frac{1}{2}$

$f_2(s_2\|\omega)$	H	L
$\varnothing$	1	1

Here agent 2 does not receive any informative signal. For the first agent, for any prior in the multiple prior set, the signal l perfectly reveals that the true state is L that is $\Pr(\omega = L | s_1 = l) = 1$. For the signal h, given a prior $\hat{\pi}$ we have:

$$\Pr(\omega = H | s_1 = h) = \frac{f(s_1 = h | \omega = H)\hat{\pi}(H)}{f(s_1 = h|\omega = H)\hat{\pi}(H) + f(s_1 = h | \omega = L)\hat{\pi}(L)} = \frac{\hat{\pi}(H)}{\hat{\pi}(H) + \frac{1}{2}\hat{\pi}(L)} = \frac{2\hat{\pi}(H)}{1 + \hat{\pi}(H)}$$

Then we have:

$$\mathcal{P}_{s_1=h} = \left\{ (\hat{\pi}_H, \hat{\pi}_L) | (\hat{\pi}_H \in \left[\frac{3}{5}, \frac{8}{11} \right] \right\}$$
$$\mathcal{P}_{s_1=l} = (\hat{\pi}_H, \hat{\pi}_L) = (0, 1)$$
$$\mathcal{P}_{s_2} = \mathcal{P}_0$$

Using 4, we have that agent 2 demands:

$$x_2 = \begin{cases} \left(\frac{4}{7} \frac{p_L}{p_H}, \frac{3}{7} \right) & \text{if } \frac{p_H}{p_L} > \frac{4}{3} \\ \left(\frac{p_L}{p_H + p_L}, \frac{p_L}{p_H + p_L} \right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{3}{4}, \frac{4}{3} \right] \\ \left(\frac{3}{7} \frac{p_L}{p_H}, \frac{4}{7} \right) & \text{if } \frac{p_H}{p_L} < \frac{3}{4} \end{cases}$$

Clearly, when $s_1 = l$, the demand for agent 1 becomes $x_1 = \left(0, \frac{p_H}{p_L}\right)$. The market clears at $p^* = \left(\frac{3}{10}, \frac{7}{10}\right)$ and the Walrasian allocations are:

$$x_1 = \left(0, \frac{3}{7}\right) \qquad x_2 = \left(1, \frac{4}{7}\right)$$

Ambiguity averse agent 2 acts as if the prior probability of high state is $\frac{3}{7}$ which is a result of the fact that ambiguity averse agents make their decision under the least favorable prior given a price vector.

On the other hand when $s_1 = l$, using theorem 4, we get that agent 1 demands:

$$x_1(s_1 = h) = \begin{cases} \left(\frac{8}{11}, \frac{3}{11} \frac{p_H}{p_L} \right) & \text{if } \frac{p_H}{p_L} > \frac{8}{3} \\ \left(\frac{p_H}{p_H + p_L}, \frac{p_H}{p_H + p_L} \right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{3}{2}, \frac{8}{3} \right] \\ \left(\frac{3}{5}, \frac{2}{5} \frac{p_H}{p_L} \right) & \text{if } \frac{p_H}{p_L} < \frac{3}{2} \end{cases} \qquad x_1(s_1 = l) = (0, 1)$$

The market clearing conditions yield a unique price vector in the simplex, $p^* = \left(\frac{10}{17}, \frac{7}{17}\right)$ and the Walrasian allocations are:

$$x_1 = \left(\frac{3}{5}, \frac{4}{7}\right) \qquad x_2 = \left(\frac{2}{5}, \frac{3}{7}\right)$$

To contrast with what would happen if there were no ambiguity, assume that the common prior was a singleton which is a subset of $\mathcal{P}_0$. Let π denote the common prior with $\pi \in \mathcal{P}_0$. Then the demand for agent 1 becomes:

$$x_1(s_1 = h) = \left(\frac{2\pi}{1+\pi}, \frac{1-\pi}{1+\pi}\frac{p_H}{p_L}\right)$$

And the demand for agent 2 becomes:

$$x_2 = \left(\pi\frac{p_L}{p_H}, 1-\pi\right)$$

The market clearing conditions yield a unique price vector in the simplex, $p^* = \left(\frac{\pi+\pi^2}{1+\pi^2}, \frac{1-\pi}{1+\pi^2}\right)$ and the Walrasian allocations are:

$$x_1 = \left(\frac{2\pi}{1+\pi}, \pi\right) \qquad x_2 = \left(\frac{1-\pi}{1+\pi}, 1-\pi\right)$$

Below we plot the Edgeworth box for Ambiguity equilibrium and the Walrasian equilibria for each prior in the multiple prior set:

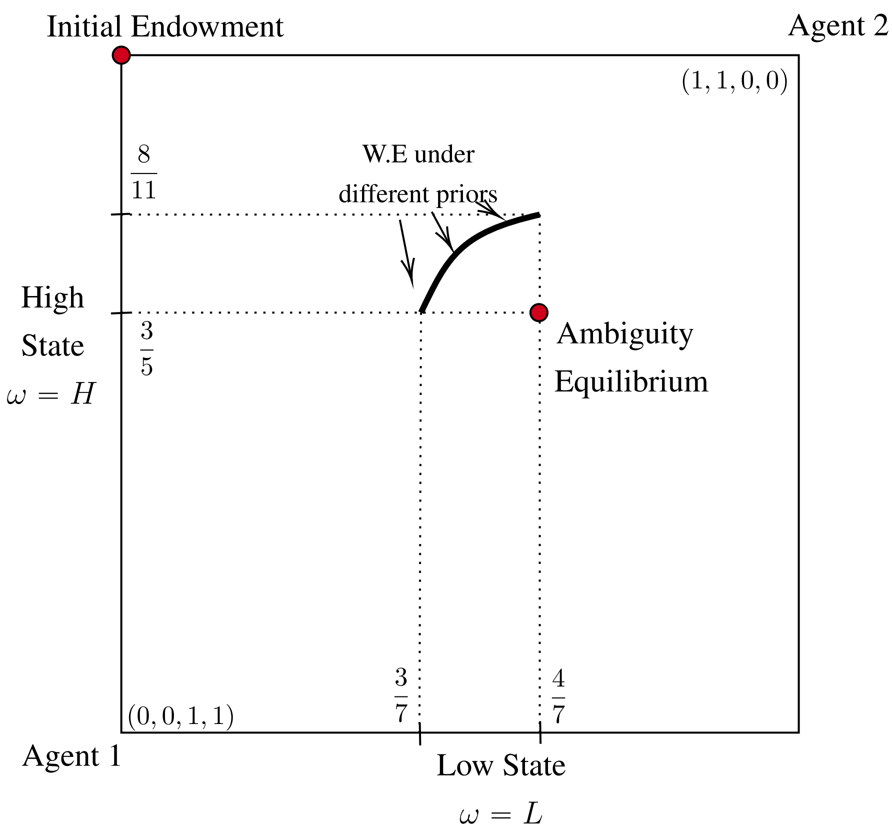

Figure 3.2: Walrasian Equilibrium and Ambiguity Equilibrium

Notice that ambiguity equilibrium is not a subset of the possible Walrasian equilibria under different priors of the multiple priors set. We can further explore the trading volume for Walrasian equilibria and the ambiguity equilibrium. Indeed, figure 3.2 demonstrates that the trading volume is higher with ambiguity regardless of the common prior assumed for the Walrasian equilibrium without ambiguity.

3.3.2 Multiplicity of Equilibria

Unfortunately, with multiple-prior ambiguity, often there is a multiplicity of equilibrium. We can observe this by modifying above example. Let the common prior set be the following:

$$\mathcal{P}_0 = \left\{(\hat{\pi}_H, \hat{\pi}_L) | (\hat{\pi}_H \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right]\right\}$$

And we keep the private signal structure from before:

$f_1(s_1\|\omega)$	H	L
h	1	$\frac{1}{2}$
l	0	$\frac{1}{2}$

$f_2(s_2\|\omega)$	H	L
$\varnothing$	1	1

For the first agent, for any prior in the multiple prior set, the signal l perfectly reveals that the true state is L that is $\Pr(\omega = L|s_1 = l) = 1$. For the signal h, given a prior $\hat{\pi}$ we have:

$$\Pr(\omega = H|s_1 = h) = \frac{f(s_1 = h|\omega = H)\hat{\pi}(H)}{f(s_1 = h|\omega = H)\hat{\pi}(H) + f(s_1 = h|\omega = L)\hat{\pi}(L)} = \frac{\hat{\pi}(H)}{\hat{\pi}(H) + \frac{1}{2}\hat{\pi}(L)} = \frac{2\hat{\pi}(H)}{1 + \hat{\pi}(H)}$$

Then we have:

$$\begin{aligned}
\mathcal{P}_{s_1=h} &= \left\{(\hat{\pi}_H, \hat{\pi}_L)|(\hat{\pi}_H \in \left[\frac{1}{2}, \frac{4}{5}\right]\right\} \\
\mathcal{P}_{s_1=l} &= (\hat{\pi}_H, \hat{\pi}_L) = (0, 1) \\
\mathcal{P}_{s_2} &= \mathcal{P}_0
\end{aligned}$$

Again using theorem 4, we get that agent 2 demands:

$$x_2 = \begin{cases} \left(\frac{2}{3}\frac{p_L}{p_H}, \frac{1}{3}\right) & \text{if } \frac{p_H}{p_L} > 2 \\ \left(\frac{p_L}{p_H+p_L}, \frac{p_L}{p_H+p_L}\right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{1}{2}, 2\right] \\ \left(\frac{1}{3}\frac{p_L}{p_H}, \frac{2}{3}\right) & \text{if } \frac{p_H}{p_L} < \frac{1}{2} \end{cases}$$

When $s_1 = l$, using theorem 4, we get that agent 1 demands:

$$x_1(s_1 = h) = \begin{cases} \left(\frac{4}{5}, \frac{1}{5}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L} > 4 \\ \left(\frac{p_H}{p_H+p_L}, \frac{p_H}{p_H+p_L}\right) & \text{if } \frac{p_H}{p_L} \in [1, 4] \\ \left(\frac{1}{2}, \frac{1}{2}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L} < 1 \end{cases} \qquad x_1(s_1 = l) = (0, 1)$$

The market clearing conditions are satisfied for any price in the simplex satisfying $p^* \in \left\{(z, 1-z)|z \in \left(\frac{1}{2}, \frac{2}{3}\right)\right\}$, and the Walrasian allocations are:

$$x_1 = (p, p) \qquad x_2 = (1-p, 1-p)$$

We can plot the edgeworth box again for above allocations:

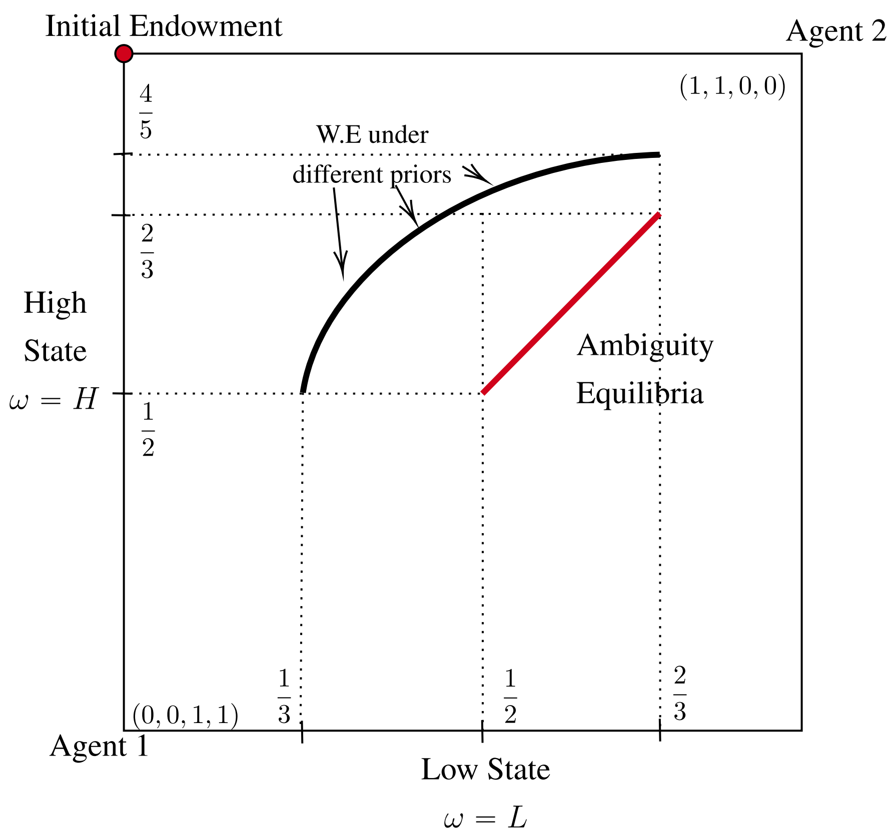

Figure 3.3: Multiplicity of Ambiguity Equilibrium

We can look for the extreme case where the multiple prior set is all possible priors, that is:

$$\mathcal{P}_0 = \{(\hat{\pi}_H, \hat{\pi}_L)|(\hat{\pi}_H \in [0, 1]\}$$

With such a multiple prior set, regardless of the private signals, the ambiguity averse decision maker won't be able to update the prior set. As a result the demand for agent 1 becomes:

$$x_1 = \left(\frac{p_H}{p_H + p_L}, \frac{p_H}{p_H + p_L}\right)$$

And similarly for agent 2, we have:

$$x_2 = \left(\frac{p_L}{p_H + p_L}, \frac{p_L}{p_H + p_L}\right)$$

Clearly any price clears the market. The price we pick for the equilibrium directly determines the allocations for each agents. The fact that any price would constitute an equilibrium means that with extreme ambiguity aversion, the usual Walrasian equilibrium won't be able to produce a non-arbitrary allocation. Figure 3.4 demonstrates this point. Interestingly, compated to Walrasian Equilibrium, ambiguity equilibria have more volume in general.

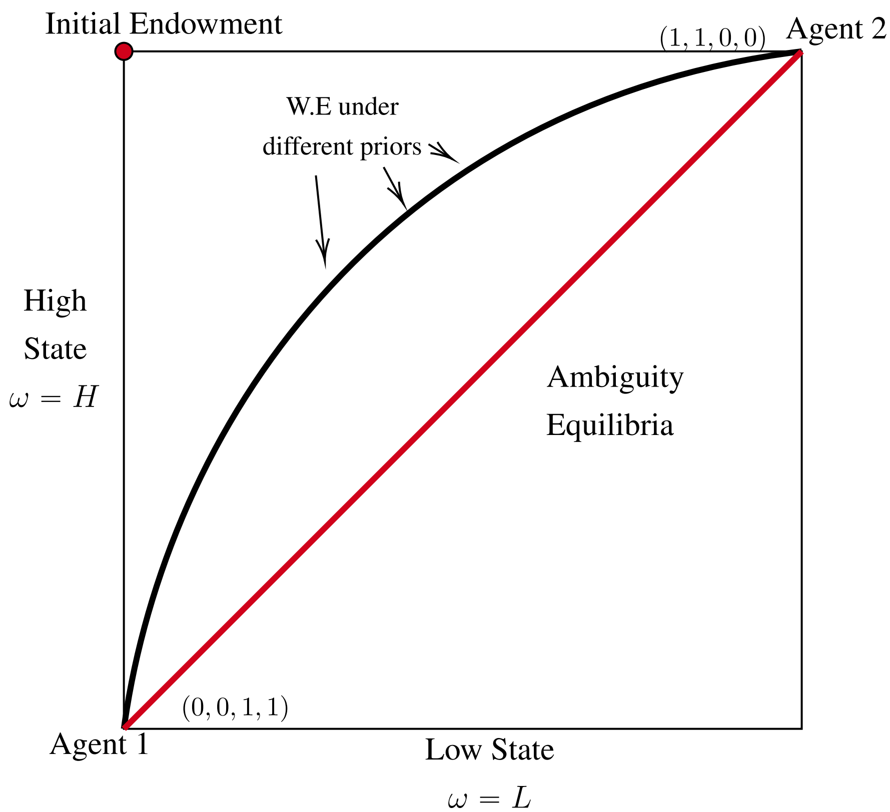

Figure 3.4: Extreme Ambiguity Aversion

We can generalize the possibility of multiplicity of equilibrium to arbitrary Bermoulli utilities, number of agents and information structures:

Theorem 5. *Take an Arrow securities economy with equal aggregate endowments in each state, $e_\omega = e_{\omega'}$ for all $\omega, \omega' \in \Omega$. Given a private signal realization $s \in \times_{i\in N} S_i$, if $\bigcap_{i\in N} \mathcal{P}_{s_i} \neq \varnothing$ then every price vector $p \in \bigcup_{i\in N} \mathcal{P}_{s_i}$ is a Walrasian equilibrium where every agent demands the same level of consumption in all states.*

Proof. Take any $p \in \bigcap_{i\in N} \mathcal{P}_{s_i}$, then for any agent $i \in N$, $p \in \mathcal{P}_{s_i}$. By theorem 4, we know that:

$$\forall i \in N, \quad x_i = (p \cdot e_i)\iota$$

where $\iota \in \mathbb{R}^k$ is defined such that $\iota = (1, 1, \cdots, 1)$. Then we have:

$$\sum_{i\in N} x_i = \sum_{i\in N} (p \cdot e_i)\iota = p \cdot \sum_{i\in N} e_i = \sum_{i\in N} e_i$$

Above last equality follows the fact that aggregate endowments are equal in each state. Therefore, any $p \in \bigcup_{i\in N} \mathcal{P}_{s_i}$ is a Walrasian equilibrium and every agent demands the same level of consumption in all states. □

3.4 Strategic Trading: Bayes Nash Equilibrium vs Nash Equilibrium

3.4.1 Bayesian Nash Equilibrium without Ambiguity

So far, we have assumed that the agents are price takers. Although it does make sense to assume price taking assumption in production economies, the predictions under price taking behavior is not satisfactory for Arrow securities. To see this point, let's consider a no-ambiguity version of our two agents example. Recall that the endowments for our two agents economy were given such that:

	H	L
1	1	0
2	0	1

Assume that the common prior is that $\Pr(\theta = H) = \frac{1}{2}$ and the private signal structure is given such that:

$f_1(s_1\|\omega)$	H	L
h	1	$\frac{1}{2}$
l	0	$\frac{1}{2}$

$f_2(s_2\|\omega)$	H	L
$\varnothing$	1	1

Now we have that $\Pr(\omega = H|s_1 = h) = \frac{2}{3}$, $\Pr(\omega = H|s_1 = l) = 0$ and $\Pr(\omega = H|s_2) = \frac{1}{2}$. Then the Walrasian demand for each agent becomes:

$$x_1(s_1 = h) = \left(\frac{2}{3}, \frac{1}{3}\frac{p_H}{p_L}\right)$$
$$x_1(s_1 = l) = \left(0, \frac{p_H}{p_L}\right)$$
$$x_2 = \left(\frac{1}{2}\frac{p_L}{p_H}, \frac{1}{2}\right)$$

Then $p^* = \left(\frac{3}{5}, \frac{2}{5}\right)$ is the walrasian equilibrium price when $s_1 = h$ and $p^* = \left(\frac{1}{3}, \frac{2}{3}\right)$ is the walrasian equilibrium price when $s_1 = l$. Then there exists a one-to-one map between the private signal of agent 1 and the equilibrium price. This implies that player 2 should be able to deduce the signal that agent 1 received. But then agent 2 won't demand what we prescribed for the Walrasian equilibrium. Instead agent 2 may set up a demand schedule such that:

$$x_2(p) = \begin{cases} \left(\frac{2}{3}\frac{p_L}{p_H}, \frac{1}{3}\right) & \text{if} \quad p_H > p_L \\ (0, 1) & \text{otherwise} \end{cases}$$

With above demand schedule the market clearing prices become $p^* = \left(\frac{2}{3}, \frac{1}{3}\right)$ when $s_1 = h$ and $p^* = (0, 1)$ is the walrasian equilibrium price when $s_1 = l$. As a result there will be no trade when $s_1 = l$. New equilibrium is clearly better for agent 2 in both states. Is it possible for agent 1 to respond to above demand schedule to preserve his informational advantage? Essentially, the only way for agent 1 to preserve his informational advantage is to make sure the same equilibrium price arise regardless of the signal he received. This can only be possible if agent 1 submits the same demand schedule regardless of her signal. However, there is a problem of commitment in the sense that an ex-ante optimal pooling strategy won't be interim optimal as it would be beneficial to deviate at least for one signal. To see this assume that agent 2 submits the demand schedule $(\beta_2(\tilde{p}), 1 - \tilde{p}\beta_2(\tilde{p})$ where $\tilde{p} = \frac{p_H}{p_L}$ and $\beta_2 : \triangle^1 \to [0, 1]$ is a continuous function. Given the demand schedule of agent 2, agent 1 will pick $(\beta_1(\tilde{p}), \tilde{p}(1 - \beta_2(\tilde{p})))$. Market clearing conditions imply that $\beta_1(\tilde{p}) + \beta_2(\tilde{p}) = 1$ where $\beta_1 : \triangle^1 \to [0, 1]$ is a continuous function. Then given the posterior belief $\Pr(\omega = H|s) = \mu_s$ objective function of agent 1 becomes:

$$\mu_s \log[\beta_1(\tilde{p})] + (1 - \mu_s) \log[\tilde{p}(1 - \beta_1(\tilde{p}))] = \mu_s \log[1 - \beta_2(\tilde{p})] + (1 - \mu_s) \log[\tilde{p}\beta_2(\tilde{p})]$$

Then finding a demand schedule $(\beta_1(\tilde{p}), \tilde{p}(1-\beta_2(\tilde{p})))$ is equivalent to solving the following for $\tilde{p}$:

$$\max_{\tilde{p}} \ \mu_s \log[1-\beta_2(\tilde{p})] + (1-\mu_s)\log[\tilde{p}\beta_2(\tilde{p})]$$

For the solution to above problem does not depend on μ_s, both $\log[1-\beta_2(\tilde{p})]$ and $(1-\mu_s)\log[\tilde{p}\beta_2(\tilde{p})]$ should be maximized at same $\tilde{p}$. To see this assume for some $\mu_s \in [0,1]$, $\tilde{p}^*$ is a solution to above problem but it is not a maximizer for either $\log[1-\beta_2(\tilde{p})]$ or $\log[\tilde{p}\beta_2(\tilde{p})]$. But then by appropriately choosing λ, the maximizer of the one of the terms will be the solution to above problem implying that the solution depends on μ_s. Then both $\log[1-\beta_2(\tilde{p})]$ and $(1-\mu_s)\log[\tilde{p}\beta_2(\tilde{p})]$ should be maximized at same $\tilde{p}$. But regardless of the function β_2, $\log[1-\beta_2(\tilde{p})]$ and $(1-\mu_s)\log[\tilde{p}\beta_2(\tilde{p})]$ can not have the same maximizer.

Then, we conclude that no pooling strategy can be interim optimal for agent 1. This implies that agent 1 is unable to benefit from her private information. Indeed, once agent's take into account the relationship between prices and signals, there will be no trade in case $s_1 = l$.

We can generalize this finding. To do this, we first need to define a demand schedule game and the appropriate Bayesian Nash equilibrium notion for this game.

The demand schedule game is a tuple, $(N, (u_i)_{i\in N}, \Omega, e, \pi_0, (S_i)_{i\in N}, (f_i(\cdot|\omega))_{i\in N})$ where N is the set of agents where $|N| = n$, u_i for agent i is the strictly increasing and strictly concave and differentiable Bernoulli utility function, Ω is the finite set of states where $|\Omega| = k$, $e \in \mathbb{R}_+^{n\times k}$ is the endowment vector, π_0 is the common prior, S_i for agent i is the set of private signals and $f_i(\cdot|\omega) : S \to [0,1]$ is the conditional distribution of the privates signals of agent i conditional on the state $\omega \in \Omega$. A strategy for agent i is a function $\beta_i : S_i \times \triangle^{k-1} \to \mathbb{R}_+^k$ determining the demand at each private signal and price tuple.

Now, we can define the Bayesian Nash equilibrium of the demand schedule game:

Definition 2. *The strategy tuple $(\beta_i)_{i\in N}$ is a Bayesian Nash equilibrium of the demand schedule game if there exists an equilibrium price mapping $p^* : \times_{i\in N} S_i \to \triangle^{k-1}$ such that:*

- $\forall s \in \times_{i\in N} S_i, \forall i \in N \ \beta_i(s_i, p^*(s)) \in \arg\max_{x_i\in X_i} \mathbb{E}[u_i(x_i)|s_i, p]$
- $\forall s \in \times_{i\in N} S_i, \ \sum_{i\in N} \beta_i(s_i, p^*(s)) = \sum_{i\in N} e_i$

where for each $i \in N$, $X_i = \{x_i \in \mathbb{R}_+ \,|p^* \cdot x_i \leqslant p^* \cdot e_i\}$.

Now that we have our notion of equilibrium, we can prove that there is no pooling equilibrium in the demand schedule game:

Theorem 6. *There can be no pooling equilibrium in the demand schedule game. That is, let*

$(\beta_i)_{i\in N}$ *be a Bayesian Nash equilibrium with the price mapping* p^*. *If* $\Pr(\cdot|s_i) \neq \Pr(\cdot|s_i')$ *for some* $s_i, s_i' \in S_i$, *then for any equilibrium price mapping* $p^* : \times_{i\in N} S_i \to \triangle^{k-1}$, *we have that* $p^*(s_i', s_{-i}) \neq p^*(s_i, s_{-i})$.

Proof. Assume that $\Pr(\cdot|s_i) \neq \Pr(\cdot|s_i')$ for some $s_i, s_i' \in S_i$ but $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$. $\Pr(\cdot|s_i) \neq \Pr(\cdot|s_i')$ implies $\Pr(\omega|s_i, p^*(s_i, s_{-i})) \neq \Pr(\omega'|s_i', p^*(s_i', s_{-i}))$ for some $\omega, \omega' \in \Omega$. Now the individual optimality implies:

$$\frac{u'(\beta_i(s_i, p^*(s_i, s_{-i}))(\omega) \Pr(\omega|s_i, p^*(s_i, s_{-i}))}{u'(\beta_i(s_i, p^*(s_i, s_{-i}))(\omega') \Pr(\omega'|s_i, p^*(s_i, s_{-i}))} = \frac{p^*(s_i, s_{-i})(\omega)}{p^*(s_i, s_{-i})(\omega')}$$

Similarly we have:

$$\frac{u'(\beta_i(s_i', p^*(s_i, s_{-i}))(\omega) \Pr(\omega|s_i', p^*(s_i', s_{-i}))}{u'(\beta_i(s_i', p^*(s_i, s_{-i}))(\omega') \Pr(\omega'|s_i', p^*(s_i', s_{-i}))} = \frac{p^*(s_i', s_{-i})(\omega)}{p^*(s_i', s_{-i})(\omega')}$$

Since $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$, above implies:

$$\frac{u'(\beta_i(s_i, p^*(s_i, s_{-i}))(\omega) \Pr(\omega|s_i, p^*(s_i, s_{-i}))}{u'(\beta_i(s_i, p^*(s_i, s_{-i}))(\omega') \Pr(\omega'|s_i, p^*(s_i, s_{-i}))} = \frac{u'(\beta_i(s_i', p^*(s_i, s_{-i}))(\omega) \Pr(\omega|s_i', p^*(s_i', s_{-i}))}{u'(\beta_i(s_i', p^*(s_i, s_{-i}))(\omega') \Pr(\omega'|s_i', p^*(s_i', s_{-i}))}$$

And since $\Pr(\omega|s_i, p^*(s_i, s_{-i})) \neq \Pr(\omega'|s_i', p^*(s_i', s_{-i}))$, above implies:

$$\frac{u'(\beta_i(s_i, p^*(s_i, s_{-i}))(\omega)}{u'(\beta_i(s_i, p^*(s_i, s_{-i}))(\omega')} \neq \frac{u'(\beta_i(s_i', p^*(s_i, s_{-i}))(\omega)}{u'(\beta_i(s_i', p^*(s_i, s_{-i}))(\omega')}$$

And together with the fact that $p^*(s_i, s_{-i})) \cdot \beta_i(s_i, p^*(s_i, s_{-i})) = p^*(s_i, s_{-i})) \cdot \beta_i(s_i', p^*(s_i, s_{-i}))$, above implies $\beta_i(s_i, p^*(s_i, s_{-i})) \neq \beta_i(s_i', p^*(s_i, s_{-i}))$. Then market clearing implies that there exists some $j \in N$ with $\beta_j(s_j, p^*(s_i, s_{-i})) \neq \beta_j(s_j, p^*(s_i', s_{-i}))$ which contradicts our assumption that $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$. Therefore, we conclude that $\Pr(\cdot|s_i) \neq \Pr(\cdot|s_i')$ for some $s_i, s_i' \in S_i$ implies $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$. $\square$

A direct result of theorem 6 is that the prices in the demand schedule game is fully revealing in the sense that every agent makes their decision as if they received not just their own private signal but private signals of all others:

Corollary 6. *Given a non-degenerate signal structure such that* $\forall i \in N$, *there is no* $s_i, s_i' \in S_i$ *with* $\Pr(\cdot|s_i) = \Pr(\cdot|s_i)$, *an equilibrium price mapping* p^* *is invertible. That is prices fully reveal the underlying signals.*

Above corollary directly yields us a version of no-trade theorems. To spell out this version of no-trade theorem, we need to introduce the notion of interim Pareto optimality.

Definition 3. *Let $x \in \mathbb{R}^{n\times k}$ be a feasible allocation of state consumption for each agent given an endowment structure $e \in \mathbb{R}^{n\times k}$. We say that x is interim Pareto optimal with respect to signal realizations $(s_1, \cdots, s_n) \in bigtimes_{i\in N} S_i$ if there exists no other allocation $x' \in \mathbb{R}^{n\times k}$ such that:*

- $\forall i \in N,\ \mathbb{E}[u(x_i')|s_1, \cdots, s_n] \geqslant \mathbb{E}[u(x_i)|s_1, \cdots, s_n]$.
- *There exists some $j \in N$ with $\mathbb{E}[u(x_j')|s_1, \cdots, s_n] > \mathbb{E}[u(x_j)|s_1, \cdots, s_n]$.*
- $\sum_i x_i' = \sum_i e_i$.

With the above definition, we are ready to spell out the no-trade theorem.

Theorem 7. *If the initial endowment $e \in \mathbb{R}^{n\times k}$ is interim Pareto optimal with respect to signal realizations $(s_1, \cdots, s_n) \in \bigtimes_{i\in N} S_i$, then in any Bayesian Nash equilibrium of the demand schedule game with equilibrium price mapping $p^* : \bigtimes_{i\in N} S_i \to \triangle^{k-1}$, for all $i \in N$, we have that $\beta_i^*(s_i, p^*(s)) = e_i$. That is, there can be no trade in equilibrium.*

Proof. Take a demand schedule game with an initial endowment $e \in \mathbb{R}^{n\times k}$ which is interim Pareto optimal with respect to signal realizations $(s_1, \cdots, s_n) \in \bigtimes_{i\in N} S_i$. Assume that there exists a Bayesian Nash equilibrium with equilibrium price mapping $p^* : \bigtimes_{i\in N} S_i \to \triangle^{k-1}$ where there exists some $j \in N$ with $\beta_j^*(s_j, p^*(s)) \neq e_j$. Since β^* is a Bayesian Nash equilbrium, we have that:

$$\forall i \in N, \quad \beta_i^*(s_i, p^*(s)) \in \arg\max_{x_i \in X_i} \mathbb{E}[u_i(x_i)|s_i, p^*]$$

By corollary 6, we have that $\mathbb{E}[u_i(x_i)|s_i, p^*] = \mathbb{E}[u(x_i)|s_1, \cdots, s_n]$ and therefore we can re-write above as:

$$\forall i \in N, \quad \beta_i^*(s_i, p^*(s)) \in \arg\max_{x_i \in X_i} \mathbb{E}[u_i(x_i)|s_1, \cdots, s_n]$$

Since $e_i \in X_i$ for all $i \in N$, then above implies that for all $i \in N$, $\mathbb{E}[u_i(\beta_i^*(s_i, p^*(s)))|s_1, \cdots, s_n] \geqslant \mathbb{E}[u_i(e_i)|s_1, \cdots, s_n]$. Now, assume that $\mathbb{E}[u_j(\beta_j^*(s_j, p^*(s)))|s_1, \cdots, s_n] = \mathbb{E}[u_j(e_j)|s_1, \cdots, s_n]$. Since $\beta_j^*(s_j, p^*(s)) \neq e_j$, for any $\lambda \in (0, 1)$ letting $x_j' = \lambda\beta_j^*(s_j, p^*(s)) + (1 - \lambda)e_j$ we have that $x_j' \neq e_j$, $x_j' \neq \beta_j^*(s_j, p^*(s))$ and $x_j' \in X_j$. But then the strict concavity of u_j implies that $\mathbb{E}[u_j(\beta_j^*(s_j, p^*(s)))|s_1, \cdots, s_n] < \mathbb{E}[u_j(x_j')|s_1, \cdots, s_n]$ contradicting that $\beta_j^*(s_j, p^*(s)) \in \arg\max_{x_j \in X_j} \mathbb{E}[u_j(x_j)|s_1, \cdots, s_n]$. Then we must have that $\mathbb{E}[u_j(\beta_j^*(s_j, p^*(s)))|s_1, \cdots, s_n] < \mathbb{E}[u_j(e_j)|s_1, \cdots, s_n]$. But this contradicts interim optimality of e. Thus we conclude that for all $i \in N$, we have that $\beta_i^*(s_i, p^*(s)) = e_i$. $\square$

Finally, we introduce ambiguity to the demand schedule game and show the main result of this chapter: Since prices are not necessarily revealing, private information is valuable in the demand schedule game under ambiguity and there is trade even if the initial endowment is interim optimal.

3.4.2 Bayesian Nash Equilibrium under Ambiguity

We start with our previous example of two agents economy with the endowment structure:

	H	L
1	1	0
2	0	1

Assume that the common multi-prior set is $\mathcal{P}_0 = \{(\hat{\pi}_H, \hat{\pi}_L | \hat{\pi} \in \left[\frac{1}{3}, \frac{2}{3}\right]\}$ and the private signal structure is given such that:

$f_1(s_1\|\omega)$	H	L
h	1	$\frac{1}{2}$
l	0	$\frac{1}{2}$

$f_2(s_2\|\omega)$	H	L
$\varnothing$	1	1

We have already established that if agent 1 receives the signal l then the market clears at $p^* = \left(\frac{3}{10}, \frac{7}{10}\right)$ and if agent 1 receives the signal h then the market clears at $p^* = \left(\frac{10}{17}, \frac{7}{17}\right)$. This implies that prices are fully revealing and agent 2 should be able to deduce the private signal of agent 1 by looking at prices.

What happens when the prices are not fully revealing? To answer this question let's consider another example of a two agents economy with the following endowment structure:

	H	L
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{2}$

Assume that the common multi-prior set is $\mathcal{P}_0 = \{(\hat{\pi}_H, \hat{\pi}_L | \hat{\pi} \in \left[\frac{1}{3}, \frac{2}{3}\right]\}$ and the private signal structure is given such that:

$f_1(s_1\|\omega)$	H	L
h	$\frac{3}{5}$	$\frac{2}{5}$
l	$\frac{2}{5}$	$\frac{3}{5}$

$f_2(s_2\|\omega)$	H	L
$\varnothing$	1	1

Above signal structure implies:

$$\mathcal{P}_{s_1=h} = \left\{(\hat{\pi}_H, \hat{\pi}_L) | (\hat{\pi}_H \in \left[\frac{3}{7}, \frac{3}{4}\right]\right\}$$

$$\mathcal{P}_{s_1=l} = \left\{(\hat{\pi}_H, \hat{\pi}_L) | (\hat{\pi}_H \in \left[\frac{1}{4}, \frac{4}{7}\right]\right\}$$

$$\mathcal{P}_{s_2} = \mathcal{P}_0$$

Using 4, we have that agent 2 demands:

$$x_2 = \begin{cases} \left(\frac{2}{3}\frac{p_H+p_L}{2p_H}, \frac{1}{3}\frac{p_H+p_L}{2p_L}\right) & \text{if } \frac{p_H}{p_L} > 2 \\ \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{1}{2}, 2\right] \\ \left(\frac{1}{3}\frac{p_H+p_L}{2p_H}, \frac{2}{3}\frac{p_H+p_L}{2p_L}\right) & \text{if } \frac{p_H}{p_L} < \frac{1}{2} \end{cases}$$

And agent 1 demands:

$$x_{1h} = \begin{cases} \left(\frac{3}{4}\frac{p_H+p_L}{2p_H}, \frac{1}{4}\frac{p_H+p_L}{2p_L}\right) & \text{if } \frac{p_H}{p_L} > 3 \\ \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{3}{4}, 3\right] \\ \left(\frac{3}{7}\frac{p_H+p_L}{2p_H}, \frac{4}{7}\frac{p_H+p_L}{2p_L}\right) & \text{if } \frac{p_H}{p_L} < \frac{3}{4} \end{cases} \qquad x_{1l} = \begin{cases} \left(\frac{4}{7}\frac{p_H+p_L}{2p_H}, \frac{3}{7}\frac{p_H+p_L}{2p_L}\right) & \text{if } \frac{p_H}{p_L} > \frac{4}{3} \\ \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{1}{3}, \frac{4}{3}\right] \\ \left(\frac{1}{4}\frac{p_H+p_L}{2p_H}, \frac{3}{4}\frac{p_H+p_L}{2p_L}\right) & \text{if } \frac{p_H}{p_L} < \frac{1}{3} \end{cases}$$

The market clears for any $p \in \{(p_H, p_L) \in \triangle^1 | \frac{p_H}{p_L} \in \left[\frac{3}{4}, 2\right]\}$ if $s_1 = h$ and for any $p \in \{(p_H, p_L) \in \triangle^1 | \frac{p_H}{p_L} \in \left[\frac{1}{2}, \frac{4}{3}\right]\}$ if $s_1 = l$. When $p \in \{(p_H, p_L) \in \triangle^1 | \frac{p_H}{p_L} \in \left[\frac{3}{4}, \frac{4}{3}\right]\}$, the prices are not revealing. However, the equilibrium allocations are the same regardless of the prices, and there is no trade. Interestingly, the initial endowment is interim Pareto optimal in this example, so this example verifies the no-trade theorem. Now let's consider an endowment structure that is not interim Pareto optimal:

	H	L
1	1	0
2	0	1

Keeping the signal structure the same, agent 2 demands:

$$x_2 = \begin{cases} \left(\frac{2}{3}\frac{p_L}{p_H}, \frac{1}{3}\right) & \text{if } \frac{p_H}{p_L} > 2 \\ \left(\frac{p_L}{p_H+p_L}, \frac{p_L}{p_H+p_L}\right) & \text{if } \frac{p_H}{p_L} \in \left[\frac{1}{2}, 2\right] \\ \left(\frac{1}{3}\frac{p_L}{p_H}, \frac{2}{3}\right) & \text{if } \frac{p_H}{p_L} < \frac{1}{2} \end{cases}$$

And agent 1 demands:

$$x_{1h}=\begin{cases}\left(\frac{3}{4},\frac{1}{4}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L}>3\\ \left(\frac{p_H}{p_H+p_L},\frac{p_H}{p_H+p_L}\right) & \text{if } \frac{p_H}{p_L}\in\left[\frac{3}{4},3\right]\\ \left(\frac{3}{7},\frac{4}{7}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L}<\frac{3}{4}\end{cases}\qquad x_{1l}=\begin{cases}\left(\frac{4}{7},\frac{3}{7}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L}>\frac{4}{3}\\ \left(\frac{p_H}{p_H+p_L},\frac{p_H}{p_H+p_L}\right) & \text{if } \frac{p_H}{p_L}\in\left[\frac{1}{3},\frac{4}{3}\right]\\ \left(\frac{1}{4},\frac{3}{4}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L}<\frac{1}{3}\end{cases}$$

Again, the market clears for any $p \in \{(p_H, p_L) \in \triangle^1 | \frac{p_H}{p_L} \in \left[\frac{3}{4}, 2\right]\}$ if $s_1 = h$ and for any $p \in \{(p_H, p_L) \in \triangle^1 | \frac{p_H}{p_L} \in \left[\frac{1}{2}, \frac{4}{3}\right]\}$ if $s_1 = l$. So when $p \in \{(p_H, p_L) \in \triangle^1 | \frac{p_H}{p_L} \in \left[\frac{3}{4}, \frac{4}{3}\right]\}$, the prices are not revealing. However, in this case, there is some trade at almost all equilibrium prices and some equilibria is better for agent 1. Indeed, if $\frac{p_H}{p_L} = \frac{4}{3}$, then agent 1 is better off compared to agent 2. This point raises the question that what would arise as Bayesian Nash equilibrium of the demand schedule game. Imagine that agent 1 wants to exploit this situation and given above demand schedule of agent 2, agent 1 submits the following regardless of his private signal:

$$\beta_1=\begin{cases}\left(\frac{4}{7},\frac{3}{7}\frac{p_H}{p_L}\right) & \text{if } \frac{p_H}{p_L}>\frac{4}{3}\\ \left(\frac{p_H}{p_H+p_L},\frac{p_H}{p_H+p_L}\right) & \text{if } \frac{p_H}{p_L}=\frac{4}{3}\\ (1,0) & \text{if } \frac{p_H}{p_L}<\frac{4}{3}\end{cases}$$

With above demand schedule, the equilibrium price will be such that $\frac{p_H}{p_L} = \frac{4}{3}$ which is better than most Walrasian equilibria above. But of course, this is not a Bayesian Nash equilibrium since given x_2, β_1 is not a best response when $x_1 = h$. Consider following demand schedules:

$$\beta_1=\begin{cases}\left(0,\frac{p_H}{p_L}\right) & \text{if } \ p_H>p_L\\ \left(\frac{1}{2},\frac{1}{2}\right) & \text{if } \ p_H=p_L\\ (1,0) & \text{otherwise}\end{cases}\qquad \beta_2=\begin{cases}(0,1) & \text{if } \ p_H>p_L\\ \left(\frac{1}{2},\frac{1}{2}\right) & \text{if } \ p_H=p_L\\ \left(\frac{p_L}{p_H},0\right) & \text{otherwise}\end{cases}$$

Now both schedules are feasible, though not necessarily optimal at each price in Walrasian sense. However, at the unique equilibrium price vector, $p_H = p_L$, the allocations are optimal for both agents. It is straightforward to see that they are best responses to each other.

Here, the non-revealing prices doesn't really help the agent 1 who has informational advantage. To generalize this observation, we need to redefine the demand schedule game to incorporate the multiple prior set. The demand schedule game is a tuple, $(N, (u_i)_{i\in N}, \Omega, e, \mathcal{P}_0, (S_i)_{i\in N}, (f_i(\cdot|\omega))_{i\in N})$ where N is the set of agents where $|N| = n$, u_i for agent i is the strictly increasing and strictly concave and differentiable Bernoulli utility function, Ω is the finite set of states where $|\Omega| = k$,

$e \in \mathbb{R}_+^{n\times k}$ is the endowment vector, $\mathcal{P}_0$ is the multiple prior set, S_i for agent i is the set of private signals and $f_i(\cdot|\omega) : S \to [0,1]$ is the conditional distribution of the privates signals of agent i conditional on the state $\omega \in \Omega$. A strategy for agent i is a function $\beta_i : S_i \times \triangle^{k-1} \to \mathbb{R}_+^k$ determining the demand at each private signal and price tuple.

Now, we can define the Bayesian Nash equilibrium of the demand schedule game:

Definition 4. *The strategy tuple $(\beta_i)_{i\in N}$ is a Bayesian Nash equilibrium of the demand schedule game if there exists an equilibrium price mapping $p^* : \times_{i\in N} S_i \to \triangle^{k-1}$ such that:*

- $\forall s \in \times_{i\in N} S_i, \forall i \in N\ \beta_i(s_i, p^*(s)) \in \arg\max_{x_i\in X_i} \min_{\hat{\pi}\in\mathcal{P}_{s_i}} \mathbb{E}[u_i(x_i)|\hat{\pi}_i, p]$
- $\forall s \in \times_{i\in N} S_i,\ \sum_{i\in N} \beta_i(s_i, p^*(s)) = \sum_{i\in N} e_i$

where for each $i \in N$, $X_i = \{x_i \in \mathbb{R}_+ | p^* \cdot x_i \leqslant p^* \cdot e_i\}$.

With the demand schedule game re-defined to incorporate multiple priors, we can state the following theorem:

Proposition 18. *Let $(\beta_i)_{i\in N}$ be a Bayesian Nash equilibrium with the price mapping p^*. If $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$ for some $s_i, s_i' \in S_i$, then $\beta_i(s_i, p^*(s_i', s_{-i})) = \beta_i(s_i, p^*(s_i, s_{-i})$*

Proof. Let $(\beta_i)_{i\in N}$ be a Bayesian Nash equilibrium with the price mapping p^* with $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$ for some $s_i, s_i' \in S_i$. Since $p^*(s_i', s_{-i}) = p^*(s_i, s_{-i})$, then $\beta_j(s_j, p^*(s_i', s_{-i})) = \beta_j(s_j, p^*(s_i, s_{-i})$ for all $j \in N$. Then market clearing conditions imply $\beta_i(s_i, p^*(s_i', s_{-i})) = \beta_i(s_i, p^*(s_i, s_{-i})$. □

Above proposition demonstrates that, every agent will have same equilibrium allocation under a non-revealing equilibrium price regardless of their private signal. This implies that trade may be possible with interim Pareto optimal allocations, but it won't improve any agent compared to the initial endowment:

Theorem 8. *If the initial endowment $e \in \mathbb{R}^{n\times k}$ is interim Pareto optimal with respect to signal realizations $(s_1, \cdots, s_n) \in \times_{i\in N} S_i$, then in any Bayesian Nash equilibrium of the demand schedule game under ambiguity with equilibrium price mapping $p^* : \times_{i\in N} S_i \to \triangle^{k-1}$, for all $i \in N$, we have that $\min_{\hat{\pi}\in\mathcal{P}_{s_i}} \mathbb{E}[u_i(\beta_i)|\hat{\pi}, p^*] = \min_{\hat{\pi}\in\mathcal{P}_{s_i}} \mathbb{E}[u_i(e_i)|\hat{\pi}, p^*]$. That is, there can be no trade that improves any agent in equilibrium.*

Proof. When prices are revealing in the sense that $p^(s) \neq p^*(s')$, then every agent making decisions based on all private signals. Then we can use arguments similar to corollary 6 and theorem 7 to show above holds. Now consider the non-revealing prices which may arise in an ambiguity setting. Propoposition 18 shows that agents have identical demands under non-revealing prices regardless of their private signal. Then those private signals become redundant and we can define a new signal structure such that $\tilde{S} = S\backslash \sim$ where $s \sim s'$ if $p^*(s) = p^*(s')$. But then the prices are fully revealing under the signal structure $\tilde{S}$. □

Thus, we have shown that ambiguity introduces non-reveling prices but it doesn't break the no-trade theorem in a meaningful way. There may be trade in Bayesian Nash equilibrium, but it won't improve any agent strictly.

3.5 Conclusion

In this chapter, we studied ambiguity aversion and it's implications for Walrasian equilibrium. The result of Condie and Ganguli (2011) showing that the prices may not be fully revealing when agents are ambiguity averse begs the question whether non-revealing prices allow agents to benefit from their private information via some pooling strategy for distinct private signals. It turns out that the answer is negative, and under ambiguity, it is still impossible to benefit from private information.

The intuition behind this result is fairly straightforward. Ambiguity does allow agents with private information to follow pooling strategies where they end up submitting demand schedules that lead to same prices and allocations although they have received different signals leading to different posteriors. But they are able to do this only because ambiguity preferences allow them to have same level utility given two different private signals. Hence they do not benefit from non-revealing prices.

Our result assumes that all agents have a set valued common prior which effectively implies that all agents are ambiguity averse. Since a study of trading with rational agents require some form of common prior, modeling ambiguity averse traders together with non-ambiguity averse traders is not straightforward. For future work, one may model traders directly using posteriors without any reference to a prior so that traders endowed with a posterior set and traders with singleton posteriors can coexist. However, this approach has two issues. First, since some of the posteriors are set valued, there is no clear way of introducing Bayesian consistency that imposes restrictions on the posteriors so that one can come up with a prior -set valued or not- that rationalizes given posteriors. Second issue is that, without a clearly defined prior, we need to impose a learning rule for agents to learn from prices about other agents' private information since usual Bayesian updating requires a prior. It may, however, be fruitful to tackle this problems and come up with a convincing framework that can deal with both type of traders and see the implications of such a framework for strategic trading.

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