Preference Heterogeneity, Aggregate Risk and the Welfare Effects of Social Security

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Abstract

This paper quantitatively studies the distributional welfare implications of a pay-as-you-go social security system in a lifecycle model with ex-ante household heterogeneity in both preferences and lifecycle income streams, classified by education levels and borrowing costs. To analyze this problem, I provide an algorithm to solve a large-scale model with ex-ante heterogeneous and long-period-lived agents, borrowing constraints, and multiple assets where state aggregation methods generate large errors in optimal choices which should be avoided for quantitative welfare analysis. Through a welfare decomposition analysis, I determine the sources of welfare benefits and costs for each educational group. In the baseline calibration, I find that high-school dropouts and college educated groups are advantaged, while the high-school graduates group is disadvantaged by the welfare system. The high-school dropouts obtain a welfare gain mostly from intragenerational redistribution from the other income richer groups through the social security system. The college graduates group experiences a welfare loss from such redistribution, but old-age consumption insurance and increases in asset returns provide a substantial offsetting welfare gain to this most patient and risk-averse group. The high-school graduates receive a welfare gain from both intragenerational redistribution and consumption insurance. However, borrowing constraints strengthen the welfare cost of the social security system for this group by preventing consumption smoothing during the early stage of life which is essential especially when wages are reduced by crowding out.

JEL classification: C68, E21, E27, E62, H31, H55

Keywords: Preference Heterogeneity, Borrowing constraint, Aggregate Shock, Social Security; Welfare

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1 Introduction

There has been little research studying the distributional welfare implications of pay-as-you-go social security systems between the rich and poor or the educated and uneducated due to computational challenges.[1] However, it is essential to identify who are welfare winners and losers and which group obtains more welfare benefits than the other in evaluating the effects of social security policy when socio-economic groups are heterogeneous. From a political economy perspective, distributional welfare analysis allows us to explain who would support the sustainability of the welfare system. It can also answer whether the policy indeed advantages the poor or uneducated groups, which was a major motivation for the establishment of the U.S. Social Security system after the great depression. Empirical papers find that education levels are correlated with income and differences in preferences, which in turn determine the overall welfare effects of a social security system. Income profiles determine whether intragenerational redistribution by the policy generates benefits or costs. Preferences can affect the size of income effects from price changes induced by social security. The poor or uneducated group obtains welfare gains from the intragenerational redistribution whereas the rich or educated group does not. However, an increase in interest rates from crowding out by social security produces a larger income effect for the latter group since they can save more, due to a higher marginal propensity to save overall, and a stronger precautionary saving motive. These opposing welfare effects require a quantitative analysis to determine the distributional welfare effects of a pay-as-you-go social security system across education strata. The goal of this paper is to pursue such a quantitative analysis by developing an efficient, accurate and stable algorithm for solving a large-scale lifecycle model with ex-ante heterogeneous households and aggregate shocks when state aggregation methods should be avoided due to their large errors. This paper finds that there are both welfare winners and losers in its calibrated model. College educated groups and high-school dropouts obtain welfare gains of 1.72% and 0.44% respectively in terms of certainty equivalent consumption variation (CEV henceforth) between the laissez-faire baseline economy and the economy with a social security system. However, the high-school graduates group receives a net welfare loss of -0.51% in terms of the same measure. This quantitative result suggests that social security does not necessarily generate advantages for the relatively poor groups.

The empirical literature has found that educated households are more patient and less willing to substitute intertemporally than uneducated households, on average.[2] In addition, the educated group has a uniformly higher income profile than its counterpart. The introduction of social security systems provide i) old-age consumption insurance, ii) change prices, and iii) implement

[1] However, there is extensive research on the welfare effects of the policy using the model with representative households (See İmrohoroglu et al. (1995); Conesa and Krueger (1999); Krueger and Kubler (2006); Hasanhodzic and Kotlikoff (2013); Harenberg and Ludwig (2018)).

[2] Lawrance (1991) finds that time preference rates vary 2 percent between college-educated families and families without a college education in the U.S., with income, age and race held constant. With the PSID panels, Cagetti (2003) finds that the rate of time preference decreases with education and is higher by $5\%-10\%$ for college-educated families than for those in lower educational groups. The risk-aversion coefficient also decreases with education in their estimation results. Alan and Browning (2010) strongly reject the hypothesis of the homogeneity of the discount factor and the elasticity of intertemporal substitution with the same data. Druedahl et al. (2017) exploit Danish longitudinal register data and conclude that the estimated values of discount factors and CRRA utility coefficients are shifted towards higher values for high-skilled households with at least a bachelor degree.

redistributive intragenerational transfers. The heterogeneity in preferences and income profiles affects welfare through each of these three channels.

For example, more risk-averse agents save more for self-insuring against volatile capital income in retirement periods. Social security systems reduce the income risk of retirees by providing transfers from the young's labor income. The insurance given by social security generates larger welfare benefits for more risk-averse households since they strongly demand the stable consumption streams which can be achieved via the income-risk hedge. More patient and risk-averse households experience large positive income effects when the rate of return rises from crowding out by social security systems because they save a large amount due to a high marginal propensity to save, and strong precautionary saving motives. The large income effects enhance the welfare of these households by increasing their mean consumption levels. The social security system also generates substitution effects by raising interest rates. Under a higher interest rate, agents shift their consumption to the future. More patient households prefer such consumption shifting more than less patient ones since they value future consumption highly. As a side effect of crowding out, wages decrease and thus households experience negative income effects. In addition, there is an intragenerational redistribution from high to low-income households via the linear tax and lump-sum transfer incorporated into the pay-as-you-go social security system. Thus, the income poor experience an increase in their lifetime income whereas the income rich face a decrease.

In each channel mentioned above, social security systems generate different welfare implications across groups with varying levels of education. Thus, the overall welfare effects of social security systems can be unequal over such groups. However, we cannot tell, a priori, which groups obtain more welfare than others and who get gains and losses from the policy since there are opposing welfare effects for all educational groups. Specifically, the educated group can obtain welfare gains from price changes induced by social security if the positive income effects from increases in interest rates dominate the negative income effects from decreases in wages. In contrast, the intragenerational redistribution offsets such possible gains. For the uneducated group, the transfer from the income rich enhances their welfare, but they can experience an overall welfare loss from the policy if the negative income effects from wage decreases dominate the relatively smaller positive income effects from rises in interest rates.

Therefore, we need a quantitative analysis to determine the distributional welfare effects of one of the largest components of government expenditure in many developed economies. However, it is challenging to compute the equilibrium in a large-scale lifecycle model with ex-ante heterogeneous and many-period-lived agents. State aggregation methods such as the one in Krusell and Smith (1998) can be a solution to compute equilibria in such models, as adopted, for example, in Storesletten et al. (2007). However, Krueger and Kubler (2004) show that the state aggregation methods can generate large errors in individual optimal choice problems if aggregate wage and asset return risks are imperfectly correlated due to stochastic depreciation, which is present in the model of this paper, and needed to study the insurance function of social security in pooling labor and capital income. For the quantitative welfare analysis, then, large errors in individual choices should be avoided, since they can lead to incorrect welfare implications.

In this paper, I provide an accurate, fast and reliable algorithm for computing equilibria in a lifecycle model with rich heterogeneity and imperfectly correlated aggregate risk. I also suggest how to computationally deal with the model with borrowing constraints and multiple assets so as not to lose accuracy and speed of convergence. With this algorithm, I study the distributional welfare implications of a pay-as-you-go social security system across educational strata. In addition, I quantify the welfare effects of each of the three channels mentioned above for each educational group through a welfare decomposition analysis. This decomposition analysis indicates which channels produce welfare benefits and costs among different educational groups.

1.1 Approach

I calibrate a lifecycle model with ex-ante heterogeneous households classified by their educational levels, business cycle risk, and borrowing costs. There are three levels of education: college graduates, high-school graduates, and high-school dropouts. The education level determines risk-aversion, time preference and permanent income profiles of households.[3] In the baseline calibration, college graduates are the most patient and most risk-averse, then high-school graduates, and high-school dropouts in order. Due to the education premium, college educated groups have higher lifecycle income profiles than the other two groups. In this model, I focus on the impact of aggregate risk as opposed to idiosyncratic risk when there are ex-ante heterogeneous households. The financial market is incomplete against business cycle risk because there are more states than assets – stocks and bonds – in the model. In addition, an aggregate shock is modeled to affect both the total factor productivity (TFP) and capital depreciation. In the calibrated model, TFP and stochastic depreciation are imperfectly correlated to be consistent with data. Then, the pay-as-you-go social security transfer can provide insurance on old-age consumption by pooling wage and asset return risks, i.e. intergenerational risk sharing, in this setting.

The fact that consumption profiles track income profiles is suggestive evidence of the presence of borrowing constraints (See Gourinchas and Parker (2002) and Fernández-Villaverde and Krueger (2007)). Thus, I let households face a borrowing constraint in the model. I introduce a soft borrowing constraint where all generations pay borrowing costs which linearly increase in the amount borrowed. Thus, soft borrowing constraints imply that there is a wedge between borrowing and lending rates, unlike the hard borrowing constraint as done in Constantinides et al. (2002), where borrowing and lending rates are the same but the young face limits on the amount borrowed. There are many reasons for introducing soft borrowing constraints. First, it is more realistic to assume that agents can borrow as much as they want if they accept high borrowing costs. For example, households denied loans by banks or under credit card borrowing limits can access non-banking financial institutions such as payday loans at high cost. Second, the soft borrowing constraints already nest hard borrowing constraints. For example, as linear borrowing cost parameters go to infinity, soft borrowing constraints degenerate to no-borrowing constraints. A crucial practical reason for working with soft borrowing constraints is that they make computation easier by allowing me to approximate the borrowing cost schedule with a smooth function. The borrowing costs apply to the total amount borrowed, whether done via trade in stocks or bonds, since real-world short-sale transactions on stock generally involve a stock loan fee or stock borrowing fee from a brokerage firm.

The presence of borrowing constraints significantly affects the welfare implications of social security. Under hump-shaped income profiles, young generations are borrowing constrained be-

[3] It is standard to estimate preference heterogeneity across different education groups, assuming homogeneity within groups in the empirical literature on preferences (See Lawrance (1991); Cagetti (2003); Alan and Browning (2010); Cooper and Zhu (2016); Druedahl et al. (2017)).

cause they have incentives to borrow against future income to increase current consumption, but borrowing constraints prevent it. Social security systems lower labor income by imposing a tax on wages and generating crowding out which reduces wages. Thus, the marginal young households should decrease consumption even when they are already borrowing constrained since their income mostly comes from supplying labor. Therefore, social security systems generate welfare costs for all groups if borrowing constraints prevent consumption smoothing in the early stages of life.

I develop an algorithm to compute the equilibrium in a large-scale lifecycle model with ex-ante heterogeneity and an imperfectly correlated aggregate shocks. Since state aggregation methods yield large errors in this environment, I instead use cash-on-hand variables across all ages and types as state variables in the policy functions. The shocks in the model are small, and thus the policy functions are well approximated by linear functions of the state variables, which is justified by a routine application of stochastic extensions of the Hartman-Grobman theorem for stochastic dynamic systems. Using linear functions makes computation feasible and generates acceptable errors. However, it still takes a long time to compute equilibria when there are multiple assets because one needs to use Newton-type methods to solve non-linear optimality conditions. To resolve this issue, I adopt the consumption policy functions from the economy with only risky capital to simulate the economy with multiple assets. This computational strategy has several advantages. First, it requires solving for only the Euler equations corresponding to risky capital. One can transform such optimality conditions into a system of linear equations in the coefficients of policy functions which is very easy to solve. Thus, it takes minimal time to find policy functions in the economy with only risky capital. Surprisingly, there are small errors in the optimality conditions evaluated at the simulation data which I generate by finding stock and bond shares in each simulation period given the consumption policy functions from the single asset economy.[4] To further increase accuracy, I solve the model twice: on the first-order Smolyak grid and the simulated Ergodic set in the spirit of Judd et al. (2011). Lastly, I approximate the soft borrowing cost schedule with a smooth function to avoid having to solve for conditional optimality depending on whether or not households borrow.

With the algorithm above, I compute an equilibrium in the laissez-faire baseline calibrated economy and the economy with a social security system. Then, I calculate certainty equivalent consumption variation from the baseline economy to the economy with the policy in terms of the ex-ante expected utility to evaluate the welfare implications of social security systems across educational groups in general equilibrium. I also consider a partial equilibrium version of the model which restricts prices from changing to isolate how the insurance and redistribution effects from the introduction of social security affect the welfare of different agents. To compute the partial equilibrium, I find policy functions under a social security system when rational agents expect future prices as in the baseline economy by injecting the law of motion from the baseline economy, and simulate these policy functions while imposing the simulated prices from the baseline economy. I also compute the CEVs from the baseline economy to the partial equilibrium economy. With the CEV values, I run a welfare decomposition analysis to determine the channels of welfare

[4] As error measures, I use the average deviations of simulated data generated by approximated policy functions from the perfect satisfaction of Euler equations. The incomplete market literature provides a possible explanation behind the good approximations with only the capital asset. The results in the literature imply that adding an asset will not change the existing incomplete markets equilibrium as long as the new asset doesn't complete the market (See Cass and Citanna (1998)).

benefits and costs for each educational group among risk-sharing, limited consumption smoothing, intragenerational redistribution, and general equilibrium effects.

1.2 Findings

The calibrated model with heterogeneous agents in this paper generates different welfare results from those obtained in the representative agent case. The latter model generates a welfare loss as noted in the previous studies whereas the heterogeneous agent model generates gains for certain educational groups from the social security system. In partial equilibrium, high-school graduates and dropouts obtain welfare gains whereas college educated groups have welfare losses from the welfare system. Specifically, certainty equivalent consumption variations are −1.68%, 1.01% and 2.36% for college graduates, high-school graduates and high-school dropouts between the baseline calibrated economy and the economy with a social security system but without price changes, respectively. The welfare results are flipped in general equilibrium where high-school dropouts and college educated groups are advantaged, while the high-school graduates group is disadvantaged by the social security system. Corresponding certainty equivalent consumption variations are 1.72%, −0.51% and 0.44% between the baseline calibrated economy and the economy with a social security system.

The insurance provided by the social security system increases the welfare of all groups because the intergenerational transfer reduces the consumption volatility in old-age periods by pooling imperfectly correlated risk in labor and capital income. Both high-school graduates and dropouts obtain a welfare gain from intragenerational redistribution from the college educated groups through the linear tax and lump-sum transfer structure of the social security system. This channel mostly advantages the high-school dropouts since they receive more than what they contribute to the system. On the other hand, the college graduates group experience a welfare loss from such redistribution. The borrowing costs lower the welfare of all groups under the social security systems because this friction prevents consumption smoothing during the young periods. Social security contribution lowers disposable income. Young households have the incentive to increase current consumption by borrowing against their higher future incomes. However, the amount borrowed is limited due to the borrowing costs and optimal intertemporal choices cannot be achieved.

In general equilibrium, increases in asset returns provide a substantial offsetting welfare gain to the college educated group. College graduates are the most patient and risk-averse in the calibrated model. Thus, they save more than the other groups because of a higher marginal propensity to save and a stronger precautionary saving motive against the possible loss of asset value due to the depreciation shocks. An increase in the rate of return from crowding out generates a substantial income effect to the college graduates group and lowers the price of future consumption. Thus, the college graduates group can increase their mean consumption and purchase old-age consumption (that they value relatively more highly than the other groups) at lower prices after the introduction of the social security system. At the same time, the social security system reduces wages by crowding out private saving. The wage reduction together with the labor income tax causes a larger set of young households to be borrowing constrained by reducing their disposable income more in general equilibrium than in partial equilibrium. Thus, the young consumption will decrease and deviate further from the optimal intertemporal choice, which generates a greater welfare cost. Therefore, the high-school graduates experience a welfare loss in general equilibrium in the end.

The welfare cost from the borrowing constraints also has an effect in the economy without the ex-ante heterogeneity. I find that there is a welfare loss in the economy with representative households from the social security system in general equilibrium along with the prior studies. However, unlike the previous studies, the social security system yields a welfare loss even in partial equilibrium in the economy with borrowing constraints, because costs from the friction dominate gains from old-age consumption insurance.

In the sensitivity analysis, I try several experiments to test the robustness of the results in the baseline calibrated model. First, I increase the share of the college-educated group, considering the growing number of college graduates in the U.S. recently. My model suggests that a larger share of college graduates implies that average welfare rises for all groups under social security, but the high-school graduates still obtain a welfare loss. The reason is that an increase in the share of college graduates enlarges the amount of transfer and thus strengthens both inter- and intragenerational redistribution between different groups and generations. Second, I expand the size of the social security systems because the current U.S. social security contribution rate is much higher than the marginal rate in the benchmark model. A moderate size social security system changes the quantitative welfare results such that all groups obtain lower welfare than the benchmark case. The welfare decrease is much more significant for high-school graduates and dropouts. It turns out that the high-school dropouts group experience a welfare loss, unlike in the baseline calibrated model. This result obtains because a larger social security system results in a significant drop in wages which leads to very low young-age consumption for the poor income group due to the limited consumption smoothing from borrowing constraints. Finally, I consider how the volatility in capital income risk affects the distributional welfare implications of social security by doubling the size of stochastic depreciation shocks to make the equity premium and Sharpe ratio from the model closer to their empirical counterparts. Under a larger shock, even high-school graduates have welfare gains because of strengthened insurance via social security. These comparative statics analyses imply that although the quantitative welfare results are affected, the key result remains the same: the certainty equivalent consumption variation is the largest for college graduates, then high-school dropouts, and high-school graduates in order.

1.3 Literature

The study of the welfare implications of social security dates back to Diamond (1977) and Merton (1983). Those papers show how social security can partially insure against aggregate shocks to increase economic welfare. Based on the insight that social security can reduce consumption risk, many papers examine its welfare effect analytically in two-period partial equilibrium models (See Gordon and Varian (1988), Shiller (1999), Bohn (2001), Ball and Mankiw (2007), and Bohn (2009)). On the other hand, there are only a few quantitative papers using general equilibrium models with aggregate risk and social security. Krueger and Kubler (2006) is a representative example. They evaluate the welfare effects of social security by comparing its partial risk-sharing benefit in incomplete markets with its welfare cost due to crowding out of capital accumulation. However, that paper ignores household heterogeneity in preferences and age/income profiles.[5] Therefore, there are substantial differences in the welfare results. Krueger and Kubler (2006) conclude that the introduction of a marginal social security system does not improve the welfare of households

[5] Only Samwick (1998) and Hosseini and Shourideh (2017) consider preferences heterogeneity in the discount factor, but their models do not have aggregate shocks.

in general equilibrium from an ex-interim welfare perspective although it does in partial equilibrium. In my paper, certain groups can obtain welfare gains from preferences and age/income profiles heterogeneity. Without these ex-ante heterogeneities, the lifecycle model with borrowing constraints generates a welfare loss due to the limited consumption smoothing in the early stage of life. Harenberg and Ludwig (2018) considers both aggregate and idiosyncratic risks and study the insurance function of social security against the amplified risk via interaction between the two shocks. However, their paper does not consider ex-ante heterogeneity and borrowing constraints. Thus, they cannot answer who is advantaged by social security systems and who is not.

In terms of computation, Hasanhodzic and Kotlikoff (2013) and Reiter (2015) provide methods for solving large-scale lifecycle models. Hasanhodzic and Kotlikoff (2013) examine a lifecycle model with long-lived agents, multiple assets and imperfectly correlated aggregate risk. They also approximate policy functions with linear functions without state space reduction. Unlike their paper, my model adds ex-ante heterogeneity and borrowing constraints which make computation more burdensome and generate different welfare implications. To resolve the computational challenge, I use the strategy of adopting policy functions from the single asset economy to the multiple assets one. Thus, I can reduce computation time significantly by transforming the problem into a system of linear equations in the economy only with risky capital. I approximate borrowing constraints with smooth functions so that I can compute equilibrium with smooth polynomials. Reiter (2015) also considers global linear approximations to large-sized lifecycle models. In his model, an imperfect correlation structure is not considered between wages and asset returns, which is significant for the insurance role of social security. Reiter (2015) deals with multiple assets through efficient implementation of quasi-Newton methods. I deal with the same issue by adopting the results from the single asset economy as discussed above.

I organize this paper as follows. Section 2 describes the quantitative model. I present the calibration in Section 3. Section 4 shows the main results of the quantitative analysis. In Section 5, I check the robustness of the results. Section 6 concludes the paper. The appendix provides the novel numerical algorithm of this paper.

2 The economic model

2.1 Aggregate shock and demography

The economy is closed. Time is discrete, labeled by $t = 0, 1, \ldots, \infty$. An aggregate shock z_t arises at the beginning of each period in the economy. z_t follows a first-order Markov chain with a finite support Z and a nonnegative transition probability matrix π_z, where $\pi_z(z_{t+1} \mid z_t)$ represents the probability of z_{t+1} given z_t and $z^t = (z_0, z_1, \ldots, z_t)$ is the history of shocks up to time t given the initial shock z_0.

M types of agents are born in every period and live a_d periods of lives. I denote the type and age of agents by $m = 1, \ldots, M$ and $a = 1, \ldots, a_d$. $N_{am,t}$ represents the population of age a and type m group in time t. Each group consists of a continuum of households. There is exogenous population growth and the growth rate is given by n. The total population at time t is denoted by $N(z^t) = \sum_{m=1}^{M}\sum_{a=1}^{a_d} N_{am}(z^t)$. I normalize the initial population to be unity, $\sum_{m=1}^{M}\sum_{a=1}^{a_d} N_{am}(z_0) = 1$ where $p_m = \sum_{a=1}^{a_d} N_{am}(z_0)$ for the share of group m in each generation and $\frac{N_{(a+1)m}(z_0)}{N_{am}(z_0)} = (1+n)$

for $\forall a = 1, \ldots, a_d - 1$. Then, the total population at time t is $N(z^t) = (1+n)^t$. There is no mortality risk. Within an age group, I assume there is a continuum of agents with a unit measure so that households take prices as given.

2.2 Heterogeneous preferences and labor productivity process

In the economy, there is initial condition heterogeneity. At birth, type m households are endowed with different time discount factor β_m and risk aversion γ_m. This initial condition heterogeneity remains the same over their lifetime. Agents are endowed with one unit of time per period and supply labor inelastically. Agents are also born with different productivities, so I let θ_{am} denote the stationary age-type specific productivity profile over the life-cycle. They retire exogenously at age $a_R < a_d$ following the statutory retirement age. Retirees do not provide labor supply. Hence, the labor supply is zero for retirees.

2.3 Borrowing costs

I introduce a soft borrowing constraint where households can borrow as much as they want and all generations face borrowing costs which increase in the amount borrowed. With this constraint, there is a wedge between borrowing and lending rates. This is a reasonable alternative to hard borrowing constraints where the young can face limits in the amount borrowed and borrowing and lending rates are the same because we observe that deposit rates are less than the interest rates on personal bank loans, and there are spreads between the borrowing and lending rates in the data. Empirical papers also support the fact that the risk-free saving interest rate is lower than the loan interest rates, which is called dual interest rates(See Athreya et al. (2012)).

When there are both risky and risk-free assets, most previous studies impose a borrowing constraint on bond holdings. However, even when investors short-sell stocks, there is a stock loan fee or stock borrowing fee that a brokerage firm charges to investors who borrow shares.[6] Thus, one should consider a borrowing cost for not only bond borrowing but also stock short-sale. When households borrow via bonds or investors short sell stocks, there are lower borrowing interest rates or low stock loan fees if there are collateral requirements. For example, investors face low short-sale costs when they use their bond holdings as a secured basis. Households get low borrowing interest rates if they use their stock holdings as collateral. Thus, I consider a borrowing cost based on the total amount of borrowing, not bond borrowing only. Therefore, a positive asset holding offsets the amount of borrowing via the other asset when calculating borrowing costs. If households borrow with both bonds and stock, then the total amount borrowed determines the borrowing cost.

The model's borrowing costs are assumed as follows:

$$f(s_{am} + b_{am}) = \begin{cases} -\lambda (s_{am} + b_{am}) & \text{if } (s_{am} + b_{am}) < 0 \\ 0 & \text{if } (s_{am} + b_{am}) \geq 0 \end{cases} \tag{1}$$

where s_{am} and b_{am} are stock and bond holdings for households in age a within type m. Here, λ is a linear borrowing cost parameter. The real borrowing costs are assumed to be disposed of by banks or brokerage firms outside of the model.

[6] In accordance with a Securities Lending Agreement, a stock loan fee must be completed before the stock is borrowed by investors such as a hedge fund or retail investor.

The borrowing costs imply that there are no costs if the total saving is positive whereas households face a linear borrowing cost if they are net borrowers no matter their portfolio composition.

Without a borrowing constraint, the young agents borrow via bonds and invest in stock markets under an imperfectly correlated shock between TFP and stochastic depreciation. Since all households prefer current to future consumption, the lifecycle consumption profiles show downward slopes for all types. The downward trends of consumption profiles are not consistent with data. As many empirical papers noted, consumption profiles also have a hump along with labor income profiles and the location of the consumption peak falls between ages $45 - 55$ (See Gourinchas and Parker (2002) and Fernández-Villaverde and Krueger (2007)).

In the economy with the borrowing constraint, there is a hump in the consumption profiles. The reason is that agents borrow less and consume more in the early ages since the effective interest rate considering borrowing costs is bigger than the time discount rate. Agents postpone their consumption to the future. Thus, consumption profiles show an upward trend in the borrowing stage. When agents save, they consider the interest rate on saving which is lower than the time discount factor. Then, agents prefer to consume now than future which results in the downward trend of consumption profiles in the saving stage.

2.4 Household problems

At any date-event z^t, households in age a within type m are characterized by both individual states – cash-on-hand variable x_{am} – and aggregate states – the state of the economy z_t and the distribution of households over age, type, and cash-on-hand Γ_t.

Households in age a within type m consume a single good c_{am} and save via stock and bond, s_{am} and b_{am}, for tomorrow. The consumer's problem is given as follows:

$$V_{am}\left(x_{am}; z, \Gamma\right) = \max_{c_{am}, s_{am}, b_{am}} \frac{{c_{am}}^{1-\gamma_m} - 1}{1-\gamma_m} + \beta_m \mathbb{E}\left[V_{(a+1)m}\left(x'_{(a+1)m}; z', \Gamma'\right)\right] \tag{2}$$

subject to

$$c_{am} + s_{am} + b_{am} = x_{am}, \tag{3}$$

$$\begin{aligned} x'_{(a+1)m} &= (1-\tau)\, w'\theta_{(a+1)m} + (1+r'_s)\, s_{am} + (1+r_b)\, b_{am} - f\left(s_{am} + b_{am}\right) && \text{if } a+1 \leq a_R \\ x'_{(a+1)m} &= (1+r'_s)\, s_{am} + (1+r_b)\, b_{am} + T' - f\left(s_{am} + b_{am}\right) && \text{if } a+1 > a_R \end{aligned}, \tag{4}$$

$$z' \sim \pi_z\left(z' \mid z\right), \tag{5}$$

and

$$\Gamma' = H\left(\Gamma, z, z'\right) \tag{6}$$

where $\Gamma = \{x_{am}\}_{am}$ is the set of cash-on-hand across all age/type groups.

For the household problem, I assume a constant relative risk aversion (CRRA) utility function. Here,τ is time-stationary social security contribution rate, $w\left(z^t\right)$ is the aggregate wage, $r_s\left(z^t\right)$ is the rate of the equity return at the date-event z^t, and $r_b\left(z^{t-1}\right)$ is the rate of the bond return at time t which is pre-determined at the date-event z^{t-1}. I denote the social security transfer by $T\left(z^t\right)$ that retirees receive after age a_R. The function $H\left(\cdot\right)$ is the law of motion for the set of cash-on-hand over all age/type groups which is derived from the exogenous shock processes and the households' optimal choices. I assume households are born with zero assets, i.e. $s_{0m} = b_{0m} = 0$.

2.5 Production, factor markets, and resource constraint

I assume there is a representative firm witha Cobb-Douglas production function with capital share α and deterministic labor-augmenting productivity growth g. The representative firm uses capital $K(z^t)$ and hires labor $L(z^t)$ to produce output $Y(z^t) = A(z_t) K(z^t)^\alpha \left((1+g)^t L(z^t)\right)^{1-\alpha}$ where $A(z_t)$ is a multiplicative TFP shock. I normalize the initial labor technology level to be 1. There is a stochastic depreciation shock $\delta(z_t)$ and thus the capital stock evolves following $K(z^t) = I(z^{t-1}) + K(z^{t-1})(1-\delta(z_{t-1}))$. The production market is perfectly competitive.

I assume that the representative firm issues both stocks and bonds to fund the accumulation of the capital stock, $S(z^t)$ and $B(z^t)$, following the finance literature where the leverage is incorporated to increase the volatility of equity returns. Thus, $K(z^t) = S(z^t) + B(z^t) = S(z^t)(1+k_f)$. k_f is an exogenous debt-equity ratio which is constant over time and states, so that the firm only chooses the aggregate capital, not the capital structure. This approach is also used in Harenberg and Ludwig (2018) to keep depreciation shocks small in calibration, while allowing for stock return volatility. They argue that introducing leverage is desirable since large depreciation shocks generate quite large variations in real economic variables. Hasanhodzic and Kotlikoff (2013) also note this fact.

The firm's problem generates the following prices:

$$r\left(z^t\right) = \alpha A(z_t)\left(\frac{K(z^t)}{(1+g)^t L(z^t)}\right)^{\alpha-1} - \delta(z_t), \tag{7}$$

and

$$w\left(z^t\right) = (1-\alpha)(1+g)^t A(z_t)\left(\frac{K(z^t)}{(1+g)^t L(z^t)}\right)^{\alpha} \tag{8}$$

where $r(z^t)$ is the rental rate of capital and $w(z^t)$ is the aggregate wage per labor supply.

From the leverage structure, I can obtain the leveraged stock return as $r_s(z^t) = r(z^t) + k_f(r(z^t) - r_b(z^t))$. This equation means the return to equity is the investment return plus the leverage ratio multiplied by the investment return net of borrowing costs via bonds. The mean and stock return volatility are $\mathbb{E}(r_{s,t}) = (1+k_f)\mathbb{E}(r_t) - k_f\mathbb{E}(r_{b,t})$ and $Var(r_{s,t}) = (1+k_f)^2 Var(r_t)$ respectively. Thus, the leverage increases mean and stock return volatility.

The labor supply by the group with age a and type m is $L_{am}(z^t) = \theta_{am} N_{am}(z^t)$. Thus, $L(z^t) = \sum_{m=1}^{M}\sum_{a=1}^{a_d} L_{am}(z^t) = \sum_{m=1}^{M}\sum_{a=1}^{a_d}\theta_{am}N_{am}(z^t)$. Likewise, the total stock investment is given by $S(z^t) = \sum_{m=1}^{M}\sum_{a=1}^{a_d-1} s_{am}(z^{t-1}) N_{am}(z^{t-1})$. $s_{am}(z^{t-1})$ is the stock investment of age a and type m agents in time $t-1$ for time t. The total bond investment or bond market clearing condition is given by $B(z^t) = \sum_{m=1}^{M}\sum_{a=1}^{a_d-1} b_{am}(z^{t-1}) N_{am}(z^{t-1})$, where $b_{am}(z^{t-1})$ is the bond investment of age a and type m agents in time t.

The resource constraint in the date-event z^t is given by:

$$C\left(z^t\right) + K\left(z^{t+1}\right) + \Lambda\left(z^t\right) = Y\left(z^t\right) + (1-\delta(z_t))K\left(z^t\right) \tag{9}$$

where $\Lambda(z^t) = \lambda\sum_{m=1}^{M}\sum_{a=1}^{a_d-1}\left(s_{am}(z^{t-1}) + b_{am}(z^{t-1})\right) I\left(s_{am}(z^{t-1}) + b_{am}(z^{t-1}) < 0\right) N_{am}(z^{t-1})$, where $I\left(s_{am}(z^{t-1}) + b_{am}(z^{t-1}) < 0\right)$ is an indicator function and equals one if $s_{am}(z^{t-1}) + b_{am}(z^{t-1}) < 0$, and $C(z^t) = \sum_{m=1}^{M}\sum_{a=1}^{a_d} c_{am}(z^t) N_{am}(z^t)$ is the aggregate consumption.

2.6 Goverment budget constraint

I assume that the financial market is sequentially incomplete, although the newly born generation and existing generations can share risk using the tradable assets. Thus, the social security system can insure against the aggregate uncertainty and the birth risk that an agent cannot insure against. The government runs a pay-as-you-go social security system. There is no debt and thus the government runs a balanced budget. I assume that the government operates lump-sum transfer following the social security literature (See e.g. Conesa and Krueger (1999); Hasanhodzic and Kotlikoff (2013); Harenberg and Ludwig (2018))

With these assumptions, the government budget constraint is given by:

$$\tau w\left(z^{t}\right)L\left(z^{t}\right)=T\left(z^{t}\right)P\left(z^{t}\right) \tag{10}$$

where $T\left(z^t\right)$ is the lump-sum transfer in history z^t and $P\left(z^t\right)$ is the number of pensioners, $P\left(z^t\right)=\sum_{m=1}^{M}\sum_{a=a_R+1}^{a_d}N_{am}\left(z^t\right)$.

2.7 Equilibrium

In a competitive general equilibrium, households and firms optimize, all goods, factor and financial markets clear and the government budget constraint is satisfied. I focus on a recursive Markov equilibrium in the computational solution. Because of both technology and population growth, I normalize all aggregate varaibles with total labor efficiency, $(1+g)^{t}L\left(z^{t}\right)$. Thus, the normalized aggregate variables are $q\left(z^t\right)=\frac{Q\left(z^t\right)}{(1+g)^tL\left(z^t\right)}$ for $Q\in\{K,Y,C,I\}$. I also detrend individual variables with the level of technology: $\tilde{o}_{am}\left(\tilde{x}_{am};z_t,\mathit{\Gamma}_t\right)=\frac{o_{am}\left(x_{am};z_t,\mathit{\Gamma}_t\right)}{(1+g)^t}$ for $o\in\{c,s,b,x,w,T\}$. where the tilde denotes normalized variables.

With these detrended variables, I can express the individual budget constraint as follows:

$$\tilde{c}_{am}+(1+g)\,\tilde{s}_{am}+(1+g)\,\tilde{b}_{am}\quad=\quad\tilde{x}_{am}, \tag{11}$$

and

$$\begin{aligned}\tilde{x}_{(a+1)m}&=(1-\tau)\,\tilde{w}\theta_{(a+1)m}+\left(1+r_s'\right)\tilde{s}_{am}+\left(1+r_b\right)\tilde{b}_{am}-f\left(\tilde{s}_{am}+\tilde{b}_{am}\right) && \text{if } a+1\leq a_R\\ \tilde{x}_{(a+1)m}&=\left(1+r_s'\right)\tilde{s}_{am}+\left(1+r_b\right)\tilde{b}_{am}+\tilde{T}'-f\left(\tilde{s}_{am}+\tilde{b}_{am}\right) && \text{if } a+1> a_R\end{aligned} \tag{12}$$

where $\tilde{w}\left(z^{t+1}\right)=(1-\alpha)\,A\left(z_{t+1}\right)k\left(z^{t+1}\right)^{\alpha}$, $\tilde{T}\left(z^{t+1}\right)=\frac{\tau\tilde{w}\left(z^{t+1}\right)L\left(z^{t+1}\right)}{P\left(z^{t+1}\right)}$ and $\tilde{\mathit{\Gamma}}_t$ is the distribution of households over age, type and the normalized cash-on-hand.

The Euler equations in the detrended variables can be written as:

$$\left(\tilde{c}_{am}\right)^{-\gamma_m}=\beta_m\mathbb{E}_t\left[\left(1+r_s\left(z^{t+1}\right)-f'\left(\tilde{s}_{am}+\tilde{b}_{am}\right)\right)\left((1+g)\,\tilde{c}_{(a+1)m}\right)^{-\gamma_m}\right] \tag{13}$$

and

$$0=\mathbb{E}_t\left[\left(r_s\left(z^{t+1}\right)-r_b\left(z^{t}\right)\right)\left(\tilde{c}_{(a+1)m}\right)^{-\gamma_m}\right] \tag{14}$$

I can also express the capital per labor efficiency level as:

$$k\left(z^{t}\right)=\frac{\sum_{m=1}^{M}\sum_{a=1}^{a_d-1}\left\{\tilde{s}_{am}\left(z^{t-1}\right)+\tilde{b}_{am}\left(z^{t-1}\right)\right\}N_{am}\left(z_0\right)}{(1+n)\sum_{m=1}^{M}\sum_{a=1}^{a_d}\theta_{am}N_{am}\left(z_0\right)} \tag{15}$$

2.8 Computational solution

I compute an equilibrium in my model with globally linear approximations for the policy functions without state space aggregation as done in Hasanhodzic and Kotlikoff (2013) and Reiter (2015). I use the set of first-order Smolyak nodes as grids on which I approximate the policy functions. I choose the lower and upper grid bounds using the maximum and minimum of deterministic steady-states assuming that each state of nature continues. I re-approximate the policy functions on the simulated nodes generated by the policy functions on the Smolyak nodes to improve accuracy. To reduce computation time, I use a derivative-free iterative method similar to the endogenous grid method developed by Carroll (2006) but based on the exogenous grids. One of the most interesting findings in this paper is that the consumption policy functions from the economy only with capital well approximates the equilibrium in the economy with both capital and bonds. Thus, I find the consumption policy functions from the former economy and simulate the latter economy with these policy functions by solving for bond holdings or shares in each period instead of finding the bond holding or bond share policy functions explicitly. Lastly, I approximate the soft borrowing cost schedule with a smooth function to avoid solving for conditional optimality conditions depending on borrowing or not. The borrowing costs on the total amount borrowed allow adopting the consumption policy functions from the economy with only risky capital to simulate the economy with multiple assets because both economies essentially have liquidity friction on the total amount borrowed. See Appendix A for the step-by-step numerical procedures of this algorithm and its computational result.

2.9 Welfare criterion

Following Harenberg and Ludwig (2018), I employ the ex-ante expected utility of a household at the start of economic life in each group to study the welfare implication of the introduction of the PAYGO system across groups. To evaluate the expected lifetime utility for each group, I calculate the unconditional average of households' expected lifetime utility on the stochastic steady-state. By appealing the Ergodic theorem, I regard the simulation data truncated for certain initial periods to remove the initial condition effect to represent the Markov distribution of the equilibrium.

I compare the welfare of economies with two different policies by calculating a certainty equivalent consumption variation for each economy and finding its ratio between the two economies with different policies. I compare the long-run welfare effects of the introduction of the PAYGO system. Although I do not include the transition between the two economies from no social security system to the PAYGO system, the introduction of the PAYGO system will increase the welfare for each group if including the welfare effects along the transition. The reason is that those generations on the transition experience the full insurance benefits but partly avoid the long-run welfare costs of crowding out moving from no social security system to the PAYGO system. Therefore, the welfare I calculate provides a lower bound on the welfare effects by ignoring the transition path.

2.10 Decomposition analyses for welfare effects

In the quantitative analysis below, I compute the welfare change from the $laissez-faire$ economy to the economy with a marginal social security system with a contribution rate of 2%, by comparing long-run equilibria in the two economies. To measure the welfare change, I calculate the certainty equivalent (CE) consumption for the two economies and obtain the certainty equivalent consumption variation from the two CE consumption values.

I decompose the welfare effect of the introduction of a marginal social security system into risk-sharing and crowding out effects. To isolate the risk-sharing effect, I calculate a CE consumption in a partial equilibrium setting where a social security system is introduced but prices such as wages and returns remain unchanged. In partial equilibrium, agents expect a change in their income via the social security tax and transfer. Thus, they alter their saving behaviors but there are no general equilibrium effects on prices due to distortions in saving behavior.

For this analysis, I impute the sequence of prices and shocks and the law of motion from general equilibrium in the economy without a social security system into an open economy with a social security system where prices are given as imputed. Then, I find policy functions for the open economy to simulate an equilibrium path with which I calculate a CE consumption in a partial equilibrium. The CEV from the risk-sharing is derived by comparing the CE consumptions in a general equilibrium without a policy and a partial equilibrium with a policy.

To infer the welfare effect of crowding out, I take the difference between the two CEVs: CEV between two general equilibria with and without a social security system and CEV between a general equilibrium without a policy and a partial equilibrium with a policy.

3 Calibration

I calibrate a set of parameters by taking their values from the literature or measuring them from the data. These exogenously determined parameters are called the set of first-stage parameters. Then, I calibrate the rest of the model's parameters by jointly matching the model-simulated moments to their corresponding moments in the data. I call this set of parameters the second-state parameters.

I summarize the baseline calibration parameter values in Table 1. In Appendix A, I provide the numerical procedure of the endogenous calibration.

3.1 Demographics

There are three types of groups characterized by education levels: college graduates (c), a group with high-school education but without a college degree (h), a group without high-school education (d). I assume that adult age starts at age 24 for all households and they retire at the statutory retirement age of 65 following Gourinchas and Parker (2002).[7] I assume agents live for 58 periods up to age 81 following the life expectancy at birth for employed women from the actuarial life table in the year 2013 from the U.S. Social Security Administration (SSA).[8] Hence, in my model, one period corresponds to one year in the real economy.

[7] Less educated workers start working at an earlier age around 20 but they also retire 3 or 4 years earlier than college graduates between 2000 and 2009 as shown in Rutledge et al. (2018). Hence, I assume all households start and end working periods simultaneously in this model.

[8] Wives tend to live longer in a household and household decisions should consider events until the death of wives. I do not differentiate the life expectancy between groups and assume no mortality risk for simplicity.

Table 1: Summary of the Baseline Calibration

Parameters	Value	Target (source)	Stage
Demographics			
Biological age at $a = 1$	24		1^{st}
Model age at retirement a_R	42	Statutory retirement age of 65 (SSA)	1^{st}
Model age maximum a_d	58	Life expectancy of 81 years (SSA)	1^{st}
Population growth n	0.011	U.S. Social Sec. Admin. (SSA)	1^{st}
Education strata (p_c, p_h, p_d)	$(0.49, 0.43, 0.08)$	BLS current population survey (CPS)	1^{st}
Households			
Discount factor $(\beta_c, \beta_h, \beta_d)$	$(0.989, 0.952, 0.948)$	Cagetti (2003)	1^{st}
Relative risk aversion $(\gamma_c, \gamma_h, \gamma_d)$	$(4.26, 3.27, 2.74)$	Cagetti (2003)	1^{st}
Age-type productivity	See Table 2	Cocco et al. (2005)	1^{st}
Technology			
Technology growth g	0.018	TFP growth (NIPA)	1^{st}
Leverage ratio k_f	0.66	Rajan and Zingales (1995)	1^{st}
Capital share α	0.3	Hubbard and Judd (1987)	1^{st}
Standard deviation of TFP $\bar{A}$	0.029	Std. of TFP, 0.029 (NIPA)	1^{st}
Transition probabilities of productivity π_A	0.941	Autocorrelation of TFP, 0.88 (NIPA)	1^{st}
Mean depreciation rate of capital δ_0	0.111	Bond return, 0.023 (Shiller)	2^{nd}
Standard deviation of depreciation $\bar{\delta}$	0.032	Std. of consumption growth, 0.017 (Shiller)	2^{nd}
Conditional prob. of depreciation shocks π_δ	0.883	Corr.(TFP, returns), 0.50 (NIPA, Shiller)	2^{nd}
Borrowing costs			
Borrowing cost parameters $\left(\psi_2, \tilde{\lambda}\right)$	$(0.5, 6)$	Consumption peak level for high and low educ., $(1.34, 1.17)$ in Fernández-Villaverde and Krueger (2007)	2^{nd}

From the BLS current population survey (CPS) in year 2013, I calculated shares across types using the number of employed workers within each group. The numbers are 0.49 for group (c), 0.43 for group (h) and 0.08 for group (d). I assume that the distribution of each group is time-invariant and households within a group share common parameter values across every cohort.

Population grows at a rate of 1.1% in the data from SSA which has been adopted in many macroeconomics papers.[9]

3.2 Borrowing constraint

In the computation, I approximate the borrowing cost with a smooth function as suggested in Hasanhodzic (2015) which is a variant of Chen and Mangasarian (1996). This smooth approximation allows computing the consumption policy functions with smooth polynomials. The smooth approximation of (1) is given by:

$$\hat{f}\left(\chi\right) = \psi_1(-\tilde{\lambda}\chi - 1 + \frac{1}{\psi_2} ln\left(1 + e^{\psi_2\tilde{\lambda}\chi + \psi_2}\right)) \tag{16}$$

where λ is a borrowing cost parameter to be calibrated to target the equity premium. ψ_1 is an intercept term which decreases $f\left(\chi\right)$ value when $\chi < 0$. ψ_2 is a smoothing parameter around $\chi = 0$. As ψ_2 increases, $f\left(\chi\right)$ rapidly increases as χ decreases below 0. Thus, a low ψ_1 and high ψ_2 allow a more exact approximation for (1), but they make (16) less smooth.

Its derivative with respect to the total saving is given by:

$$\hat{f}'\left(\chi\right) = \psi_1\tilde{\lambda}(-1 + \frac{e^{\psi_2\tilde{\lambda}\chi + \psi_2}}{1 + e^{\psi_2\tilde{\lambda}\chi + \psi_2}}) \tag{17}$$

Gourinchas and Parker (2002) have shown that household age-consumption profiles exhibit a statistically significant hump after controlling both economic growth and family size. The actual consumption increases during ages $20-40$ and falls off during ages $50-70$. The consumption peak occurs around age 45 in their estimates. Fernández-Villaverde and Krueger (2007) also provide estimates for the household age-consumption profiles for high and low education groups adjusted for age, cohort, time effects, and household size. The high education group refers to households with at least some college. The low education group is defined by households with a high school degree or less. In the context of my paper, the high education group is consistent with college graduates group and the low education groups is corresponding to high school graduates and dropouts. Following Bullard and Feigenbaum (2007), I use the ratio of peak consumption to age 30 consumption as a metric that can be used to gauge the nature of the hump. In Fernández-Villaverde and Krueger (2007), the ratio estimates are 1.34 for the high education group and 1.17 for the low education group based on adult-equivalent non-durable consumption data.

I set the value of ψ_1 at 0.1 similar to Hasanhodzic (2015). I calibrate the parameters ψ_2 and $\tilde{\lambda}$ at 0.5 and 6 to match the ratios of consumption peak to age 30 consumption for both high and low educations groups. For the ratio for the low education group, I calculate a weighted average of such ratios for high school graduates and dropouts with their population shares.

[9] See e.g. Krueger and Kubler (2006); Conesa et al. (2009); Kitao (2014); Peterman and Sommer (2014); Harenberg and Ludwig (2018).

Table 2: Deterministic labor income process from Cocco et al. (2005)

	c	h	d
constant	–4.3148	–2.1700	–2.1361
Age	0.3194	0.1682	0.1684
$\text{Age}^2/10$	–0.0577	-0.0323	-0.0353
$\text{Age}^3/100$	0.0033	0.0020	0.0023

3.3 Households

There are many empirical papers which estimate time discount rates and risk-aversion across education strata. (Gourinchas and Parker 2002; Cagetti 2003; Alan and Browning 2010; Cooper and Zhu 2016). Among these, Cagetti (2003) provides the estimates of time discount rates and risk-aversion under the assumption that preferences are CRRA, by matching the simulated median wealth profiles in each education group with those observed in the Panel Study of Income Dynamics and the Survey of Consumer Finances in a life cycle model of wealth accumulation. Following Cagetti (2003), I also assume CRRA preferences and adopt their time discount factor and risk-aversion parameter estimates for each type as follows: $(\beta_c, \beta_h, \beta_d) = (0.989, 0.952, 0.948)$ and $(\gamma_c, \gamma_h, \gamma_d) = (4.26, 3.27, 2.74)$. Cooper and Zhu (2016) obtain similar estimates using the same dataset and Alan and Browning (2010) derive a similar qualitative pattern for the preference parameters in education although their numbers are quantitatively different. These preference parameter values imply that more educated workers are more patient and more risk-averse. This result fits with general intuition and abundant empirical evidence which shows that patient and risk-averse youth are more likely to enter and graduate from college to realize high-income profiles and low earnings volatility after entering the labor market (See e.g. Cadena and Keys 2015).

I take the estimated values for deterministic age-type specific labor productivity from Cocco et al. (2005) which uses the PSID. For this, they split data into three education groups: c, h, and d. They control household fixed effects, household size, and marital status. They fitted a third-order polynomial to the age dummies to obtain the age-productivity profile for each group (looking at Figure 1 in their paper). I adopt their hump-shaped deterministic age profiles for $\log(\theta_{am})$ in this paper as described in Table 2.

3.4 Technology

Based on the total factor productivity, the deterministic trend growth rate is set at 1.8% which is in line with other studies. I set the leverage in the firm sector at 0.66 as in Rajan and Zingales (1995). The capital share of output equals 0.3 following the literature (See e.g. Hubbard and Judd (1987)). This number is consistent with the direct estimates of total compensation as a fraction of GDP using the NIPA data.

Following Krueger and Kubler (2006) and Harenberg and Ludwig (2018), I assume an aggregate shock given by a four-state Markov chain with a transition matrix π_z where $Z = \{z_1, z_2, z_3, z_4\}$. This aggregate shock affects TFP and depreciation. z_1 and z_2 are recession periods where wages are low and thus $A_{z_1} = A_{z_2} = 1 - \bar{A}$ where I normalize average productivity level to unity. In boom periods, $A_{z_3} = A_{z_4} = 1 + \bar{A}$ assuming symmetry on the size of the shock. Depreciation shocks $\delta_{z_1} = \delta_{z_3} = \delta_0 + \bar{\delta}$ and $\delta_{z_2} = \delta_{z_4} = \delta_0 - \bar{\delta}$ allow for an imperfect correlation between TFP and stochastic depreciation, where δ_0 denotes average depreciation rate.

I also introduce symmetry in the transition matrix in line with Harenberg and Ludwig (2018) as: $\pi_A = \pi\left(A' = 1 - \bar{A} \mid A = 1 - \bar{A}\right) = \pi\left(A' = 1 + \bar{A} \mid A = 1 + \bar{A}\right)$ and $1 - \pi_A = \pi\left(A' = 1 + \bar{A} \mid A = 1 - \bar{A}\right) = \pi\left(A' = 1 - \bar{A} \mid A = 1 + \bar{A}\right)$. Let π_δ be the probability of being in the high (low) depreciation state conditional on being in the low (high) technology state. I also assume symmetric conditional probability given by $\pi_\delta = \pi\left(\delta' = \delta_0 + \bar{\delta} \mid A' = 1 - \bar{A}\right) = \pi\left(\delta' = \delta_0 - \bar{\delta} \mid A' = 1 + \bar{A}\right)$. Under these assumptions, I obtain the following transition matrix:

$$\pi_z = \begin{bmatrix} \pi_A \pi_\delta & \pi_A\left(1 - \pi_\delta\right) & \left(1 - \pi_A\right)\left(1 - \pi_\delta\right) & \left(1 - \pi_A\right)\pi_\delta \\ \pi_A \pi_\delta & \pi_A\left(1 - \pi_\delta\right) & \left(1 - \pi_A\right)\left(1 - \pi_\delta\right) & \left(1 - \pi_A\right)\pi_\delta \\ \left(1 - \pi_A\right)\pi_\delta & \left(1 - \pi_A\right)\left(1 - \pi_\delta\right) & \pi_A\left(1 - \pi_\delta\right) & \pi_A \pi_\delta \\ \left(1 - \pi_A\right)\pi_\delta & \left(1 - \pi_A\right)\left(1 - \pi_\delta\right) & \pi_A\left(1 - \pi_\delta\right) & \pi_A \pi_\delta \end{bmatrix}$$

I set $\bar{A} = 0.029$ and $\pi^A = 0.941$ to match the standard deviation and autocorrelation of TFP of 0.029 and 0.88 from Harenberg and Ludwig (2018).[10] I calibrate $\delta_0 = 0.111$ to produce the long run real return on 10 years the U.S. government bonds over the period $1960 - 2007$ which was 2.3%. I set $\bar{\delta}$ at 0.032 to match the standard deviation of consumption growth between 1947 and 2007, 0.017.[11] Lastly, I set $\pi_\delta = 0.883$ to target the correlation of the cyclical component of TFP with risky returns of 0.5 following the estimate in Harenberg and Ludwig (2018).

4 Results

In this section, I analyze the effects of social security systems on aggregate variables and welfare across different educational groups. Following Krueger and Kubler (2006) and Harenberg and Ludwig (2018), I consider the marginal introduction of the PAYG social security system with its initial contribution rate in the United States, 2%.

4.1 Aggregate effects and long-run welfare implication

In Table 3, I summarize the effects of introducing social security with a contribution rate at 2% on aggregate capital, output, wages, returns, and welfare.

[10] Note that the standard deviation of TFP is just $\bar{A}$ no matter what the transition matrix is and its autocorrelation is $\left(2\pi^A - 1\right)$ given the structure of the Markov transition matrix.

[11] I use the data on Robert Shiller's website: http://www.econ.yale.edu//˜shiller/data.htm.

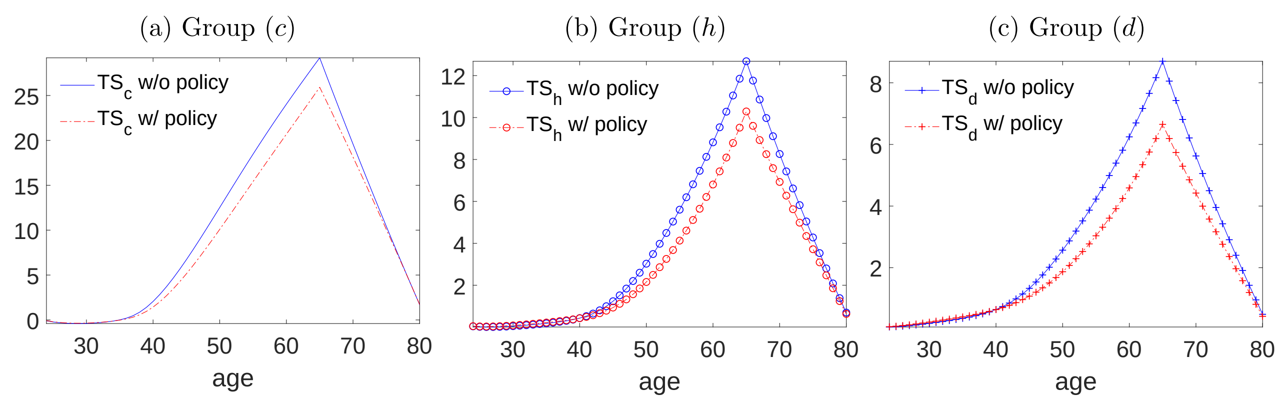
Table 3: The effects of the social security introduction on aggregate variables

Variable	Change
Average capital k	$\triangle k/k = -15.30\%$
Average output y	$\triangle y/y = -4.84\%$
Average normalized wage $\tilde{w}$	$\triangle \tilde{w}/\tilde{w} = -4.84\%$
Average stock return r_s	$\triangle r_s = 1.66\%$
Average bond return r_b	$\triangle r_b = 1.66\%$
CEV_c^{GE}	$\triangle CE_c^{GE}/CE_c^{GE} = 1.72\%$
CEV_h^{GE}	$\triangle CE_h^{GE}/CE_h^{GE} = -0.51\%$
CEV_d^{GE}	$\triangle CE_d^{GE}/CE_d^{GE} = 0.44\%$

- I denote capital and output per efficiency level.
- $\triangle X/X = \frac{\mathbb{E}(X_t|\tau=2\%)-\mathbb{E}(X_t|\tau=0\%)}{\mathbb{E}(X_t|\tau=0\%)}$
- $\triangle x = \mathbb{E}\left(x_t \mid \tau = 2\%\right) - \mathbb{E}\left(x_t \mid \tau = 0\%\right)$
- CEV denotes certainty equivalent consumption variation for groups (c), (h) and (d).

The marginal introduction of social security leads to capital stock (per efficiency level) reduction by 15.30% on average because the social security transfer leads households to reduce total saving between age 40 and 80 as seen in Figure 1. In Figure 1, the blue lines and red lines describe average total saving profiles in the economy without and with the social security system, respectively.[12] At the peak saving age 65, total saving decreases by 11.24% for group (c), 18.89% for group (h), 23.54% for group (d) after the introduction of a marginal social security system. The high-school dropout group reduces their saving the most at the peak age because the replacement of the social security to their private saving is the highest. Note that there is less change in the total saving before age 40. The borrowing constraint causes agents to consume their cash-on-hand due to the cost of borrowing to smooth consumption over the lifecycle. Thus, households start saving after reaching the point that consuming cash-on-hand yields low enough marginal utility around age 40 no matter their types. A social security tax reduces disposable labor income, which affects households' incentives to borrow but by a limited amount, and there are less total saving changes after the system is established.

Figure 1: Average total saving profiles



[12] I take the average consumption simulation data conditional on age.

The capital reduction leads to an average output (per efficiency level) reduction of 4.84% and thus the normalized wage decreases exactly up to 4.84%. The lower capital stock increases the marginal productivity of capital which raises the stock return by 1.66%. The average bond return also increases and its extent is 1.66% for two reasons. The social security transfer lowers the bond saving along with total saving. The insurance provided by the social security system causes households to increase their stock investment shares in middle-age as seen in Figure 2. The reduction in bond demand decreases bond price and increases its return.

Figure 2: Average stock shares from age 47

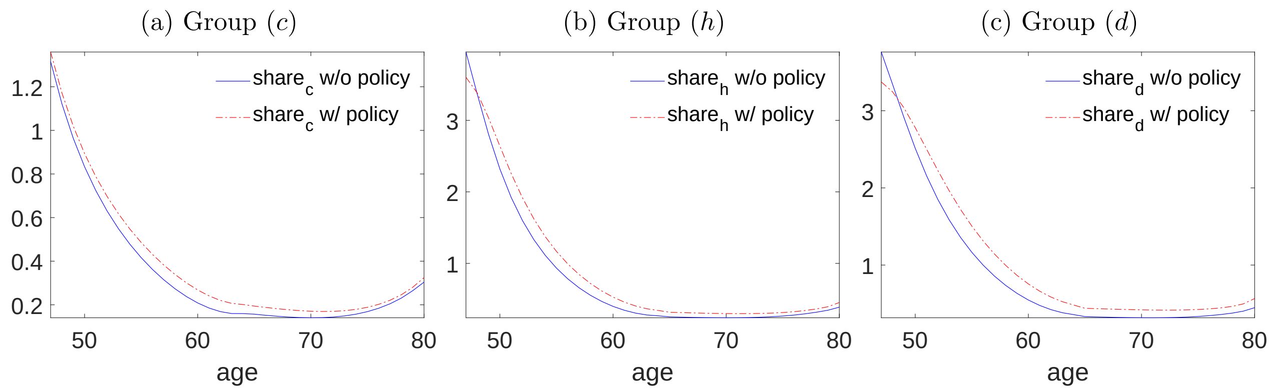


Table 3 shows that the college educated group benefits the most from the marginal social security introduction in the baseline calibrated model. The high-school dropout group also benefits from the social security system. However, the high-school graduate group experiences a welfare loss from the policy. I measure the welfare gain/loss with the certainty equivalent consumption variation as discussed in Subsection 2.10. I first calculate the certainty equivalent consumptions for the two economies with and without the social security system which generate the same utility with the long-run equilibria in each economy. Then, I compute the percentage change between the CE consumptions after the marginal social security introduction. The values of CEV are 1.72% for college graduates group, -0.51% for the group of high school graduates without a college degree and a CEV of 0.44% for high school dropouts group.

One significant implication from the result is that the social security system separates welfare winning and losing groups across education strata. Previous studies cannot identify specific policy implication for each educational group since they focus on ex-ante identical agents. The quantitative welfare result in this paper can answer the questions in the long-run: who is advantaged from the social security policy and is the welfare result according to the goal of such policy? Finally, from the perspective of a utilitarian social planner, the changes indicated by the model would constitute an overall welfare improvement from the introduction of the social security system.

4.2 Welfare effects from insurance versus crowding out

In this section, I dissect the welfare results in Table 3 into risk-sharing and crowding out. I examine the welfare implication of the social security system in partial equilibrium where I remove the effect of price changes on allocations by fixing prices at those in the baseline economy without the social security system. To find the policy functions in partial equilibrium, I let agents expect future wages and returns with the law of motion from the baseline economy without a tax. Given the exogeneous law of motion, I compute the households' policy functions and simulate these function given the sequence of prices from the baseline economy. Then, I calculate the CEV for partial equilibrium, CEV^{PE}, which represents the welfare gain/loss from the insurance function of the social security system. I regard $CEV^{GE} - CEV^{PE}$ as the welfare gain/loss from the crowding out, CEV^{CO}, because the difference considers price changes from the social security system.

Table 4: Welfare effects from insurance versus crowding out

	group (c)	group (h)	group (d)
CEV^{GE}	1.72%	−0.51%	0.44%
CEV^{PE}	−1.68%	1.01%	2.36%
CEV^{CO}	3.41%	−1.52%	−1.92%

Table 4 shows the welfare effects of insurance and crowding out. For the college graduates group, there is a welfare loss of 1.68% from the insurance function of the social security system because the system implements an intragenerational redistribution from the high-income group to the other lower income groups. However, the college graduates group gets a large welfare benefit from the price changes induced by a social security system. The welfare gain from crowding out is 3.41%. A social security system increases the returns of both stocks and bonds. Since the college graduates group is patient, their marginal propensity to save is higher than the other two groups and they weigh future consumption highly. An increment in the rate of return enlarged the value of their savings via an income effect and thus future consumption. Thus, this group faces a welfare benefit from the price changes induced by the introduction of a social security system which dominates the welfare loss from the intragenerational redistribution of the social security system. This result suggests that the interplay of general equilibrium and risk-sharing effects across heterogeneous groups can be significantly larger than those obtained in homogeneous agent models, such as that examined by Henriksen and Spear (2012).

Both high school graduates and dropouts groups obtain welfare benefits in partial equilibrium up to 1.01% and 2.36% respectively because of the intragenerational redistribution from the college graduates group to these groups via the social security system. This welfare benefit is much larger for the high school dropouts group because they experience the most transfer benefit from their lowest income. In other words, they receive in old-age transfers more than they contribute to the social security system during the working periods. However, both groups get welfare losses from the induced price changes by the social security system of −1.52% and −1.92%, respectively. The reason is that these groups are relatively impatient and thus save less and value current consumption more. An increase in the rate of return is not favorable for these groups because they face lower income effects from less saving. On the other hand, a wage reduction decreases their welfare by lowering the value of their disposable labor incomes and consumption when young significantly under the borrowing constraints.

In the subsequent sections, I dig further into the welfare results included in CEV^{PE} and CEV^{CO} by examining changes in consumption allocations from several directions.

4.3 Partial equilibrium analysis

There are changes in partial equilibrium consumption with the social security policy compared to the baseline economy in three dimensions: average lifecycle consumption profiles, consumption volatility conditional on age, and mean consumption level. For example, the intragenerational transfer increases the mean consumption level for the poor income group at the cost of reducing consumption for the income rich group. The old-age consumption volatility declines as the social security system pools the labor and capital incomes under an imperfectly correlated shock. In the calibrated model, wage rates and equity returns are weakly positively correlated, exhibiting a correlation coefficient close to 0.2. Agents reduce their saving in the middle-aged periods and increase stock share because the social security system partly secures old-age consumption. A higher stock share can increase the mean consumption level.

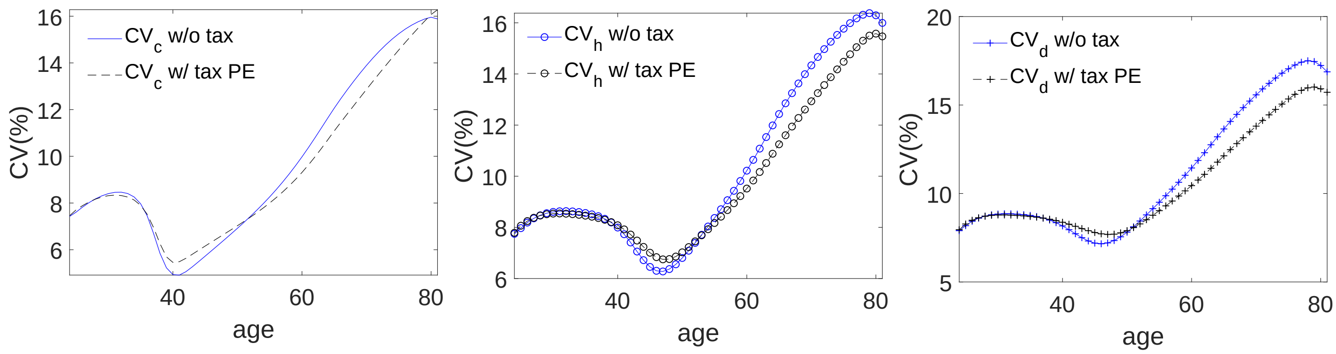
To isolate the welfare gain from the insurance function of social security, I calculate the certainty equivalent consumption variation using the certainty equivalent consumptions computed with average lifecycle consumption profiles for each economy. I denote the certainty equivalent consumption variation using average consumption data conditional on age in partial equilibrium setting by $DCEV^{PE}$. If $CEV^{PE} > DCEV^{PE}$, there is a welfare gain from the social security system by reducing consumption variation. $DCEV^{PE}$ represents changes in both mean consumption level from a tax/transfer and altered saving patterns and average lifecycle consumption profiles.

The average lifecycle consumption profiles generated in the model yield $DCEV_c^{PE} = -1.70\%$ for the college graduates group, $DCEV_h^{PE} = 0.84\%$ for the high-school graduates group, and $DCEV_h^{PE} = 2.15\%$ for the high-school dropouts group. These $DCEV^{PE}$ values are lower than the CEV^{PE} values, indicating that the social security system increases the welfare of all groups by providing insurance against fluctuations in old-age consumption.

Figure 3 shows the insurance role of the social security system. In the figure, I draw the coefficient of variation profiles in the economy without the policy and the economy with the policy without price changes. After retirement, all educational groups experience reduced old-age consumption volatility from the social security transfer because it hedges the private saving risk. The coefficient of variation profiles show a U-shaped curve. The young consume most of their disposable labor income because of the borrowing constraint. Thus, the consumption volatility is proportional to the disposable income for the young. The old finance their consumption from saving. The stochastic depreciation can make both the principal and the return on assets uncertain and households can lose their investement principle. Thus, in the $laixxez - faire$ environment old age consumption is highly volatile. The middle-aged can hedge consumption risk by rebalancing investment portfolios over time due to not being bound by liquidity constraints, and thus this age faces the lowest consumption volatility.

Figure 3: The coefficient of variation profiles in partial equilibrium

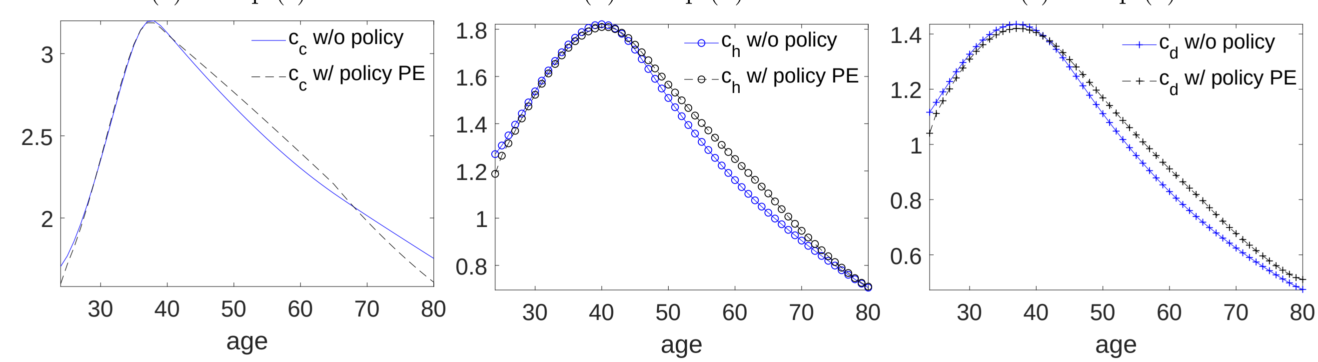
(a) Group (c) (b) Group (h) (c) Group (d)



The social security system implements an intragenerational redistribution among different educational groups. In the calibrated model, there is a wealth transfer from the college educated group to the other two groups. This wealth transfer generates positive $DCEV^{PE}$ for both high-school graduates and dropouts. Figure 4 shows that income poor groups consume more in the economy with a social security system in their middle-aged and old periods. On the other hand, $DCEV_c^{PE}$ is negative for the income rich group because their wealth ends up being transferred to the lower income groups. In addition, disposable labor income decreases during working periods because of the social security tax. The reduction in labor income decreases the young consumption before they start saving around age 40 because young households want to borrow but the amount borrowed is limited due to the borrowing cost (See Figure 4). These two effects significantly lower the welfare of the college-educated groups in partial equilibrium.[13]

Figure 4: Average consumption profiles in partial equilibrium

(a) Group (c) (b) Group (h) (c) Group (d)

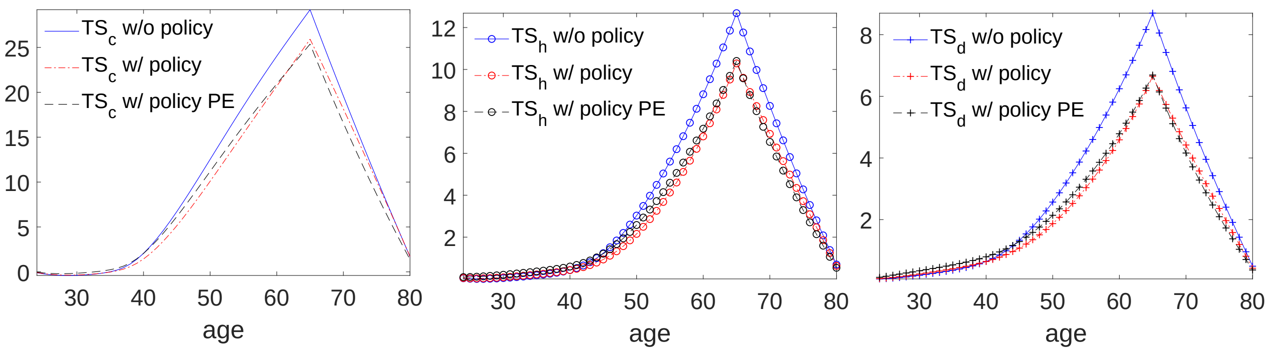


In Figure 4, the average consumption profile shows a hump because of the borrowing constraint. One thing to note is that the consumption of the college-educated groups decreases rapidly during retirement in partial equilibrium. The insurance provided through the social security system weakens the precautionary saving motive for this risk-averse group and thus they reduce saving significantly as seen in Figure 5. Therefore, the old-age consumption declines rapidly.

[13] Hubbard and Judd (1987) also point out the welfare cost from borrowing constraints when a social security system is introduced.

Figure 5: Average total saving profiles including partial equilibrium

(a) Group (c) (b) Group (h) (c) Group (d)



4.4 Crowding out effect analysis

This section analyzes how lower wages and higher returns affect the consumption volatility, lifecycle consumption profiles and mean consumption level in general equilibrium.

As I did in the previous section, I also isolate the welfare gain from the insurance function of social security in general equilibrium environment by computing $DCEV^{GE}$. Here, $DCEV^{GE}$ is the certainty equivalent consumption variation using average consumption data conditional on age from the economy without the social security policy to the economy with the policy in general equilibrium. If $CEV^{GE} > DCEV^{GE}$, there is a welfare gain from the social security system by reducing the consumption variations in general equilibrium. More importantly, if $CEV^{GE} - DCEV^{GE} > CEV^{PE} - DCEV^{PE}$, then there is a stronger insurance effect in general equilibrium than in partial equilibrium. With the simulation data, I find that $DCEV_c^{GE} = 1.14\%$, $DCEV_h^{GE} = -0.82\%$, and $DCEV_d^{GE} = 0.15\%$. This result implies that $CEV^{GE} - DCEV^{GE} > CEV^{PE} - DCEV^{PE}$ holds and the difference is large for the college graduates, high-school graduates, and high-school dropouts in order: $\left(CEV^{GE} - DCEV^{GE}\right) - \left(CEV^{PE} - DCEV^{PE}\right) = 0.56\%$, 0.14% and 0.08% for groups $(c), (h)$ and (d) respectively.

Figure 6: The coefficient of variation profiles

(a) Group (c) (b) Group (h) (c) Group (d)

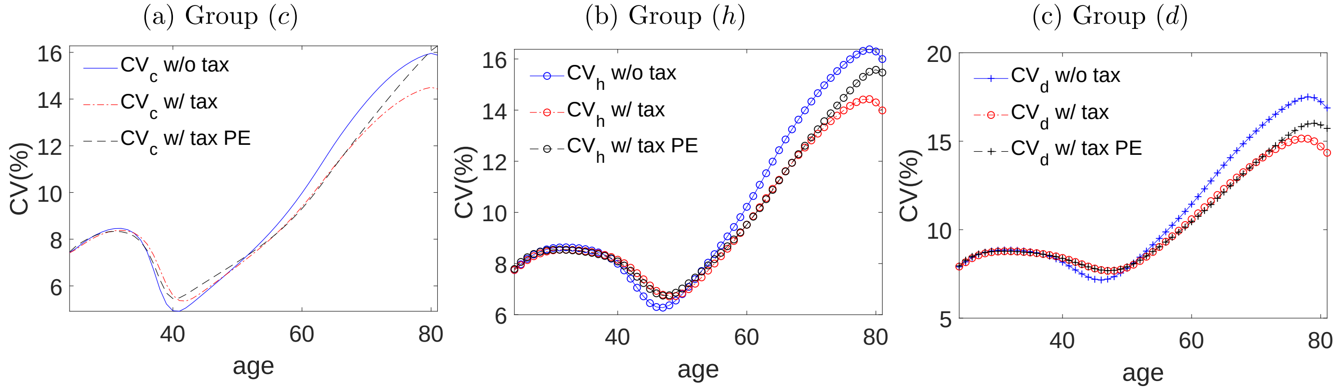
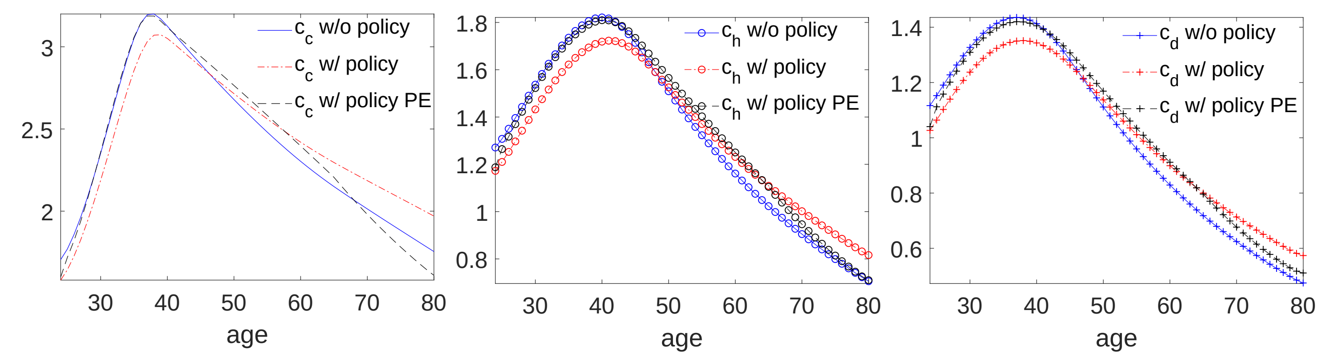


Figure 6 shows that the college educated group experiences a large reduction in the old-age consumption volatility in general equilibrium compared to the other groups. To understand this, one should note that the total saving after retirement is larger in general equilibrium environment than in partial equilibrium one because of the substitution and income effects from an increase in the rate of return. A higher rate of return implies cheaper future consumption and increases the value of saving which results in large total saving as shown in Figure 5. More saving in general equilibrium than partial equilibrium leads to higher old-age consumption (See Figure 7). This higher old-age consumption causes a lower coefficient of variation in retirement periods. Since the college educated groups are the most risk-averse, they obtain the most welfare gain in moving from partial equilibrium to general equilibrium setting, where old-age consumption volatility decreases.

Figure 7: Average consumption profiles

(a) Group (c) (b) Group (h) (c) Group (d)



After shifting terms, I obtain $\left(DCEV_c^{GE} - DCEV_c^{PE}\right) = \left(CEV_c^{GE} - CEV_c^{PE}\right) - 0.56\% = 2.85\%$ for the college graduates group. Following the same approach, $\left(DCEV_h^{GE} - DCEV_h^{PE}\right) = -1.66\%$ and $\left(DCEV_d^{GE} - DCEV_d^{PE}\right) = -2.00\%$. These results indicate that only the college educated group experiences a welfare gain moving from the average consumption profile in partial equilibrium setting to general equilibrium one. The reason is that there is less welfare transfer from the college graduates group to the other two groups because of a lower general equilibrium wage. The lower wage encourages this most risk-averse group to save more due to the precautionary saving motive, as seen in Figure 5. In addition, the college educates group is more patient than the other two groups. Thus, their total saving is much larger than the other groups. If the rate of return increases, there are larger income effects for this group from increased saving. The substitution effect from an increase in returns also shifts consumption to the old-periods. The most patient group values future consumption more than the other groups. Therefore, the college graduates group receive a positive welfare effect from the general equilibrium effects induced by a social security system (see Kuhle (2012) which provides an analytical demonstration of this effect).

On the other hand, the high-school graduates and dropouts groups get a smaller overall transfer due to the general equilibrium effect and relatively weaker income effects from their lower saving. They also value current consumption more than the college educated group. Thus, an increase in returns is not favorable to this group. More importantly, a lower wage reduces consumption during the young periods under the borrowing constraint. Since high-school graduates and dropouts groups already have low young consumption, an additional decrease in young-age consumption lowers the welfare of those groups significantly.

4.5 Welfare analysis with representative households

For this comparison, I assume there is no heterogeneity in either preferences and permanent age/productivity profiles. I perform the same welfare analysis as above to highlight the role of heterogeneous preference and income streams in the welfare implication of the social security system. In the lifecycle model with representative households, I take the quarterly subjective discount factor 0.99 widely used in the literature. Then, the annual discount factor is 0.96 for one-period in the model. Following Harenberg and Ludwig (2018), I set the coefficient of relative risk aversion of the representative household at 3. I calculate the weighted income profiles using the income profiles for each group with their employed population ratio in the year 2013 from the BLS CPS data. For the other first-stage parameters, I take values from the heterogeneous model above and recalibrate the parameters of mean depreciation rate, the standard deviation of depreciation, the conditional probability of a depreciation shock on the TPF state and borrowing costs to match the empirical targets discussed in Section 3. The consumption peak relative to age 30 is 1.21 for the representative agent model following the estimates in Fernández-Villaverde and Krueger (2007) using adult-equivalent non-durable consumption data for the benchmark model without educational heterogeneity.

Table 5: The effects of the social security introduction on aggregate variables in the representative agent model

Variable	Change
Average capital k	$\triangle k/k = -12.16\%$
Average output y	$\triangle y/y = -3.81\%$
Average normalized wage $\tilde{w}$	$\triangle \tilde{w}/\tilde{w} = -3.81\%$
Average stock return r_s	$\triangle r_s = 1.41\%$
Average bond return r_b	$\triangle r_b = 1.41\%$

- I denote capital and output per efficiency level.
- $\triangle X/X = \frac{\mathbb{E}(X_t|\tau=2\%)-\mathbb{E}(X_t|\tau=0\%)}{\mathbb{E}(X_t|\tau=0\%)}$
- $\triangle x = \mathbb{E}(x_t \mid \tau = 2\%) - \mathbb{E}(x_t \mid \tau = 0\%)$

Table 5 shows the effects of the marginal social security system on aggregate variables in the lifecycle model without cohort heterogeneity. Compared to the heterogeneous agents model, the aggregate capital level decreases less in the representative agent model than in the heterogeneous agents model. In that model, the change in capital was$\triangle k/k = -15.30\%$, while in the representative agent model it is -12.16%. This smaller reduction in the aggregate capital leads to a smaller decrease in output and wage and a smaller increase in the rate of return relative to the base model with preference and income heterogeneity.

In both models, agents decrease private saving because the social security transfers guarantee old-age consumption and replace the role of private saving. In addition, the insurance provided via the social security system weakens the precautionary saving motive in private saving. The more risk-averse households are, the more they reduce saving. The risk-aversion coefficient of the representative agent is lower than that of the high-school graduates in the heterogeneous agents model. Thus, the precautionary saving motive is relatively weaker in the representative agent model which leads to a smaller decrease in aggregate saving. Other than the precautionary saving motive channel, there are general equilibrium effects in the total saving amount. An increase in returns from the crowding out generates a substitution effect. This effect causes agents to save more for future consumption. Under the CRRA preferences, the elasticity of intertemporal substitution is proportional to the inverse of the risk-aversion coefficient. Thus, less-risk averse households raise saving more than more-risk averse ones when the rates of return go up. This general equilibrium effect also contributes to a smaller decrease in the total saving in the representative agent model.

Table 6: Welfare effects from insurance versus crowding out in the representative agent model

	Welfare change
CEV^{GE}	-1.19%
CEV^{PE}	-0.51%
CEV^{CO}	-0.68%

In Table 6, I summarize the welfare decomposition for the introduction of the social security system in the representative agent case. In general equilibrium, there is a welfare loss of 1.19% measured in CEV from the marginal social security system. This result is consistent with the previous studies such as Krueger and Kubler (2006) and Harenberg and Ludwig (2018) in which the introduction of a social security system results in a welfare loss in the lifecycle model with only aggregate shocks.

Unlike the previous studies, introduction of a social security system generates a welfare loss even in partial equilibrium setting, because of no intragenerational redistribution and the borrowing constraint. The social security system lowers the disposable income of the young from their labor supply. Under a hump-shaped income stream, young workers want to borrow to smooth consumption because the marginal utility of consumption is very high if they just consume cash-on-hand from their disposable labor income. Under the borrowing constraint, it is costly to smooth consumption over the lifecycle by borrowing. The welfare cost from low consumption in the early periods of life can dominate the welfare gain from the insurance on old-age consumption. Thus, the social security system can yield a welfare loss even in partial equilibrium. If borrowing costs are reduced, then CEV^{PE} increases.

5 Sensitivity Analysis

5.1 Increasing the share of college graduates

In the population survey data, the share of the college-educated group has grown over time mostly because more children tend to enter universities expecting higher future income streams or lower earnings volatility. Thus, I ask how the growing population of the income rich affects the implications of social security in aggregate variables and welfare. With the BLS CPS in the year 2017, I reset the shares of each educational group as 0.52 for college graduates group, 0.41 for high-school graduates group and 0.07 for high-school dropouts group.

In Table 7 and 8, I summarize the comparative statics analysis results in aggregate variables and households welfare respectively. Table 7 shows that a higher share of the college-educated group decreases aggregate capital 0.2% more than in the benchmark calibrated model. As a result, both output and wages decline 0.07% more and both stock and bond returns increase by 0.01% more. In the comparative statics analysis, the share of population shifts to the group with a higher risk aversion and a lower intertemporal elasticity of substitution. The insurance provided by social security reduces the saving of more risk-averse households more significantly than less risk-averse groups. In addition, households with a low intertemporal elasticity of substitution increase their saving less in response to an increase in the rate of return from crowding out than ones with a high intertemporal elasticity of substitution.

Table 7: The aggregate effects of social security under a higher share of the college-educated group

Variable	Change
Average capital k	$\triangle k/k = -15.50\%$
Average output y	$\triangle y/y = -4.91\%$
Average normalized wage $\tilde{w}$	$\triangle \tilde{w}/\tilde{w} = -4.91\%$
Average stock return r_s	$\triangle r_s = 1.67\%$
Average bond return r_b	$\triangle r_b = 1.67\%$

Table 8: Welfare effects under a higher share of the college-educated group

	group (c)	group (h)	group (d)
CEV^{GE}	1.96%	−0.36%	0.60%
CEV^{PE}	−1.60%	1.18%	2.59%
CEV^{CO}	3.56%	−1.54%	−1.99%

The welfare decomposition analysis is displayed in Table 7 for the economy with a higher share of college graduates. In partial equilibrium, all groups obtain either more welfare gains or less welfare losses compared to the benchmark case. The certainty equivalent consumption variation in partial equilibrium increase by 0.08%, 0.17%, and 0.23% for groups (c), (h) and (d), respectively. In the economy with more college graduates, both high-school graduates and dropouts receive more intragenerational wealth transfers from the income rich because of an increase in government tax revenue. A larger amount of transfer reinforces the degree of insurance on the old-age consumption. This intergenerational redistribution also improves the welfare of the college-educated group.

A further decrease in wages due to crowding out generates a larger welfare loss for high-school graduates and dropouts groups under borrowing constraints than the benchmark economy. However, more welfare gains from stronger intragenerational redistribution dominate the welfare losses from the reduction in wages. Thus, the economy with more college graduates enhances the welfare of groups (h) and (d) than the benchmark model. The college-educated group experiences stronger income effects from the increase in returns, which improves their welfare in this economy as well.

5.2 Expanding the social security system

I increase the size of social security systems from the benchmark model by making the contribution rate double at 4%. The tax rate increase will strengthen both the insurance function and crowding out effects from social security.

Table 9 displays the aggregate effects of social security when the social security contribution rate rises up to 4%. Compared to the benchmark case, aggregate capital drops more significantly, by about 26%. The larger social security system provides stronger insurance and replaces a larger part of private saving. Thus, all types of households decrease saving more than the benchmark case. The substantial capital crowding out also changes prices markedly. Wages decline about 9% and both stock and bond returns rise about 3%.

Table 9: The aggregate effects of social security under a higher contribution rate

Variable	Change
Average capital k	$\triangle k/k = -26.10\%$
Average output y	$\triangle y/y = -8.64\%$
Average normalized wage $\tilde{w}$	$\triangle \tilde{w}/\tilde{w} = -8.64\%$
Average stock return r_s	$\triangle r_s = 3.17\%$
Average bond return r_b	$\triangle r_b = 3.17\%$

Table 10: Welfare effects under a higher contribution rate

	group (c)	group (h)	group (d)
CEV^{GE}	1.10%	−2.62%	−1.01%
CEV^{PE}	−2.67%	1.97%	4.23%
CEV^{CO}	3.76%	−4.59%	−5.24%

In Table 10, I summarize the welfare decomposition analysis results in the economy with a higher contribution rate. As expected, the larger social security system increases the welfare of high-school graduates and dropouts in partial equilibrium relative to the benchmark case by strengthening the wealth transfer from the income rich to the income poor. Thus, the college graduates group experience more welfare losses.

Although groups (h) and (d) obtain more welfare gains in partial equilibrium, their welfare in general equilibrium is lower than in the benchmark general equilibrium case. In this larger social security system, the high-school dropouts indeed obtain welfare losses, unlike the benchmark case. There are two reasons behind the reversed welfare results. First, there is a weaker intragenerational transfer in general equilibrium than in partial equilibrium because of a lower amount of transfer due to a decrease in wages. In addition, an almost 9% decrease in wages reduces the disposable income of the young significantly, and they have stronger incentives to borrow. However, the borrowing constraint limits the amount of consumption in the early lifetime periods. The welfare cost from the limited consumption smoothing dominates the welfare gain from stronger intergenerational redistribution. For the college graduates, they also have substantial welfare costs from the borrowing limits. Coupled with a higher tax rate, they face a welfare decrease in general equilibrium from the benchmark case.

This comparative statics analysis implies a larger social security system can overturn the results from the benchmark model that a pay-as-you-go social security can improve the welfare of certain groups. In the economy with a high contribution rate, the income poor high-school dropouts can have welfare losses from significantly reduced young-age consumption under the borrowing constraint although they receive more welfare transfers. College graduates might also experience welfare losses from stronger redistribution which can dominate the positive income effects from increases in returns.

5.3 Double shocks model

In this section, I consider what happens to the welfare effects of social security when shocks are made larger. The base calibrated model produces an equity premium of 0.21% and a Sharpe ratio of 0.034 well below their empirical counterparts.[14] These results are consistent with other studies which calibrate a low risk-aversion with small-size aggregate shocks to match consumption volatility (See Hasanhodzic and Kotlikoff (2013) and Harenberg and Ludwig (2018)). Thus, I examine the economy with stochastic depreciation shocks doubled.

In Table 11, I show the effects of social security on the aggregate variables when stochastic depreciation shocks are doubled. Aggregate capital decreases slightly more than in the base model because households reduce their private saving more when they hold more assets from strong self-insurance motives against large shocks. Average output and wages decline up to 5.53%. The returns on stocks and bonds increase by 1.88% and 1.86% respectively.

Table 11: The aggregate effects of social security under double shocks

Variable	Change
Average capital k	$\triangle k/k = -17.51\%$
Average output y	$\triangle y/y = -5.53\%$
Average normalized wage $\tilde{w}$	$\triangle \tilde{w}/\tilde{w} = -5.53\%$
Average stock return r_s	$\triangle r_s = 1.88\%$
Average bond return r_b	$\triangle r_b = 1.86\%$

Table 12: Welfare effects under double shocks

	group (c)	group (h)	group (d)
CEV^{GE}	3.44%	0.08%	0.93%
CEV^{PE}	−1.16%	1.53%	3.00%
CEV^{CO}	4.60%	−1.44%	−2.07%

Table 12 summarizes the welfare decomposition analysis. I stress that all groups obtain welfare gains in the economy with double shocks, unlike the benchmark case. Insurance provided via social security enhances the welfare of households by significantly reducing consumption volatility against large shocks. Thus, all groups experience higher welfare in partial equilibrium in this economy than in the base one. In general equilibrium, the welfare costs from wage decreases and borrowing constraints generate a larger negative value of CEV^{CO} for high-school dropouts. However, the gains from insurance against large shocks surpass such costs. Thus, the high-school dropouts group obtains a welfare gain. Interestingly, not only college graduates but also high-school graduates have higher values of CEV^{CO}. More asset holding from a strong precautionary saving motive generates larger positive income effects even for the high-school graduates group. Hence, high-school graduates have a slightly positive welfare gain in general equilibrium.

From this comparative statics analysis, I observe that the quantitative welfare results are substantially affected by the size of shocks. However, the qualitative results are quite robust in the

[14] From empirical studies, equity premiums range between 4% and 6% and Sharpe ratios range between 0.28 and 0.33.

sense that certainty equivalent consumption variation is the largest for college graduates, then high-school dropouts, and high-school graduates in order.

6 Conclusion

This paper studies the distributional welfare implications of a pay-as-you-go social security system across education strata. The ex-ante heterogeneous agent model generates gains for certain educational groups from the social security system. In the baseline calibrated economy, I find that high-school dropouts and college educated groups obtain welfare gains, while the high-school graduates group have welfare losses from the social security system. The welfare advantages under social security for the high-school dropouts group derive mostly from intragenerational redistribution. The general equilibrium effects provide a substantial welfare gain to the college-educated groups. Borrowing constraints strengthen the welfare cost from the social security system by preventing consumption smoothing during the early stage of life. For high-school graduates, the cost from the financial friction dominates gains from intragenerational redistribution and old-age consumption insurance. The existence of welfare winners and losers provides an explanation for why a pay-as-you-go social security system can still survive in developed countries facing fiscal crises, even using ex-ante expected utility as the welfare criteria, unlike the standard social welfare function in the political economy literature on social security, which considers only the welfare of currently living generations. Moreover, the quantitative analysis implies that social security can be more advantageous for the rich or educated groups, in contrast to the original arguments made after the great depression to motivate its adoption.

In this paper, I abstract from many potentially interesting features such as idiosyncratic earning, health and mortality risks and preference heterogeneity within educational groups. Incorporating these features might contribute to the literature in two directions. First, the model with substantial idiosyncratic risks and continuum of ex-ante heterogenous households may significantly affect the distributional welfare effects of social security systems. Second, solving such models will require the development of an advanced algorithm to deal with the continuum of ex-ante heterogeneous agents. I also restrict policy design experiments by focusing on insurance through certain taxes and transfers structures in a balanced budget constraint. One can consider other policy schemes with different policy structures and government debt to generate more policy implications in the model with ex-ante heterogeneity. I leave these tasks for future research.

A Appendix

A.1 Computational Method

I approximate the Markov policy rules with globally linear functions on the set of cash-on-hand across all types and ages. In other words, I use all the state variables information without a state space reduction or using only low-order moments. The reason is that Krueger and Kubler (2004) and Hasanhodzic and Kotlikoff (2013) show that there is an inaccuracy issue when representing the Markov policy functions with low-order moments if there is an imperfectly correlated TFP and depreciation shocks affecting the income of the young and old differently. Cozzi (2011) argues that low-order moments cannot well describe the law of motion under risk-aversion heterogeneity because different types of households show a distinct marginal propensity to save and thus aggregation methods cannot capture equilibrium dynamics in the economy inhabited by ex-ante heterogeneous agents. To make computation feasible in a model of almost 80 periods in this paper, I approximate the Markov policy rules with linear polynomials as done in Hasanhodzic and Kotlikoff (2013) and Reiter (2015).

As a computational contribution, this paper finds that consumption policy functions in an economy only with capital (or stock) well capture the equilibrium consumption path in an economy with both stock and bond when cash-on-hand is used as state variables instead of asset holdings. In other words, there are small Euler equation errors when I apply the consumption policy rules from an economy only with capital into the simulation of the economy with two assets by solving for bond and capital shares in each period given the consumption rules. The order of error is almost the same as the error when one uses the consumption policy functions from explicitly solving for the economy with two assets into the simulation. One possible reason is that households keep a similar marginal propensity to consume no matter the number of assets is. If one is given the same amount of cash-on-hand, she spends a similar amount for consumption today between economies with only capital or both. This similarity result has an important computational implication that one can reduce computation time when finding consumption policy functions by solving for the economy with capital instead of the one with both stock and bond. The former economy requires to solve equations (13) and (14) to find consumption policy functions whereas the latter economy needs to solve only equation (13). Thus, using the similarity result reduces one equation for all age/type groups in every grid which facilitates computing the consumption policy functions.

Another advantage of solving for only equation (13) is that one can construct the system of Euler equations across all age/type groups in every grid as a system of linear equations in the coefficients of the globally linear policy functions. Thus, one can avoid using a Newton-type non-linear equation solver and rather use a derivative-free iterative method as in the endogenous grid method developed by Carroll (2006). However, I keep using exogeneous grids in my computation with a technical trick to transform the problem into a system of linear equations. A derivative-free iterative method reduces computation time significantly.

I describe the algorithm in this paper step by step with details as follows. The algorithm consists of inner and outer loops. In the inner loop, I solve for the Markov consumption policy functions given a set of calibrated parameters. In the outer loop, I find calibrated parameters values to match the target moments.

Inner loop.

(i) Construct linear policy functions for the Markov consumption rules with cash-on-hand over all groups.

- The consumption policy functions are given by $\tilde{c}_{am} = \left[\begin{array}{cc} 1, & \tilde{\Gamma} \end{array}\right] \eta_{am}$ for $\forall am$. Γ is a row vector of normalized cash-on-hand across all types and age from 2 to $a_d = 58$ with an abuse of notation, i.e. $\tilde{\Gamma} = [\tilde{x}_{2c}, \ldots, \tilde{x}_{a_d c}, \tilde{x}_{2h}, \ldots, \tilde{x}_{a_d h}, \tilde{x}_{2d}, \ldots, \tilde{x}_{a_d d}]$. η_{am} is the coefficient of the policy function correponding to the constant and state variables.

(ii) Then, make an initial guess for the coefficients in the linear policy functions given a set of calibrated parameters.

- To set a good initial guess for the coefficients of the policy functions, I find out their first-order Taylor approximation at a deterministic steady state in the economy only with capital using Dynare. As discussed above, I find out consumption policy functions by solving for the economy with single asset and apply the approximations into the simulation of the economy with two assets. In a deterministic setting, solving for the economy with two assets generates an indeterminacy issue because stock and bond are identical due to no-arbitrage property. Thus, I obtain a local linear approximation in the single asset economy.
- The result from Dynare shows the coefficients of a local linear approximation for consumption on capital instead of cash-on-hand: $\tilde{c}_{am} = \bar{\bar{c}}_{am} + \left(\tilde{K} - \bar{\bar{\tilde{K}}}\right) \phi_{am}$ where $\bar{\bar{c}}_{am}$ is the deterministic steady-state value of consumption for age a and type m, $\tilde{K}$ is the vector of capital holdings of age 2 to a_d for all types from the previous period and $\bar{\bar{\tilde{K}}}$ is the vector of deterministic steady-state values for $\tilde{K}$. Thus, one should transform the local linear approximation for consumption as a function of cash-on-hand. As an example for the economy without a tax and a borrowing cost,

$$\bar{\bar{c}}_{am}+\left(\tilde{K} - \bar{\bar{\tilde{K}}}\right) \phi_{am} = \bar{\bar{c}}_{am}+\left(\frac{\tilde{\Gamma} - \tilde{w}\Theta}{(1+r)} - \bar{\bar{\tilde{K}}}\right) \phi_{am} = \left(\bar{\bar{c}}_{am} - \frac{\tilde{w}\Theta\phi_{am}}{(1+r)} - \bar{\bar{\tilde{K}}}\phi_{am}\right)+\tilde{\Gamma}\frac{\phi_{am}}{(1+r)}$$

where the second equation comes from the budget constraint in a vector form, $\tilde{\Gamma} = \tilde{w}\Theta + (1+r)\tilde{K}$ in which Θ is the vector of θ_{am} across all types and age from 2 to $a_d = 58$. This transformation implies:

$$\eta_{am} = \left[\begin{array}{c} \bar{\bar{c}}_{am} - \frac{\tilde{w}\Theta\phi_{am}}{(1+r)} - \bar{\bar{\tilde{K}}}\phi_{am} \\ \frac{\phi_{am}}{(1+r)} \end{array}\right]$$

(iii) Generate a set of grids on which I find the coefficients of consumption policy functions satisfying the Euler equations, (13).

- There are $1 + M(a_d - 1) = 1 + 3 \times 57 = 172$ coefficients. Thus, the number of grids should be larger than 172. As a candidate, I generate the first-order Smolyak grids in the space $\left[\begin{array}{cc} -1, & 1 \end{array}\right]^{M(a_d-1)}$ and transform them into the $\left[\begin{array}{cc} \tilde{k}_{1c,min}, & \tilde{k}_{1c,max} \end{array}\right] \times \ldots \times \left[\begin{array}{cc} \tilde{k}_{a_d d,min}, & \tilde{k}_{a_d d,max} \end{array}\right]$. The number of first-order Smolyak grids is $1 + 2M(a_d - 1) = 1 + 6 \times 57 = 343$. Thus, there are more nodes than are required.

- To obtain $\tilde{k}_{am,min}$ and $\tilde{k}_{am,max}$ for $\forall am$, I find the nonstochastic steady state in the economy under each state $z \in Z$. Then, I calculate the maximum and minimum values of $\left\{\bar{\tilde{k}}_{am,z}\right\}_z$ for $\forall am$ where $\bar{\tilde{k}}_{am,z}$ are the deterministic steady state when state z continues. Under the stochastic environment, there is a precautionary saving motive and thus the maximum and minimum values of $\tilde{k}_{am}$ is not just the maximum and minimum values of $\left\{\bar{\tilde{k}}_{am,z}\right\}_z$. Thus, I properly scale the maximum and minimum values among the deterministic steady states to derive $\tilde{k}_{am,min}$ and $\tilde{k}_{am,max}$. Given each node of capital holdings, I calculate prices, transfer and cash-on-hand variables to calculate the next period capital holding and current period consumption.

(iv) Find the coefficients of the consumption policy functions with a derivative-free iterative method.

- I first find the value of $\tilde{c}_{am}$in each grid by transforming the Euler equation, (13) in the single asset economy as follows:

$$(\tilde{c}_{am})^{-\gamma_m} = \beta_m \mathbb{E}\left[\left(1 + r' - f'\left(\hat{\tilde{k}}_{am}\right)\right)\left\{(1+g)\,\hat{\tilde{c}}_{(a+1)m}\right\}^{-\gamma_m}\right] \tag{18}$$

where $\hat{\tilde{k}}_{am} = \frac{\left(\tilde{x}_{am} - \hat{\tilde{c}}_{am}\right)}{1+g} = \frac{\left(\tilde{x}_{am} - \left[\begin{array}{cc} 1, & \tilde{\Gamma} \end{array}\right]\eta_{am}\right)}{1+g}$ and $\hat{\tilde{c}}_{(a+1)m} = \left[\begin{array}{cc} 1, & \hat{\tilde{\Gamma}}' \end{array}\right]\eta_{am}$ where $\tilde{x}_{am}$ and $\hat{\tilde{x}}'_{am}$ in $\hat{\tilde{\Gamma}}'$ are given by for the example without a tax and a borrowing cost:

$$\tilde{x}_{am} = \tilde{w}\theta_{am} + (1+r)\,\tilde{k}_{am} \tag{19}$$

and

$$\hat{\tilde{x}}'_{am} = \tilde{w}'\theta_{am} + (1+r')\,\hat{\tilde{k}}_{am} \tag{20}$$

in which $\tilde{w} = (1-\alpha)\,A\tilde{k}^{\alpha}$, $r = \alpha A\tilde{k}^{\alpha-1} - \delta$, $\tilde{w}' = (1-\alpha)\,A\hat{\tilde{k}}^{\alpha}$ and $r' = \alpha A\hat{\tilde{k}}^{\alpha-1} - \delta$.

- Then, regress the set of $\{\tilde{c}_{am}\}$ corresponding to grids on $\left[\begin{array}{cc} 1, & \tilde{\Gamma} \end{array}\right]$ to update η_{am}. When updating η_{am}, I use the dampening strategy. For example. $\eta^{i+1}_{am} = \sigma\hat{\eta}^{i+1}_{am} + (1-\sigma)\,\eta^{i}_{am}$ where η^{i}_{am} is the coefficient parameter values in the i-th iteration, $\hat{\eta}^{i+1}_{am}$ is the regression coefficient above and σ is a dampening parameter. The coefficient parameter values in the $(i+1)$-th iteration, η^{i+1}_{am}, is determined as a convex combination of $\hat{\eta}^{i+1}_{am}$ and η^{i}_{am}.
- Iterate the steps above until convergence is achieved by satisfying:

$$\sup\left|\left\{\hat{\tilde{c}}^{i+1}_{am}\right\}_{am} - \left\{\hat{\tilde{c}}^{i}_{am}\right\}_{am}\right| < \epsilon \tag{21}$$

where $\left\{\hat{\tilde{c}}^{i}_{am}\right\}_{am}$ is the set of $\hat{\tilde{c}}^{i}_{am}$ for all ages and types in all grids and ϵ is a predetermined convergence measure.

(v) Solve the model again based on the Ergodic set.

- Simulate the model with the consumption policy functions above and use the simulated nodes to reset the upper and lower bounds of grids. The simulated nodes provide information about the Ergodic set where the stochastic steady-state of the model exists.

Outer loop:

(i) Change the second stage parameters to match empirical moments in the data.

  (a) To obtain the model statistics, simulate the economy for with $11,000$ periods the approximated consumption policy functions from above. Throw away the first 1000 periods, so that $10,000$ simulation periods are left. Given the consumption policy functions from the single asset economy, find the total saving for the next period. Then, find bond holdings by solving the equation (14) in each simulation period given the total saving.

  (b) Let $\mathcal{T}$ be the target statistics in the data and $\mathcal{P}$ be the second stage parameters. From the simulation, I can calculate the simulated statistics $\hat{\mathcal{T}}(\mathcal{P})$. Then, I solve for $\mathcal{P}$ as a root of $\mathcal{T} - \hat{\mathcal{T}}(\mathcal{P}) = 0$.

(ii) Increase the social security contribution rate given the calibrated parameters and find the new general equilibrium by repeating steps above.

The algorithm is well converged, it takes less than 1 minute to find the consumption policy functions in the single asset economy. The computation time is about 30 minutes to simulate $11,000$ periods in the economy with two assets. The average Euler equation errors are less than 0.1%. I implement this algorithm using Matlab 2018 in a desktop with Intel Core i7-6700K CPU @ 4.00GHz.

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Singular Invariant Markov Equilibrium in Stochastic Overlapping Generations Models*

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This paper examines the invariant Markov distribution associated with the rational expectations equilibrium in multi-period stochastic overlapping generations models under pure exchange. We prove that a simple linear iterated function system can generate the invariant distribution in these models under small discrete aggregate shocks. We provide sufficient conditions for the linear iterated function system to generate a singular invariant measure. We also characterize the set of economies which satisfy the sufficient conditions. The attractors of the singular invariant measure in our models exhibit fractal patterns. The existence of the linear iterated function system implies that an algorithm based on its structure can allow computing equilibria in long-period stochastic overlapping generations models with heterogeneity.

JEL classification: C61, C62, C63 and D51

Keywords: Singular Continuous Invariant Measure, Linear Iterated Function System, Stochastic Overlapping Generations Model, Fractal Self-Affinity, Cantor Distribution

1 Introduction

The multi-period stochastic overlapping generations (SOLG) model, extending the work of Samuelson (1958), has been widely used in macroeconomics, finance, and policy-making as an important workhorse model. Despite its importance, theoretical results in this model have been restricted to the existence, uniqueness, and stability of the stochastic steady state equilibria in the literature (see Grandmont and Hildenbrand 1974; Laitner 1981; Spear and Srivastava 1986; Wang 1993; Duffie et al. 1994; Wang 1994; Morand and Reffett 2007 and many others).

The steady state in a stochastic model generates an invariant distribution over a fixed subset of the space of state variables. The deterministic OLG model, as a special case by collapsing a

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shock to a certain state, has the simplest invariant measure which concentrates on a single point with probability one.[1] Due to their simplicity, it is acceptable to limit attention to the properties described above when examining the deterministic steady states.

In contrast to deterministic OLG models, the steady state of a SOLG model is more complicated in the sense that the support of the invariant measure can be an uncountable subset of the state space even under a small aggregate shock.[2] In very simple stochastic models, such as that originally examined by Lucas (1972), one can generally find equilibria in which prices only depend on the contemporaneous aggregate shock, which we call strongly stationary equilibria. Spear (1985), however, showed that these strongly stationary, or short memory equilibria do not exist in SOLG models where two-period lived agents trade multiple goods and have intertemporally non-separable preferences.[3] Citanna and Siconolfi (2007) and Henriksen and Spear (2012) obtain the same result for two-period lived OLG models with time-separable utilities and cohort heterogeneity, and three-period lived OLG models populated by representative agents, respectively.

For general SOLG models without these kinds of stochastic equilibria, Duffie et al. (1994) showed that there will exist stationary Markov equilibria once one includes an appropriate set of lagged endogenous state variables. Citanna and Siconolfi (2010) showed that it was generically sufficient to consider recursive equilibria, in which the endogenous state variables are taken as the wealth distribution.

The exclusion of the memoryless equilibria and the existence of the Markov equilibria imply uncountable support for an invariant measure in general SOLG models. However, neither of these papers provides any characterizations of the nature of the continuous invariant measure. Therefore, an important question arises as to the nature of the invariant measure of the Markov equilibria as the stochastic steady-state equilibria.

By the Lebesgue decomposition theorem, measures on a Euclidean space can be decomposed into the sum of two measures, one absolutely continuous with respect to the Lebesgue measure, and the other singular. The *singular continuous* distribution is defined as a probability measure which assigns probability one to a Lebesgue measure zero set. A probability measure is *absolutely continuous* with respect to the Lebesgue measure if it assigns a positive probability to Lebesgue measure non-zero sets. It is well-known that a simple shock process such as a symmetric Bernoulli distribution can produce either a singular or an absolutely continuous invariant measure in a stochastic model (see Mitra et al. 2003 for instance).

To complete the theoretical characterizations of the steady-state equilibria in SOLG models, we study the continuity properties of their invariant distributions. Examining this feature is theoretically interesting since there exists a bifurcation into the two types of the measure continuity, depending on the parameters of the models for small deviations from a deterministic model. Knowing the continuity features of an invariant measure can help determine the existence of its density function. Specifically, an absolutely continuous measure has a density with

[1] Benhabib and Day (1982) discuss the possibility of deterministic OLG models generating continuous invariant measures if erratic paths exist. However, the invariant measure of interest in this paper is one generated by iterating a dynamical system where the contraction mapping principle is applied. Such an invariant measure in deterministic OLG models should be a degenerate probability measure on a stationary point.

[2] In our model, aggregate uncertainty has a finite number of states. A small aggregate shock, in this case, means a short distance between the realized values of stochastic fundamentals in each state.

[3] The short memory equilibria depend on a finite history of shocks. As a special case of the equilibria, the strongly stationary equilibria is only affected by the current exogenous shock. Therefore, the cardinality of the support of the short-memory equilibria is finite.

respect to the Lebesgue measure whereas a singular continuous measure does not. One needs only know the parameters identifying its density function to describe an absolutely continuous measure. On the other hand, representing a singular continuous measure requires a large set of information: the value of the distribution for every point in its support.

Despite its significance, there has been little attention, in the literature, paid to the problem of characterizing the continuity features of invariant measures in stochastic models. (Exceptions are Mitra et al. (2003) and Mitra and Privileggi (2009).) A key reason for the paucity of research is that researchers lack knowledge of the equilibrium process in these stochastic models. Even in models where one makes specific assumptions about preferences and other primitives, characterizing the equilibrium law of motion for the stochastic economy can be daunting. Furthermore, even if one is aware of the functional form of an equilibrium mapping, it can be difficult to determine under what conditions singular or absolutely continuous measures arise, and how they depend on economic environments unless the mapping is of a simple form.

Therefore, another main goal of this paper is to study whether there exists a simple form for the equilibrium mappings in SOLG models. We then provide a sufficient condition for the invariant measure of the equilibrium mapping to be singular. We also investigate under what economic parameter values the equilibrium mapping generates a singular invariant measure by satisfying the sufficient condition.

We approach the study of continuity properties by working from a simple OLG model to a general one, step by step. For analytical tractability, we first study a monetary model where agents live three periods, have logarithmic preferences, consume a single good, and save via accumulations of money holdings. There is an independent and identically distributed (I.I.D.) shock to the total amount of money which follows a simple Bernoulli process. This tractable model yields a closed form equilibrating process. One noticeable point is that the law of motion for the state variable, the money holding of the young, is linear in the lagged state variable as long as the aggregate shock is sufficiently small. The price of the single good is also linear in the state variable. Therefore, a linear iterated function system (LIFS) can represent the dynamics of Markov equilibria (ME).[4]

The closed form solution, in this case, states that the slope parameter of the LIFS is only affected by the endowment ratio between ages, not the total amount of endowment, the size of money shock or the probability of shocks. The Lipschitz constant of the LIFS belongs to $\left(0, \frac{1}{2}\right)$. A one-dimensional LIFS with this scaling factor satisfies the so-called "no-overlap" property. A non-overlapped LIFS creates a Cantor-like invariant distribution whose support is a Cantor-like invariant set. The Cantor-like invariant distribution is singular with respect to the Lebesgue measure if generated by a LIFS and the Cantor-like attractor shows fractal self-affinity.[5] Therefore, the log-linear monetary model has a singular continuous Markov measure under small aggregate shocks. This example shows that a multi-period SOLG model can generate fractal patterns in the rational expectation equilibrium.

In a three-period SOLG model with a simple Lucas tree asset and general preferences, we cannot obtain a closed-form solution as in the log-linear monetary model. Hence, we apply

[4] The Markov equilibria of interest in this paper are also called 'simple Markov equilibria' to contrast with generalized Markov equilibria where the state space adds variables such as the last period's marginal utility or a Lagrange multiplier (Kydland and Prescott, 1980), asset prices and individual consumption (Duffie et al., 1994), and continuation utilities (Feng et al., 2014). We avoid the possibility of multiple solutions issue given the minimal state variables by restricting to the equilibria around the deterministic steady state of interest instead of expanding the state space.

[5] A self-affine object is invariant under an anisotropic transformation (non-uniform scaling).

the implicit function theorem (IFT) to infer that a LIFS can generate ME around a deterministic steady state for small shocks. We show that the slope parameters of the LIFS correspond to the eigenvalues of the Jacobian matrix for the locally linearized price dynamics evaluated at the deterministic steady state as long as the aggregate shock is small enough. Based on this relationship, we numerically study how structural parameters affect the Lipschitz constant and thus determine an open set of economies where the no-overlap property holds and hence, a singular invariant measure arises. We find that a model with low risk-aversion and small dividend income share can generate a singular Markov measure. This result highlights why there can exist a unique singular measure in the monetary model above, which is characterized by a zero dividend and risk-aversion set at one.

Finally, we extend the model to a more realistic version by allowing agents to live many-period lives, introducing permanent heterogeneity and letting the number of states of nature be arbitrary but finite. As in the simpler models, we show the existence of a simple but high-dimensional LIFS in a neighborhood of the deterministic steady state as long as the aggregate shocks are sufficiently small. We provide a condition under which the LIFS has identical affine matrices across states. As we did in the three-period model, we find a relationship between the affine matrices of the LIFS and the stable eigenvalues of the locally linearized price dynamics for both determinate and indeterminate equilibria. We give a weak sufficient condition for the model to generate the singular invariant measure. This condition is closely related to the size of the spectral radius of the affine matrices. We numerically check a similar relation between the largest eigenvalue and structural parameters as in the three-period model. Lastly, we produce some examples where multi-dimensional singular Markov measures appear when satisfying our weak sufficient condition for singularity.

The empirical relevance of our theoretical results can be found in the literature on fractal phenomena observed in financial data. The self-affinity aspect of stock prices was one of the topics studied in both the non-linear dynamics and time-series econometrics literature for over a decade between the mid-1980's and '90's. In the former literature, earlier researchers analyzed low-dimensional deterministic dynamic systems as data generating processes since the attractors of the chaotic dynamic systems can have self-affinity features. They claimed to have found evidence for the existence of low-dimensional chaotic attractors in the time-series stock prices data (see Baumol and Benhabib 1989; Scheinkman and Lebaron 1989). However, later work with large data sets concluded that the previous results were misderived due to the paucity of data (see Vassilicos et al. 1993). The consensus now is that there is very little evidence for low-dimensional deterministic systems as stock price generating processes (see Vassilicos et al. 1993; LeBaron 1994). These results, then, require (at a minimum) high-dimensional chaotic systems or stochastic systems to account for the empirical features of stock prices.

In the econometrics literature, many stochastic models have been developed based on the theory of fractal measure and fractional Brownian motion processes pioneered by Mandelbrot (1963) and Mandelbrot (1967). The time-series econometrics models capture the self-affinity properties observed in the financial time-series data. However, they lack an economic mechanism behind the data-generating processes. To fill this gap, our paper shows how stochastic economic models can generate the self-affine features in their rational expectation equilibria via high-dimensional recursive policy functions.

This paper contributes to the literature on several dimensions. On the theoretical side, as far as we know, this is the first paper to study the continuity property of an invariant measure in multi-period SOLG models by providing mechanisms and conditions for singular measures to appear. This paper also identifies the set of economies satisfying these conditions. This, in turn, extends the theoretical characterization of SOLG models beyond the well-known results on existence, uniqueness, and convergence.

There are a few papers in the literature which also examine singular invariant distributions in dynamic general equilibrium models. Mitra et al. (2003) characterizes the invariant Markov distribution in terms of singularity versus absolute continuity in a one-sector stochastic growth model with logarithmic preferences and Cobb-Douglas production functions. Thus, the stylized model has an explicit law of motion. On the other hand, we study not only a tractable monetary OLG model and but also a general model without preference specifications. Hence, we have to deal with the equilibrating process described by a system of implicit difference equations. We show simple linear policy functions can generate ME under certain conditions via an application of the IFT.

Mitra and Privileggi (2009) shows similar results with ours in the sense that they provide the characteristics of structural parameters to generate a singular invariant measure without parametric assumptions. One main difference is their paper analyzes a one-sector growth model whereas this paper deals with SOLG models. More importantly, we specify the high-dimensional law of motion explicitly even for general preferences and shock processes, so that we examine the continuity property of a multi-dimensional invariant measure.

Gardini et al. (2009) shows the existence of a Cantor-like limiting distribution of forward-looking equilibria in OLG models. However, their paper concentrates on two-period deterministic OLG models and thus, the invariant measure of interest in their paper is essentially for sunspot equilibria when multiple equilibria exist due to a backward bending offer curve. The nature of the equilibria in this paper is different in terms of dimensionality and the cause of stochasticity. This paper treats high-dimensional stochastic steady state generated by an intrinsic shock such as a shock to the endowment, dividend and asset quantity.

Another theoretical contribution of the paper is to explicitly show that the LIFS takes the eigenvalues of its corresponding full price dynamic system as its own in multi-period OLG models with cohort heterogeneity. Thus, given information on the eigenvalues of price dynamics, one can infer the contractivity and singularity property of the LIFS. To the best of our knowledge, this is the first paper to clearly show how the two systems are related via eigenvalues in general OLG models.

The results in this paper also have computational implications from the existence of a LIFS as the equilibrating process for SOLG models with heterogeneous agents. Researchers can simply adopt the structure of the LIFS and iterate this functional specification to convergence to find Markov policy rules. Since the LIFS requires only the first-order term for each endogenous state variable in the approximated policy functions, it will make computing very long-period lived SOLG models feasible by reducing a lot of unknown coefficients.

Since we prove the existence of a LIFS for an arbitrary shock process, a linear form approximation for the equilibrium process can be accurate even under an imperfectly correlated shock between labor and capital incomes. Under this type of shock, quasi-aggregation methods such as Krusell and Smith (1998) might generate large approximation errors since the marginal propensity to save can vary greatly by age. Therefore, our results offer the foundation of an alternative algorithm to deal with the limitations of existing ones in computing equilibrium in SOLG models.

2 Log-Linear Monetary Model

We first study a standard three-period SOLG model with fiat money under logarithmic preferences. Time is discrete labeled by $t = 0, 1, 2, \ldots$. Agents live three periods: youth, middle-aged and retired denoted by y, m, and r respectively. They consume a single good and can save via accumulations of the fiat money.

We assume an aggregate shock has a two states of support and follows an I.I.D. process with the probability of state s occurring equally to π^s where $0 < \pi^s < 1$ for $s \in \{h, l\}$ and $\pi^h + \pi^l = 1$. The history of the aggregate shocks given the initial shock realization, s_0, is represented by $S^t = \{s_1, s_2, \ldots, s_t \mid s_0\} \in \Sigma^t = \{h, l\}^t$ where we let l represent the low shock and h the high shock. We let $\left(S^t, s_{t+1}\right)_{s_{t+1} \in \{h,l\}}$ denote the set of nodes for one period after the histroy S^t. We make $\left(S^t, s_{t+1}, s_{t+2}\right)_{s_{t+1}, s_{t+2} \in \{h,l\}^2}$ denote the set of nodes for two periods after the histroy S^t. The total money supply in time t is stochastic in the amount M^{s_t} which depends only on the realization of the current shock.

If the economy moves from the low state to the high state, the total money supply increases. In the opposite case, the total money supply decreases. In these changes of the total money supply, we assume the government sells lump sums of the new money or buys lump sums of the old money by trading with the single good. If the government increases or decreases the money holding of existing agents proportionately, we recover the strong stationary equilibrium in parallel to the results in Lucas (1972). To have history dependent equilibria, we need the lump sum transaction mechanism which generates a real effect as above.

Agents are endowed with non-stochastic consumption goods in the amounts $\left(\omega_y, \omega_m, 0\right)$. The consumption plan of the representative household born in time t and history S^t is denoted by $c\left(S^t\right) = \left(c_y\left(S^t\right), \left(c_m\left(S^t, s_{t+1}\right)\right)_{s_{t+1} \in \{h,l\}}, \left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right)_{s_{t+1}, s_{t+2} \in \{h,l\}^2}\right)$ where $c_y\left(S^t\right)$ is consumption when young in node S^t, $c_m\left(S^t, s_{t+1}\right)$ is consumption when middle-aged in node $\left(S^t, s_{t+1}\right)$, and $c_r\left(S^t, s_{t+1}, s_{t+2}\right)$ is the old period consumption in node $\left(S^t, s_{t+1}, s_{t+2}\right)$. Similarly, the money holdings of the household is given by $m\left(S^t\right) = \left(m_y\left(S^t\right), \left(m_m\left(S^t, s_{t+1}\right)\right)_{s_{t+1} \in \{h,l\}}\right)$ where $m_y\left(S^t\right)$ and $m_m\left(S^t, s_{t+1}\right)$ denote money holdings when young and middle-aged in node S^t and $\left(S^t, s_{t+1}\right)$, respectively.

Lifetime preferences are additively time-separable logarithmic given by a von Neumann-Morgenstern utility function $U : \mathbb{R}^7_+ \to \mathbb{R}$. U is specified by:

$$(1)\quad E_t U\left(c\left(S^t\right)\right) = \ln\left(c_y\left(S^t\right)\right) + \sum_{s_{t+1} \in \{h,l\}} \pi^{s_{t+1}} \left\{ \ln\left(c_m\left(S^t, s_{t+1}\right)\right) + \sum_{s_{t+2} \in \{h,l\}} \pi^{s_{t+2}} \ln\left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right) \right\}$$

where the discount factor sets at one, $\beta = 1$, for analytical tractability. Sequential budget constraints are given by:

$$(2)\quad \begin{aligned} c_y\left(S^t\right) &= \omega_y - \tilde{p}\left(S^t\right) m_y\left(S^t\right) \\ c_m\left(S^t, s_{t+1}\right) &= \omega_m + \tilde{p}\left(S^t, s_{t+1}\right) m_y\left(S^t\right) - \tilde{p}\left(S^t, s_{t+1}\right) m_m\left(S^t, s_{t+1}\right) \ \text{ for } s_{t+1} \in \{h, l\} \\ c_r\left(S^t, s_{t+1}, s_{t+2}\right) &= \tilde{p}\left(S^t, s_{t+1}, s_{t+2}\right) m_m\left(S^t, s_{t+1}\right) \ \text{ for } \left(s_{t+1}, s_{t+2}\right) \in \{h, l\}^2 \end{aligned}$$

where $\tilde{p}\left(S^t\right)$, $\tilde{p}\left(S^t, s_{t+1}\right)$ and $\tilde{p}\left(S^t, s_{t+1}, s_{t+2}\right)$ are the price of the fiat money in terms of the single good in node S^t, $\left(S^t, s_{t+1}\right)$ and $\left(S^t, s_{t+1}, s_{t+2}\right)$, respectively. Note that we do not specify the problems of the initial middle-aged and old generations in time 0 since we focus on the stochastic steady state in all models analyzed in this paper.

Agents maximize the expected utility subject to the sequential budget constraints. This yields the following three first-order conditions (FOC):

$$\frac{\tilde{p}\left(S^t\right)}{c_y\left(S^t\right)} = \frac{\pi^h \tilde{p}\left(S^t, h\right)}{c_m\left(S^t, h\right)} + \frac{\pi^l \tilde{p}\left(S^t, l\right)}{c_m\left(S^t, l\right)} \tag{3}$$

and

$$\frac{\tilde{p}\left(S^t, s_{t+1}\right)}{c_m\left(S^t, s_{t+1}\right)} = \frac{\pi^h \tilde{p}\left(S^t, s_{t+1}, h\right)}{c_r\left(S^t, s_{t+1}, h\right)} + \frac{\pi^l \tilde{p}\left(S^t, s_{t+1}, l\right)}{c_r\left(S^t, s_{t+1}, l\right)} \tag{4}$$

for $s_{t+1} \in \{h, l\}$.

The money market clearing condition requires:[6]

$$M^{s_t} = m_y\left(S^t\right) + m_m\left(S^t\right) \tag{5}$$

for $s_t \in \{h, l\}$.

Since there is a change in the total money supply between periods, the current money supply can be rewritten with the previous money supply as:

$$M^{s_t} = m_y\left(S^t\right) + m_m\left(S^t\right) = M^{s_{t-1}} + \triangle M_{t-1,t} = m_y\left(S^{t-1}\right) + m_m\left(S^{t-1}\right) + \triangle M_{t-1,t} \tag{6}$$

where $M^{s_{t-1}}$ is the total money supply in time $t-1$ under state s_{t-1}, M^{s_t} is the one in time t under state s_t, and $\triangle M_{t-1,t}$ denotes a change in the total money supply between time $t-1$ and t.

With the notations above, we define two equilibrium concepts for this monetary model: the competitive equilibrium and the recursive ME.

Definition 1. The competitive equilibrium in the monetary model is a sequence of the money holdings, consumption and asset prices in all nodes starting in time 0: $\left\{m\left(S^t\right), c\left(S^t\right), \tilde{p}\left(S^t\right)\right\}$ for $\forall S^t$ and $t \geq 0$. The competitive equilibrium requires:

- Individuals maximize their expected utility under budget constraints given the sequence of the price of the fiat money.
- The money market clears and the aggregate resource constraint holds.

The existence of a competitive equilibrium can be verified using a standard truncation method used in Balasko and Shell (1981).

Definition 2. The recursive ME is defined by time-homogeneous policy functions for money holdings, consumption and asset prices: $\left\{m_y(\chi), m_m(\chi), c_y(\chi), c_m(\chi), c_r(\chi), \tilde{p}(\chi)\right\}$ which solve the household problem and clear the money and consumption markets. $\chi = \left[m_{y,-1}, s\right] \in \hat{\Sigma} \subset \mathbb{R}^2$ represents the minimal state variables: the lagged money holdings of the young and the realization of the current aggregate uncertainty.

6 By Walras's law, we can ignore the market clearing for the consumption good.

For the recursive ME, we can re-write the equilibrium conditions as follows:

$$\frac{\tilde{p}\left(m_{y,t-1},s_t\right)}{c_y\left(m_{y,t-1},s_t\right)}=\frac{\pi^h\tilde{p}\left(m_{y,t},h\right)}{c_m\left(m_{y,t},h\right)}+\frac{\pi^l\tilde{p}\left(m_{y,t},l\right)}{c_m\left(m_{y,t},l\right)} \tag{7}$$

$$\frac{\tilde{p}\left(m_{y,t},s_{t+1}\right)}{c_m\left(m_{y,t},s_{t+1}\right)}=\frac{\pi^h\tilde{p}\left(m_{y,t+1},h\right)}{c_r\left(m_{y,t+1},h\right)}+\frac{\pi^l\tilde{p}\left(m_{y,t+1},l\right)}{c_r\left(m_{y,t+1},l\right)} \tag{8}$$

for $s_{t+1}\in\{h,l\}$ and

$$M^{s_t}=m_y\left(m_{y,t-1},s_t\right)+m_m\left(m_{y,t-1},s_t\right) \tag{9}$$

We impose the following two linear functions for prices and money holdings that the agents use to forecast future variables:

$$\begin{aligned} m_y\left(m_{y,t-1},s_t\right)&=\bar{m}^{s_t}-\gamma^{s_t}\left(m_{y,t-1}-\bar{m}^{s_t}\right)\\ q\left(m_{y,t-1},s_t\right)&=\bar{q}^{s_t}+\rho^{s_t}\left(m_{y,t-1}-\bar{m}^{s_t}\right)\end{aligned} \tag{10}$$

for $s_t\in\{h,l\}$ where $q=\frac{1}{\tilde{p}}$, i.e. the price of the consumption good in terms of the fiat money. The first and second equations in (10) are the forecast functions for the money holding of the young and the price of the single good respectively given the lagged money holding of the young and the realization of the current shock. By plugging these forecast functions into the FOCs and combining them with the market clearing condition, we can solve this monetary model and get a LIFS generating ME as long as the size of the aggregate shock measured by M^h-M^l is sufficiently small. We summarize these results in the following proposition.

Proposition 1. *In a three-period monetary OLG model with $u(c)=ln(c)$ and $\beta=1$, there will be ME for the model generated by a LIFS given by (10) for sufficiently small shocks. The slope parameters in (10) are parallel between states : $\gamma=\gamma^h=\gamma^l$ and $\rho=\rho^h=\rho^l$. A closed form solution for γ and ρ is given by:*

$$\gamma=-\frac{1}{2}-\frac{\omega_y}{\omega_m}+\sqrt{\left(\frac{1}{2}+\frac{\omega_y}{\omega_m}\right)^2+\frac{\omega_y}{\omega_m}} \tag{11}$$

and

$$\rho=\frac{2\gamma-1}{\omega_m} \tag{12}$$

Proof. *See Appendix A*

The policy functions for the money holding for the young determine the LIFS in the monetary model. The closed form solution for the Lipschitz constant of the LIFS, γ, is only affected by the endowment ratio between the young and the middle-aged, not the total amount of endowment. On the other hand, the slope of the price function is influenced by the total endowment because there is ω_m in the denominator of the closed form solution for ρ. Likewise, the state-contingent stationary points for the money holding function, $\{\bar{m}^s\}$, are independent of the total amount of endowment but dependent on the endowment ratio while the ones for the price function, $\{\bar{q}^s\}$, are affected by the total endowment. (see the proof of Proposition 1 in Appendix A.)

The intuitions behind these results are as follows. Since the logarithmic utility function is inter-temporally homothetic, optimal consumption allocations are proportional to the wealth of agents. When the total endowment changes but the endowment ratio between generations remains constant, the equilibrium price will move in proportion to the total endowment variation. Thus, the same money holding across generations can attain the optimal consumption choice. However, the money holding demands will alter if there are any variations in the endowment ratio between generations since some generations want more money and others not. Therefore, only the ratio of endowment matters for the money holding demand whereas both the total endowment and its ratio affect the price of consumption.

The slope parameters defined by (11) and (12) do not depend on the total money supply in each state and the probability distribution of shocks at least if the shock process is I.I.D. The money supply and probabilities do, however, affect the state-contingent intercept parameters. These findings imply that the rates of variations in the current money holding for the young and the price in response to variations in the lagged money holding for the young are constant whatever the size of the stochastic money supply and its probability distribution. These results accord closely with the findings in Mitra et al. (2003) that the amplitude of an exogenous shock and its probability distribution do not have an effect on the Lipschitz constant of the LIFS in a one-sector stochastic growth model with a logarithmic utility function and a Cobb-Douglas production function.

With the closed form solution above, we know the slope of the LIFS is increasing in the endowment ratio, $\frac{\omega_y}{\omega_m}$. The slope converges to 0 as the endowment ratio goes to 0 and it converges to $\frac{1}{2}$ as the endowment ratio goes to infinity. Under this range of the Lipschitz constant, the images of two parallel functions in the LIFS on the smallest open interval containing its attractor have an empty intersection set. Hence, the law of motion in the monetary model always satisfies the no-overlap property. This result is presented in the following corollary.

Corollary 1. *The Lipschitz constant of the LIFS, γ, is between 0 and $\frac{1}{2}$ for the entire set of economies in the monetary model. Hence, both contractivity and no-overlap property hold always so that there exists a unique Markov measure which is a Cantor-like distribution.*

Proof. *From the well-known results for the one-dimensional homogeneous LIFS with two states.*

A one-dimensional non-overlapped LIFS creates a unique Cantor-like invariant distribution of which support is a unique Cantor-like invariant set. The Cantor-like invariant distribution is singular with respect to the Lebesgue measure if it is generated by a LIFS because the images of the iterates of the LIFS decreases proportionally by a factor of $(1-2\gamma)$ and the limiting set has Lebesgue measure zero. Therefore, there exists a unique singular Markov measure for the entire economies in the log-linear monetary model under a small aggregate shock. Moreover, the monetary example here shows a multi-period SOLG model can generate fractal patterns in the rational expectation equilibrium since the Cantor-like invariant set or attractor has a self-affine structure. (see Appendix A to see how a one-dimensional non-overlapped generates a singular invariant distribution in detail.)

Finally, the fact that money holdings are a sufficient statistic for the history of shocks is particularly apparent here given that the Cantor set is itself homeomorphic to the space of the history of shocks from the minus infinity to the present. We denote this space of infinite histories by Σ. From the results in Woodford (1986), we also know the price and savings will be given by a unique function of the histories of shocks at least for small shocks. One can define a function, known as the coding map, $\pi(S)$ where $S \in \Sigma$, which maps infinite histories into the realizations of allocations. By the results on the contractive LIFS in Atkins et al. (2010), one can also show that the lagged asset holdings in the model will remain sufficient statistics for the history of shocks even in cases where the LIFS doesn't satisfy the no-overlap condition and the invariant measure is absolutely continuous. Their result implies that the coding map is point fiber when a LIFS is contractive and thus, there is a unique history of shocks leading to points in the attractor. This in turn tells us that the prices generated in the stochastic equilibrium can be written as $\pi_q(S_{-1}s) = q(m_{-1}, s) = q(\pi_m(S_{-1}), s)$ where π_q and π_m are coding maps for the price and money holdings respectively, the money holdings coding map is point fiber, S_{-1} is the history of shocks up to the previous period and s denote the state of the current shock. (see Appendix A for the definitions of the coding map and fiber in detail.)

3 Lucas-Tree Model

In this section, we replace the fiat money with a Lucas-tree to study an invariant Markov measure in a more general setting. Thus, agents save via accumulations of equity shares in a tree asset. We keep the assumption that aggregate uncertainty is given by two states of nature, $s \in \{h, l\}$. However, we assume here that the shock follows a first-order Markov process with a transition probability matrix $\prod = \begin{bmatrix} \pi^{hh} & \pi^{hl} \\ \pi^{lh} & \pi^{ll} \end{bmatrix}$ where $\pi^{ss'} \geq 0$ for $(s, s') \in \{h, l\}^2$ and $\pi^{sh} + \pi^{sl} = 1$ for $s \in \{h, l\}$. The tree asset in time t delivers dividends stochastically in the amount δ^{s_t} which depend only on the current shock.

Agents born in time t and history S^t are endowed with stochastic consumption goods in the amounts $\left(\omega_y^{s_t}, \omega_m^{s_{t+1}}, \omega_r^{s_{t+2}}\right)$ which also depends only on the current shock. The lifetime asset portfolio of the representative households is given by $e\left(S^t\right) = \left(e_y\left(S^t\right), \left(e_m\left(S^t, s_{t+1}\right)\right)_{s_{t+1} \in \{h,l\}}\right)$ where $e_y\left(S^t\right)$ is equity holding when young in node S^t and $e_m\left(S^t, s_{t+1}\right)$ is equity holding when middle-aged in node $\left(S^t, s_{t+1}\right)$.

Lifetime preferences are additively time-separable given by a von Neumann-Morgenstern utility function $U : \mathbb{R}_+^7 \to \mathbb{R}$. U is specified by:

(13)

$$E_t U\left(c\left(S^t\right)\right) = u\left(c_y\left(S^t\right)\right) + \beta \sum_{s_{t+1} \in \{h,l\}} \pi^{s_t s_{t+1}} \left\{ u\left(c_m\left(S^t, s_{t+1}\right)\right) + \beta \sum_{s_{t+2} \in \{h,l\}} \pi^{s_{t+1} s_{t+2}} u\left(c_r\left(S^t, s_{t+1}, s_{t+2}\right)\right) \right\}$$

where $\beta \in (0, 1]$. We assume utility functions $u(\cdot)$ satisfy regular conditions: $u'(c) > 0$, $u''(c) < 0$, and $u'(0) = +\infty$.

Sequential budget constraints are given by:

$$\begin{aligned} c_y\left(S^t\right) &= \omega_y^{s_t} - p\left(S^t\right) e_y\left(S^t\right) \\ c_m\left(S^t, s_{t+1}\right) &= \omega_m^{s_{t+1}} + \left(p\left(S^t, s_{t+1}\right) + \delta^{s_{t+1}}\right) e_y\left(S^t\right) - p\left(S^t, s_{t+1}\right) e_m\left(S^t, s_{t+1}\right) \text{ for } s_{t+1} \in \{h, l\} \\ c_r\left(S^t, s_{t+1}, s_{t+2}\right) &= \omega_r^{s_{t+2}} + \left(p\left(S^t, s_{t+1}, s_{t+2}\right) + \delta^{s_{t+2}}\right) e_m\left(S^t, s_{t+1}\right) \text{ for } \left(s_{t+1}, s_{t+2}\right) \in \{h, l\}^2 \end{aligned} \tag{14}$$

where $p\left(S^t\right)$, $p\left(S^t, s_{t+1}\right)$ and $p\left(S^t, s_{t+1}, s_{t+2}\right)$ are the price of the equity in terms of the single good in node S^t, $\left(S^t, s_{t+1}\right)$ and $\left(S^t, s_{t+1}, s_{t+2}\right)$, respectively. Agents maximize expected utility subject to the sequential budget constraints. This yields the following three first-order conditions conditional on the shock history:

$$p\left(S^t\right)u'\left(c_y\left(S^t\right)\right)=\beta\sum_{s_{t+1}\in\{h,l\}}\pi^{s_ts_{t+1}}\left(p\left(S^t,s_{t+1}\right)+\delta^{s_{t+1}}\right)u'\left(c_m\left(S^t,s_{t+1}\right)\right) \tag{15}$$

and

$$p\left(S^t,s_{t+1}\right)u'\left(c_m\left(S^t,s_{t+1}\right)\right)=\beta\sum_{s_{t+2}\in\{h,l\}}\pi^{s_{t+1}s_{t+2}}\left(p\left(S^t,s_{t+1},s_{t+2}\right)+\delta^{s_{t+2}}\right)u'\left(c_r\left(S^t,s_{t+1},s_{t+2}\right)\right) \tag{16}$$

for $s_{t+1}\in\{h,l\}$.

Under standard assumptions on preferences and endowments, Eq. (15) and (16) yield asset demand functions:

$$e_y\left(S^t\right)=e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right) \tag{17}$$

and

$$e_m\left(S^t,s_{t+1}\right)=e_m\left(\boldsymbol{P}_t^{t+2}\left(S^t\right);s_{t+1}\right) \tag{18}$$

for $s_{t+1}\in\{h,l\}$ where $\boldsymbol{P}_t^{t+2}\left(S^t\right)=\left\{p\left(S^t\right),\left(p\left(S^t,s_{t+1}\right)\right)_{s_{t+1}\in\{h,l\}},\left(p\left(S^t,s_{t+1},s_{t+2}\right)\right)_{s_{t+1},s_{t+2}\in\{h,l\}^2}\right\}$.

The demands for the equity are the functions of $\boldsymbol{P}_t^{t+2}\left(S^t\right)$. We let $\left(p\left(S^t,s_{t+1}\right)\right)_{s_{t+1}\in\{h,l\}}$ denote the set of equity prices that agents should expect over all possible paths of shock histories one period after S^t. We make $\left(p\left(S^t,s_{t+1},s_{t+2}\right)\right)_{s_{t+1},s_{t+2}\in\{h,l\}^2}$ denote the one two periods after S^t. Thus, $\boldsymbol{P}_t^{t+2}\left(S^t\right)$ is the set of all equity prices that agents born in time t and node S^t have to forecast over their lifetime. The equity demand functions are indexed by the possible history of shocks realized after the first age in the lifetime. For example, we index $e_m\left(\boldsymbol{P}_t^{t+2}\left(S^t\right);s_{t+1}\right)$ by s_{t+1}.

The market-clearing condition then requires that:[7]

$$e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right)+e_m\left(\boldsymbol{P}_{t-1}^{t+1}\left(S^{t-1}\right);s_t\right)=1 \tag{19}$$

When convenient, we will also allow the total number of assets to vary stochastically, so that the market-clearing condition becomes:

$$e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right)+e_m\left(\boldsymbol{P}_{t-1}^{t+1}\left(S^{t-1}\right);s_t\right)=\bar{a}^{s_t} \tag{20}$$

With the notations above, we define two equilibrium concepts as in the monetary model.

Definition 3. The competitive equilibrium in the Lucas-tree model is a sequence of the equity holdings, consumptions and asset prices in all nodes starting in time 0, $\left\{e\left(S^t\right),c\left(S^t\right),p\left(S^t\right)\right\}$ for $\forall S^t$ and $t\geq 0$, such that:

- Individuals maximize their expected utility under budget constraints given the sequence of the price of the equity.

[7] By Walras's law, we also ignore the market clearing for the consumption good here.

- The asset market clears and the aggregate resource constraint holds.

We can show the existence of a competitive equilibrium using a standard method as stated in the monetary model.

Definition 4. The recursive ME is defined by time-homogeneous policy functions for the equity holdings, consumptions and asset prices: $\{e_y(\chi), e_m(\chi), c_y(\chi), c_m(\chi), c_r(\chi), p(\chi)\}$ which solve the household problem and clear both the asset and consumption markets. $\chi = [e_{y,-1}, s] \in \hat{\Sigma} \subset \mathbb{R}^2$ represents the minimal state variables: the lagged equity holdings of the young and the realization of the current aggregate uncertainty.

For the recursive ME, we write the following equilibrium conditions:

$$e_y\left(\boldsymbol{P}_t^{t+2}\left(S^t\right)\right) = e_y\left(e_{y,t-1}, s_t\right) \tag{21}$$
$$1 = e_y\left(e_{y,t-1}, s_t\right) + e_m\left(\boldsymbol{P}_{t-1}^{t+1}\left(S^{t-1}\right); s_t\right)$$

where the first equation represents the optimality condition for the household problem and the second one is the asset market clearing condition.

The analysis we are undertaking hereafter is parallel to that of Woodford (1986), particularly with respect to Theorem 2 of his paper. This paper looks at a two-period lived model with multiple commodities and shows, via a functional application of the IFT that for a SOLG economy with small shocks, there is a unique equilibrium price function which depends on the infinite history of shocks to endowments. We will use a similar functional application of the IFT to show the existence of a LIFS as the ME mapping where the lagged endogenous state variables are, in fact, sufficient statistics for the shock history around the deterministic steady-states under small aggregate shocks.

For this analysis, we impose the following linear forecast functions:

$$\begin{aligned} e_y\left(e_{y,t-1}, s_t\right) &= \bar{e}^{s_t} - \gamma\left(e_{y,t-1} - \bar{e}^{s_t}\right) = G\left(e_{y,t-1}, s_t\right) \\ p\left(e_{y,t-1}, s_t\right) &= \bar{p}^{s_t} + \rho\left(e_{y,t-1} - \bar{e}^{s_t}\right) = H\left(e_{y,t-1}, s_t\right) \end{aligned} \tag{22}$$

for $s_t \in \{h, l\}$. Note that we let the affine coefficients in (22) be independent of the state of the current shock according to the result in Proposition 1.

To show that the recursive ME can be implemented by (22), we will proceed two steps. First, we show that the linear forecast functions in a three-period deterministic OLG model will be the steady-state values of the equity holding for the young and the equity price when evaluated at the steady states. This is the content of the following lemma.

Lemma 1. *In a three-period deterministic OLG model with a single long-lived asset, the linear forecast functions will be the steady-state values of e_y and p when $e_{y,-1} = \bar{e}$. , i.e. $G\left(e_{y,-1}\right) = \bar{e} - \gamma\left(e_{y,-1} - \bar{e}\right) = \bar{e}$ and $H\left(e_{y,-1}\right) = \bar{p} + \rho\left(e_{y,-1} - \bar{e}\right) = \bar{p}$ at $e_{y,-1} = \bar{e}$.*

Proof. *See Appendix A*

With the result in Lemma 1, we apply the IFT to show the existence of a LIFS in this three-period SOLG model in a neighborhood of the deterministic steady state if the aggregate shock is sufficiently small. This result is presented in the following proposition.

Proposition 2. *In a three-period SOLG model with a single long-lived asset, there will be ME generated by a LIFS given by (22) in a neighborhood of the deterministic steady state for sufficiently small shocks.*

Proof. *See Appendix A*

To prove Proposition 2, we do not specify any particular deterministic steady states in applying the IFT. Thus, there exists a LIFS around all possible deterministic steady states. Proposition 2 does not essentially need any specific assumptions on structural parameters such as preferences, endowments, dividends, total asset quantities and the Markov shock processes other than the size of the shock measured by max $\left[\left\{\left|\omega_i^h - \omega_i^l\right|\right\}_{i\in\{y,m,r\}}, \left|\delta^h - \delta^l\right|\right]$. The transversal density theorem in the proof of Proposition 2 indicates that the result in this proposition holds for almost all asset quantities. Thus, the results in Proposition 2 apply for a fairly broad set of economies in a three-period SOLG model with a single long-lived asset.

To study the continuity property of an invariant Markov measure, we first have to examine the characteristics of reduced form parameters in the linear forecast functions, γ and ρ. By the IFT results in Proposition 2, the slope parameters of a LIFS in a stochastic model are close to the corresponding ones in a deterministic model by continuity as long as the size of an exogenous shock is sufficiently small. Here, we show that the slopes of the LIFS correspond to the stable eigenvalues of the Jacobian matrix at the steady state for the underlying price dynamics in the deterministic model. We show this relationship by constructing a first-order forecast function. (as in Kehoe and Levine (1985) which restricts the forward dynamics to the stable manifold of the steady-state equilibrium.) This will then guarantee that the LIFS derived in Proposition 2 is contractive and that the resulting equilibrium stochastic process is not transient.

To demonstrate this relationship, we first note that, generically, the underlying price dynamics for the three-period deterministic OLG model in a neighborhood of the steady state will take the form:

(23) $$p_{t+1} = z\left(p_t, p_{t-1}, p_{t-2}\right)$$

This is determined from the solutions for p_{t+1} in the following equilibrium conditions by applying the IFT under the generic assumption that $\frac{\partial e_y}{\partial p_{t+1}} \neq 0$:

(24) $$e_y\left(p_{t-1}, p_t, p_{t+1}\right) + e_m\left(p_{t-2}, p_{t-1}, p_t\right) = 1$$

Letting $\hat{q}_t = \left(p_t, p_{t-1}, p_{t-2}\right)$, we can write the third-order equilibrium law of motion for the prices as the first-order vector system:

(25) $$\hat{q}_{t+1} = \hat{z}\left(\hat{q}_t\right) = \begin{bmatrix} z\left(p_t, p_{t-1}, p_{t-2}\right) \\ p_t \\ p_{t-1} \end{bmatrix}$$

The Jacobian matrix for this system at the steady-state, $\bar{q} = [\bar{p}, \bar{p}, \bar{p}]$, takes the form:

(26) $$D_{\hat{q}}\hat{z} = Z = \begin{bmatrix} z_1 & z_2 & z_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where $z_i = \frac{\partial z}{\partial p_{t+1-i}}\left|_{\hat{q}_t = \bar{q}}\right.$ for $\forall i \in \{1, 2, 3\}$. The local stability of the steady state (equivalently, the determinacy of the steady-state) depends on the eigenvalues of the matrix Z.

We obtain the first-order forecast function following Kehoe and Levine (1985) by requiring that:

(27) $$p_{t+1} = f\left(p_t\right)$$

Given such a forecast function, we let:

$$\hat{e}_y(p_t) = e_y(p_t, f(p_t), f \circ f(p_t)) \tag{28}$$

Then, we define:

$$\hat{e}'_y = \frac{\partial e_y}{\partial p_t} + \frac{\partial e_y}{\partial p_{t+1}} f' + \frac{\partial e_y}{\partial p_{t+2}} f'^2 \,|_{p_t = \bar{p}} \tag{29}$$

With these notations, we can derive the model-consistent specification of the values of $-\gamma$ and ϱ which we summarize in the following proposition.

Proposition 3. *In a three-period deterministic OLG model with a single long-lived asset, the slope parameter of the LIFS, $-\gamma$, coincides with the stable eigenvalue of Z at the corresponding steady-states. The slope parameter of the equity price function, ρ, is defined by:*

$$\varrho = \frac{-\gamma}{\hat{e}'_y} \tag{30}$$

Proof. *See Appendix A*

The main implication of Proposition 3 is that the stable eigenvalue of the Jacobian matrix for the price dynamics system determines the slope parameter of the LIFS in the neighborhood of the steady-state.

According to the conditions in Blanchard and Kahn (1980), one can characterize the local determinacy and stability of equilibrium dynamics around steady-states. In a three-period OLG model, there are three eigenvalues for Z and one predetermined price variable. Thus, if the number of the stable eigenvalues of Z, whose moduli lie inside the unit circle, is exactly one, then the equilibrium dynamics converging to the steady state is locally determinate. This locally unique equilibrium is called saddle-path stable. Locally indeterminate equilibrium arises when there are more than one stable eigenvalues for Z. In this case, there are a continuum of equilibria converging to the steady-states.

For the determinate case, there is one contractive LIFS and thus there exists a unique invariant measure around the deterministic steady states by the contraction mapping theorem as long as the size of a shock is small. On the other hand, for the indeterminate case, there can be multiple contractive LIFSs taking different stable eigenvalues as their Lipschitz constant. Since each contractive LIFS generates a unique invariant measure, this result implies there are possibly multiple invariant measures for an indeterminate steady state under a small shock.

If we don't impose the stability restriction to the LIFS, we can end up generating an equilibrium stochastic process which is transient in the sense that under repeated shocks, the forward equilibrium trajectory eventually leaves any open neighborhood of the deterministic steady-state.

If there are no stable eigenvalues, then the equilibrium is called explosive in the sense that it diverges from the deterministic steady-states unless it starts at the stationary points. In this case with potential bubbles, there can be a stable equilibrium trajectory via the backward dynamics of the model rather than the forward dynamics.

Since the stable eigenvalues of Z determine the no-overlap property of the one-dimensional LIFS as well, we now use eigenvalues information on Z to classify economies with the Lucas tree as having singular or absolutely continuous measures instead of finding a LIFS explicitly.

We also find a relationship between the structural parameters in this model and the slope of the LIFS. Since this problem is not analytically tractable, we instead use a numerical analysis under certain parametric specifications. We introduce a constant relative risk aversion (CRRA) utility function, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, where σ is the coefficient of relative risk aversion. We normalize $\omega_y + \omega_m + \delta$ to be 1. Lastly, we assume a symmetric I.I.D. shock with the two states, i.e. $\pi^h = \pi^l = 0.5$.

In this three-period Lucas-tree OLG model with the parametric assumptions above, we numerically find that there exists a unique steady state where the equity price is positive in the set of parameters of interest. For the unique steady-state, there exist one real eigenvalue inside the unit circle and two complex eigenvalues outside the unit circle. Following the Blanchard-Kahn eigenvalue condition, we can classify this unique steady state as a determinate or locally saddle-path stable equilibrium. We restrict the LIFS to the real eigenvalue of the unique steady state since we want the equilibrium mapping to generate a stable unique invariant measure.

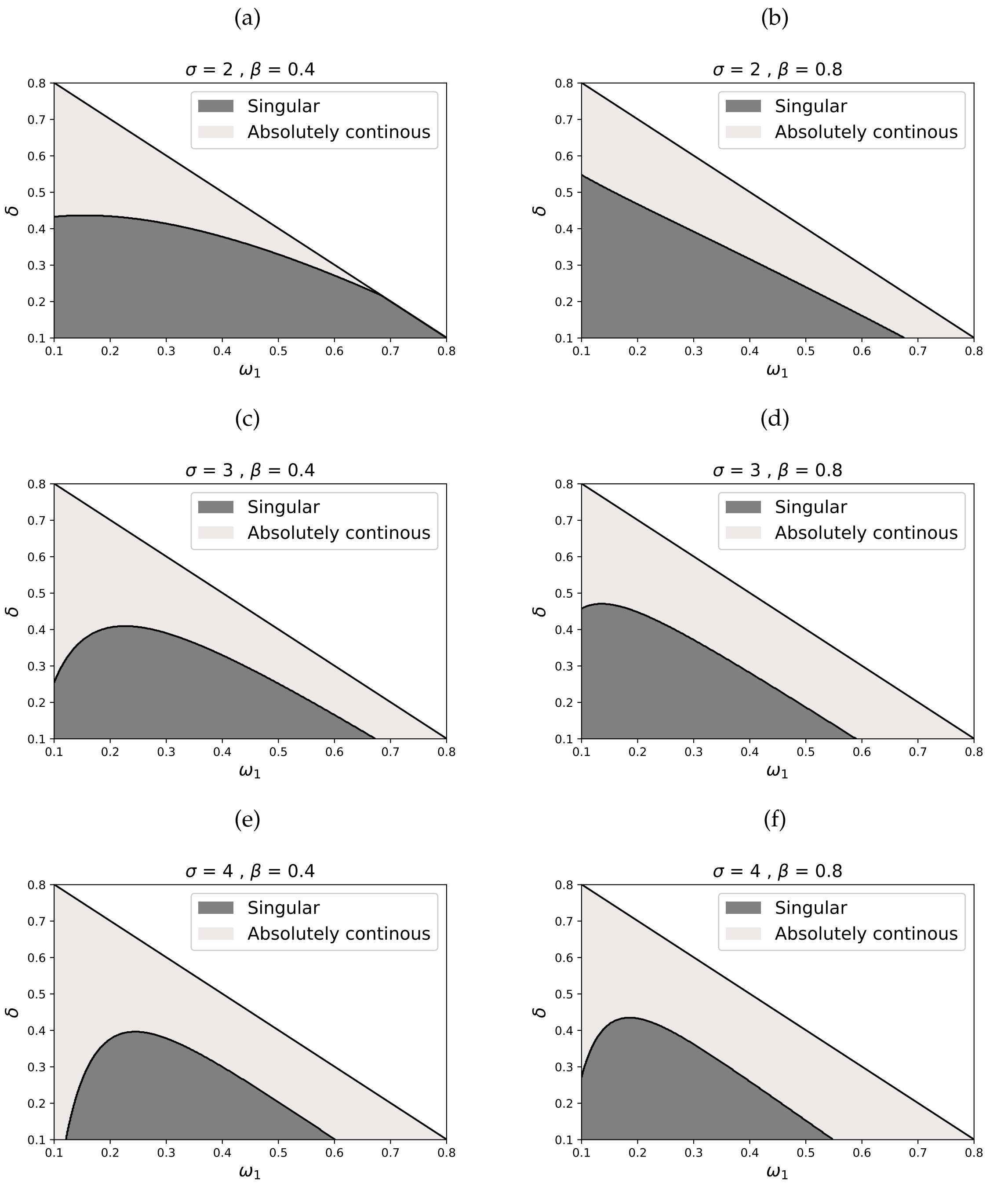
For the case where the aggregate shock has two states with equal probability, the no-overlap condition will hold if $\gamma < 1/2$. Since $-\gamma$ sets equal to the real eigenvalue of Z, we classify the set of economies where the absolute value of the real eigenvalue is less than $1/2$ as singularly continuous and the rest of economies as absolutely continuous. We should emphasize that there can exist a singular continuous measure even when the no-overlap property is not satisfied, i.e. $\gamma \in (1/2, 1)$. The set of γ values in $(1/2, 1)$ generating such essentially singular measure has Lebesgue measure zero relative to $\mathbb{R}$. These results justify classifying the economies where $\gamma \in (1/2, 1)$ as being absolutely continuous in our numerical analysis. (We refer the interested readers to Appendix A for more details about the essentially singular measure.)

Figure 1 shows the classification of economies into the two types. From all panels in the figure, we observe that γ increases as δ increases. This means that the current young's equity holding drops more sharply as the lagged equity holding rises under a higher dividend share. As δ and $e_{y,t-1}$ rise, the change in capital income for the middle-aged can be seen in $(p_t + \delta + \triangle\delta)\left(e_{y,t-1} + \triangle e_{y,t-1}\right)$ from the middle-aged budget constraint. The interaction term $\triangle\delta \cdot \triangle e_{y,t-1}$ indicates the current middle-aged takes a larger income share out of the total resource as $e_{y,t-1}$ increases under a higher δ. Thus, the middle-aged asset demand increases which makes the slope of the equity policy function steeper.

Figure 1 also implies γ increases as ω_1 rises when ω_1 is high enough. This result can be explained by a price change accompanied by a change in the endowment structure. As ω_1 increases, the aggregate demand for equity increases and thus, the equity prices will rise. Unlike the hump-shaped endowment structure where young agents might want to borrow, both the young and the middle-aged will save under the decreasing endowment structure with $\omega_1 > \omega_2 > \omega_3 = 0$. From the budget constraint for the middle-aged, an increment in the equity price can be reflected in this way: $(p_t + \triangle p_t + \delta)\left(e_{y,t-1} + \triangle e_{y,t-1}\right)$. The interaction term $\triangle p_t \cdot \triangle e_{y,t-1}$ implies the current middle-aged get a larger income share in the decreasing endowment structure due to the price hike for their previous equity holdings. Thus, γ increases, just as it does for an increase in δ.

Another interesting result is that an increase in ω_1 rather leads to a decline in γ when ω_1 is low enough. In this hump-shaped endowment structure, the young agents can borrow via short-selling the equity asset, assuming there are no borrowing constraints. As above, an increase in ω_1 still leads to a price hike as the demand for the equity by the middle-aged increases whereas the young short-sells the equity less. Under a higher equity price, the middle-aged's income share rises more sharply as their previous equity holdings increase by $\triangle e_{y,t-1}$ and thus,

FIGURE 1. The invariant Markov measure for different parameter configurations



they are willing to buy more assets. However, the current equilibrium asset holdings of the middle-aged might not grow faster, since the young agents short-sell the equity less under a higher ω_1. The higher equity price further reduces the amount of short-sales by the young since they can consume as much as they want even with a lower amount of short-sales under that price. Therefore, γ decreases even when ω_1 increases if ω_1 is low enough.

By the comparison of two panels with the same relative risk aversion but different time discount factors, we see that the area for the singular measure contracts from the right but expands to the left. For the contraction, the mechanism is quite close to the one in the case with an increase in ω_1 when ω_1 is already large because both the young and the middle-aged are willing to save more as β goes up. Thus, the equity price will rise, making the middle-aged wealthier as their previous equity holdings increase. For the expansion, we can borrow the intuition from the case with an increase in ω_1 when ω_1 sets low enough since the young will borrow less whereas the middle-aged will demand more in a higher β. The equity price will rise as well in this case, which raises the value of the previous equity holdings of the middle-aged and thus they demand more assets. However, they cannot buy more assets in equilibrium because the young short-sells the asset less when β is higher. Hence, γ goes up as β increases when ω_1 is high enough whereas it is in reverse when ω_1 is sufficiently low.

As the relative risk aversion increases, the set of parameters for the singular invariant measure shrinks from both the left and right sides. For the right side contraction, as σ goes up, both the young and the middle-aged demand the equity more because of a strong consumption smoothing motivation. Thus, the equity price will be higher relative to the case with a lower σ under the same ω_1. The wealthier middle-aged will increase its demand for the asset resulting in a higher γ. For the left side contraction, as σ goes up, the young will borrow more to smooth consumption, requiring more short-sales. This allows the middle-aged to purchase more equity in equilibrium and thus γ becomes higher.

In summary, we find that a low δ and σ expand the area for the singular measure. This result sheds light on why the unique Markov measure in the monetary model is singular since the dividend and relative risk aversion degenerate to 0 and 1, respectively. An increase in ω_1 and β converts a singular measure into an absolutely continuous measure when ω_1 is high enough. On the other hand, when ω_1 is low enough, a rise in ω_1 and β converts an absolutely continuous measure into a singular measure. The U-shape of γ in ω_1 means the singular measure is likely to arise under a hump-shaped endowment structure.

We run a simulation for a three-period SOLG model with the Lucas tree to verify the implications of the results in Figure 1. For this simulation, we approximate the policy functions with high-order Chebyshev polynomials. As the numerical analysis above, we use the CRRA utility function and set $\sigma = 2$ following Henriksen and Spear (2012). We let $\beta = 0.54 = 0.97^{60/3}$ where a one-year time-discount factor is 0.97 as commonly used in the applied macroeconomic literature and we regard one period in this model as 20 years. In the deterministic version of the model, we assume the total endowment is 1. The labor's share of the total endowment, ω, is $\frac{2}{3}$ and the ratio of labor income between the young and the middle-aged $\frac{\omega_y}{\omega_m} = \frac{3}{5}$. Hence, $\delta = \frac{1}{3}$, $\omega_y = \frac{1}{4}$ and $\omega_m = \frac{5}{12}$. The total asset quantity is assumed to be 1. We introduce shocks on the dividend and the total asset quantity which follow a symmetric I.I.D. Bernoulli process with two states. The realizations on the two stochastic parts are perfectly correlated as next: $\{\delta^l, \delta^h\} = \{0.99, 1.01\}$ and $\{a^l, a^h\} = \{0.99, 1.01\}$[8].

Proposition 2 and Figure 1 imply that the invariant Markov distribution can be generated

[8] These shocks are multiplicative. For example, the realized total dividend is $1.01 \times \frac{1}{3} = 0.3367$ in the high state.

by a LIFS under a small shock and it should be singular under these parameter values. Figure 2 graphically shows that the LIFS is indeed the equilibrium mapping under the small shock we set above. Its slope in this example is less than 1/2. Therefore, the ergodic distribution of this numerical model is a Cantor-like distribution as seen in Figure 3.

FIGURE 2. The equity policy function in the three-period SOLG economy with a single asset

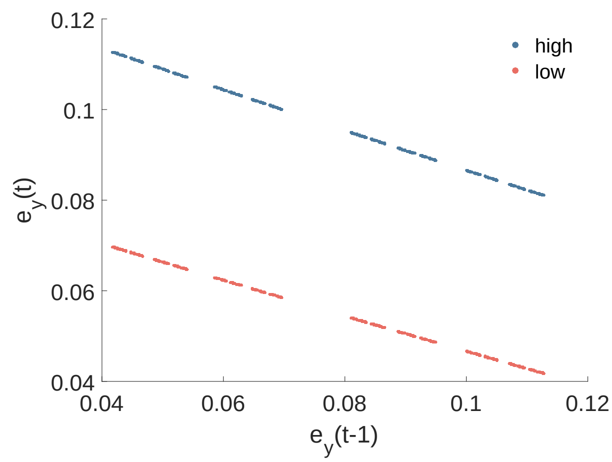
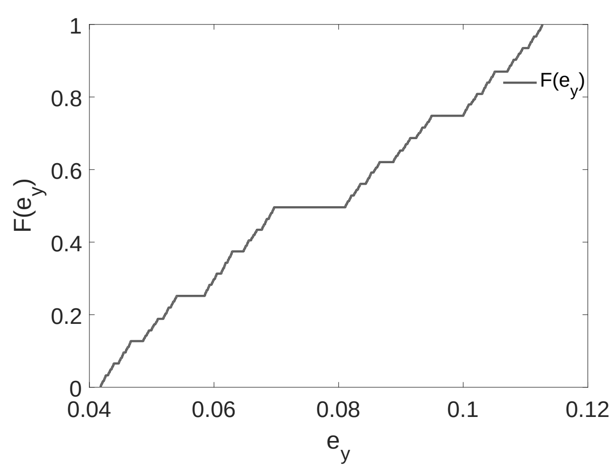


FIGURE 3. The invariant measure of the three-period SOLG economy with a single asset



Finally, it is natural to ask if there is an intuitive reason why the ME should be implementable via a LIFS. From the mathematics of the model, it is clear that the LIFS gives us enough variables to solve the model's equilibrium equations. It is also clear that the LIFS will generate the correct equilibrium relationships between the history of shocks and prices/allocations. Furthermore, any equivalent (possibly non-linear) policy functions would necessarily have to deliver the same relationships, since the results in Woodford (1986) imply that equilibrium prices as functions of the histories of shocks are locally unique for small shocks around the steady state. Thus, in this sense, the linear system is the simplest mechanism for allowing the lagged endogenous state variables to act as sufficient statistics for the infinite histories. In the simple exchange setting, the perturbations of asset holdings generated by the equilibrium will have no effect on the overall resources of the economy given the state, and so will only have an impact on welfare via allocative effects. Since we already know that the competitive equilibria in these models are not Pareto optimal (see Henriksen and Spear 2012), there would seem to be nothing to be gained by working with more general policy functions. This is indeed what comes out of the simulations of the model when we allow for flexible functional forms.

4 Generalized Model

In this section, we extend the model to be more realistic by allowing an arbitrary lifetime, heterogeneity, and a general shock process. The goal of this section is to show the existence of a simple but high dimensional LIFS and then study the invariant Markov measure in the complicated model with the help of the LIFS.

Agents live L-period lives (say $L = 70$ as in Ríos-Rull (1996)) from the youngest age 1 to the oldest age L, with $M \geq 1$ different types of agents born each period who differ in terms of preferences and endowments. We assume there is a continuum of each type of agent, and thus they take prices as given.

For generality, there are $S > 1$ states of exogenous shocks, $s_t \in \{z_1, \ldots, z_S\}$. The aggregate shock follows the first-order Markov process with a stationary Markov chain $\prod$ with support $s \in \{z_1, \ldots, z_S\}$ under which $\pi(s' \mid s) \geq 0$ for $(s, s') \in \{z_1, \ldots, z_S\}^2$ and $\sum_{s' \in \{z_1, \ldots, z_S\}} \pi(s' \mid s) = 1$ for $\forall s \in \{z_1, \ldots, z_S\}$. $\pi(s' \mid s)$ denotes the probability that state s' occurs given state s in the previous time. $\pi(S^\tau \mid S^t)$ implies the probability that the node S^τ occurs in time τ conditional on the node S^t in time t.

We maintain the assumption that there is only one asset, either money or a tree asset. Individuals consume a single good, and can save via accumulations of the single asset. The total asset quantity varies stochastically in the amount $\bar{a}^{s_t}$ for $s_t \in \{z_1, \ldots, z_S\}$. The single asset delivers dividends stochastically in the amount δ^{s_t} for $s_t \in \{z_1, \ldots, z_S\}$ if the single asset is the Lucas tree. Note that both the total asset quantity and the dividend depend only on the realization of the current shock.

The single asset delivers dividends stochastically in the amount δ^{s_t} for $s_t \in \{z_1, \ldots, z_S\}$ if the single asset is the Lucas tree.

Type-j agents born in time t are endowed with stochastic consumption goods in the amounts $\omega_j = \left\{\omega_{i,j}^{s_{t+i-1}}\right\}_{i=1}^{L}$ where $\omega_{i,j}^{s_{t+i-1}}$ is the endowment of the type-j agent in age i in time $t + i - 1$ which depends only on the current shock realization s_{t+i-1}. The consumption stream of a type-j agent born in time t and history S^t is denoted by $c_j(S^t) = \left\{\left(c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L}$ where $c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)$ is the consumption of a type-j agent in age i given a path of shocks for $(i-1)$ periods after the history S^t, $S_{t+1}^{t+i-1} = (s_{t+1}, \ldots, s_{t+i-1})$ for $i \geq 1$. When $i = 1$, we define $c_{1,j}(S^t, S_{t+1}^t) = c_{1,j}(S^t)$. Similarly, the lifetime portfolio of a type-j household is denoted by $e_j(S^t) = \left\{\left(e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L-1}$ where $e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)$ is the equity holding of a type-j agent in age i in node S^{t+i-1}. As above, we define $e_{1,j}(S^t, S_{t+1}^t) = e_{1,j}(S^t)$. Note that households do not save in the last age, L.

The lifetime expected utility for a type-j individual is given by a von Neumann-Morgenstern utility function $U : \mathbb{R}_{+}^{\left(S^{L}-1\right)/(S-1)} \to \mathbb{R}$:

$$E_t U_j\left(c_j\left(S^t\right)\right) = E_t U_j\left(\left\{\left(c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right)\right)_{S_{t+1}^{t+i-1}}\right\}_{i=1}^{L}\right) \tag{31}$$

where the certainty utility function satisfies the following regularity conditions, $U_j'(\cdot) > 0$, $U_j''(\cdot) < 0$, and $U_j'(\cdot) = +\infty$ for all arguments.

Type-j agents maximize the lifetime expected utility subject to a sequence of budget constraints as follows:

$$\begin{aligned} c_{1,j}\left(S^t\right) &= \omega_{1,j}^{s_t} - p\left(S^t\right) e_{1,j}\left(S^t\right) \\ c_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) &= \omega_{i,j}^{s_{t+i-1}} + \left(p\left(S^t, S_{t+1}^{t+i-1}\right) + \delta^{s_{t+i-1}}\right) e_{(i-1),j}\left(S^t, S_{t+1}^{t+i-2}\right) \\ &\quad - p\left(S^t, S_{t+1}^{t+i-1}\right) e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) \text{ for } \forall S_{t+1}^{t+i-1} \text{ and } i \in \{2, \ldots, L-1\} \\ c_{L,j}\left(S^t, S_{t+1}^{t+L-1}\right) &= \omega_{L,j}^{s_{t+L-1}} + \left(p\left(S^t, S_{t+1}^{t+L-1}\right) + \delta^{s_{t+L-1}}\right) e_{(L-1),j}\left(S^t, S_{t+1}^{t+L-2}\right) \text{ for } \forall S_{t+1}^{t+L-1} \end{aligned} \tag{32}$$

where $p\left(S^t, S_{t+1}^{t+i-1}\right) = p\left(S^{t+i-1}\right)$ is the price of the equity in terms of the single good in node S^{t+i-1}.

Solving the optimization problem for type-j agents yields asset demand functions:

$$e_{i,j}\left(S^t, S_{t+1}^{t+i-1}\right) = e_{i,j}\left(\boldsymbol{P}_t^{t+L-1}\left(S^t\right); S_{t+1}^{t+i-1}\right) \tag{33}$$

for $\forall S_{t+1}^{t+i-1}$ and $i \in \{1, \ldots, L-1\}$ where $\boldsymbol{P}_t^{t+L-1}\left(S^t\right) = \left\{p\left(S^t\right), \ldots, \left(p\left(S^t, S_{t+1}^{t+L-1}\right)\right)_{S_{t+1}^{t+L-1}}\right\}$. The equity demand functions are indexed by the history of shocks realized after the first period in the lifetime, S_{t+1}^{t+i-1} for $i > 1$. When $i = 1$, we define $e_{1,j}\left(S^t\right) = e_{1,j}\left(\boldsymbol{P}_t^{t+L-1}\left(S^t\right)\right)$.

For the extended model, the asset market-clearing at time t and node S^t requires that:[9]

$$\sum_{i=1}^{L-1} \sum_{j=1}^{M} e_{i,j}\left(\boldsymbol{P}_{t+1-i}^{t+L-i}\left(S^{t+1-i}\right); S_{t+2-i}^{t}\right) = \bar{a}^{s_t} \tag{34}$$

With the notations above, we define two equilibrium concepts as in models above.

Definition 5. The competitive equilibrium in the generalized model is a sequence of the asset holdings and consumptions for all types and asset prices in all nodes starting in time 0, $\left\{\left\{e_j\left(S^t\right), c_j\left(S^t\right)\right\}_j, p\left(S^t\right)\right\}$ for $\forall S^t$ and $t \geq 0$, satisfying:

- Individuals maximize their expected utility under budget constraints given the sequence of asset prices.
- The asset market clears and the aggregate resource constraint holds.

[9] By Walras's law, we ignore the market clearing for the consumption good here as well.

We can show the existence of a competitive equilibrium for this generalized model using a standard truncation method as well.

For the recursive ME that we will define below, we take all but one of the asset demand quantities as the lagged endogenous state variables via the asset market clearing condition. For specificity, we exclude the asset demand of the type-M agent in the second oldest cohort with age $L-1$ and let the set of the lagged state variables in time t:

$$\xi_{t-1} = \left\{ \left\{ e_{i,j,t-1} \right\}_{j=1,\ldots,M} \right\}_{i=1,\ldots,L-1} \setminus \left\{ e_{(L-1),M,t-1} \right\} \tag{35}$$

With an abuse of notation, we also denote ξ_{t-1} as the $((L-1)M-1)$ vector of the lagged endogenous state variables in time t.

Definition 6. The recursive ME is defined by time-homogeneous policy functions for the asset holdings, consumptions and asset prices: $\left\{ \left\{ e_j(\chi), c_j(\chi) \right\}_j, p(\chi) \right\}$ which solve the household problem and clear both the asset and consumption markets. $\chi = [\xi_{-1}, s] \in \hat{\Sigma} \subset \mathbb{R}^{(L-1)M}$ represents the minimal state variables: the lagged asset holdings distribution and the realization of the current aggregate uncertainty.

For the recursive ME, we re-write the equilibrium conditions as follows:

$$\begin{aligned} \boldsymbol{E}\left(S^t\right) &= \xi\left(\xi_{t-1}, s_t\right) \\ \bar{a}^{s_t} &= \iota^T \xi\left(\xi_{t-1}, s_t\right) + e_{(L-1),M}\left(\boldsymbol{P}_{t-L+2}^{t+1}\left(S^{t-L+2}\right); S_{t-L+3}^t \right) \end{aligned} \tag{36}$$

where ι is an $((L-1)M-1)$ vector of ones to sum the asset holdings distribution except for the demand by the type-M and age-$(L-1)$ agent and

$$\boldsymbol{E}\left(S^t\right) = \begin{bmatrix} \left\{ e_{(L-1),j}\left(\boldsymbol{P}_{t-L+2}^{t+1}\left(S^{t-L+2}\right); S_{t-L+3}^t \right) \right\}_{j=1,\ldots,M-1} \\ \left\{ e_{(L-2),j}\left(\boldsymbol{P}_{t-L+3}^{t+2}\left(S^{t-L+3}\right); S_{t-L+4}^t \right) \right\}_{j=1,\ldots,M} \\ \vdots \\ \left\{ e_{1,j}\left(\boldsymbol{P}_{t}^{t+L-1}\left(S^{t}\right) \right) \right\}_{j=1,\ldots,M} \end{bmatrix} \tag{37}$$

is the $((L-1)M-1)$ vector of demand functions for all but the second oldest type-M agents in node S^t.

As we did in Section 3, we show the existence of a LIFS as the ME mapping around the deterministic steady-states under small aggregate shocks in this generalized model via a functional application of the IFT.

For this analysis, we specify the law of motion for the endogenous state variables as:

$$\xi\left(\xi_{t-1}, s_t\right) = \bar{\xi}^{s_t} + \Gamma^{s_t}\left(\xi_{t-1} - \bar{\xi}^{s_t}\right) = G\left(\xi_{t-1}, s_t\right) \tag{38}$$

for $s_t \in \{z_1, \ldots, z_S\}$ where Γ^{s_t} is a $((L-1)M-1) \times ((L-1)M-1)$ coefficient matrix given state s_t in time t with $\rho\left(\Gamma^{s_t}\right)$ – the spectral radius of Γ^{s_t} – less than one by focusing on either determinate or indeterminate case as discussed later.

Next, we let the law of motion for prices be:

$$p\left(\xi_{t-1}, s_t\right) = \bar{p}^{s_t} + \left(\Lambda^{s_t}\right)^T\left(\xi_{t-1} - \bar{\xi}^{s_t}\right) = H\left(\xi_{t-1}, s_t\right) \tag{39}$$

for $s_t \in \{z_1, \ldots, z_S\}$ where Λ^{s_t} is a $((L-1)M-1)$ coefficient vector given state s_t in time t. Note that we allow the affine matrices, Γ^{s_t} and Λ^{s_t}, to vary over the state since we allow an arbitrary number of states which can be more than two in this generalized model.

As the analysis in the three-period SOLG model above, we first show in Lemma 2 that when the shocks are zero, the linear forecast functions satisfy the equilibrium conditions for the extended model at the steady states.

Lemma 2. *In a general deterministic OLG model with a single long-lived asset, the linear forecast functions will be the steady-state values of ξ and p when $\xi_{-1} = \bar{\xi}$, i.e. $G(\xi_{-1}) = \bar{\xi} + \Gamma\left(\xi_{-1} - \bar{\xi}\right) = \bar{\xi}$ and $H(\xi_{-1}) = \bar{p} + \Lambda^T\left(\xi_{-1} - \bar{\xi}\right) = \bar{p}$ at $\xi_{-1} = \bar{\xi}$.*

Proof. *See Appendix A*

With Lemma 2, we apply the functional version of the IFT to show the existence of a high-dimensional LIFS in this extended SOLG model in a neighborhood of the deterministic steady state under a small aggregate shock. This is the main content of the following proposition.

Proposition 4. *In a general SOLG model with a single long-lived asset, there will be ME generated by a LIFS given by (38) and (39) in a neighborhood of the deterministic steady state for sufficiently small shocks.*

Proof. *See Appendix A*

As in the three-period model, we do not specify the deterministic steady states to prove Proposition 4. Thus, there exists a LIFS around all possible deterministic steady states. Proposition 4 also does not require any specific assumptions on structural parameters other than the size of the shock.[10] Therefore, the results in Proposition 4 apply to an extensive set of economies in a general SOLG model with a single long-lived asset.

We now numerically check the assertion of Proposition 4. For this analysis, we simulate four different multi-period SOLG models with high-degree Chebyshev polynomials: four-period lived agents model, five-period lived agents model, six-period lived agents model and four-period lived and two types of agents model. These four different models have two, three, four and five lagged endogenous state variables, respectively.

In these simulations, we keep using the CRRA preference. $\sigma = 5$ and $\beta = 0.97^{60/L}$ for one period. There is an I.I.D. imperfectly correlated multiplicative shock between endowment and dividend as follows: $\{\delta^{z_1}, \omega^{z_1}\} = \{0.95, 0.95\}$, $\{\delta^{z_2}, \omega^{z_2}\} = \{1.05, 0.95\}$, $\{\delta^{z_3}, \omega^{z_3}\} = \{0.95, 1.05\}$, $\{\delta^{z_4}, \omega^{z_4}\} = \{1.05, 1.05\}$ where the probability $\pi^s = 0.25$ for $s \in \{z_1, z_2, z_3, z_4\}$. Since each model has a different number of overlapping generations and types within a cohort, we set a different distribution of endowment shares across ages and types for each model. The endowment distributions are summarized in Table 1.

It is worth noting that for the four-period lived and two types of agents model, we let the endowment ratio between types in a cohort changes as the cohort ages to reflect the kind of income mobility observed in data. Otherwise, a constant endowment ratio between types over ages gives rise to linear dependence between the asset holdings of heterogeneous agents in each generation because of the inter-temporally homothetic utility function. Under this type of preference, consumption allocations are proportional to the agents' endowment shares, and thus excess good demands as well. Since excess good supplies imply the agents' saving via holding assets, the agents' asset holdings are also in proportion to their endowment shares.

[10] The size of a shock is measured by $\max_{(s,s')}\left[\left\{\left|\omega_{i,j}^{z_s} - \omega_{i,j}^{z_{s'}}\right|\right\}_{(i,j)}, \left|\delta^{z_s} - \delta^{z_{s'}}\right|, \left|\bar{a}^{z_s} - \bar{a}^{z_{s'}}\right|\right]$.

Table 1. Distributions of endowment shares in simulated models

	(1) 4pd1type age	(2) 5pd1type age	(3) 6pd1type age	(4) 4pd2type age	(4) 4pd2type type
ω_1	0.1833	0.1487	0.1233	0.1833	$(0.5609, 0.4391)$
ω_2	0.2400	0.1753	0.1493	0.2400	$(0.5716, 0.4284)$
ω_3	0.2433	0.1847	0.1560	0.2433	$(0.5363, 0.4637)$
ω_4	0	0.1580	0.1253	0	-
ω_5	-	0	0.1127	-	-
ω_6	-	-	0	-	-
Sum	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	

- The subscript numbers in the leftmost column indicate age.
- The endowment profiles set for each model to be hump-shaped following data.
- The parentheses denote the endowment ratio between types in each age.
- The endowments across ages sum up to the labor's share of the total endowment, $\frac{2}{3}$.

Therefore, the asset holdings as lagged endogenous state variables are linearly dependent on each other under the constant endowment ratio between types across ages if preferences are described by the CRRA utility function.

To check the linearity property of the equilibrium laws of motion, we run a standard ordinary least square (OLS) regression on the simulated data from the four different models. For each model, we simulate the economy for 21,000 periods and ignore the first 1000 periods to avoid the effect of initial conditions on the results and make sure the data used in the regression analysis lies on the equilibrium attractor. Tables 2 – 5 summarize the results by regressing the equilibrium allocations on the lagged endogenous state variables for each state of nature in the four distinct models with the corresponding simulated data. Tables 2 and 3 show respectively the R-squared value and the standard deviation of the error from the regression of the equity holdings of the age-2 group on the lagged endogenous state variables.[11] Tables 4 and 5 display R^2 and $\hat{\sigma}$ respectively from the regression of the equity price on the lagged state variables. In the four tables, rows and columns represent states and models, respectively. As Proposition 4 indicates, these numerical results show that the LIFS can represent the recursive ME in a neighborhood of a deterministic steady state because R^2 and $\hat{\sigma}$ are almost 1 and 0, respectively.

Table 2. R^2 from equity holdings of age-2 group on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	1.0000	1.0000	1.0000	1.0000
$\{1.05, 0.95\}$	1.0000	1.0000	1.0000	1.0000
$\{0.95, 1.05\}$	1.0000	1.0000	1.0000	1.0000
$\{1.05, 1.05\}$	1.0000	1.0000	1.0000	1.0000

Endogenous variables are all significant with the p value of 0.001.

[11] One can get similar regression results for the equity holdings of other age groups in all models examined here.

Table 3. $\hat{\sigma}$ from equity holdings of age-2 group on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	0.0000	0.0000	0.0000	0.0000
$\{1.05, 0.95\}$	0.0001	0.0000	0.0000	0.0000
$\{0.95, 1.05\}$	0.0000	0.0000	0.0000	0.0000
$\{1.05, 1.05\}$	0.0000	0.0000	0.0000	0.0000

Endogenous variables are all significant with the p value of 0.001.

Table 4. R^2 from equity price on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	0.9997	0.9996	0.9995	0.9997
$\{1.05, 0.95\}$	0.9997	0.9996	0.9995	0.9997
$\{0.95, 1.05\}$	0.9997	0.9996	0.9995	0.9997
$\{1.05, 1.05\}$	0.9997	0.9996	0.9994	0.9997

Endogenous variables are all significant with the p value of 0.001.

Table 5. $\hat{\sigma}$ from equity price on endogenous state variables

$\{\delta, \omega\}$	(1) 4pd1type	(2) 5pd1type	(3) 6pd1type	(4) 4pd2type
$\{0.95, 0.95\}$	0.0001	0.0001	0.0001	0.0001
$\{1.05, 0.95\}$	0.0001	0.0001	0.0001	0.0001
$\{0.95, 1.05\}$	0.0001	0.0001	0.0001	0.0001
$\{1.05, 1.05\}$	0.0001	0.0001	0.0002	0.0001

Endogenous variables are all significant with the p value of 0.001.

Another purpose of the numerical analysis is to check the extent of an aggregate shock under which an economy can have a system of linear functions as the equilibrating process. We stress that structural parameters can affect the bounds of the shock support needed to achieve the linearity result. A higher σ generates a smaller R^2 fixing the size of the shock due to imposing more curvatures in the problem. In other words, a smaller size of shock is required for a model with a high σ to achieve the same level of R^2 in a model with a low σ. Similarly, a lower β generates a smaller R^2 fixing the shock size. These results imply that the extent of an aggregate shock enlarges in models with a lower σ and a higher β.

Our numerical results indicate that we can observe an acceptable linear law of motion determined by an OLS regression in all the four different models with a high relative risk aversion – $\sigma = 5$ – and a moderate size of imperfectly correlated shock. In the applied macro literature, widely used values of the relative risk aversion are between 1 and 5. Existing estimates using micro-level data support this range since they report the elasticity of intertemporal substitution lies between 0.2 and 2 which implies the relative risk aversion is between 0.5 and 5 assuming the CRRA utility function (see Havránek 2013). We indeed find many examples adopting a small aggregate shock and $\sigma \leq 5$ for calibration in the macro literature (see Krusell and Smith 1998; Storesletten et al. 2007; Hasanhodzic and Kotlikoff 2013).

Thus, one can adopt the algorithm based on the structure of the LIFS to compute equilibria in a broad set of SOLG models with an aggregate shock with a finite support.[12] The algorithm has some important advantages. First, it will generate small approximation errors under a moderate size of imperfectly correlated shock between labor and capital incomes as the linearity results hold under this type of shock. Moreover, the algorithm will make computing a very long-period lived SOLG model with heterogeneity feasible since it needs to include only the first-order term for each endogenous state variable in approximating functions, not any higher orders or interaction terms between endogenous variables.[13]

Therefore, this algorithm will resolve the issue that arises in applying the algorithm of Krusell and Smith (1998) to the long-period lived SOLG models with heterogeneity where generations are affected disparately by an imperfectly correlated shock. Their algorithm exploits low-order moments including aggregate wealth or the mean of wealth distribution over age and type to predict future prices. However, the low-order moments are not the sufficient statistics to summarize agents' choice when marginal propensities to save are distinct across generations and types due to different effects on labor and capital incomes from the imperfectly correlated shock. Thus, the forecasts based on the low-order moments deviate substantially from the actual equilibrium in this case so that the Krusell and Smith algorithm might generate relatively large approximation errors (see Krueger and Kubler (2004)).

Our findings show that very accurate forecasts might require information about the wealth for each age and type since different generations and types exhibit heterogeneous saving behaviors under the imperfectly correlated shock. Tracing the asset holdings of all ages and types seems to resurrect the curse of dimensionality issue. However, this issue can be partly avoided by the fact that the ME can be generated by the simple LIFS, and thus it is enough to perform the first-order approximation in the projection method. We stress that our results do not contradict the Krusell and Smith method, but rather complement their work by providing a possible alternative in certain circumstances under which their algorithm does not provide a good approximation.

We now analyze the condition under which there exists a homogeneous LIFS as seen in the three-period model with the Lucas-tree. This is the main argument of the following proposition.

Corollary 2. *In a general SOLG model with a single long-lived asset, there will be ME generated by a homogeneous LIFS with $\Lambda^s = \Lambda$ and $\Gamma^s = \Gamma$ for $\forall s$ in a neighborhood of the deterministic steady state for sufficiently small shocks, if $S \leq (L-1)\, M\, ((L-1)\, M - 1) + 1$.*

Proof. *See Appendix A*

[12] Note that there are many papers in the insurance, asset pricing, and social security literature which use SOLG models with a discrete shock (see Ríos-Rull 1994; Ríos-Rull 1996; Storesletten et al. 2007; Krueger and Kubler 2006 and others).

[13] To show this point, one would examine how the projection method based on the structure of the LIFS reduces the number of unknown polynomial coefficients compared to the standard one with tensor products. The standard projection method requires $(L-1)\, MS\, ((L-1)\, M - 1)^{d+1}$ unknowns where d is the degree of the approximating polynomials, whereas the LIFS structure requires only $((L-1)\, M)^2\, S$ number of unknowns. As an example, think about a SOLG model where a representative agent lives10 periods and there are two states. In this case, the LIFS algorithm and the standard one yield 162 and 9216 numbers of unknowns respectively, assuming $d = 2$.

According to Corollary 2, one can find a LIFS having constant affine matrices over states if $S \leq (L-1)\,M\,((L-1)\,M-1)+1$. In the three-period lived SOLG model with $s \in \{h, l\}$ above, $L = 3$ and $M = 1$, and thus $S = 2 < (L-1)\,M\,((L-1)\,M-1)+1 = 3$. Hence, there exists a homogeneous LIFS in this model.

Next, we study the continuity property of an invariant Markov measure in this generalized model. We first examine the characteristics of reduced form parameters in the linear law of motion. As we did for the three-period model, we restrict our attention to the case where a homogeneous LIFS exists. By the IFT results in Proposition 4 and Corollary 2, the affine matrices of a LIFS in a stochastic model are close to the corresponding ones in a deterministic model by continuity if the size of an aggregate shock is sufficiently small. In the deterministic model, we show that the affine matrices take a subset of the stable eigenvalues of the Jacobian matrix of the price dynamics at the steady state as their own.

As with the three-period model, the key to showing the dynamic consistency of the ME is the construction of a forecast function which is lower order than the full price dynamic forecast. We first obtain the full price dynamic system by applying the IFT to the asset market clearing conditions in a neighborhood of the steady state for the generalized deterministic OLG model:

$$p_{t+1} = z\left(\hat{q}_t\right) \tag{40}$$

where $\hat{q}_t = (p_t, \ldots, p_{t-2L+4})$.

Then, we can write this as the first-order vector system:

$$\hat{q}_{t+1} = \hat{z}\left(\hat{q}_t\right) = \begin{bmatrix} z\left(\hat{q}_t\right) \\ p_t \\ \vdots \\ p_{t-2L+5} \end{bmatrix} \tag{41}$$

where $\hat{z} : \mathbb{R}^{2L-3}_{++} \to \mathbb{R}^{2L-3}_{++}$.

We consider general forecasts which depend on the $(L-2)$ predetermined price variables at time $t+1$, $q_t = (p_t, \ldots, p_{t-L+3})$. We denote these forecast functions as:

$$p_{t+1} = f\left(q_t\right) \tag{42}$$

These forecasts restrict the forward dynamics of the prices to the stable manifold of the steady state as we will show later. Similar to $\hat{z}\left(\hat{q}_t\right)$, we define:

$$q_{t+1} = \hat{f}\left(q_t\right) = \begin{bmatrix} f\left(q_t\right) \\ p_t \\ \vdots \\ p_{t-L+4} \end{bmatrix} \tag{43}$$

where $\hat{f} : \mathbb{R}^{L-2}_{++} \to \mathbb{R}^{L-2}_{++}$.

Let Z and F be given by:

$$Z = \begin{bmatrix} Dz \\ J_{(2L-3)} \end{bmatrix} \text{ and } F = \begin{bmatrix} Df \\ J_{(L-2)} \end{bmatrix} \tag{44}$$

where Z is the Jacobian matrix of $\hat{z}\left(\hat{q}_t\right)$ with respect to $\hat{q}_t^T$ and F is the Jacobian matrix of $\hat{f}\left(q_t\right)$ with respect to q_t^T evaluated at the steady state. $Dz = \frac{\partial z(\hat{q}_t)}{\partial \hat{q}_t^T}$ and $Df = \frac{\partial f(q_t)}{\partial q_t^T}$ at the steady state. $J_n = \begin{bmatrix} I_{n-1} & \mathbf{0} \end{bmatrix}$ where I_{n-1} is a $(n-1)$ dimensional identity matrix and $\mathbf{0}$ is a $(n-1)$ dimensional zero column vector.

For the model without heterogeneity, let the vector of the asset holdings for ages from 1 to $(L-2)$ in time $t+1$ be:

$$\xi\left(p_{t+L}, \ldots, p_{t+1}, p_t, \ldots, p_{t-L+4}\right) = \xi\left(\hat{q}_{t+L}\right) \tag{45}$$

Given the general forecast function, one can write this asset holdings vector as:

$$\hat{\xi}\left(q_{t+1}\right) = \xi\left(f\left(q_{t+L-1}\right), \ldots, f\left(q_{t+1}\right), q_{t+1}\right) \tag{46}$$

We define the derivative of (46) with respect to q_{t+1}^T evaluated at the steady state:

$$\hat{\Xi} = \frac{\partial \hat{\xi}\left(q_{t+1}\right)}{\partial q_{t+1}^T} = \frac{\partial \xi\left(\hat{q}_{t+L}\right)}{\partial \hat{q}_{t+L}^T} \frac{\partial \hat{q}_{t+L}}{\partial q_{t+1}^T} = \Xi K \tag{47}$$

where $\hat{\Xi}$ is the derivative of $\hat{\xi}\left(q_{t+1}\right)$ with respect to q_{t+1}^T, Ξ is the derivative of $\xi\left(\hat{q}_{t+L}\right)$ with respect to $\hat{q}_{t+L}^T$, and K is the derivative of $\hat{q}_{t+L}$ with respect to q_{t+1}^T evaluated at the steady state. These matrices are given by:

$$\Xi = \begin{bmatrix} \frac{\partial \xi_1}{\partial p_{t+L}} & \frac{\partial \xi_1}{\partial p_{t+L-1}} & \cdots & \frac{\partial \xi_1}{\partial p_{t-L+4}} \\ \frac{\partial \xi_2}{\partial p_{t+L}} & \frac{\partial \xi_2}{\partial p_{t+L-1}} & \cdots & \frac{\partial \xi_2}{\partial p_{t-L+4}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi_{L-2}}{\partial p_{t+L}} & \frac{\partial \xi_{L-2}}{\partial p_{t+L-1}} & \cdots & \frac{\partial \xi_{L-2}}{\partial p_{t-L+4}} \end{bmatrix} \text{ and } K = \begin{bmatrix} DfF^{L-2} \\ \vdots \\ DfF \\ Df \\ I_{(L-2)} \end{bmatrix} \tag{48}$$

(see Kim and Spear (2017) for the derivation of the matrix K in detail.)

For the model with heterogeneity, one can define $\hat{\Xi}$ similar with the one without heterogeneity but it is a $((L-1)M-1) \times (L-2)$ matrix in the heterogeneity case since there are $((L-1)M-1)$ lagged asset holdings as the endoegenous state variables.

With these notations, we can examine a relationship between the eigenvalues of Γ and Z matrices for both the determinate and indeterminate cases in the general model. We also find a functional relationship between Γ and Λ. These results are summarized in the following proposition.

Proposition 5. *In a general deterministic OLG model with a single long-lived asset with and without cohort heterogeneity, Γ takes a subset of the stable eigenvalues for the Z matrix as a part of its eigenvalues. There is a functional relationship between Γ and Λas:*

$$\Lambda^T = DfF^{-1}\hat{\Xi}^{-1}\Gamma \tag{49}$$

for the model without heterogeneity and

$$DfF = \Lambda^T \Gamma \hat{\Xi} \tag{50}$$

for the model with heterogeneity.

Proof. *See Appendix A*

As we noted in the three-period model, we can also classify the general model as locally determinate, indeterminate and explosive cases based on the Blanchard and Kahn condition. In the determinate case, there are the same number of stable eigenvalues for Z with the predetermined variables. One can find stable eigenvalues for Z more than the number of the predetermined variables in the indeterminate case. Lastly, the explosive case lacks the stable eigenvalues and thus its number is less than the number of the predetermined variables.

For the model without heterogeneity, the eigenvalues of Γ can exactly match the stable eigenvalues of Z in the determinate case since Γ is a $(L-2) \times (L-2)$ matrix and there are $(L-2)$ stable eigenvalues for Z. Thus, there is one contractive LIFS which generates a unique invariant measure around the deterministic steady states by the contraction mapping theorem as long as the size of a shock is small.

For the indeterminate case in the model without heterogeneity, the number of the stable eigenvalues of Z is more than $(L-2)$. Γ can take a subset of the stable eigenvalues as its own. Thus, one can construct multiple contractive LIFSs. This result indicates there are possibly multiple invariant measures for an indeterminate steady state under a small shock.

Finally, in the explosive case, there are fewer than $(L-2)$ stable eigenvalues for Z. One cannot find a LIFS of which Γ has all eigenvalues inside the unit circle. In this case, one can find a stable equilibrium trajectory by working in the backward dynamics of the model, as we noted in the three-period model.

For the model with heterogeneity, the stable eigenvalues of Z determine only a subset of the eigenvalues of Γ in all cases since the cohort heterogeneity expands the dimension of Γ.

A strong sufficient condition for a high-dimensional LIFS in a general model to generate a singular invariant distribution is that the spectral radii of the affine matrices $\{\Gamma^s\}_s$ are sufficiently close to zero. Figure 2 provides an intuition behind how the no-overlap property can be satisfied when the spectral radius or the Lipschitz constant is small enough in a one-dimensional LIFS. As the spectral radius goes to zero, the linear maps in Figure 2 become flatter and the images of the LIFS are less likely to be overlapped. Analogously, for the high-dimensional LIFS, as the spectral radiuses of $\{\Gamma^s\}_s$ converge to zero, open sets containing the images of individual functions in the LIFS shrink and degenerate to $\{\bar{\xi}^s\}_s$ in the limit case. Therefore, the high-dimensional LIFS will be non-overlapped.

However, although the images of individual functions in the high-dimensional LIFS have both overlap and gaps, its attractor can be a Lebesgue measure zero set as long as there are gaps in the images of the LIFS when first iterating on an open set containing its attractor. Through iterations, the gaps fill out the open set and thus, the attractor of the LIFS will be a Lebesgue measure zero set and its invariant measure will be singular in the limit (see Jorgensen et al. 2007).

Based on this result, we study a relatively weak sufficient condition for a high-dimensional LIFS to generate a singular invariant measure. For this analysis, we assume that Γ^s is diagonalizable with linearly independent eigenvectors, $\{e_i^s\}_{i=1}^{(L-1)M-1}$, for $\forall s$. With this assumption, we can transform the high-dimensional LIFS, $G_s : \mathbb{R}^{(L-1)M-1} \to \mathbb{R}^{(L-1)M-1}$ for $s \in \{z_1, \ldots, z_S\}$, into the following system:

$$\sum_{i=1}^{(L-1)M-1} \theta_i e_i^s = \sum_{i=1}^{(L-1)M-1} \left(\theta_{-1,i} - \bar{\theta}_i^s\right) \lambda_i^s e_i^s + \sum_{i=1}^{(L-1)M-1} \bar{\theta}_i^s e_i^s \tag{51}$$

where λ_i^s is the i-th eigenvalue of Γ^s for $i \in \{1, \ldots, (L-1)\,M-1\}$.

Since we focus on the set of orthogonal eigenvectors, the coefficients of each eigenvector in both sides of (51) should be equivalent. Thus, the system in (51) reduces to:

$$\boldsymbol{\theta} = \boldsymbol{\lambda}^s \left[\boldsymbol{\theta}_{-1} - \bar{\boldsymbol{\theta}}^s\right] + \bar{\boldsymbol{\theta}}^s \tag{52}$$

where $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{(L-1)M-1} \end{bmatrix}$, $\boldsymbol{\theta}_{-1} = \begin{bmatrix} \theta_{-1,1} \\ \theta_{-1,2} \\ \vdots \\ \theta_{-1,(L-1)M-1} \end{bmatrix}$, $\bar{\boldsymbol{\theta}}^s = \begin{bmatrix} \bar{\theta}_1^s \\ \bar{\theta}_2^s \\ \vdots \\ \bar{\theta}_{(L-1)M-1}^s \end{bmatrix}$ and $\boldsymbol{\lambda}^s$ is a diagonal matrix with eigenvalues as entries. For the homogeneous LIFS, $\Gamma^s = \Gamma$ for $\forall s$ and thus, the superscript for the eigenvalues drops out, i.e. $\lambda_i^s = \lambda_i$ for $\forall s$ and $\forall i$.

Lastly, let us define S_i as the number of distinct i-th row elements in $\left\{\bar{\boldsymbol{\theta}}^s\right\}_s$. For example, $\bar{\boldsymbol{\theta}}^{z_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,$\bar{\boldsymbol{\theta}}^{z_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,$\bar{\boldsymbol{\theta}}^{z_3} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ when $S = 3$. In this case, $S_1 = 2$ - $\{0,1\}$ - and $S_2 = 3$ - $\{0,1,2\}$.

With these notations, we can define a weak sufficient condition for the system of maps $\{G_s\}_s$ to generate a singular invariant measure. We summarize this condition in the following proposition.

Proposition 6. *The attractor of a high-dimensional LIFS is a Lebesgue measure zero set and its corresponding invariant measure is singular with respect to the Lebesgue measure if the following conditions are satisfied: i) $S \leq (L-1)M-1$ or ii) if $S > (L-1)M-1$, there exists i such that $\max_s \left\{\lambda_i^s\right\} < \frac{1}{S_i}$ for the heterogeneous LIFS and there exists i such that$\lambda_i < \frac{1}{S_i}$ for the homogeneous LIFS.*

Proof. *See Appendix A*

As for the three-period model, we analyze a relationship between the structural parameters and the existence of a singular invariant measure in this general model via the weak sufficient condition in Proposition 6. The weak sufficient condition also implies that the spectral radii of the affine matrices are the single most important measure for determining singularity, as the affine coefficient is in the three-period model. Thus, we examine the effects of the structural parameters on the spectral radiuses which we can find by rather calculating the stable eigenvalue of Z in the general model due to the result in Proposition 5.

In this general model, it is hard to classify entire economies into having singular or absolutely continuous measures even when assuming the CRRA preference since a longer lifetime increases the number of endowment parameters which enlarges the dimension of the parameters to be analyzed. Instead, we select a subset of economies enough to check the relationship between the structural parameters and the spectral radii.

Via the numerical analysis of this sample, we find a similar relationship as in the three-period model. Specifically, an increase in δ increases the spectral radius. The spectral radius is small under a hump-shaped endowment profile no matter what the lifetime length is. A lower σ leads to a smaller maximal eigenvalue. Hence, a singular measure will arise in a general model with a low δ and σ under a hump-shaped endowment because the LIFS in the model can satisfy the weak sufficient condition in Proposition 6 which needs small spectral radii.

We provide an intuition for how a high δ leads to a large maximal eigenvalue as follows. One can find similar intuitions for how other parameters affect the spectral radius. Additional lagged equity holdings for other ages reduce an individual's current asset holdings more sharply under a higher δ since other ages become wealthier and can purchase more assets. On the other hand, additional lagged equity holdings for an agent rather raises her current asset holdings more sharply in this case because she now gets wealthier and can buy more assets. Thus, a high δ increases the absolute values of all elements in the affine matrices which makes the spectral radius larger.

Now, we study invariant measures in two four-period lived representative agent models with the CRRA utility functions to illustrate the results in Proposition 6. The first model has two states of nature whereas the second one has four states of nature. Both models satisfy the condition in Corollary 2 because $(L-1)\,M\,((L-1)\,M-1)+1 = 7$. Thus, there exists a homogenous LIFS in all these models. The first model has a singular invariant measure since the number of states is two which is equal to the dimension of the affine matrix. We call this measure a trivial singular measure. On the other hand, the second model has a singular measure by satisfying the eigenvalues condition in Proposition 6 given that the number of states is greater than the dimension of the affine matrix. We call this measure a non-trivial singular measure.

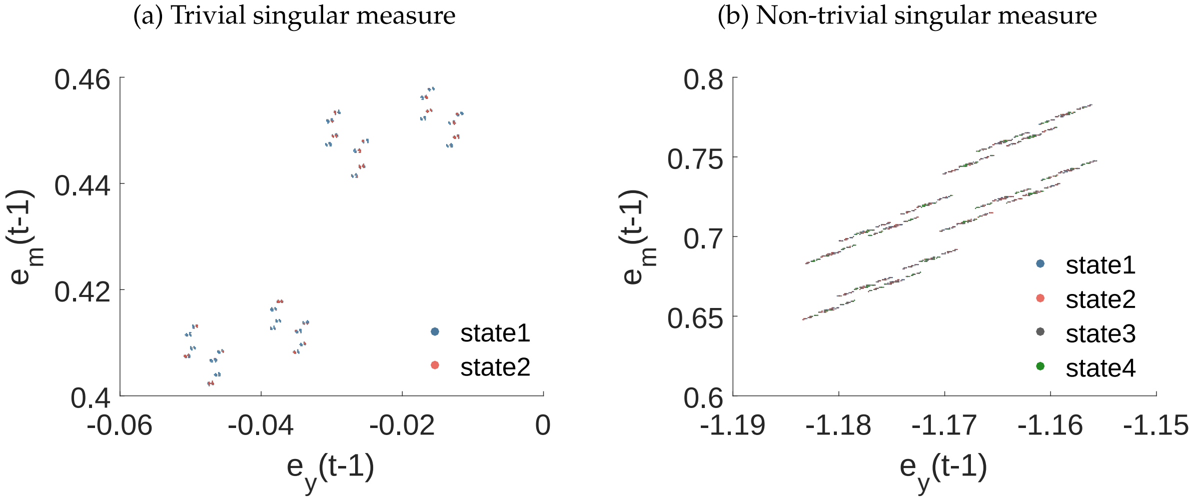
For the model generating a trivial singular measure, we set $\sigma = 2$, $\beta = 0.97^{60/4}$, $\delta = \frac{1}{3}$ and $\omega = \frac{2}{3}$. The lifetime endowment stream is denoted by $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \omega \times \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0\right\}$. For this model, we assume an I.I.D. shock with two states having an equal probability. This multiplicative shock affects the dividend and total asset quantity and its realizations in each state are given by $\{\delta^{z_1}, a^{z_1}\} = \{0.95, 0.95\}$ and $\{\delta^{z_2}, a^{z_2}\} = \{1.05, 1.05\}$.

For the model generating a non-trivial singular measure, we use the logarithmic preference, set the dividend share very small – $\delta = \frac{1}{10}$ – and make the endowment profile quite hump-shaped – $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \omega \times \left\{\frac{1}{16}, \frac{10}{16}, \frac{5}{16}, 0\right\}$ – to obtain a low spectral radius which can satisfy the eigenvalues condition in Proposition 6. We maintain the same time discount factor with the first model. In this model, we introduce an I.I.D. shock with four states having an equal probability. This shock is also multiplicative and affects the dividend and total asset quantity. Its realizations in each state are defined by $\{\delta^{z_1}, a^{z_1}\} = \{0.95, 0.95\}$, $\{\delta^{z_2}, a^{z_2}\} = \{1.05, 0.95\}$, $\{\delta^{z_3}, a^{z_3}\} = \{0.95, 1.05\}$, $\{\delta^{z_4}, a^{z_4}\} = \{1.05, 1.05\}$.

Figures 4a and 4b show the attractor set for each trivial and non-trivial singular measure in the two four period-lived SOLG models.[14] The attractor sets in these figures are sparse in the two-dimensional state space for the lagged asset holdings because the attractor of any singular measures is a Lebesgue measure zero set. As seen in the three-period model, the attractors in these models also exhibit fractal self-affinity.

[14] To draw these attractor sets, we plot the simulation data from the two different models into the state space ignoring the first 1000 periods of observations. Based on the ergodic theorem, the set of time-series data generated by equilibrium mappings represents their invariant set.

FIGURE 4. The attractor sets of two four-period lived SOLG models



5 Conclusion

This paper studies the invariant Markov distribution associated with the rational expectations equilibrium in diverse SOLG models under pure exchange: a log-linear monetary model, a three-period model with a Lucas tree asset, and a generalized model with cohort heterogeneity. We are especially interested in the singular measure in these models since the support of this distribution exhibits self-affinity which is consistent with the fractal patterns observed in macro-finance data. We prove that the local ME in all the models of this paper can be generated by a LIFS under arbitrary shock processes if the aggregate shock is sufficiently small. Therefore, we examine the continuity property of the invariant measure by characterizing the sufficient conditions on the LIFS for its distribution to be singular with respect to the Lebesgue measure.

For the log-linear monetary model, we derive a closed form solution for the slope parameters of the LIFS under a small shock. This form implies that only the endowment ratio between ages matters and the absolute value of the Lipschitz constant is less than $1/2$ for the entire parameter space. Thus, the LIFS in this model under small aggregate shocks is non-overlapped and generates a Cantor-like invariant distribution all the times. The attractor of the LIFS is a Cantor-like invariant set showing fractal self-affinity.

In the three-period model with a long-lived asset and general preferences, we demonstrate that the slope parameters of the LIFS correspond to the stable eigenvalues of the Jacobian matrix for the price dynamics evaluated at the deterministic steady state if the aggregate shock is small. Based on this relationship, we numerically examine how structural parameters affect the stable eigenvalues of the price system and under what parameters values they have an absolute value less than $1/2$. Assuming a CRR preference, we find that a decrease in risk-aversion and dividend income share decreases the absolute value of the stable eigenvalues. Thus, the LIFS will be non-overlapped and its invariant Markov distribution can be a Cantor-like distribution for a model inhabited by low risk-averse agents with a low dividend share.

We extend the results above to the general model with a longer lifespan and heterogeneity. Similar to the three-period model, we show that the affine matrices of the LIFS take, as their own eigenvalues, the stable eigenvalues of the Jacobian matrix for the price dynamics evaluated at the steady states under the small aggregate shocks. We provide a sufficient condition for the high-dimensional LIFS to generate a singular measure which is closely related to the spectral radius of the affine matrices. Thus, information on the stable eigenvalues of the price systems can allow one to infer the singularity property of the invariant measures generated by the LIFS. The sufficient condition implies that we can find a trivial singular measure if the dimension of the essential state space for policy functions is larger than or equal to the number of states of nature. To obtain a non-trivial singular measure, the number of the shock states should be larger than the dimension of the state space and the eigenvalues of the affine matrices should be small enough. We numerically find a similar relationship between the largest eigenvalue and structural parameters in this general model as in the three-period model. Thus, it requires a low risk-aversion and a low dividend share in total income to produce a non-trivial singular measure in the general model.

The existence of the LIFS implies that an algorithm based on this structure allows computing equilibria in a very long-period SOLG models with heterogeneous agents at least for small aggregate shocks, thus avoiding the well-known curse of dimensionality. Since we prove the existence of the LIFS under arbitrary shock processes, an approximation adopting the LIFS structure might work well under an imperfectly correlated shock between labor and capital incomes as an alternative to quasi-aggregation methods which might generate large approximation errors in this case.

Finally, we should note that our results have some limitations that merit further research. First, we do not provide a necessary and sufficient condition for a singular measure to arise from a linear stochastic equilibrium mapping. Studying such conditions for a high-dimensional LIFS is a very difficult problem well-known in the dynamic system literature. As next, the invariant Markov measure of interest in this paper is one generated by the models under small aggregate shocks. To examine the continuity property of the invariant measure under a large size shock, one should study a possibly non-linear IFS since the existence of a LIFS as the equilibrium mapping will not hold in this case. We deal with the possibility of multiple solutions issue given the minimal state variables by restricting to the equilibria around the deterministic steady state of interest. To avoid such restriction, one can study sunspot-like equilibria where a sunspot variable picks one of the multiple solutions. Otherwise, one should provide conditions for the equilibria given the minimal state space to be unique. We also leave the question to develop and test the algorithm based on the implication of the LIFS. Since our analytical findings hold under small aggregate shocks, it is worth analyzing the size and types of shocks where algorithms adopting the LIFS structure generate acceptable approximation errors.

A Supplementary Material

One can find the proofs of propositions, lemmas, and corollaries, background on iterated function system, and numerical algorithm for computing equilibria in the models of this paper in supplementary material related to this article. The supplementary material is uploaded online at https://doi.org/10.1184/R1/5881105.v1.

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Income, Price Dispersion and Risk Sharing

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Abstract

In this paper, we study how income-dependent prices under imperfect competition affect consumption across states and ages in a life-cycle model with the incomplete market. We show that the income-dependent prices can reduce risk sharing by biasing consumption toward the rich than the perfectly competitive economy. Thus, there exists additional consumption volatility when the good markets are imperfectly competitive along with the financial market friction. In numerical analysis, we quantify the welfare loss from the additional volatility in a parametrized version of the model and find out that it takes about 50% of the welfare loss solely from the incomplete market. We also show that the income-dependent prices over the life-cycle can depress consumption smoothing behavior and, in fact, generate correlated consumption/income profiles without other frictions. We conduct a policy analysis which confirms that fiscal policy can reduce consumption variation across ages and states. In contrast, monetary policy can mitigate consumption volatility between states, but it might generate more varying consumption profiles over the life-cycle via an intertemporal wedge.

JEL classification: C72; D43; D52; E21

Keywords: imperfect competition; income-dependent prices; incomplete market; risk-sharing; consumption smoothing

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1 Introduction

Most consumption risk-sharing papers assume that the commodity markets are competitive where all agents buy identical goods at the same prices without having market power on affecting prices. However, we can easily find some anecdotal evidence to show that prices are dispersed over the income of buyers for the same goods because of bulk discounts, bargaining power, and many other reasons. For example, investors can get higher interest rates if they save a large amount of deposit and consumers can get price discounts if they purchase more than a certain limit, as in the business models of Costco or Walmart in the U.S.. Beyond the anecdotal evidence, there is ample empirical finding that unit prices are heterogeneous across households according to their income in developing countries. With Indian villages data, Rao (2000) shows that the poor pay more for the same goods than the rich because of quantity premiums that they have to pay when buying goods in small quantities. He shows that the Gini coefficients of real incomes are indeed greater than that of nominal incomes in the Indian villages economies if considering the income-dependent price heterogeneity.

Under income-dependent heterogeneous prices, more consumptions will be allocated to the group facing relatively lower prices. Thus, the imperfectly competitive good markets can affect consumption inequality and risk-sharing. In this paper, we examine whether imperfect competition will increase consumption volatility and reduce risk sharing under the incomplete financial market compared to its counterpart, competitive economy. We also check that additional consumption volatility from the imperfect competition generates a significant welfare loss. In addition, we study government policy to improve long-run welfare when both frictions are present and quantify its welfare effects.

In the Permanent Income Hypothesis literature, it is well known that consumption and income profiles closely parallel (see Carroll and Summers 1991; Carroll 1997 and many others). There are two strands of models to explain this stylized fact. The first model is the life-cycle model with liquidity constraints and precautionary saving motives. The other one is the Keynesian model inhabited by impatient or hand-to-mouth consumers who exhaust their disposable income in every period. Under imperfectly competitive markets, overlapped cohorts can face different prices for identical goods due to the hump-shaped life-cycle income profiles. In particular, the middle-aged generation might experience lower prices than the young or retired. Hence, we also examine that a model incorporating imperfect competition can give rise to the correlated consumption/income profiles over the life-cycle without any other frictions.

To address these issues, we embed the Shapley-Shubik market game in a standard stochastic overlapping generations (SOLG) model where agents use fiat money to save and insure against endowment risk (see Shapley and Shubik 1977; Dubey and Shubik 1978 for more details about the Shapley-Shubik market game). We assume that agents live three periods: youth, middle-aged and retired periods. This three-period SOLG model remains tractable but still captures an inverse-U shape endowment structure consistent with the life-cycle income profile in the data. In the Shapley-Shubik market game, players trade with other anonymous players by making offers of consumption goods and bids of money in a centralized trading post. Each player is allocated a proportion of the aggregate commodity offer in the proportion that her bid bears to the aggregate bid. Similarly, she is assigned a share of the aggregate money bid in the proportion that her offer has to the aggregate offer of the good. The good price is defined by dividing the total money bid by the total good supply. Under this trading mechanism, strategic agents can affect the price of a good and its allocations via bidding process.[1] Therefore, fiat money in our model works not only as a store of value and a medium of exchange but also as an instrument to hedge risk and to exercise market power over the good price when there are both incomplete market and imperfect competition frictions.

In this paper, we obtain that short memory equilibria do not exist in the SOLG model with strategic interactions as Henriksen and Spear (2012) derive the same results under perfect competition. Thus, we focus on the pure strategy Markov Nash equilibrium as an equilibrium concept for numerical analysis.[2] We show that this recursive equilibrium can be generated by linear policy functions under a sufficiently small shock, similar to the findings in Kim and Spear (2017) which focuses on a perfectly competitive market.

We confirm that overlapped cohorts face different *effective marginal* prices for identical goods in our model depending on market power characterized by household income.[3] The Shapley-Shubik market game mechanism generates a feedback effect that players return a portion of their own bids by offering the good simultaneously. Players with large offers experience high return rates which imply that they need to give up less saving to increase current bid for an additional unit of the good. Therefore, those with large offers can purchase identical goods at lower marginal prices. Under the sell-all strategy, the more one has endowments, the more she will offer goods. Thus, the *effective marginal* prices are distributed according to household income.

In the deterministic version of the model, we analytically show that perfect consumption smoothing fails because the inverse-U shape endowment structure yields a price heterogeneity over one's life-cycle. Agents face the lowest marginal prices in the middle-aged period when they offer the largest amount of the good and have the highest return rate. To intertemporally optimize, they transfer their lifetime wealth to the second period of life to consume more at a cheaper price. Thus, one's consumption flow will follow her endowment stream in our model. This result indicates that a model with imperfect competition provides a distinctive explanation for the correlated consumption/income profiles on the top of the existing models with capital market constraints or impatient consumers.

The primary result of this paper is that imperfect competition increases consumption volatility and reduces risk sharing compared to its counterpart, competitive economy. The consumption risk-sharing requires a transfer from high-income group to low-income one under an idiosyncratic endowment shock. However, the rich income group faces a lower marginal price than the poor one due to a larger endowment offer in our model. This price inequality leads to skewed allocations to the high-income group relative to the competitive economy. Hence, imperfect competition above the incomplete market worsens risk sharing between agents against an idiosyncratic shock and increases their risk-exposure of consumptions. In numerical analysis, we check that within-age consumption volatility increases 4.0% for the young, 2.9% for the middle-aged and 2.6% for the old when adding imperfect competition to the incomplete

[1] The price of a good is the inverse of the price of money when there is a single perishable good. Thus, if agents have market power in the commodity market, they have market power with respect to the price of money as well.

[2] The Markov equilibrium does not allow agents to monitor the past actions of others so that they condition their decisions only on current states.

[3] We introduce here the distinction between average prices – those given by dividing aggregate bids by aggregate offers – and marginal or effective prices – which equal agents' marginal rates of substitution at any best response, because it is important for our analysis of the effects of imperfect competition.

market. This additional consumption volatility generates a complementary welfare loss. In other words, the welfare loss under imperfect competition and the incomplete market is bigger than the sum of welfare loss from each friction. As the size of shocks increases, the additional consumption volatility and its supermodular welfare loss grow as well. Our numerical analysis implies that the complementary welfare loss takes about 50% of the welfare loss solely from the incomplete market friction.

To check the robustness of the results, we work with other parameter values for time discount factor and risk aversion. Any changes in these parameters still induce the same qualitative result of the complementary welfare loss from both frictions, although they can attenuate such welfare loss quantitatively.

We find that the structure of an endowment shock matters for the size of the additional consumption volatility. Unlike an idiosyncratic endowment shock, an aggregate endowment shock generates small additional consumption volatility because endowments are positively correlated across agents, and they experience similar movements in marginal prices. Thus, there are no specific groups which can purchase identical goods at significantly lower prices. The within-age consumption volatility increases only 1.52% for the young, 0.47% for the middle-aged and 0.8% for the old under an aggregate shock in a parametrized version of the model.

We analyze two types of government policies to improve social welfare. We measure social welfare in the *ex-ante* expected utility of an individual being born into an economy. Expansionary monetary policy transfers new money to low-income group who faces a bad endowment shock. This policy reduces consumption risk between states, but it does not mitigate the welfare loss from the imperfect competition because of inability in adjusting the effective marginal prices. When both frictions are present, the monetary policy might rather decrease social welfare by generating more consumption variation over one's life-cycle via an intertemporal wedge although it decreases the within-age consumption volatility.[4] Therefore, we study fiscal policy with linear endowment tax and lump-sum transfer. We show that this policy pools the consumption risk and weakens the welfare loss from the imperfect competition by reducing the share of consumptions traded under the strategic interactions. As income tax rates rise, there are more smooth consumptions over the life-cycle and less volatility of consumption between states. Thus, the fiscal policy can improve the social welfare even if both frictions are present. From this analysis, we note that the introduction of imperfect competition generates an asymmetry between the effects of fiscal and monetary policy actions, unlike the competitive economy.

We also consider a possible extension of the model by incorporating the search activity of agents for a price discount opportunity. In this extended model, we assume that both offering goods and exercising search effort reduce effective marginal prices. Whether the imperfect competition increases or decreases consumption risk-exposure depends on which effect dominates given the inverse relationship of search effort with income from the opportunity cost of search such as wage.

We organize this paper as follows. Section 2 briefly describes the three-period SOLG model incorporating the Shapley-Shubik market game. In this section, we show the short memory equilibria do not exist. In Section 3, we examine how the imperfect competition generates non-

[4] The imperfect competition already makes non-smooth consumptions over the life-cycle. Further consumption variation over ages will reduce welfare significantly under a concave utility function.

smooth consumption allocations over the life-cycle in the deterministic version of the model. Section 4 runs diverse numerical analyses to study how the imperfect competition increases consumption volatility additionally and calculate its welfare loss. In Section 5, we check the robustness of our results. We examine two types of government policies to improve social welfare in Section 6. In Section 7, we extend the model by integrating the search behavior of agents. Finally, Section 8 concludes this paper. The appendix provides proofs and numerical algorithms.

2 Model

In this section, we develop a pure exchange overlapping generation model incorporating the Shapley-Shubik market game. Time is discrete and indexed by t from 1 to infinity. In each period, $n > 0$ agents are born and live three-periods labeled as young, middle-aged and old. There is no population growth. We assume that n agents within the same cohort are identical in both preferences and endowments and thus, we focus on symmetric Nash equilibria. In period 1, there are n middle-aged consumers who live in period 1 and 2 and n old consumers who live only in period 1.

In every period, there are a single perishable commodity and fiat money. The initial middle-aged and old carry money holdings of $m_{1,0}$ and $m_{2,0}$ respectively where $n\left(m_{1,0} + m_{2,0}\right) = nM$. We assume that the aggregate supply of money is fixed at nM from period 1 onward in the base model.

There is an exogenous shock with two states of nature, $s \in \{\alpha, \beta\}$, which affects endowments. The shock process is assumed to be independent and identically distributed (IID) across time with the state probability given by $0 < \pi^s < 1$ for $s \in \{\alpha, \beta\}$, where $\pi^\alpha + \pi^\beta = 1$. Agents' endowment profiles are given by a stochastic nonnegative vector $\boldsymbol{\omega}^s = \left(\omega_1^s, \omega_2^{s'}, \omega_3^{s''}\right)$ where ω_1^s is endowment when young in state s, $\omega_2^{s'}$ is endowment when middle-aged in state s' and $\omega_3^{s''}$ is endowment when old in state s''. We assume $\omega_i^s \gg 0$ for $\forall i$ and $\forall s$. Note that endowments in each age depend only on the current realization of an exogenous shock.

A consumption vector$\boldsymbol{c}_t^s = \left(c_{1,t}^s, c_{2,t+1}^{s'}, c_{3,t+2}^{s''}\right)$ is for the representative agent born in state s at time t. $c_{1,t}^s$ is the first-period consumption given state s in time t, $c_{2,t+1}^{s'}$ is the second-period consumption given state s' in time $t+1$, and $c_{3,t+2}^{s''}$ is the last period consumption given state s'' in time $t+2$. Hereafter, we denote $x_{i,j}^s$ as the value of x in the i-th stage of an agent's life given state s at time j.

The consumer preferences are given by a time-separable utility function $U : \mathbb{R}_+^7 \to \mathbb{R} \cup \{-\infty\}$. U is specified by:

$$(1) \qquad U\left(\boldsymbol{c}_t^s\right) = u\left(c_{1,t}^s\right) + \delta \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u\left(c_{2,t+1}^{s'}\right) + \delta^2 \sum_{(s',s'') \in \{\alpha,\beta\}^2} \pi^{s'} \pi^{s''} u\left(c_{3,t+2}^{s''}\right)$$

where the one-period utility function $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is C^3, strictly increasing, strictly concave, $u'''(c) > 0$ for $\forall c > 0$ and satisfies the Inada condition, and $0 < \delta \leq 1$ is the time discount factor.

Agents trade the single commodity with outside money in a central trading post under strategic interactions. The identical agents born in state s at time t make the same non-negative lifetime offers of goods $\boldsymbol{q}_t^s = \left(q_{1,t}^s, q_{2,t+1}^{s'}, q_{3,t+2}^{s''}\right) \gg \mathbf{0}$ to receive money and the same non-negative lifetime bids of money $\boldsymbol{b}_t^s = \left(b_{1,t}^s, b_{2,t+1}^{s'}, b_{3,t+2}^{s''}\right) \gg \mathbf{0}$ to buy goods. They also hold the same amount of money today and tomorrow to save for the next period represented by $\boldsymbol{m}_t^s = \left(m_{1,t}^s, m_{2,t+1}^{s'}\right)$. Therefore, $\left\{(\boldsymbol{q}_t^s, \boldsymbol{b}_t^s, \boldsymbol{m}_t^s) \in \mathbb{R}^{17} \mid \boldsymbol{\omega}^s \gg \mathbf{0}\right\}$ denotes the strategy set of the identical households born in state s at time t. We denote $\left(Q_{1,t}^s, Q_{2,t+1}^{s'}, Q_{3,t+2}^{s''}\right) = \left(nq_{1,t}^s, nq_{2,t+1}^{s'}, nq_{3,t+2}^{s''}\right)$ and $\left(B_{1,t}^s, B_{2,t+1}^{s'}, B_{3,t+2}^{s''}\right) = \left(nb_{1,t}^s, nb_{2,t+1}^{s'}, nb_{3,t+2}^{s''}\right)$ as the sum of identical lifetime offers and bids of n agents born in state s in period t.

The aggregate offer of good in time t is the sum of offers made in period t by all consumers born in periods $t-2$, $t-1$ and t: $Q_t^s = Q_{3,t}^s + Q_{2,t}^s + Q_{1,t}^s$. Likewise, the aggregate bid of money in time t is the sum of the bids made in period t by all consumers born in periods $t-2$, $t-1$ and t: $B_t^s = B_{3,t}^s + B_{2,t}^s + B_{1,t}^s$. Given offers and bids, the trading mechanism under the Shapley-Shubik market game allocates goods and money as follows. Each consumer is allocated a proportion of the aggregate offer of the commodity in the proportion that her bid bears to the aggregate bid. Similarly, each consumer is assigned a share of the aggregate bid of money in the proportion that her offer has to the aggregate offer.

We write the budget constraints faced by identical agents born in time t and state s:

$$\begin{aligned} b_{1,t}^s + m_{1,t}^s &= \frac{q_{1,t}^s}{Q_t^s} B_t^s \\ b_{2,t+1}^{s'} + m_{2,t+1}^{s'} - m_{1,t}^s &= \frac{q_{2,t+1}^{s'}}{Q_{t+1}^{s'}} B_{t+1}^{s'} \\ b_{3,t+2}^{s''} - m_{2,t+1}^{s'} &= \frac{q_{3,t+2}^{s''}}{Q_{t+2}^{s''}} B_{t+2}^{s''} \end{aligned} \tag{2}$$

where $\frac{B_t^s}{Q_t^s}$ can be interpreted as the price of the single good in terms of the money in period t and state s.

The identical households born in time t and state s consume goods as follows under the trading mechanism of the market game:

$$\begin{aligned} c_{1,t}^s &= \omega_1^s - q_{1,t}^s + \frac{b_{1,t}^s}{B_t^s} Q_t^s \\ c_{2,t+1}^{s'} &= \omega_2^{s'} - q_{2,t+1}^{s'} + \frac{b_{2,t+1}^{s'}}{B_{t+1}^{s'}} Q_{t+1}^{s'} \\ c_{3,t+2}^{s''} &= \omega_3^{s''} - q_{3,t+2}^{s''} + \frac{b_{3,t+2}^{s''}}{B_{t+2}^{s''}} Q_{t+2}^{s''} \end{aligned} \tag{3}$$

We are interested in the pure strategy symmetric Nash equilibria and thus, we express prices or the inverse of prices in (2) and (3) in terms of the offers and bids of other agents. For this, we introduce new variables: $B_{t,-i}^s = B_t^s - b_{i,t}^s$ and $Q_{t,-i}^s = Q_t^s - q_{i,t}^s$ for $\forall i \in \{1,2,3\}$. With these notations, we can rewrite (2):

$$
\begin{aligned}
b^{s}_{1,t} + m^{s}_{1,t} &= \left(\frac{B^{s}_{t,-1} - m^{s}_{1,t}}{Q^{s}_{t,-1}} \right) q^{s}_{1,t} \\
b^{s'}_{2,t+1} + m^{s'}_{2,t+1} - m^{s}_{1,t} &= \left(\frac{B^{s'}_{t+1,-2} - m^{s'}_{2,t+1} + m^{s}_{1,t}}{Q^{s'}_{t+1,-2}} \right) q^{s'}_{2,t+1} \\
b^{s''}_{3,t+2} - m^{s'}_{2,t+1} &= \left(\frac{B^{s''}_{t+2,-3} + m^{s'}_{2,t+1}}{Q^{s''}_{t+2,-3}} \right) q^{s''}_{3,t+2}
\end{aligned}
\tag{4}
$$

We obtain by equating the right-hand sides of (2) and (4):

$$
\frac{Q^{s}_{t}}{B^{s}_{t}} = \frac{Q^{s}_{t,-1}}{B^{s}_{t,-1} - m^{s}_{1,t}}, \; \frac{Q^{s'}_{t+1}}{B^{s'}_{t+1}} = \frac{Q^{s'}_{t+1,-2}}{B^{s'}_{t+1,-2} - m^{s'}_{2,t+1} + m^{s}_{1,t}} \text{ and } \frac{Q^{s''}_{t+2}}{B^{s''}_{t+2}} = \frac{Q^{s''}_{t+2,-3}}{B^{s''}_{t+2,-3} + m^{s'}_{2,t+1}}
\tag{5}
$$

We get by substituting (4) and (5) into (3):

$$
\begin{aligned}
c^{s}_{1,t} &= \omega^{s}_{1} - \left(\frac{Q^{s}_{t,-1}}{B^{s}_{t,-1} - m^{s}_{1,t}} \right) m^{s}_{1,t} \\
c^{s'}_{2,t+1} &= \omega^{s'}_{2} - \left(\frac{Q^{s'}_{t+1,-2}}{B^{s'}_{t+1,-2} - m^{s'}_{2,t+1} + m^{s}_{1,t}} \right) \left(m^{s'}_{2,t+1} - m^{s}_{1,t} \right) \\
c^{s''}_{3,t+2} &= \omega^{s''}_{3} + \left(\frac{Q^{s''}_{t+2,-3}}{B^{s''}_{t+2,-3} + m^{s'}_{2,t+1}} \right) m^{s'}_{2,t+1}
\end{aligned}
\tag{6}
$$

The problem of identical agents is defined as given the offers and bids of all other agents:

$$
\begin{aligned}
\max_{(\boldsymbol{q}^{s}_{t}, \boldsymbol{b}^{s}_{t}, \boldsymbol{m}^{s}_{t})} U\left(\boldsymbol{c}^{s}_{t}\right) = u & \left(\omega^{s}_{1} - \left(\frac{Q^{s}_{t,-1}}{B^{s}_{t,-1} - m^{s}_{1,t}} \right) m^{s}_{1,t} \right) \\
& + \delta \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u \left(\omega^{s'}_{2} - \left(\frac{Q^{s'}_{t+1,-2}}{B^{s'}_{t+1,-2} - m^{s'}_{2,t+1} + m^{s}_{1,t}} \right) \left(m^{s'}_{2,t+1} - m^{s}_{1,t} \right) \right) \\
& + \delta^{2} \sum_{(s',s'') \in \{\alpha,\beta\}^{2}} \pi^{s'} \pi^{s''} u \left(\omega^{s''}_{3} + \left(\frac{Q^{s''}_{t+2,-3}}{B^{s''}_{t+2,-3} + m^{s'}_{2,t+1}} \right) m^{s'}_{2,t+1} \right)
\end{aligned}
\tag{7}
$$

subject to (4)

The first order conditions with respect to $m^{s}_{1,t}$ and $m^{s'}_{2,t+1}$ are respectively:

$$
\begin{aligned}
& u'\left(c^{s}_{1,t}\right) \left(\frac{Q^{s}_{t,-1}}{B^{s}_{t,-1} - m^{s}_{1,t}} + \frac{Q^{s}_{t,-1}}{\left(B^{s}_{t,-1} - m^{s}_{1,t}\right)^{2}} m^{s}_{1,t} \right) \\
&= \delta \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u'\left(c^{s'}_{2,t+1}\right) \left(\frac{Q^{s'}_{t+1,-2}}{B^{s'}_{t+1,-2} - m^{s'}_{2,t+1} + m^{s}_{1,t}} + \frac{Q^{s'}_{t+1,-2}}{\left(B^{s'}_{t+1,-2} - m^{s'}_{2,t+1} + m^{s}_{1,t}\right)^{2}} \left(m^{s'}_{2,t+1} - m^{s}_{1,t} \right) \right)
\end{aligned}
\tag{8}
$$

and

$$
\begin{aligned}
&u'\left(c_{2,t+1}^{s'}\right)\left(\frac{Q_{t+1,-2}^{s'}}{B_{t+1,-2}^{s'}-m_{2,t+1}^{s'}+m_{1,t}^{s}}+\frac{Q_{t+1,-2}^{s'}}{\left(B_{t+1,-2}^{s'}-m_{2,t+1}^{s'}+m_{1,t}^{s}\right)^{2}}\left(m_{2,t+1}^{s'}-m_{1,t}^{s}\right)\right)\\
&=\delta\sum_{s''\in\{\alpha,\beta\}}\pi^{s''}u'\left(c_{3,t+2}^{s''}\right)\left(\frac{Q_{t+2,-3}^{s''}}{B_{t+2,-3}^{s''}+m_{2,t+1}^{s'}}-\frac{Q_{t+2,-3}^{s''}}{\left(B_{t+2,-3}^{s''}+m_{2,t+1}^{s'}\right)^{2}}m_{2,t+1}^{s'}\right)
\end{aligned}
\tag{9}
$$

Note that once the money demands are determined by (8) and (9), either the offers or bids of agents are indeterminate in (4). In other words, a household's net trade can be achieved in infinite combinations of offers and bids given other agents' offers and bids (see Peck et al. 1992). For example, when increasing offers more than one's endowments, raising bids buys back the additional offers so that (4) is satisfied. Thus, either offers or bids should be exogenously determined to avoid the indeterminacy issue. Following a standard assumption in the market game literature, we focus on the offer constrained game in which agents must offer all their endowments, so-called sell-all strategy, for the rest of the paper. As shown in Peck et al. (1992), the equilibria of the offer constrained game would be those of the unconstrained game, as long as exogeneous offers yield interior bids.

We summarize the offer constrained market game under sell-all strategy in SOLG models in the following definition.

Definition 1. The following elements describe the offer-constrained SOLG market game with sell-all strategy.

1. $3n$ players in each period
2. A finite set $\Phi = \{\alpha, \beta\}$ of states of shocks. Shocks follow an independent and identically distributed process with the state probability, $0 < \pi^s < 1$ for $s \in \{\alpha, \beta\}$ where $\pi^\alpha + \pi^\beta = 1$
3. Stochastic endowments $\boldsymbol{\omega}^s = \left(\omega_1^s, \omega_2^{s'}, \omega_3^{s''}\right)$ where $(s, s', s'') \in \{\alpha, \beta\}^3$ for all periods
4. The time-separable von Neumann-Morgenstern utility function U
5. The strategy set $\left\{(\boldsymbol{q}_t^s, \boldsymbol{b}_t^s, \boldsymbol{m}_t^s) \in \mathbb{R}^{17} \mid \boldsymbol{\omega}^s \gg \mathbf{0}\right\}$
6. Offers constrained at endowments, $\boldsymbol{q}_t^s = \boldsymbol{\omega}^s$ for $\forall t$

In this model, we are interested in symmetric Nash equilibria because we assume that identical agents within the same cohort bid and save equal amounts. We further concentrate on monetary Nash equilibria where the price of money is positive. Thus, we assume appropriate endowment profiles such that the resulting aggregate money demands are positive.

We now define a monetary symmetric Nash equilibrium in an offer constrained market game under sell-all strategy in SOLG models.

Definition 2. For the offer constrained market game under sell-all strategy in the three-period SOLG models, a monetary symmetric Nash equilibrium in pure strategies is a sequence of bids and money demands $\left(b_{2,1}^{s_1}, b_{3,1}^{s_1}, b_{3,2}^{s_2}, m_{2,1}^{s_1}, \boldsymbol{b}_t^{s_t}, \boldsymbol{m}_t^{s_t}\right)_{t=1,2,\ldots}$ such that:

1. Offers are given at endowments $\left(q_{2,1}^{s_1}, q_{3,1}^{s_1}, q_{3,2}^{s_2}, \boldsymbol{q}_t^{s_t}\right)_{t=1,2,\ldots} = \left(\omega_2^{s_1}, \omega_3^{s_1}, \omega_3^{s_2}, \boldsymbol{\omega}^{s_t}\right)_{t=1,2,\ldots}$ where s_t is the state realization in period t.

2. Every agent's strategies $\left(b_{2,1}^{s_1}, b_{3,1}^{s_1}, b_{3,2}^{s_2}, m_{2,1}^{s_1}, \boldsymbol{b}_t^{s_t}, \boldsymbol{m}_t^{s_t}\right)_{t=1,2,\ldots}$ are the best response to the actions of other agents taken as given.

3. For $\forall t$, $n\left(m_{1,t}^{s_t} + m_{2,t}^{s_t}\right) = nM$, where nM is the stock of fiat money from period 1 onward.

2.1 Short Memory and Recursive Equilibria

In this subsection, we show the non-existence of short memory monetary Nash equilibria. Then, we prove the existence of recursive Markov monetary Nash equilibria under a sufficiently small shock. We compute the latter equilibria when we implement the welfare analysis below. For these proofs, we first simplify the parentheses in the consumption good allocation rule in (6) and the first-order conditions in (8) and (9) with (5):

$$\begin{aligned} c_{1,t}^{s} &= \omega_1^{s} - \frac{Q_t^{s}}{B_t^{s}} m_{1,t}^{s} \\ c_{2,t+1}^{s'} &= \omega_2^{s'} - \frac{Q_{t+1}^{s'}}{B_{t+1}^{s'}}\left(m_{2,t+1}^{s'} - m_{1,t}^{s}\right) \\ c_{3,t+2}^{s''} &= \omega_3^{s''} + \frac{Q_{t+2}^{s''}}{B_{t+2}^{s''}} m_{2,t+1}^{s'} \end{aligned} \tag{10}$$

$$\begin{aligned} & u'\left(c_{1,t}^{s}\right)\frac{B_{t,-1}^{s}}{Q_{t,-1}^{s}}\left(\frac{Q_t^{s}}{B_t^{s}}\right)^2 \\ = \delta \sum_{s'\in\{\alpha,\beta\}} & \pi^{s'} u'\left(c_{2,t+1}^{s'}\right)\frac{B_{t+1,-2}^{s'}}{Q_{t+1,-2}^{s'}}\left(\frac{Q_{t+1}^{s'}}{B_{t+1}^{s'}}\right)^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned} & u'\left(c_{2,t+1}^{s'}\right)\frac{B_{t+1,-2}^{s'}}{Q_{t+1,-2}^{s'}}\left(\frac{Q_{t+1}^{s'}}{B_{t+1}^{s'}}\right)^2 \\ = \delta \sum_{s''\in\{\alpha,\beta\}} & \pi^{s''} u'\left(c_{3,t+2}^{s''}\right)\frac{B_{t+2,-3}^{s''}}{Q_{t+2,-3}^{s''}}\left(\frac{Q_{t+2}^{s''}}{B_{t+2}^{s''}}\right)^2 \end{aligned} \tag{12}$$

where $\frac{Q_t^{s}}{B_t^{s}}$, $\frac{Q_{t+1}^{s'}}{B_{t+1}^{s'}}$ and $\frac{Q_{t+2}^{s''}}{B_{t+2}^{s''}}$ are the inverses of good prices in time t, $t+1$ and $t+2$ respectively.

As the number of identical agents converges to infinity, $\frac{B_{t,-1}^{s}}{Q_{t,-1}^{s}}$ goes to $\frac{B_t^{s}}{Q_t^{s}}$ and it will be canceled out with $\frac{Q_t^{s}}{B_t^{s}}$ in (11). Likewise, $\frac{B_{t+1,-2}^{s'}}{Q_{t+1,-2}^{s'}}$ and $\frac{B_{t+2,-3}^{s''}}{Q_{t+2,-3}^{s''}}$ will be canceled out with $\frac{Q_{t+1}^{s'}}{B_{t+1}^{s'}}$ and $\frac{Q_{t+2}^{s''}}{B_{t+2}^{s''}}$ in (11) and (12). Thus, the first-order conditions in an imperfectly competitive economy degenerate to the usual optimality conditions in the perfect competition model in the limit.

We call $\frac{B^s_{t,-i}}{Q^s_{t,-i}}\left(\frac{Q^s_t}{B^s_t}\right)^2$ the *effective marginal price* of money for age i in time t and state s and its inverse is the *effective marginal price* of the commodity that age-i agents pay to purchase an additional good in time t and state s. The derivation of the *effective marginal price* is straightforward from the allocation rule and individual budget constraints. In time t and state s, age-i agents should bid $\frac{\triangle c^s_{i,t}(B^s_t)^2}{Q^s_t B^s_{t,-i}-\triangle c^s_{i,t}B^s_t}$ to get additional consumptions up to $\triangle c^s_{i,t}$ from the allocation rule. The budget constraint implies that the age-i agents return back their own bids partially via offers to their shares of the aggregate offer, $\frac{q^s_{i,t}}{Q^s_t}$. Thus, the age-i agents need to give up $\frac{Q^s_{t,-i}}{Q^s_t}$ amount of money for saving to increase the current bid by 1. From these results, we know that the age-i agents have to reduce $\frac{(B^s_t)^2}{Q^s_t B^s_{t,-i}-\triangle c^s_{i,t}B^s_t}\frac{Q^s_{t,-i}}{Q^s_t}$ amount of saving to raise the current consumption by 1. $\frac{(B^s_t)^2}{Q^s_t B^s_{t,-i}-\triangle c^s_{i,t}B^s_t}\frac{Q^s_{t,-i}}{Q^s_t} = \frac{Q^s_{t,-i}}{B^s_{t,-i}}\left(\frac{B^s_t}{Q^s_t}\right)^2 \frac{Q^s_t B^s_{t,-i}}{Q^s_t B^s_{t,-i}-\triangle c^s_{i,t}B^s_t}$ after some calculations. As $\triangle c^s_{i,t} \longrightarrow 0$, this expression reduces to $\frac{Q^s_{t,-i}}{B^s_{t,-i}}\left(\frac{B^s_t}{Q^s_t}\right)^2$ which we call the *effective marginal price* of the good for age i in time t and state s. We stress here that the offer-dependent return rates are the main cause of the heterogeneous *effective marginal prices* across agents.

We now show the non-existence of short memory monetary Nash equilibria in the following proposition by counting the number of variables and equations in (10), (11) and (12) assuming the short memory Nash equilibria.

Proposition 1. There are no short memory (or T-memory) monetary Nash equilibria for an open and dense set of the offer constrained market game under sell-all strategy in the three-period SOLG models.

Proof. See Appendix A.1. □

The non-existence of short memory monetary Nash equilibria implies that any rational expectations equilibrium must include lagged endogenous state variables. Here, we take the distribution of asset holdings across agents as the endogenous state variables. This type of equilibrium is generally referred to as a recursive Markov equilibrium. We state the definition of the recursive Markov equilibrium in our model below.

Definition 3. For the offer constrained market game under sell-all strategy in the three-period SOLG models, recursive Markov monetary Nash equilibria in pure strategies consist of policy functions for bids and money demands, $\{b_1(\sigma_t), b_2(\sigma_t), b_3(\sigma_t), m_1(\sigma_t), m_2(\sigma_t)\}$, which are the best responses to the actions of other agents taken as given and clear the money market. $\sigma_t = \left[m^{s_{t-1}}_{1,t-1}, s_t\right] \in \Sigma \subset R \times \{\alpha, \beta\}$ represents the minimal state variables: the lagged money holdings carried by the current middle-aged and the realization of a current shock.

Note that the lagged money holdings of the current old can be ignored in the space of the endogenous state variables by the money market clearing condition. In the Markov equilibrium, there is no trigger strategy monitoring the past actions of others so that agents condition their decisions only on current states.

We now show that the recursive Markov equilibria can be generated by a set of linear policy functions for the offer constrained market game under sell-all strategy in the three-period SOLG economies if the exogenous shock is sufficiently small.[5] For this proof, we impose the following linear forecast functions:

$$\begin{aligned} m_1\left(m_{1,t-1}^{s_{t-1}}, s_t\right) &= \bar{m}_1^{s_t} - \gamma\left(m_{1,t-1}^{s_{t-1}} - \bar{m}_1^{s_t}\right) = G\left(m_{1,t-1}^{s_{t-1}}, s_t\right) \\ b_1\left(m_{1,t-1}^{s_{t-1}}, s_t\right) &= \bar{b}_1^{s_t} + \rho\left(m_{1,t-1}^{s_{t-1}} - \bar{m}_1^{s_t}\right) = H\left(m_{1,t-1}^{s_{t-1}}, s_t\right) \end{aligned} \tag{13}$$

for $s_t \in \{\alpha, \beta\}$.

We stress that the affine coefficients in (13) are assumed to be independent of the state of the current shock following the result in Kim and Spear (2017) which shows that homogeneous linear forecast functions can generate the recursive Markov equilibria in a three-period competitive SOLG model. We do not consider the bid policy functions of the middle-aged and old because they can be represented with $m_{1,t}^{s_t}$ and $b_{1,t}^{s_t}$ from the individual budget constraint as following:

$$b_{2,t}^{s_t} = U\left(b_{1,t}^{s_t}, m_{1,t}^{s_t}, m_{1,t-1}^{s_{t-1}}\right) = \frac{q_{2,t}^{s_t}}{q_{1,t}^{s_t}} b_{1,t}^{s_t} + \left(1 + \frac{q_{2,t}^{s_t}}{q_{1,t}^{s_t}}\right) m_{1,t}^{s_t} - M + m_{1,t-1}^{s_{t-1}} \tag{14}$$

and

$$b_{3,t}^{s_t} = V\left(b_{1,t}^{s_t}, m_{1,t}^{s_t}, m_{1,t-1}^{s_{t-1}}\right) = \frac{q_{3,t}^{s_t}}{q_{1,t}^{s_t}}\left(b_{1,t}^{s_t} + m_{1,t}^{s_t}\right) + M - m_{1,t-1}^{s_{t-1}} \tag{15}$$

We now write the equilibrium conditions that the linear forecast functions should satisfy as in the following form:

(16)

$$\begin{aligned} m_1\left(m_{1,t-1}^{s_{t-1}}, s_t\right) &= m_1\left(B_{t,-1}^{s_t}, \left\{B_{t+1,-2}^{s_{t+1}}\right\}_{s_{t+1}\in\{\alpha,\beta\}}, \left\{B_{t+2,-3}^{s_{t+2}}\right\}_{(s_{t+1},s_{t+2})\in\{\alpha,\beta\}^2}\right) \\ M &= m_1\left(m_{1,t-1}^{s_{t-1}}, s_t\right) + m_2\left(B_{t-1,-1}^{s_{t-1}}, \left\{B_{t,-2}^{s_t}\right\}_{s_t\in\{\alpha,\beta\}}, \left\{B_{t+1,-3}^{s_{t+1}}\right\}_{(s_t,s_{t+1})\in\{\alpha,\beta\}^2}\right) \end{aligned}$$

where the bid arguments inside the big parentheses in (16) represent the set of total bids net of an agent's bid in each age. One should consider such bids over all possible paths of shocks after her birth. The bid arguements, $B_{t,-i}^{s_t}$ for $\forall t, s_t$ and i , are implicitely expressed with $b_1\left(m_{1,t-1}^{s_{t-1}}, s_t\right)$, $U\left(b_{1,t}^{s_t}, m_{1,t}^{s_t}, m_{1,t-1}^{s_{t-1}}\right)$ and $V\left(b_{1,t}^{s_t}, m_{1,t}^{s_t}, m_{1,t-1}^{s_{t-1}}\right)$.

The first equation in (16) indicates that the linear forecast functions should equal the optimal money holding. The second equation in (16) implies that the linear forecast functions should clear the money market. One can derive optimal bids from the budget constraint if optimal money demands are known. Thus, it is enough to show the consistency of these two equilibrium conditions with the linear forecast functions in (13) to conclude that those functions generate the recursive Markov equilibria.

We proceed two steps to show that the linear forecast functions satisfy (16) under a sufficiently small shock. First, we check that such forecast functions can be a solution to the equilibrium system at the deterministic steady states. This is the content of the following lemma.

[5] We measure the size of the exogenous shock by $\max\left\{\left|\omega_i^{\alpha} - \omega_i^{\beta}\right|\right\}_{i\in\{1,2,3\}}$.

Lemma 1. The linear forecast functions in (13) satisfy (16) at the steady-states in the deterministic case of the offer constrained OLG market game under the sell-all strategy.

Proof. See Appendix A.2 □

With the result in Lemma 1, we apply the implicit function theorem (IFT) to show the existence of linear forecast functions which generate the recursive Markov equilibrium in a neighborhood of the deterministic steady states if the exogenous shock is sufficiently small. We present this result in the following proposition.

Proposition 2. There exist recursive Markov Nash equilibria generated by linear forecast functions given by (13) in a neighborhood of the deterministic steady states for the offer constrained market game under sell-all strategy in a three-period SOLG model with sufficiently small shocks.

Proof. See Appendix A.3 □

To prove Proposition 2, we do not specify particular deterministic steady states and assumptions on structural parameters such as preferences, endowments, total money quantities and shock processes. Thus, there exist linear forecast functions around all possible deterministic steady states for a fairly broad set of economies as long as the shock is small enough.

One of the limitations of the results in Proposition 2 is that the size of the shock should be sufficiently small. However, we execute welfare analysis later on the recursive equilibrium for not only small but also moderate sizes of shocks. Kubler and Polemarchakis (2004) examine the existence of (stationary) Markov ϵ- equilibria in OLG economies which clear the market and are within ϵ-bound of true utility maximizing choices. They show that such Markov ϵ-equilibria exist for all $\epsilon > 0$ and converge to competitive equilibria as $\epsilon \longrightarrow 0$. They stress out that only Markov ϵ- equilibria can be computed in numerical work because of rounding and truncation errors. With the justification of Kubler and Polemarchakis (2004), we focus on computing Markov ϵ- equilibria in the welfare analysis. We numerically check that such Markov ϵ- equilibria can be found in our model with strategic interactions under any sizes of shocks for any ϵ-error bounds.

3 Correlated consumption/income profiles

We analytically show that perfect consumption smoothing fails over the life-cycle and one's consumptions rather follow her endowment stream in the model with strategic interactions. To show this argument more clearly, we focus on the deterministic version of the model above with $\delta = 1$.

Under these restrictions, we observe equal lifetime consumptions at the deterministic steady-states if the good markets are competitive because overlapped cohorts face the same marginal prices for identical goods. However, agents can experience a price heterogeneity across the life-cycle according to their income as seen in (11) and (12). Thus, agents might consume more in a certain age which leads to unequal lifetime consumptions. This is the main content of the following proposition.

Proposition 3. Perfectly smoothed consumptions cannot be the stationary allocations in the deterministic OLG economy with $\delta = 1$ if there are strategic interactions. Instead, the steady-state allocations are characterized by more consumptions in ages with higher endowments.

Proof. See Appendix A.4. □

The Shapley-Shubik market game mechanism generates a feedback effect that agents return a portion of their own bids by offering the good at the same time. The rate of return equals one's share of the aggregate offer. Thus, there are higher return rates for the ages with larger endowments under the sell-all strategy. A high return rate allows agents to give up less saving to increase current bid for an additional unit of the good. This result implies that households can purchase identical goods at lower marginal prices in the periods with larger endowments. The intertemporally optimizing consumers transfer their wealth to the ages with lower marginal prices to increase consumption. Hence, we observe that consumption growth closely parallels income growth at the stationary equilibria.

The correlated consumption/income profiles lead to the failure of perfect consumption smoothing in the model with strategic interactions. Even under equal lifetime incomes, the perfectly smoothed consumptions cannot be the outcome of the steady states because such allocations violate the money market clearing condition as noted in the proof of Proposition 3.

To be consistent with data, we now restrict our attention to inverse-U shape lifetime income profiles. We assume that agents retire in the old period and receive zero endowments. The result in Proposition 3 indicates that the lifetime consumptions will also show a hump-shaped distribution. In the following corollary, we summarize this finding and other interesting features for the stationary allocations under the endowment structures of interest.

Corollary 1. In the deterministic OLG economy with strategic interactions, the stationary consumption allocations are hump-shaped if the endowment structures are inverse-U shaped. Assuming $\delta = 1$ and $\omega_3 = 0$, the steady-states allocations satisfy the relationship that $c_1 \gtreqless \frac{1}{3}(\omega_1 + \omega_2)$, $c_2 > \frac{1}{3}(\omega_1 + \omega_2)$, and $c_3 < \frac{1}{3}(\omega_1 + \omega_2)$. $c_1 > \frac{1}{3}(\omega_1 + \omega_2)$ if ω_1 is smaller than but close enough to $\frac{\Omega}{2n}$. $c_1 < \frac{1}{3}(\omega_1 + \omega_2)$ if ω_1 is larger than but sufficiently close to $\frac{\Omega}{3n}$.

Proof. See Appendix A.5. □

This result implies that a model with imperfect competition provides a distinctive explanation for the correlated consumption/income profiles even with patient consumers and without capital market imperfections such as liquidity constraints and precautionary saving motives. It is worth recognizing how the income-dependent prices produce a welfare loss in the OLG market game economy. In the competitive economy, prices adjust to balance the purchasing power of overlapped generations so that they face the same marginal prices for identical goods. The income-neutral prices allow the perfect consumption smoothing. On the other hand, the price heterogeneity generates variations in the life-cycle consumptions which leads to a welfare loss in the imperfectly competitive economy.

4 Consumption Volatility and Risk Sharing

In this section, we examine how the exposure of consumption to endowment risk strengthens when adding the imperfect competition on the top of the financial market incompleteness.

To be specific, we quantitatively study how and how much strategic interactions reduce risk-sharing among overlapped generations by increasing consumption volatility between states. We also calculate the welfare loss from the additional consumption volatility compared to the benchmark, deterministic competitive economy.

We first consider an idiosyncratic shock across generations where the young and middle-aged receive stochastic endowments in the opposite direction. We maintain the assumption that $\delta = 1$. These assumptions will make the welfare analysis clearer because the benchmark allocations in the frictionless economy will be the perfectly smoothed consumptions over ages and states and one can identify the welfare loss by comparing the certainty equivalent consumptions in the frictional economies with the benchmark ones. We also consider an aggregate shock and a standard time discount factor from the macroeconomics literature in the robustness check section.

For the numerical analysis, we assume the constant relative risk aversion utility function, $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma = 2$. We assume that there is one representative agent born in each generation – $n = 1$ – to highlight the effect of strategic interactions on the risk-sharing. The total money quantity, M, is normalized to be 1. The total endowment is also normalized to be 1 and the shares of the endowments are assumed to be $\frac{3}{8}$, $\frac{5}{8}$ and 0 for the young, the middle-aged and the retired in the deterministic economy: $\{\omega_1, \omega_2, \omega_3\} = \left\{\frac{3}{8}, \frac{5}{8}, 0\right\}$.

The idiosyncratic shock follows a simple IID Bernoulli process with $\pi^\alpha = 0.5$ and $\pi^\beta = 0.5$. In our base case, we consider the stochastic endowment streams given by $\left\{\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha\right\} = \left\{\frac{5}{16}, \frac{11}{16}, 0\right\}$ and $\left\{\omega_1^\beta, \omega_2^\beta, \omega_3^\beta\right\} = \left\{\frac{7}{16}, \frac{9}{16}, 0\right\}$ where ω_i^s is the endowment of age i in state s. Thus, the baseline idiosyncratic shock generates the standard deviation of $\frac{1}{16}$ for the stochastic endowments. The young cohort is advantaged in state β whereas the middle-aged cohort is advantaged in state α because they receive more endowments in such states than in the deterministic economy.

Given the parameters values above, we compute the recursive Markov Nash equilibria for a welfare analysis in the three types of economies with only the incomplete market, only the imperfect competition, and both frictions.[6] We simulate the three economies for 21,000 periods and ignore the first 1000 periods to avoid the effect of initial conditions on the results. Time averages and cross-sectional averages will be the same because of the ergodicity in the recursive Markov equilibria.[7] This property allows us to calculate the ex-ante expected utility of the stochastic steady-states with the simulation data. Then, we find certainty equivalent (CE) consumptions to achieve the ex-ante expected utility in the frictional economies. The welfare loss from each friction is measured with the difference between its CE consumption and the benchmark perfect smoothing consumption as the percentage of the benchmark allocation. One can view the welfare loss measure or the CE consumption loss rate as a relative risk-premium. By comparing the welfare losses in the three economies, we can identify the complementary welfare loss from the additional consumption volatility via interactions between the incomplete market and imperfect competition. We define such supermodular welfare loss as the welfare loss in the economy with both frictions net of the sum of the welfare losses in the other two economies with each friction.

[6] We describe the algorithm computing the recursive equilibria in Appendix B.

[7] The proof of this property can be obtained by a straightforward generalization of the technique introduced by Duffie et al. (1994) for competitive Markov equilibria.

Table 1: Welfare analysis under the base economy

Benchmark consumptions without frictions				
Age	State α	State β	Mean	CV (%)
Young	0.3333	0.3333	0.3333	0.00%
Middle-aged	0.3333	0.3333	0.3333	0.00%
Old	0.3333	0.3333	0.3333	0.00%
Equilibrium consumptions only with imperfect competition				
Age	State α	State β	Mean	CV (%)
Young	0.3286	0.3286	0.3286	0.00%
Middle-aged	0.4007	0.4007	0.4007	0.00%
Old	0.2707	0.2707	0.2707	0.00%
Equilibrium consumptions only with incomplete market				
Age	State α	State β	Mean	CV (%)
Young	0.3146	0.3455	0.3301	4.68%
Middle-aged	0.3639	0.3064	0.3352	8.58%
Old	0.3214	0.3480	0.3347	3.98%
Equilibrium consumptions with both frictions				
Age	State α	State β	Mean	CV (%)
Young	0.2975	0.3544	0.3260	8.72%
Middle-aged	0.4493	0.3567	0.4030	11.48%
Old	0.2532	0.2889	0.2711	6.58%
Welfare analysis				
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3249	0.3310	0.3214
CE consumption as (%) of benchmark	100%	97.48%	99.29%	96.42%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

Table 1 summarizes the welfare analysis results for the three types of economy mentioned above relative to the benchmark economy under the baseline parameters values. In the second and third columns, we display the mean of consumption simulation data for each age conditional on the state of the current shock. The fourth column represents the average consumption level of each cohort without conditioning on the current shock state. To express the consumption volatility faced by each generation, we use the coefficient of variation (CV) in percentage. CV is calculated by dividing the standard deviation of unconditional consumptions with its mean and then multiplying it with 100%. We write the CV values of each cohort in the last column. In the last panel, we address CE consumptions and their percentages out of the benchmark CE consumption for all economies to evaluate the welfare loss from each friction.

In the economy only with imperfect competition, the lifetime consumption profile exhibits a hump-shaped structure following the endowment profile which is consistent with the implication in Corollary 1. The middle-aged face the lowest marginal prices because of their highest offers. Thus, the young agents save more and the middle-aged save less than what they do in the competitive economy to increase consumption in a period with lower prices. This disparate saving behavior leads to a consumption variation across ages observed in the inverse U-shaped consumption profile. Such a deviation from the perfectly smoothed consumptions generates a welfare loss of 2.52% in the economy with the strategic interaction.

There is a very little variation on the lifetime consumption allocations in the economy only with the incomplete market since overlapped generations face the same prices for the identical goods. However, there exists a consumption volatility between states because the fiat money alone cannot perfectly share the birth-date and successive exogenous risk. The consumptions of the young are higher in state β than in state α whereas the middle-aged consume more in state α than in state β because the young are advantaged in state β and the middle-aged are advantaged in state α given the shock structure. Our numerical analysis tells a welfare loss of 0.71% from the consumption volatility arising under the financial market incompleteness.

In the economy with both frictions, the equilibrium consumptions are volatile across both ages and states. One of the interesting results in this economy is that the consumption volatility between states is much larger for all cohorts than that of the economy with only the incomplete market friction. Specifically, the within-age consumption volatility increases by 4.04% for the young, 2.9% for the middle-aged, and 2.6% for the old. This additional consumption volatility generates a supermodular welfare loss of 0.35% since the total welfare loss is 3.58% in the economy with both frictions whereas the sum of welfare losses from each friction is 3.23%.

The additional consumption volatility implies that the imperfect competition reduces the risk sharing between states among overlapped generations on the top of the incomplete market. Indeed, the state-dependent marginal prices allocate more consumptions to the advantaged generations – who receive a good shock – under the imperfect competition because the market game mechanism assigns lower marginal prices to those offering more and thus, the advantaged cohort can consume at cheaper prices than the disadvantaged one. This biased allocation conflicts with the requirements for perfect risk-sharing which necessitates a transfer from the advantaged cohort to the disadvantaged one. Thus, the imperfect competition increases the risk-exposure of consumption to the endowment shock.

Now, we consider two other shocks with standard deviations (SD) $\frac{2}{16}$ and $\frac{3}{16}$ to check how the size of shocks affects the complementary welfare loss. In the former shock, the stochastic endowment profile is described by $\{\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha\} = \left\{\frac{4}{16}, \frac{12}{16}, 0\right\}$ and $\left\{\omega_1^\beta, \omega_2^\beta, \omega_3^\beta\right\} = \{\frac{8}{16}, \frac{8}{16}, 0\}$. For the latter shock, it is given by $\{\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha\} = \left\{\frac{3}{16}, \frac{13}{16}, 0\right\}$ and $\left\{\omega_1^\beta, \omega_2^\beta, \omega_3^\beta\right\} = \{\frac{9}{16}, \frac{7}{16}, 0\}$.

Table 2: Welfare analysis under an idiosyncratic shock with SD $\frac{2}{16}$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with incomplete market				
Young	0.2884	0.3537	0.3211	10.17%
Middle-aged	0.3970	0.2834	0.3402	16.70%
Old	0.3147	0.3629	0.3388	7.11%
Equilibrium consumptions with both friction				
Young	0.2612	0.3764	0.3188	18.07%
Middle-aged	0.5034	0.3167	0.4101	22.77%
Old	0.2354	0.3069	0.2712	13.18%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3249	0.3237	0.3104
CE consumption as (%) of benchmark	100%	97.48%	97.12%	93.13%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

Comparing Tables 1, 2, and 3, one can notice that the welfare loss from the incomplete market increases as the size of shock increases: 2.88% for the shock with SD $\frac{2}{16}$ and 6.73% for the shock with SD $\frac{3}{16}$. This is straightforward because larger shocks raise the consumption volatility within ages. Here, an interesting finding is that the welfare loss from the financial market incompleteness grows roughly by four or ten times although we amplify the size of shock by two or three times from the base case due to the concavity of the utility function.

We also stress that the larger the size of the shock, the higher the welfare loss from the complementarity effect between imperfect competition and incomplete market: 1.47% for the shock with SD $\frac{2}{16}$ and 3.19% for the shock with SD $\frac{3}{16}$. As the size of shock rises, the gap between effective prices across generations expands. Thus, the favored cohort can buy goods at much cheaper prices under a larger shock than a smaller shock which increases the additional consumption volatility. In the base case, the additional consumption volatilities are about 4.04%, 2.9%, and 2.6% for the young, middle-aged and old, respectively. When the size of shock doubles, those volatilities are 7.9%, 6.07%, and 6.07%. If the shock rises up to three times, the corresponding volatilities are 11.87%, 9.03%, and 9.62%. Therefore, the welfare loss from the supermodular effect is multiplied as the size of shock increases. Lastly, we emphasize that the complementary welfare loss takes about 50% of the welfare loss solely from the incomplete market in any sizes of shocks as seen in comparing the results above.

Table 3: Welfare analysis under an idiosyncratic shock with SD $\frac{3}{16}$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with incomplete market				
Young	0.2557	0.3606	0.3081	17.03%
Middle-aged	0.4326	0.2644	0.3485	24.14%
Old	0.3117	0.3750	0.3434	9.22%
Equilibrium consumptions with both friction				
Young	0.2192	0.3973	0.3082	28.90%
Middle-aged	0.5618	0.2819	0.4219	33.17%
Old	0.2191	0.3208	0.2699	18.84%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3249	0.3109	0.2918
CE consumption as (%) of benchmark	100%	97.48%	93.27%	87.54%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

5 Robustness Check of the Results

In this section, we change parameters to check the robustness of our results in this paper. First, we set the time discount factor to be consistent with the literature. Then, we consider the high risk-aversion case. Lastly, we examine a different shock structure.

5.1 Changing the Time Discount Factor

We set a new time discount factor at 0.54 because the annual subjective discount factor is 0.97 from the literature and one period in our model takes 20 years in the real economy. We keep other parameters as the base case above.

The stationary consumption allocations are decreasing in ages in the economy without any frictions but a low time discount factor because agents prefer the present consumption more than the future one. However, the middle-aged consume as much as the young in an economy with imperfect competition due to facing the lowest effective prices as seen above. Thus, the correlation between life-cycle consumptions and endowments is 0.88 at the steady-state of this economy which is much larger than the one in the frictionless economy, 0.52. This result supports that the strategic interaction makes consumptions correlated with incomes over the life-cycle even with a realistic time discounting factor.

Looking at the welfare loss from each friction, there is an interesting finding that the low time discount factor attenuates the welfare loss from imperfect competition, moving it from 2.52% to 2.04%, but intensifies the welfare loss due to the incomplete market from 0.71% to 1.12%. One can get an intuition for these results by considering the extreme case of $\delta = 0$. In

Table 4: Welfare analysis under the economy with $\delta = 0.54$

Age	State α	State β	Mean	CV (%)
Benchmark consumptions without frictions				
Young	0.4396	0.4396	0.4396	0.00%
Middle-aged	0.3230	0.3230	0.3230	0.00%
Old	0.2374	0.2374	0.2374	0.00%
Equilibrium consumptions only with imperfect competition				
Young	0.4071	0.4071	0.4071	0.00%
Middle-aged	0.3913	0.3913	0.3913	0.00%
Old	0.2016	0.2016	0.2016	0.00%
Equilibrium consumptions only with incomplete market				
Young	0.4086	0.4572	0.4329	5.62%
Middle-aged	0.3648	0.2890	0.3269	11.60%
Old	0.2266	0.2538	0.2402	5.66%
Equilibrium consumptions with both friction				
Young	0.3670	0.4375	0.4023	8.76%
Middle-aged	0.4468	0.3423	0.3946	13.25%
Old	0.1862	0.2202	0.2032	8.39%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3539	0.3467	0.3500	0.3417
CE consumption as (%) of benchmark	100%	97.96%	98.88%	96.53%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

this extreme, agents care about consumption only in youth even under the imperfectly competitive market. Thus, one's saving behavior and consumption allocations are not affected by the heterogenous effective prices across ages from the market game mechanism. This result implies that there is a less room for the strategic interaction to generate welfare losses. Hence, by continuity, a lower time discount factor yields a relatively smaller welfare loss from imperfect competition.

If agents regard future consumption as being less valuable than current consumption under a low δ, then it limits the amount of borrowing and saving among adjacent generations. For example, the middle-aged generation facing a bad shock cannot borrow much from the young generation who discount future consumption significantly. Thus, a low time discount factor reduces risk-sharing over one's life-cycle and makes consumption more volatile. The consumption inequality within each age indeed increases by 1-3% compared to the base case. Therefore, the limited risk-sharing under a lower δ results in a larger welfare loss from the incomplete market.

The additional consumption volatility between states is smaller for all ages under the low time discount factor compared to the base case because consumption allocations are less distorted by the strategic interaction as δ decreases as discussed above. Those volatilities are roughly 3.14% for the young, 1.65% for the middle-aged, and 2.73% for the old which are about 1% lower than the base case. Therefore, the complementary welfare loss is 0.31% in the discounting economy which is smaller than 0.35% in the base case and takes about 27% of the welfare loss solely from the incomplete market. This result implies that the imperfect competition still increases the exposure of consumption to endowment risk under a realistic time discount factor although its quantitative welfare effect can be smaller.

5.2 Changing the risk aversion

Now, we examine how the imperfect competition affects the consumption variation across ages and states under a high-risk aversion, $\sigma = 6$. We also keep other parameters as in the base case.

After increasing the risk aversion, the life-cycle consumption profile becomes closer to the perfectly smoothed one even under the strategic interaction, but we still observe the correlated consumption/income life-cycle profiles. The consumption volatility between states gets also lower compared to the base case. This less consumption variation across both ages or states arises from the strong demand of more risk-averse households for perfect risk sharing and smoothed consumptions.

The welfare losses show different patterns in each friction. Under a high risk-aversion, the welfare loss by the imperfect competition decreases from 2.52% to 1.01% whereas it increases from 0.71% to 0.91% under the financial market incompleteness. To explain the discrepancy in the welfare results, we should note that a high risk-aversion reinforces the disutility against the consumption volatility although it decreases consumption variation across both ages or states. The latter effect dominates the former one in the economy only with the strategic interaction and thus, there is a less welfare loss. It is the opposite when there exists only the incomplete market friction which results in a larger welfare loss.

Under a high-risk aversion, the imperfect competition adds consumption volatility up to 1.64% for the young, 2.02% for the middle-aged, and 1.41% for the old. These volatilities are

Table 5: Welfare analysis under the economy with $\sigma = 6$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with imperfect competition				
Young	0.3321	0.3321	0.3321	0.00%
Middle-aged	0.3591	0.3591	0.3591	0.00%
Old	0.3088	0.3088	0.3088	0.00%
Equilibrium consumptions only with incomplete market				
Young	0.3224	0.3373	0.3298	2.27%
Middle-aged	0.3546	0.3150	0.3348	5.91%
Old	0.3231	0.3477	0.3354	3.67%
Equilibrium consumptions with both friction				
Young	0.3154	0.3410	0.3282	3.91%
Middle-aged	0.3894	0.3322	0.3608	7.93%
Old	0.2952	0.3268	0.3110	5.08%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3296	0.3303	0.3254
CE consumption as (%) of benchmark	100%	98.89%	99.09%	97.64%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %

- CE consumption represents certainty equivalent consumption

much smaller than in the base case because of the strong demand for perfect risk sharing by more risk-averse households. However, since agents with a higher risk-aversion experience more significant disutility to the same consumption variations, the complementary welfare loss is 0.34% which is comparable to 0.35% in the base case. Such welfare loss takes roughly 37% of the welfare loss solely from the incomplete market. The results so far support that the strategic interaction generates larger risk-exposure of consumptions to the shock in a high-risk aversion as well.

5.3 Changing the Shock Structure

In this subsection, we examine how the structure of the endowment shocks matters for the welfare implication of the imperfect competition. For this, we consider an aggregate shock where the stochastic endowments streams are positively correlated between overlapped generations: $\{\omega_1^\alpha, \omega_2^\alpha, \omega_3^\alpha, \Omega^\alpha\} = \left\{\frac{5}{16}, \frac{9}{16}, 0, \frac{14}{16}\right\}$ and $\left\{\omega_1^\beta, \omega_2^\beta, \omega_3^\beta, \Omega^\beta\right\} = \left\{\frac{7}{16}, \frac{11}{16}, 0, \frac{18}{16}\right\}$ in which Ω^s is the total endowment in state s. Other parameters set following the base case. Note that this aggregate shock has the same SD with the idiosyncractic shock in the base case from the viewpoints of each age.

From the welfare analysis, we find out that the strategic interaction reduces the risk-sharing in a very limited manner if the exogenous uncertainty impacts overlapped generations in the same direction.

Table 6: Welfare analysis under the economy with an aggregate shock

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with incomplete market				
Young	0.2939	0.3672	0.3306	11.09%
Middle-aged	0.3096	0.3499	0.3297	6.11%
Old	0.2715	0.4079	0.3397	20.07%
Equilibrium consumptions with both friction				
Young	0.2850	0.3673	0.3262	12.61%
Middle-aged	0.3711	0.4234	0.3972	6.58%
Old	0.2188	0.3342	0.2765	20.87%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %

Under the aggregate shock, we also observe consumption variation and correlated consumption/endowment profiles over the life-cycle if agents have market powers to the price. However, the strategic interaction does not increase the consumption inequality in each age as much as its counterpart idiosyncratic shock case. Indeed, the within-age consumption volatility rises by 1.52% for the young, 0.47% for the middle-aged, and 0.8% for the old when adding the imperfect competition on the top of the incomplete market. The intuition behind these results is that the aggregate shock does not differentiate advantaged and disadvantaged cohorts and thus overlapped generations experience similar changes in their effective good prices under the shock. Hence, consumption allocations are not biased toward a certain age which restricts the extent of the additional consumption volatility.

From Table 1 and 6, the consumption inequality defined by CV is larger for the young and the old and smaller for the middle-aged under the aggregate shock than the idiosyncratic shock, given the same SD. These different patterns in CV come mainly from the distinct behaviors of price between the two models. In the model with an aggregate shock, the value of money is volatile between states because the aggregate saving is low in state α but high in state β. However, it is quite stable in the model with an idiosyncratic shock since the aggregate saving is reliable across states due to the same total endowments.

The young households save less in the good state when facing volatile prices than when facing stable prices because of downward risk on the price of money in the next period. On the other hand, they save more in the bad state under volatile prices than stable ones due to the upward price risk of money. Thus, saving variation within the young is smaller in the aggregate shock than the idiosyncratic shock which results in a larger consumption variation for this age. The consumption variation of the old age group is solely determined by the volatility in the value of money holdings because they receive zero endowments. The large price fluctuations under the aggregate shock make the value of the old's asset holdings volatile although the saving of the middle-aged is stable. Therefore, we observe a large consumption variation for this age under the aggregate shock than the idiosyncratic shock. Since the young and old age groups absorb the risk-exposure of consumptions to the endowment shocks, the middle-aged hedge the consumption risk significantly and face a smaller consumption variation under the aggregate shock. It is the opposite in the model with the idiosyncratic shock. The young and old households rather share the consumption risk extensively and thus, the middle-aged absorb the largest portion of the risk and show a larger consumption variation.

5.4 Further Discussion

In the robustness checks above, we do not consider an idiosyncratic shock within cohorts because the welfare implications will be straightforward based on the results we have derived so far. Ex-ante identical agents become ex-post heterogenous under this idiosyncratic shock. Households receiving a good shock can purchase goods at a cheaper price whereas those receiving a bad shock will pay more to buy the good in the imperfectly competitive market. Therefore, the strategic interaction will also increase the consumption volatility within ages under this type of shock.

We have concentrated on the symmetric shocks with equal probabilities. As long as the exogenous uncertainty differentiates gaining and losing cohorts given a state, there exists marginal price dispersion for identical goods over generations. Thus, the strategic interaction will add consumption volatility even under an asymmetric shock. However, the asymmetry can affect the size of the welfare loss. For example, a probability distribution skewed to one state has a small variance and then the imperfect competition generates a less complementary welfare loss under a small size of shock as seen above.

6 Policy Analysis

In this section, we study monetary and fiscal policies to improve ex-ante expected utility of the stationary equilibria in the base economy in Section 4. We measure the welfare improvement by how much the CE consumption increases from the base economy to an economy with policy interventions.

6.1 Monetary Policy

To improve welfare, we consider an expansionary monetary policy under which the government issues new money and transfers it to the age group receiving a bad shock among the young and middle-aged. We assume that the total money supply grows at a constant rate in every period. This policy can improve social welfare by rebalancing the amount of money holdings and promoting risk-sharing between the gaining and losing cohorts. One can interpret this monetary policy as redistributing wealth implicitly from those who get a good shock to their counterpart by inducing an inflation tax.

Under the active monetary policy, a new money market clearing condition is given by:

$$M_t = M_{t-1} + \Delta m_t = (1+g)\, M_{t-1} \tag{17}$$

and the household budget constraints become:

$$\begin{aligned}
\hat{b}^s_{1,t} + \hat{m}^s_{1,t} &= \frac{q^s_{1,t}}{Q^s_t}\hat{B}^s_t + \Delta\hat{m}_t I\left(\omega^s_1 = \omega^B_1\right) \\
\hat{b}^{s'}_{2,t+1} + \hat{m}^{s'}_{2,t+1} - \frac{\hat{m}^s_{1,t}}{(1+g)} &= \frac{q^{s'}_{2,t+1}}{Q^{s'}_{t+1}}\hat{B}^{s'}_{t+1} + \Delta\hat{m}_{t+1} I\left(\omega^{s'}_2 = \omega^B_2\right) \\
\hat{b}^{s''}_{3,t+2} - \frac{\hat{m}^{s'}_{2,t+1}}{(1+g)} &= \frac{q^{s''}_{3,t+2}}{Q^{s''}_{t+2}}\hat{B}^{s''}_{t+2}
\end{aligned} \tag{18}$$

where M_t is the total money supply at time t, Δm_t denotes new money supply transferred to the age facing a bad shock, and g is the growth rate of the total money supply. $I(\cdot)$ is an indicator function and has one if the age i is in the bad endowment state, ω^B_i, and zero otherwise. For computation, we normalize variables by dividing them with the total money supply in their corresponding times and mark normalized variables with the hat symbol,$\hat{}$. For example, $\hat{b}^s_{1,t} = \frac{b^s_{1,t}}{M_t}$. From the equation (17), $\Delta\hat{m}_t = \frac{g}{1+g}$.

From the market game trading mechanism, a household born in time t obtains the following consumption allocations:

$$\begin{aligned}
c^s_{1,t} &= \omega^s_1 - q^s_{1,t} + \frac{\hat{b}^s_{1,t}}{\hat{B}^s_t} Q^s_t \\
c^{s'}_{2,t+1} &= \omega^{s'}_2 - q^{s'}_{2,t+1} + \frac{\hat{b}^{s'}_{2,t+1}}{\hat{B}^{s'}_{t+1}} Q^{s'}_{t+1} \\
c^{s''}_{3,t+2} &= \omega^{s''}_3 - q^{s''}_{3,t+2} + \frac{\hat{b}^{s''}_{3,t+2}}{\hat{B}^{s''}_{t+2}} Q^{s''}_{t+2}
\end{aligned} \tag{19}$$

As in (4), we can rewrite (18) as:

$$\begin{aligned}
\hat{b}^s_{1,t} + \hat{m}^s_{1,t} &= \left(\frac{\hat{B}^s_{t,-1} - \hat{m}^s_{1,t} + \Delta\hat{m}_t I_1}{Q^s_{t,-1}} \right) q^s_{1,t} + \Delta\hat{m}_t I_1 \\
\hat{b}^{s'}_{2,t+1} + \hat{m}^{s'}_{2,t+1} - \frac{\hat{m}^s_{1,t}}{(1+g)} &= \left(\frac{\hat{B}^{s'}_{t+1,-2} - \hat{m}^{s'}_{2,t+1} + \frac{\hat{m}^s_{1,t}}{(1+g)} + \Delta\hat{m}_{t+1} I_2}{Q^{s'}_{t+1,-2}} \right) q^{s'}_{2,t+1} + \Delta\hat{m}_{t+1} I_2 \\
\hat{b}^{s''}_{3,t+2} - \frac{\hat{m}^{s'}_{2,t+1}}{(1+g)} &= \left(\frac{\hat{B}^{s''}_{t+2,-3} + \frac{\hat{m}^{s'}_{2,t+1}}{(1+g)}}{Q^{s''}_{t+2,-3}} \right) q^{s''}_{3,t+2}
\end{aligned} \tag{20}$$

where $I_1 = I\left(\omega^s_1 = \omega^B_1\right)$ and $I_2 = I\left(\omega^{s'}_2 = \omega^B_2\right)$ with a convenient abuse of notation.

By equating the right-hand sides of (18) and (20), we derive an equation similar to (5):

(21)

$$\frac{Q^s_t}{\hat{B}^s_t} = \frac{Q^s_{t,-1}}{\hat{B}^s_{t,-1} - \hat{m}^s_{1,t} + \Delta\hat{m}_t I_1}, \quad \frac{Q^{s'}_{t+1}}{\hat{B}^{s'}_{t+1}} = \frac{Q^{s'}_{t+1,-2}}{\hat{B}^{s'}_{t+1,-2} - \hat{m}^{s'}_{2,t+1} + \frac{\hat{m}^s_{1,t}}{(1+g)} + \Delta\hat{m}_{t+1} I_2}, \quad \frac{Q^{s''}_{t+2}}{\hat{B}^{s''}_{t+2}} = \frac{Q^{s''}_{t+2,-3}}{\hat{B}^{s''}_{t+2,-3} + \frac{\hat{m}^{s'}_{2,t+1}}{(1+g)}}$$

Substituting these equations with (20) into (19), we generate the following consumption allocations presented by new money supply:

(22)

$$\begin{aligned}
c^s_{1,t} &= \omega^s_1 - \left(\frac{Q^s_{t,-1}}{\hat{B}^s_{t,-1} - \hat{m}^s_{1,t} + \Delta\hat{m}_t I_1} \right) \left(\hat{m}^s_{1,t} - \Delta\hat{m}_t I_1 \right) \\
c^{s'}_{2,t+1} &= \omega^{s'}_2 - \left(\frac{Q^{s'}_{t+1,-2}}{(1+g)\left(\hat{B}^{s'}_{t+1,-2} - \hat{m}^{s'}_{2,t+1} + \Delta\hat{m}_{t+1} I_2 \right) + \hat{m}^s_{1,t}} \right) \left((1+g)\left(\hat{m}^{s'}_{2,t+1} - \Delta\hat{m}_{t+1} I_2 \right) - \hat{m}^s_{1,t} \right) \\
c^{s''}_{3,t+2} &= \omega^{s''}_3 + \left(\frac{Q^{s''}_{t+2,-3}}{(1+g)\,\hat{B}^{s''}_{t+2,-3} + \hat{m}^{s'}_{2,t+1}} \right) \hat{m}^{s'}_{2,t+1}
\end{aligned}$$

Under the expansionary monetary policy, the first-order conditions with respect to $\hat{m}^s_{1,t}$ and $\hat{m}^{s'}_{2,t+1}$ are given by:

$$\begin{aligned}
& u'\left(c^s_{1,t}\right) \frac{\hat{B}^s_{t,-1} Q^s_{t,-1}}{\left(\hat{B}^s_{t,-1} - \hat{m}^s_{1,t} + \Delta\hat{m}_t I_1 \right)^2} \\
&= \delta \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u'\left(c^{s'}_{2,t+1}\right) \frac{\hat{B}^{s'}_{t+1,-2} Q^{s'}_{t+1,-2}}{(1+g)\left(\left(\hat{B}^{s'}_{t+1,-2} - \hat{m}^{s'}_{2,t+1} + \Delta\hat{m}_{t+1} I_2 \right) + \frac{\hat{m}^s_{1,t}}{(1+g)} \right)^2}
\end{aligned} \tag{23}$$

and

$$
\begin{aligned}
& u'\left(c^{s'}_{2,t+1}\right) \frac{\hat{B}^{s'}_{t+1,-2} Q^{s'}_{t+1,-2}}{\left(\left(\hat{B}^{s'}_{t+1,-2} - \hat{m}^{s'}_{2,t+1} + \Delta\hat{m}_{t+1} I_2\right) + \frac{\hat{m}^{s}_{1,t}}{(1+g)}\right)^2} \\
(24) \quad & = \delta \sum_{s'' \in \{\alpha,\beta\}} \pi^{s''} u'\left(c^{s''}_{3,t+2}\right) \frac{\hat{B}^{s''}_{t+2,-3} Q^{s''}_{t+2,-3}}{(1+g)\left(\hat{B}^{s''}_{t+2,-3} + \frac{\hat{m}^{s'}_{2,t+1}}{(1+g)}\right)^2}
\end{aligned}
$$

Using the equation (21), one can simplify (22), (23) and (24) as follows:

$$
\begin{aligned}
c^{s}_{1,t} &= \omega^{s}_{1} - \frac{Q^{s}_{t}}{\hat{B}^{s}_{t}} \left(\hat{m}^{s}_{1,t} - \Delta\hat{m}_t I_1\right) \\
(25) \quad c^{s'}_{2,t+1} &= \omega^{s'}_{2} - \frac{Q^{s'}_{t+1}}{\hat{B}^{s'}_{t+1}} \left(\left(\hat{m}^{s'}_{2,t+1} - \Delta\hat{m}_{t+1} I_2\right) - \frac{\hat{m}^{s}_{1,t}}{(1+g)}\right) \\
c^{s''}_{3,t+2} &= \omega^{s''}_{3} + \frac{Q^{s''}_{t+2}}{\hat{B}^{s''}_{t+2}} \frac{\hat{m}^{s'}_{2,t+1}}{(1+g)}
\end{aligned}
$$

$$
\begin{aligned}
& u'\left(c^{s}_{1,t}\right) \frac{\hat{B}^{s}_{t,-1}}{Q^{s}_{t,-1}} \left(\frac{Q^{s}_{t}}{\hat{B}^{s}_{t}}\right)^2 \\
(26) \quad & = \delta \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u'\left(c^{s'}_{2,t+1}\right) \frac{1}{(1+g)} \frac{\hat{B}^{s'}_{t+1,-2}}{Q^{s'}_{t+1,-2}} \left(\frac{Q^{s'}_{t+1}}{\hat{B}^{s'}_{t+1}}\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
& u'\left(c^{s'}_{2,t+1}\right) \frac{\hat{B}^{s'}_{t+1,-2}}{Q^{s'}_{t+1,-2}} \left(\frac{Q^{s'}_{t+1}}{\hat{B}^{s'}_{t+1}}\right)^2 \\
(27) \quad & = \delta \sum_{s'' \in \{\alpha,\beta\}} \pi^{s''} u'\left(c^{s''}_{3,t+2}\right) \frac{1}{(1+g)} \frac{\hat{B}^{s''}_{t+2,-3}}{Q^{s''}_{t+2,-3}} \left(\frac{Q^{s''}_{t+2}}{\hat{B}^{s''}_{t+2}}\right)^2
\end{aligned}
$$

With the new equilibrium conditions we derived so far, we compute the recursive monetary Nash equilibria under the active monetary policy. In the numerical exercises, we keep other parameters but the growth rates of the total money supply as in the base case. We test three different growth rates: 5%, 10%, and 20%. Three tables below summarize the welfare results for each growth rate.

The monetary policy reduces consumption volatility between states by implicitly redistributing wealth from the endowment rich cohort to the poor cohort via an inflation tax. However, it increases consumption variation over the lifetime by generating a declining consumption-age profile. The expansionary monetary policy yields an intertemporal wedge on saving because the new money supply in every period increases the good price tomorrow compared to today and thus decreases gross interest rates. This wedge discourages household to save which brings about the declining consumptions over the life-cycle.

Table 7: Welfare analysis under the economy with $g = 5\%$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with incomplete market				
Young	0.3260	0.3522	0.3391	3.87%
Middle-aged	0.3585	0.3105	0.3345	7.17%
Old	0.3156	0.3373	0.3264	3.33%
Equilibrium consumptions with both friction				
Young	0.3066	0.3591	0.3329	7.88%
Middle-aged	0.4443	0.3604	0.4024	10.43%
Old	0.2490	0.2805	0.2648	5.94%
	Benchmark	**Inc. Mk.**	**Both**	
CE consumption	0.3333	0.3316	0.3212	
CE consumption as (%) of benchmark	100%	99.48%	96.35%	

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

Table 8: Welfare analysis under the economy with $g = 10\%$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with incomplete market				
Young	0.3370	0.3584	0.3477	3.08%
Middle-aged	0.3540	0.3139	0.3340	6.01%
Old	0.3090	0.3277	0.3183	2.94%
Equilibrium consumptions with both friction				
Young	0.3150	0.3636	0.3393	7.17%
Middle-aged	0.4394	0.3638	0.4016	9.40%
Old	0.2457	0.2726	0.2591	5.19%
	Benchmark	**Inc. Mk.**	**Both**	
CE consumption	0.3333	0.3318	0.3207	
CE consumption as (%) of benchmark	100%	99.53%	96.21%	

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

Table 9: Welfare analysis under the economy with $g = 20\%$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with incomplete market				
Young	0.3568	0.3702	0.3635	1.84%
Middle-aged	0.3453	0.3203	0.3328	3.75%
Old	0.3095	0.2979	0.3037	1.91%
Equilibrium consumptions with both friction				
Young	0.3302	0.3720	0.3511	5.94%
Middle-aged	0.4305	0.3702	0.4004	7.53%
Old	0.2393	0.2578	0.2485	3.73%
	Benchmark	**Inc. Mk.**	**Both**	
CE consumption	0.3333	0.3311	0.3191	
CE consumption as (%) of benchmark	100%	99.33%	95.72%	

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %

- CE consumption represents certainty equivalent consumption

Compared to the base case, the active monetary policy improves long-run social welfare for all money growth rates by promoting risk-sharing when there is only the incomplete market friction. 10% growth rate advances welfare most and then 5% and 20 % in order. The larger the growth rate, the more risk-sharing but larger consumption variation over the life-cycle. Thus, the welfare improvement can be worst when the money growth rate is 20% because such a high rate will generate the largest intertemporal wedge among the three rates which dominates the benefit of mitigating the consumption volatility between states.

If both frictions are present, all expansionary monetary policies worsen long-run social welfare compared with the base economy. Imperfect competition already induces consumption variation over the life-cycle as seen in the correlated consumption/endowment profile. Further consumption variation generated by the intertemporal wedge will drop welfare significantly under a concave utility function. Thus, the expansionary monetary policy can decrease the social welfare under both frictions although it reduces the consumption volatility between states. For this reason, 5% money growth rate generates the least welfare loss from the base case among the three rates by distorting the life-cycle consumptions least.

In this numerical analysis, we do not study the role of the monetary policy in the economy only with the imperfect competition because there is no consumption volatility between states that the policy should reduce and the policy cannot adjust the effective prices which are solely determined by offers. We can easily conjecture that introducing the monetary policy will aggravate welfare in this case by producing intertemporal wedges.

One can consider another type of monetary policy which transfers a constant fraction of new money supply to the old in both states. At first glance, this policy seems to flatten the declining life-cycle consumption trend by increasing the old-age consumption. In fact, this policy is ineffective because households react to the policy by crowding out savings for the old-period due to the money transfer. The new monetary policy still generates an intertemporal wedge and is less effective in shrinking the consumption volatility between states because of a less transfer to the endowment poor than the previous policy. Therefore, transferring new money supply to the old is not a good policy option.

6.2 Fiscal Policy

In the previous section, we showed that monetary policy enhances the inter-generational risk-sharing against the idiosyncratic risk. However, it does not reduce the welfare loss from the strategic interaction because it cannot adjust the heterogeneous effective marginal prices. More importantly, the active monetary policy produces an intertemporal wedge and further distorts consumption variation over the life-cycle if both frictions are present. Thus, the social welfare can actually decrease.

Therefore, in this section, we propose a fiscal policy to improve welfare even when both frictions are present. We focus on a time-invariant linear endowment tax and lump-sum transfer to all living generations. The government balances its budget constraint. We assume that households offer only the after-tax endowments, not the transfer. Thus, the timeline is as follows: the government collects tax revenue first, agents offer all endowments left under the sell-all strategy, and then they receive a transfer.

Under the fiscal policy, there are no changes in the individual budget constraints, and the consumption allocation rules take the form:

$$
\begin{aligned}
c_{1,t}^{s} &= (1-\tau)\,\omega_1^{s} - q_{1,t}^{s} + \frac{b_{1,t}^{s}}{B_t^{s}} Q_t^{s} + T^{s} \\
c_{2,t+1}^{s'} &= (1-\tau)\,\omega_2^{s'} - q_{2,t+1}^{s'} + \frac{b_{2,t+1}^{s'}}{B_{t+1}^{s'}} Q_{t+1}^{s'} + T^{s'} \\
c_{3,t+2}^{s''} &= (1-\tau)\,\omega_3^{s''} - q_{3,t+2}^{s''} + \frac{b_{3,t+2}^{s''}}{B_{t+2}^{s''}} Q_{t+2}^{s''} + T^{s''}
\end{aligned} \tag{28}
$$

where τ is a time-invariant proportional tax and $T^s = \frac{\tau \Omega^s}{3n}$ denotes a lump-sum transfer in state s. The after-tax offers are given by $q_{i,t}^{s} = (1-\tau)\,\omega_i^{s}$ and $Q_t^{s} = (1-\tau)\,\Omega^{s}$ for $\forall\,(i,s)$.

We can transform the consumption allocations above into:

$$
\begin{aligned}
c_{1,t}^{s} &= (1-\tau)\,\omega_1^{s} - \left(\frac{Q_{t,-1}^{s}}{B_{t,-1}^{s} - m_{1,t}^{s}}\right) m_{1,t}^{s} + T^{s} \\
c_{2,t+1}^{s'} &= (1-\tau)\,\omega_2^{s'} - \left(\frac{Q_{t+1,-2}^{s'}}{B_{t+1,-2}^{s'} - m_{2,t+1}^{s'} + m_{1,t}^{s}}\right) \left(m_{2,t+1}^{s'} - m_{1,t}^{s}\right) + T^{s'} \\
c_{3,t+2}^{s''} &= (1-\tau)\,\omega_3^{s''} + \left(\frac{Q_{t+2,-3}^{s''}}{B_{t+2,-3}^{s''} + m_{2,t+1}^{s'}}\right) m_{2,t+1}^{s'} + T^{s''}
\end{aligned} \tag{29}
$$

It is well-known in the market game literature that allowing wash-sales decreases the welfare loss from the strategic interaction by equating the effective marginal prices among households with different levels of endowments (see Peck and Shell (1990)).[8] However, if wash-sales are prohibited because of commitment issues or other exogenous constraints, then a linear endowment tax and transfer can be an alternative. This fiscal policy reduces the amount of goods traded in the imperfect market. Thus, it limits the extent of distortion by the strategic interaction. In addition, the fiscal policy can share the idiosyncratic risk by intergenerational redistribution and reduce the consumption volatility between states.

We check the conjecture that the fiscal policy can increase the social welfare even if both frictions are present via numerical exercises. For this, we compute the recursive monetary Nash equilibria under the tax/transfer policy with the new consumption allocation rules above. We keep parameter values as in the base economy. We examine three different tax rates: 10%, 20%, and 30%. In the following three tables, we summarize the welfare results from different tax rates.

Table 10: Welfare analysis under the economy with $\tau = 10\%$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with imperfect competition				
Young	0.3289	0.3289	0.3289	0.00%
Middle-aged	0.3991	0.3991	0.3991	0.00%
Old	0.2720	0.2720	0.2720	0.00%
Equilibrium consumptions only with incomplete market				
Young	0.3160	0.3452	0.3306	4.42%
Middle-aged	0.3613	0.3086	0.3350	7.86%
Old	0.3227	0.3462	0.3344	3.51%
Equilibrium consumptions with both friction				
Young	0.3000	0.3538	0.3269	8.24%
Middle-aged	0.4437	0.3587	0.4012	10.59%
Old	0.2563	0.2874	0.2719	5.72%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3253	0.3313	0.3223
CE consumption as (%) of benchmark	100%	97.58%	99.39%	96.69%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

In the tables above, consumption variations fall across both ages and states as tax rates

[8] When agents trade under the heterogeneous marginal prices, they optimize decisions at different marginal rates of substitution. Thus, Pareto-improving allocations can exist for the equilibrium under the market game. If the wash-sales are allowed, then households can offer more than their endowments. If they increase offers, the return rates of bidding will get closer between households. In the limit, the effective marginal prices will be identical, and thus the market game allocations will converge to Pareto-efficient ones.

Table 11: Welfare analysis under the economy with $\tau = 20\%$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with imperfect competition				
Young	0.3293	0.3293	0.3293	0.00%
Middle-aged	0.3973	0.3973	0.3973	0.00%
Old	0.2735	0.2735	0.2735	0.00%
Equilibrium consumptions only with incomplete market				
Young	0.3173	0.3448	0.3310	4.16%
Middle-aged	0.3582	0.3110	0.3346	7.06%
Old	0.3245	0.3442	0.3344	2.95%
Equilibrium consumptions with both friction				
Young	0.3023	0.3532	0.3278	7.75%
Middle-aged	0.4372	0.3608	0.3990	9.57%
Old	0.2605	0.2860	0.2732	4.67%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3257	0.3317	0.3233
CE consumption as (%) of benchmark	100%	97.71%	99.50%	96.98%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

Table 12: Welfare analysis under the economy with $\tau = 30\%$

Age	State α	State β	Mean	CV (%)
Equilibrium consumptions only with imperfect competition				
Young	0.3297	0.3297	0.3297	0.00%
Middle-aged	0.3950	0.3950	0.3950	0.00%
Old	0.2753	0.2753	0.2753	0.00%
Equilibrium consumptions only with incomplete market				
Young	0.3189	0.3442	0.3315	3.82%
Middle-aged	0.3555	0.3134	0.3344	6.30%
Old	0.3256	0.3424	0.3340	2.52%
Equilibrium consumptions with both friction				
Young	0.3049	0.3523	0.3286	7.20%
Middle-aged	0.4303	0.3625	0.3964	8.55%
Old	0.2852	0.2647	0.2750	3.72%
	Benchmark	**Imp. Comp.**	**Inc. Mk.**	**Both**
CE consumption	0.3333	0.3262	0.3320	0.3243
CE consumption as (%) of benchmark	100%	97.86%	99.60%	97.28%

- CV stands for the coefficient of variation which is the standard deviation of consumption divided by its mean measured in %
- CE consumption represents certainty equivalent consumption

rise. Thus, the fiscal policy decreases the welfare loss from each friction. Moreover, the social welfare goes up when both frictions exist. These results confirm our predictions above that the fiscal policy mitigates the welfare loss from the strategic interaction and incomplete market by shrinking the volume of goods transacted in the trading post and redistributing wealth from the rich cohort to the poor one. However, we should stress out that there is no distortion from the linear tax in the endowment economy. Thus, taxing all endowments and distributing equally to living generations can achieve the stationary allocations in the frictionless economy. It is natural to derive the result that the higher the tax rates, the closer to the benchmark allocation and the higher the welfare improvement.

If labor supply is elastic and taxed, then there will be an upper bound for the tax rate to advance the long-run ex-ante expected utility because of an efficiency cost in the labor supply distortion. However, the labor income tax will not generate an intertemporal wedge unlike the monetary policy and still smooth consumptions over the life-cycle as the endowment tax. Thus, a labor distorting tax/transfer policy also has an asymmetric welfare effect with the monetary policy under the strategic interaction.[9]

We have assumed that households offer only the endowments net of taxes under the sell-all strategy. Even if we allow them to offer the transfer as well, the results will remain the same. The fiscal policy redistributes wealth from the working ages to the retired and the rich to the poor. Thus, it smoothes offers across ages and states if the offers include both after-tax endowment and transfer. Then, there will be a less gap between the effective marginal prices of different generations. Hence, the fiscal policy, in this case, will also reduce the consumption variations over both ages and states.

7 Extension

In developing economies, the effective prices largely depend on the income of buyers due to the quantity premium as seen in Rao (2000). However, Aguiar and Hurst (2007) show that the low-income group can pay less for identical goods than its counterpart in developed countries because of their intensive search to find better prices under the low opportunity cost of search from their low wage. Hence, the welfare implication of this paper should be carefully applied to the developed economies.

In this section, we extend the current model to incorporate the effect of searching activity on the marginal prices to obtain the implications of the imperfect competition to the developed economies. For simplicity, one can assume that the searching effort is exogenously given as the good endowment. The distributions of search endowment can be set consistent with the opportunity cost of searching activity across types and ages following the results in Aguiar and Hurst (2007). For example, low-wage workers receive a large search endowment whereas high-wage earners take a small one. One can think of a reduced form model where both offering consumption goods and conducting search behavior drop the effective marginal prices. Thus, we write new individual budget constraints in which both price-adjusting factors affect the

[9] The fiscal policy does not affect the consumption-age profile in a perfectly competitive economy because there is a limited distortion to reduce over the life-cycle. However, it reduces the consumption variation over age under the imperfect competition whereas the monetary policy increases it. We regard this discrepancy as the asymmetric welfare effects of fiscal and monetary policy actions

return rates of bids as follows:

$$(30)\qquad \begin{aligned} b^{s}_{1,t} + m^{s}_{1,t} &= \left(\lambda \frac{q^{s}_{1,t}}{Q^{s}_{t}} + (1-\lambda)\frac{s^{s}_{1,t}}{S^{s}_{t}}\right) B^{s}_{t} \\ b^{s'}_{2,t+1} + m^{s'}_{2,t+1} - m^{s}_{1,t} &= \left(\lambda \frac{q^{s'}_{2,t+1}}{Q^{s'}_{t+1}} + (1-\lambda)\frac{s^{s'}_{2,t+1}}{S^{s'}_{t+1}}\right) B^{s'}_{t+1} \\ b^{s''}_{3,t+2} - m^{s'}_{2,t+1} &= \left(\lambda \frac{q^{s''}_{3,t+2}}{Q^{s''}_{t+2}} + (1-\lambda)\frac{s^{s''}_{3,t+2}}{S^{s''}_{t+2}}\right) B^{s''}_{t+2} \end{aligned}$$

where we assume that agents offer all their good and search endowments. Note that the shares out of total bids for each agent linearly depend on both good offer and search effort. The weight parameter λ determines the significance of quantity premium compared to the search effort on the returns rates and marginal prices.

After some calculations, one can derive the following new marginal prices that generation i faces in time t:

$$(31)\qquad \frac{\left(1 - \lambda \frac{q^{s}_{i,t}}{Q^{s}_{t}} - (1-\lambda)\frac{s^{s}_{i,t}}{S^{s}_{t}}\right) Q^{s}_{t} B^{s}_{t,-i}}{\left(c^{s}_{i,t} - Q^{s}_{t}\right)^{2}}$$

From this equation, one can note that more one offers goods and search endowment, lower their effective prices are. Under the negative correlation between the good and search endowment distributions, we obtain similar results as derived above if λ is high enough. On the other hand, if λ is low enough then the search activity is a more significant factor than the income effect. In this case, the life-cycle price heterogeneity will be more favorable toward low wage earners because they can consume the same goods at cheaper prices and thus, the imperfect competition rather advances the consumption risk-sharing.

Whether the search effect dominates the income effect is an empirical question that we do not pursue in this paper. One can apply our model to data to estimate the parameter λ to answer this question. Instead of the linear form, one can use a more general non-linear share formula given by $f\left(q^{s}_{i,t}, s^{s}_{i,t}; Q^{s}_{t}, S^{s}_{t}\right)$, where f increases in both individual good and search endowments offers.

8 Conclusion

In this paper, we study how imperfect competition can increase consumption variation across states and ages by interacting with the incomplete financial market. We show that income-dependent prices for identical goods under imperfect competition might bias consumptions toward agents who receive high-income shocks and thus mitigate risk-sharing by generating additional consumption volatility. We quantify the additional consumption volatility and its welfare loss in a parameterized version of the model. Our numerical analysis states that the complementary welfare loss adds about 50% of the welfare loss solely from the incomplete market. We also find that the price heterogeneity across the life-cycle in our model can break down the perfect consumption smoothing result and rather brings about the correlated consumption/income profiles without other frictions such as capital market imperfection or impatient consumers. To check the robustness of our results, we try other values for the time discount factor and risk-aversion parameters and different shock structures. We observe that the implication of imperfect competition on welfare remains the same qualitatively, but the quantitative welfare implications can vary according to the parameters values. From a policy analysis, we find that both monetary and fiscal policy shrinks consumption volatility between states. However, monetary policy increases consumption variation over the lifetime by generating an intertemporal wedge, but fiscal policy decreases it by reducing the share of goods traded under the strategic interactions. Thus, the introduction of the imperfect competition results in an asymmetric welfare effect between fiscal and monetary policy, unlike the competitive models.

A Proofs

A.1 Proposition 1

We prove the non-existence of strongly stationary monetary Nash equilibria (no-recall equilibria) with the method of contradiction. It is straightforward to extend this result to short memory monetary Nash equilibria following arguments in Citanna and Siconolfi (2007) and Henriksen and Spear (2012).

We assume that there is a strongly stationary monetary Nash equilibrium where bids and money holdings only depend on the state of the current shock. Then, we can write the consumption allocations, the first-order conditions and the market clearing condition under this type of equilibria:

$$\begin{aligned} c_1^s &= \omega_1^s - \left(\frac{Q_{-1}^s}{B_{-1}^s - m_1^s}\right) m_1^s \\ c_2^{ss'} &= \omega_2^{s'} - \left(\frac{Q_{-2}^{s'}}{B_{-2}^{s'} - m_2^{s'} + m_1^s}\right)\left(m_2^{s'} - m_1^s\right) \\ c_3^{s's''} &= \omega_3^{s''} + \left(\frac{Q_{-3}^{s''}}{B_{-3}^{s''} + m_2^{s'}}\right) m_2^{s'} \end{aligned} \tag{A.1}$$

where $\{s, s', s''\} \in \{\alpha, \beta\}^3$.

$$\begin{aligned} & u'\left(c_1^\alpha\right) \frac{B_{-1}^\alpha}{Q_{-1}^\alpha}\left(\frac{Q^\alpha}{B^\alpha}\right)^2 \\ = \delta & \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u'\left(c_2^{\alpha s'}\right) \frac{B_{-2}^{s'}}{Q_{-2}^{s'}}\left(\frac{Q^{s'}}{B^{s'}}\right)^2 \end{aligned} \tag{A.2}$$

$$\begin{aligned} & u'\left(c_1^\beta\right) \frac{B_{-1}^\beta}{Q_{-1}^\beta}\left(\frac{Q^\beta}{B^\beta}\right)^2 \\ = \delta & \sum_{s' \in \{\alpha,\beta\}} \pi^{s'} u'\left(c_2^{\beta s'}\right) \frac{B_{-2}^{s'}}{Q_{-2}^{s'}}\left(\frac{Q^{s'}}{B^{s'}}\right)^2 \end{aligned} \tag{A.3}$$

$$\begin{aligned} & u'\left(c_2^{s\alpha}\right) \frac{B_{-2}^\alpha}{Q_{-2}^\alpha}\left(\frac{Q^\alpha}{B^\alpha}\right)^2 \\ = \delta & \sum_{s'' \in \{\alpha,\beta\}} \pi^{s''} u'\left(c_3^{\alpha s''}\right) \frac{B_{-3}^{s''}}{Q_{-3}^{s''}}\left(\frac{Q^{s''}}{B^{s''}}\right)^2 \end{aligned} \tag{A.4}$$

and

$$\text{(A.5)} \qquad u'\left(c_2^{s\beta}\right)\frac{B_{-2}^{\beta}}{Q_{-2}^{\beta}}\left(\frac{Q^{\beta}}{B^{\beta}}\right)^2 = \delta\sum_{s''\in\{\alpha,\beta\}}\pi^{s''}u'\left(c_3^{\beta s''}\right)\frac{B_{-3}^{s''}}{Q_{-3}^{s''}}\left(\frac{Q^{s''}}{B^{s''}}\right)^2$$

where $s \in \{\alpha, \beta\}$.

$$\text{(A.6)} \qquad (m_1^s + m_2^s) = M \text{ for } s \in \{\alpha, \beta\}$$

where $s \in \{\alpha, \beta\}$.

The equations above have the following implications. The young age consumptions depend on only the current shock state whereas the middle-aged and old period consumptions resort to both the current and lagged shock realizations because agents' money holdings are affected by the state in which the asset was purchased. One interesting finding is that the right-hand-sides of the first-order conditions of the middle-aged in (A.4) and (A.5) are independent of the lagged state, s. Thus, $c_2^{\alpha\alpha} = c_2^{\beta\alpha} = c_2^{\alpha}$ and $c_2^{\alpha\beta} = c_2^{\beta\beta} = c_2^{\beta}$ which implies that the middle-aged period consumptions also depend only on the current shock realizations. The good market clearing condition requires that $c_1^s + c_2^s + c_3^{s's} = \omega_1^s + \omega_2^s + \omega_3^s$ for $\{s', s\} \in \{\alpha, \beta\}^2$. Therefore, we also obtain that $c_3^{\alpha\alpha} = c_3^{\beta\alpha} = c_3^{\alpha}$ and $c_3^{\alpha\beta} = c_3^{\beta\beta} = c_3^{\beta}$.

We now demonstrate that the money holdings must be state-independent. We derive the following equations by imposing the results that $c_2^{\alpha s} = c_2^{\beta s}$ and $c_3^{\alpha s} = c_3^{\beta s}$ for $s \in \{\alpha, \beta\}$ on (A.1):

$$\text{(A.7)} \qquad \omega_2^s - \left(\frac{Q_{-2}^s}{B_{-2}^s - m_2^s + m_1^{\alpha}}\right)(m_2^s - m_1^{\alpha}) = \omega_2^s - \left(\frac{Q_{-2}^s}{B_{-2}^s - m_2^s + m_1^{\beta}}\right)\left(m_2^s - m_1^{\beta}\right)$$

and

$$\text{(A.8)} \qquad \omega_3^s + \left(\frac{Q_{-3}^s}{B_{-3}^s + m_2^{\alpha}}\right)m_2^{\alpha} = \omega_3^s + \left(\frac{Q_{-3}^s}{B_{-3}^s + m_2^{\beta}}\right)m_2^{\beta}$$

To satisfy these equations, $m_1^{\alpha} = m_1^{\beta} = m_1$ and $m_2^{\alpha} = m_2^{\beta} = m_2$. The state-independent money holdings simplify the budget constraints faced by the overlapped generations:

$$\text{(A.9)} \qquad \begin{aligned} b_1^s + m_1 &= \frac{q_1^s}{Q^s}B^s \\ b_2^s + m_2 - m_1 &= \frac{q_2^s}{Q^s}B^s \\ b_3^s - m_2 &= \frac{q_3^s}{Q^s}B^s \end{aligned}$$

where $s \in \{\alpha, \beta\}$.

The three equations in (A.9) are linearly dependent on each other since we obtain an identical equation by summing these equations and multiplying both sides with n:

$$nb_1^s + nb_2^s + nb_3^s = \frac{nq_1^s}{Q^s}B^s + \frac{nq_2^s}{Q^s}B^s + \frac{nq_3^s}{Q^s}B^s = B^s \tag{A.10}$$

where $s \in \{\alpha, \beta\}$.

In summary, there is a system of 15 equations: 6 equations from the consumption allocation rule, 4 equations from the first-order conditions, 4 equations from the budget constraints and 1 equation from the money market clearing condition. However, there are 14 variables: $c_1^\alpha, c_1^\beta, c_2^\alpha, c_2^\beta, c_3^\alpha, c_3^\beta, b_1^\alpha, b_1^\beta, b_2^\alpha, b_2^\beta, b_3^\alpha, b_3^\beta, m_1$ and m_2. There are more equations than variables and thus, the strongly stationary monetary Nash equilibria do not exist generically following the arguments in Spear (1985) and Citanna and Siconolfi (2007).

A.2 Lemma 1

We write the optimal money holding of young agents born in time $t-1$ as:

$$m_{1,t-1} = m_1\left(B_{t-1,-1}, B_{t,-2}, B_{t+1,-3}\right) \tag{A.11}$$

The money market clearing condition is in time t:

$$m_1\left(B_{t,-1}, B_{t+1,-2}, B_{t+2,-3}\right) + m_2\left(B_{t-1,-1}, B_{t,-2}, B_{t+1,-3}\right) = M \tag{A.12}$$

At the deterministic steady states, equations (A.11) and (A.12) reduce to:

$$\bar{m}_1 = m_1\left(\bar{B}_{-1}, \bar{B}_{-2}, \bar{B}_{-3}\right) \tag{A.13}$$

and

$$m_1\left(\bar{B}_{-1}, \bar{B}_{-2}, \bar{B}_{-3}\right) + m_2\left(\bar{B}_{-1}, \bar{B}_{-2}, \bar{B}_{-3}\right) = M \tag{A.14}$$

where $\bar{m}_1$ and $\bar{B}_{-i}$ for $\forall i$ are the money holding for the young and the total bid net of the bid of an individual in cohort i at the steady-states, respectively.

We now replace the bids arguments in (A.11) and (A.12) with $b_{2,t} = U\left(b_{1,t}, m_{1,t}, m_{1,t-1}\right)$ and $b_{3,t} = V\left(b_{1,t}, m_{1,t}, m_{1,t-1}\right)$ and plug the two linear forecast functions into $m_{1,t}$ and $b_{1,t}$ by properly adjusting the time subscripts to obtain:

(A.15)

$$G(m) = m_1\begin{pmatrix} (n-1)H(m) + nU(H(m), G(m), m) + nV(H(m), G(m), m), \\ nH(G(m)) + (n-1)U\left(H(G(m)), G^2(m), G(m)\right) + nV\left(H(G(m)), G^2(m), G(m)\right), \\ nH\left(G^2(m)\right) + nU\left(H\left(G^2(m)\right), G^3(m), G^2(m)\right) + (n-1)V\left(H\left(G^2(m)\right), G^3(m), G^2(m)\right) \end{pmatrix}$$

and

(A.16)

$$G^2(m) + m_2\begin{pmatrix} (n-1)H(m) + nU(H(m), G(m), m) + nV(H(m), G(m), m), \\ nH(G(m)) + (n-1)U\left(H(G(m)), G^2(m), G(m)\right) + nV\left(H(G(m)), G^2(m), G(m)\right), \\ nH\left(G^2(m)\right) + nU\left(H\left(G^2(m)\right), G^3(m), G^2(m)\right) + (n-1)V\left(H\left(G^2(m)\right), G^3(m), G^2(m)\right) \end{pmatrix} = M$$

where $m = m_{1,t-2}$, $G^2(m) = G \circ G(m)$ and $G^3(m) = G \circ G \circ G(m)$.

Equations (A.15) and (A.16) always hold at $m = \bar{m}_1$ because they reduce to:

$$\text{(A.17)} \qquad \bar{m}_1 = m_1\left((n-1)\bar{b}_1 + n\bar{b}_2 + n\bar{b}_3,\ n\bar{b}_1 + (n-1)\bar{b}_2 + n\bar{b}_3,\ n\bar{b}_1 + n\bar{b}_2 + (n-1)\bar{b}_3 \right)$$
$$\iff \bar{m}_1 = m_1\left(\bar{B}_{-1}, \bar{B}_{-2}, \bar{B}_{-3}\right)$$

and

$$\text{(A.18)}$$
$$\bar{m}_1 + m_2\left((n-1)\bar{b}_1 + n\bar{b}_2 + n\bar{b}_3,\ n\bar{b}_1 + (n-1)\bar{b}_2 + n\bar{b}_3,\ n\bar{b}_1 + n\bar{b}_2 + (n-1)\bar{b}_3 \right) = M$$
$$\iff \bar{m}_1 + m_2\left(\bar{B}_{-1}, \bar{B}_{-2}, \bar{B}_{-3}\right) = M$$

which hold true from (A.13) and (A.14).

Hence, the two linear forecast functions can be the solution to the equilibrium conditions at the steady-states in the deterministic case of the offer constrained OLG market game under the sell-all strategy.

A.3 Proposition 2

As we did in the deterministic case, we replace the bids arguments and plug the linear forecast functions in the equilibrium conditions under the exogenous shocks.

This procedure re-writes the optimal money holding of young agents born in time $t-1$ as:

$$\text{(A.19)} \quad G(m, s_{t-1})$$
$$= m_1 \left(\begin{array}{c} (n-1)H(m,s_{t-1}) + nU(H(m,s_{t-1}), G(m,s_{t-1}), m) + nV(H(m,s_{t-1}), G(m,s_{t-1}), m), \\ nH(G(m,s_{t-1}),h) + (n-1)U(H(G(m,s_{t-1}),h), G(G(m,s_{t-1}),h), G(m,s_{t-1})) \\ +nV(H(G(m,s_{t-1}),h), G(G(m,s_{t-1}),h), G(m,s_{t-1})), \\ nH(G(m,s_{t-1}),l) + (n-1)U(H(G(m,s_{t-1}),l), G(G(m,s_{t-1}),l), G(m,s_{t-1})) \\ +nV(H(G(m,s_{t-1}),l), G(G(m,s_{t-1}),l), G(m,s_{t-1})), \\ nH(G(G(m,s_{t-1}),h),h) + nU(H(G(G(m,s_{t-1}),h),h), G(G(G(m,s_{t-1}),h),h), G(G(m,s_{t-1}),h)) \\ +(n-1)V(H(G(G(m,s_{t-1}),h),h), G(G(G(m,s_{t-1}),h),h), G(G(m,s_{t-1}),h)), \\ nH(G(G(m,s_{t-1}),h),l) + nU(H(G(G(m,s_{t-1}),h),l), G(G(G(m,s_{t-1}),h),l), G(G(m,s_{t-1}),h)) \\ +(n-1)V(H(G(G(m,s_{t-1}),h),l), G(G(G(m,s_{t-1}),h),l), G(G(m,s_{t-1}),h)), \\ nH(G(G(m,s_{t-1}),l),h) + nU(H(G(G(m,s_{t-1}),l),h), G(G(G(m,s_{t-1}),l),h), G(G(m,s_{t-1}),l)) \\ +(n-1)V(H(G(G(m,s_{t-1}),l),h), G(G(G(m,s_{t-1}),l),h), G(G(m,s_{t-1}),l)), \\ nH(G(G(m,s_{t-1}),l),l) + nU(H(G(G(m,s_{t-1}),l),l), G(G(G(m,s_{t-1}),l),l), G(G(m,s_{t-1}),l)) \\ +(n-1)V(H(G(G(m,s_{t-1}),l),l), G(G(G(m,s_{t-1}),l),l), G(G(m,s_{t-1}),l)) \end{array} \right)$$

for $s_{t-1} \in \{\alpha, \beta\}$ where $m = m_{1,t-2}$.

The money market clearing condition in time t becomes:

$$\text{(A.20)}\quad M-G\left(G\left(m,s_{t-1}\right),s_{t}\right)$$
$$=m_{2}\begin{pmatrix}\left(n-1\right)H\left(m,s_{t-1}\right)+nU\left(H\left(m,s_{t-1}\right),G\left(m,s_{t-1}\right),m\right)+nV\left(H\left(m,s_{t-1}\right),G\left(m,s_{t-1}\right),m\right),\\ nH\left(G\left(m,s_{t-1}\right),h\right)+\left(n-1\right)U\left(H\left(G\left(m,s_{t-1}\right),h\right),G\left(G\left(m,s_{t-1}\right),h\right),G\left(m,s_{t-1}\right)\right)\\ +nV\left(H\left(G\left(m,s_{t-1}\right),h\right),G\left(G\left(m,s_{t-1}\right),h\right),G\left(m,s_{t-1}\right)\right),\\ nH\left(G\left(m,s_{t-1}\right),l\right)+\left(n-1\right)U\left(H\left(G\left(m,s_{t-1}\right),l\right),G\left(G\left(m,s_{t-1}\right),l\right),G\left(m,s_{t-1}\right)\right)\\ +nV\left(H\left(G\left(m,s_{t-1}\right),l\right),G\left(G\left(m,s_{t-1}\right),l\right),G\left(m,s_{t-1}\right)\right),\\ nH\left(G\left(G\left(m,s_{t-1}\right),h\right),h\right)+nU\left(H\left(G\left(G\left(m,s_{t-1}\right),h\right),h\right),G\left(G\left(G\left(m,s_{t-1}\right),h\right),h\right),G\left(G\left(m,s_{t-1}\right),h\right)\right)\\ +\left(n-1\right)V\left(H\left(G\left(G\left(m,s_{t-1}\right),h\right),h\right),G\left(G\left(G\left(m,s_{t-1}\right),h\right),h\right),G\left(G\left(m,s_{t-1}\right),h\right)\right),\\ nH\left(G\left(G\left(m,s_{t-1}\right),h\right),l\right)+nU\left(H\left(G\left(G\left(m,s_{t-1}\right),h\right),l\right),G\left(G\left(G\left(m,s_{t-1}\right),h\right),l\right),G\left(G\left(m,s_{t-1}\right),h\right)\right)\\ +\left(n-1\right)V\left(H\left(G\left(G\left(m,s_{t-1}\right),h\right),l\right),G\left(G\left(G\left(m,s_{t-1}\right),h\right),l\right),G\left(G\left(m,s_{t-1}\right),h\right)\right),\\ nH\left(G\left(G\left(m,s_{t-1}\right),l\right),h\right)+nU\left(H\left(G\left(G\left(m,s_{t-1}\right),l\right),h\right),G\left(G\left(G\left(m,s_{t-1}\right),l\right),h\right),G\left(G\left(m,s_{t-1}\right),l\right)\right)\\ +\left(n-1\right)V\left(H\left(G\left(G\left(m,s_{t-1}\right),l\right),h\right),G\left(G\left(G\left(m,s_{t-1}\right),l\right),h\right),G\left(G\left(m,s_{t-1}\right),l\right)\right),\\ nH\left(G\left(G\left(m,s_{t-1}\right),l\right),l\right)+nU\left(H\left(G\left(G\left(m,s_{t-1}\right),l\right),l\right),G\left(G\left(G\left(m,s_{t-1}\right),l\right),l\right),G\left(G\left(m,s_{t-1}\right),l\right)\right)\\ +\left(n-1\right)V\left(H\left(G\left(G\left(m,s_{t-1}\right),l\right),l\right),G\left(G\left(G\left(m,s_{t-1}\right),l\right),l\right),G\left(G\left(m,s_{t-1}\right),l\right)\right)\end{pmatrix}$$

for $\left(s_{t-1},s_t\right)\in\{\alpha,\beta\}^2$ where $m=m_{1,t-2}$.

The equilibrium conditions in (A.19) and (A.20) are a system of six equations in the six variables, $\left\{\bar{m}_1^{\alpha},\bar{m}_1^{\beta},\bar{b}_1^{\alpha},\bar{b}_1^{\beta},\gamma,\rho\right\}$. If we denote the equilibrium system by $Z:\mathbb{R}^6_{++}\to\mathbb{R}^6_{++}$, we want to show that the Jacobian matrix, DZ, has a full rank, i.e. $Z\pitchfork 0$, at the steady states under no exogenous shocks since we know that the linear forecast functions can be the solution to the equilibrium system in this case by Lemma 1. Then, the IFT will guarantee the existence of linear forecast functions in a neighborhood of the deterministic steady states in the offer constrained OLG market game under the sell-all strategy with a sufficiently small shock.

We now calculate the rank of DZ. To simplify the calculations, we allow the total money quantities to vary over each state, which we denote by M^{α} and M^{β} respectively. We include these variables in the rank calculation. Then, the transversal density theorem infers that $Z\pitchfork 0$ or the IFT applies for almost all money quantities. We introduce the following variables to describe the Jacobian matrix evaluated at the deterministic steady states where $m=\bar{m}_1^{\alpha}=\bar{m}_1^{\beta}=\bar{m}_1$ and $\bar{b}_1^{\alpha}=\bar{b}_1^{\beta}=\bar{b}_1$:

$$\begin{aligned}\text{(A.21)}\quad A_{\alpha\alpha}=&1-\gamma^2+m_{2,1}\left(-\rho\left(\frac{Q-q_1}{q_1}\right)+\left(1+\gamma\right)\frac{Q}{q_1}\right)\\ &+m_{2,2\alpha}\left(\rho\gamma\left(\frac{Q-q_2}{q_1}\right)+\left(1-\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-\left(1+\gamma\right)\right)\\ &+m_{2,2\beta}\left(\rho\left(1+\gamma\right)\left(\frac{Q-q_2}{q_1}\right)-\left(\gamma+\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-\left(1+\gamma\right)\right)\\ &+m_{2,3\alpha\alpha}\left(\left(-\rho\gamma^2+\left(1+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\ &+m_{2,3\alpha\beta}\left(\left(\rho\left(1-\gamma^2\right)-\left(\gamma-\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\ &+m_{2,3\beta\alpha}\left(\left(-\rho\left(1+\gamma+\gamma^2\right)+\left(1+\gamma+\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\ &+m_{2,3\beta\beta}\left(\left(-\rho\gamma\left(1+\gamma\right)+\left(\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\end{aligned}$$

$$
\begin{aligned}
A_{\alpha\beta} =& -\gamma-\gamma^2+m_{2,1}\left(-\rho\left(\frac{Q-q_1}{q_1}\right)+(1+\gamma)\frac{Q}{q_1}\right)\\
&+m_{2,2\alpha}\left(\rho\gamma\left(\frac{Q-q_2}{q_1}\right)+\left(1-\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&+m_{2,2\beta}\left(\rho\left(1+\gamma\right)\left(\frac{Q-q_2}{q_1}\right)-\left(\gamma+\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&+m_{2,3\alpha\alpha}\left(\left(-\rho\gamma^2+\left(1+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
&+m_{2,3\alpha\beta}\left(\left(\rho\left(1-\gamma^2\right)-\left(\gamma-\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
&+m_{2,3\beta\alpha}\left(\left(-\rho\left(1+\gamma+\gamma^2\right)+\left(1+\gamma+\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&+m_{2,3\beta\beta}\left(\left(-\rho\gamma\left(1+\gamma\right)+\left(\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
A_{\beta\alpha} =& 1+\gamma+m_{2,2\alpha}\left(-\rho\left(\frac{Q-q_2}{q_1}\right)+(1+\gamma)\left(\frac{Q-q_1-q_2}{q_1}\right)\right)\\
&+m_{2,3\alpha\alpha}\left(\left(\rho\gamma+\left(1-\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&+m_{2,3\alpha\beta}\left(\left(\rho\left(1+\gamma\right)-\left(\gamma+\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&+m_{2,3\beta\alpha}\left(-\rho+(1+\gamma)\right)\left(\frac{Q-q_3}{q_1}\right)\\
A_{\beta\beta} =& m_{2,2\alpha}\left(-\rho\left(\frac{Q-q_2}{q_1}\right)+(1+\gamma)\left(\frac{Q-q_1-q_2}{q_1}\right)\right)\\
&+m_{2,3\alpha\alpha}\left(\left(\rho\gamma+\left(1-\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&+m_{2,3\alpha\beta}\left(\left(\rho\left(1+\gamma\right)-\left(\gamma+\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&+m_{2,3\beta\alpha}\left(-\rho+(1+\gamma)\right)\left(\frac{Q-q_3}{q_1}\right)\\
B_{\alpha\alpha} =& m_{2,2\beta}\left(-\rho\left(\frac{Q-q_2}{q_1}\right)+(1+\gamma)\left(\frac{Q-q_1-q_2}{q_1}\right)\right)\\
&+m_{2,3\alpha\beta}\left(-\rho+(1+\gamma)\right)\left(\frac{Q-q_3}{q_1}\right)\\
&+m_{2,3\beta\alpha}\left(\left(\rho\left(1+\gamma\right)-\left(\gamma+\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&+m_{2,3\beta\beta}\left(\left(\rho\gamma+\left(1-\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)
\end{aligned}
$$

$$
\begin{aligned}
B_{\alpha\beta} =& 1+\gamma+m_{2,2\beta}\left(-\rho\left(\frac{Q-q_2}{q_1}\right)+(1+\gamma)\left(\frac{Q-q_1-q_2}{q_1}\right)\right)\\
&+m_{2,3\alpha\beta}\left(-\rho+(1+\gamma)\right)\left(\frac{Q-q_3}{q_1}\right)\\
&+m_{2,3\beta\alpha}\left(\left(\rho\left(1+\gamma\right)-\left(\gamma+\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&+m_{2,3\beta\beta}\left(\left(\rho\gamma+\left(1-\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
B_{\beta\alpha} =& -\gamma-\gamma^2+m_{2,1}\left(-\rho\left(\frac{Q-q_1}{q_1}\right)+(1+\gamma)\frac{Q}{q_1}\right)\\
&+m_{2,2\alpha}\left(\rho\left(1+\gamma\right)\left(\frac{Q-q_2}{q_1}\right)-\left(\gamma+\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&+m_{2,2\beta}\left(\rho\gamma\left(\frac{Q-q_2}{q_1}\right)+\left(1-\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&+m_{2,3\alpha\alpha}\left(\left(-\rho\gamma\left(1+\gamma\right)+\left(\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&+m_{2,3\alpha\beta}\left(\left(-\rho\left(1+\gamma+\gamma^2\right)+\left(1+\gamma+\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&+m_{2,3\beta\alpha}\left(\left(\rho\left(1-\gamma^2\right)-\left(\gamma-\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
&+m_{2,3\beta\beta}\left(\left(-\rho\gamma^2+\left(1+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
B_{\beta\beta} =& 1-\gamma^2+m_{2,1}\left(-\rho\left(\frac{Q-q_1}{q_1}\right)+(1+\gamma)\frac{Q}{q_1}\right)\\
&+m_{2,2\alpha}\left(\rho\left(1+\gamma\right)\left(\frac{Q-q_2}{q_1}\right)-\left(\gamma+\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&+m_{2,2\beta}\left(\rho\gamma\left(\frac{Q-q_2}{q_1}\right)+\left(1-\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&+m_{2,3\alpha\alpha}\left(\left(-\rho\gamma\left(1+\gamma\right)+\left(\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&+m_{2,3\alpha\beta}\left(\left(-\rho\left(1+\gamma+\gamma^2\right)+\left(1+\gamma+\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&+m_{2,3\beta\alpha}\left(\left(\rho\left(1-\gamma^2\right)-\left(\gamma-\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
&+m_{2,3\beta\beta}\left(\left(-\rho\gamma^2+\left(1+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
F_\alpha =& m_{2,1} + m_{2,2\alpha}\left(\frac{Q-q_2}{q_1}\right) + m_{2,3\alpha\alpha}\left(\frac{Q-q_3}{q_1}\right) + m_{2,3\beta\alpha}\left(\frac{Q-q_3}{q_1}\right) \\
F_\beta =& m_{2,2\beta}\left(\frac{Q-q_2}{q_1}\right) + m_{2,3\alpha\beta}\left(\frac{Q-q_3}{q_1}\right) + m_{2,3\beta\beta}\left(\frac{Q-q_3}{q_1}\right) \\
G_\alpha =& m_{2,2\alpha}\left(\frac{Q-q_2}{q_1}\right) + m_{2,3\alpha\alpha}\left(\frac{Q-q_3}{q_1}\right) + m_{2,3\beta\alpha}\left(\frac{Q-q_3}{q_1}\right) \\
G_\beta =& m_{2,1} + m_{2,2\beta}\left(\frac{Q-q_2}{q_1}\right) + m_{2,3\alpha\beta}\left(\frac{Q-q_3}{q_1}\right) + m_{2,3\beta\beta}\left(\frac{Q-q_3}{q_1}\right) \\
L_\alpha =& 0 \\
L_\beta =& 0 \\
M_{\alpha\alpha} =& 0 \\
M_{\alpha\beta} =& 0 \\
M_{\beta\alpha} =& 0 \\
M_{\beta\beta} =& 0 \\
H_\alpha =& 1+\gamma - m_{1,1}\left(-\rho\left(\frac{Q-q_1}{q_1}\right) + (1+\gamma)\frac{Q}{q_1}\right) \\
& - m_{1,2\alpha}\left(\rho\gamma\left(\frac{Q-q_2}{q_1}\right) + \left(1-\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right) - (1+\gamma)\right) \\
& - m_{1,2\beta}\left(\rho(1+\gamma)\left(\frac{Q-q_2}{q_1}\right) - \left(\gamma+\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right) - (1+\gamma)\right) \\
& - m_{1,3\alpha\alpha}\left(\left(-\rho\gamma^2 + \left(1+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right) + \left(1-\gamma^2\right)\right) \\
& - m_{1,3\alpha\beta}\left(\left(\rho\left(1-\gamma^2\right) - \left(\gamma-\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right) + \left(1-\gamma^2\right)\right) \\
& - m_{1,3\beta\alpha}\left(\left(-\rho\left(1+\gamma+\gamma^2\right) + \left(1+\gamma+\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right) - \left(\gamma+\gamma^2\right)\right) \\
& - m_{1,3\beta\beta}\left(\left(-\rho\gamma(1+\gamma) + \left(\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right) - \left(\gamma+\gamma^2\right)\right) \\
H_\beta =& - m_{1,2\alpha}\left(-\rho\left(\frac{Q-q_2}{q_1}\right) + (1+\gamma)\left(\frac{Q-q_1-q_2}{q_1}\right)\right) \\
& - m_{1,3\alpha\alpha}\left(\left(\rho\gamma + \left(1-\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right) + (1+\gamma)\right) \\
& - m_{1,3\alpha\beta}\left(\left(\rho(1+\gamma) - \left(\gamma+\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right) + (1+\gamma)\right) \\
& - m_{1,3\beta\alpha}\left(-\rho + (1+\gamma)\right)\left(\frac{Q-q_3}{q_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
I_\alpha =& - m_{1,2\beta}\left(-\rho\left(\frac{Q-q_2}{q_1}\right)+(1+\gamma)\left(\frac{Q-q_1-q_2}{q_1}\right)\right)\\
&- m_{1,3\alpha\beta}\left(-\rho+(1+\gamma)\right)\left(\frac{Q-q_3}{q_1}\right)\\
&- m_{1,3\beta\alpha}\left(\left(\rho\left(1+\gamma\right)-\left(\gamma+\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
&- m_{1,3\beta\beta}\left(\left(\rho\gamma+\left(1-\gamma^2\right)\right)\left(\frac{Q-q_3}{q_1}\right)+(1+\gamma)\right)\\
I_\beta =& 1+\gamma - m_{1,1}\left(-\rho\left(\frac{Q-q_1}{q_1}\right)+(1+\gamma)\frac{Q}{q_1}\right)\\
&- m_{1,2\alpha}\left(\rho\left(1+\gamma\right)\left(\frac{Q-q_2}{q_1}\right)-\left(\gamma+\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&- m_{1,2\beta}\left(\rho\gamma\left(\frac{Q-q_2}{q_1}\right)+\left(1-\gamma^2\right)\left(\frac{Q-q_1-q_2}{q_1}\right)-(1+\gamma)\right)\\
&- m_{1,3\alpha\alpha}\left(\left(-\rho\gamma\left(1+\gamma\right)+\left(\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&- m_{1,3\alpha\beta}\left(\left(-\rho\left(1+\gamma+\gamma^2\right)+\left(1+\gamma+\gamma^2+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)-\left(\gamma+\gamma^2\right)\right)\\
&- m_{1,3\beta\alpha}\left(\left(\rho\left(1-\gamma^2\right)-\left(\gamma-\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
&- m_{1,3\beta\beta}\left(\left(-\rho\gamma^2+\left(1+\gamma^3\right)\right)\left(\frac{Q-q_3}{q_1}\right)+\left(1-\gamma^2\right)\right)\\
J_\alpha =& - m_{1,1} - m_{1,2\alpha}\left(\frac{Q-q_2}{q_1}\right) - m_{1,3\alpha\alpha}\left(\frac{Q-q_3}{q_1}\right) - m_{1,3\beta\alpha}\left(\frac{Q-q_3}{q_1}\right)\\
J_\beta =& - m_{1,2\alpha}\left(\frac{Q-q_2}{q_1}\right) - m_{1,3\alpha\alpha}\left(\frac{Q-q_3}{q_1}\right) - m_{1,3\beta\alpha}\left(\frac{Q-q_3}{q_1}\right)\\
K_\alpha =& - m_{1,2\beta}\left(\frac{Q-q_2}{q_1}\right) - m_{1,3\alpha\beta}\left(\frac{Q-q_3}{q_1}\right) - m_{1,3\beta\beta}\left(\frac{Q-q_3}{q_1}\right)\\
K_\beta =& - m_{1,1} - m_{1,2\beta}\left(\frac{Q-q_2}{q_1}\right) - m_{1,3\alpha\beta}\left(\frac{Q-q_3}{q_1}\right) - m_{1,3\beta\beta}\left(\frac{Q-q_3}{q_1}\right)\\
N_\alpha =& 0\\
N_\beta =& 0\\
O_\alpha =& 0\\
O_\beta =& 0
\end{aligned}
$$

where $m_{i,1} = \frac{\partial m_i}{\partial B^{s_t}_{t,-1}}$, $m_{i,2s} = \frac{\partial m_i}{\partial B^{s_{t+1}}_{t+1,-2}|_{s_{t+1}=s}}$, and $m_{i,3ss'} = \frac{\partial m_i}{\partial B^{s_{t+2}}_{t+2,-3}|_{(s_{t+1},s_{t+2})=(s,s')}}$ for $\forall i, s$ and s'.

We note the following relationships among variables above: $A_{\alpha\alpha} - A_{\alpha\beta} = A_{\beta\alpha} - A_{\beta\beta} = B_{\alpha\beta} - B_{\alpha\alpha} = B_{\beta\beta} - B_{\beta\alpha} = 1+\gamma$, $A_{\alpha\beta} - A_{\beta\beta} = B_{\beta\beta} - B_{\alpha\beta}$, $F_\alpha - F_\beta = G_\beta - G_\alpha = m_{m1}$, $H_\alpha - H_\beta = I_\beta - I_\alpha$ and $J_\beta - J_\alpha = K_\alpha - K_\beta = m_{y1}$.

With the notations above, we write the Jacobian matrix with respect to the parameters in the linear forecast functions and the total money quantity parameters evaluated at the deterministic steady states:

$$\text{(A.22)} \qquad DZ = \begin{array}{c} \begin{matrix} \partial\bar{m}_1^\alpha & \partial\bar{m}_1^\beta & \partial\bar{b}_1^\alpha & \partial\bar{b}_1^\beta & \partial\rho & \partial\gamma & \partial\bar{M}^\alpha & \partial\bar{M}^\beta \end{matrix} \\ \begin{bmatrix} A_{\alpha\alpha} & B_{\alpha\alpha} & F_\alpha & G_\alpha & L_\alpha & M_{\alpha\alpha} & -1 & 0 \\ A_{\alpha\beta} & B_{\alpha\beta} & F_\alpha & G_\alpha & L_\alpha & M_{\alpha\beta} & 0 & -1 \\ A_{\beta\alpha} & B_{\beta\alpha} & F_\beta & G_\beta & L_\beta & M_{\beta\alpha} & -1 & 0 \\ A_{\beta\beta} & B_{\beta\beta} & F_\beta & G_\beta & L_\beta & M_{\beta\beta} & 0 & -1 \\ H_\alpha & I_\alpha & J_\alpha & K_\alpha & N_\alpha & O_\alpha & 0 & 0 \\ H_\beta & I_\beta & J_\beta & K_\beta & N_\beta & O_\beta & 0 & 0 \end{bmatrix} \end{array}$$

We can re-write this matrix as:

$$\text{(A.23)} \qquad DZ = \begin{bmatrix} A_{\alpha\alpha} & B_{\alpha\alpha} & F_\alpha & G_\alpha & 0 & 0 & -1 & 0 \\ A_{\alpha\beta} & B_{\alpha\beta} & F_\alpha & G_\alpha & 0 & 0 & 0 & -1 \\ A_{\beta\alpha} & B_{\beta\alpha} & F_\beta & G_\beta & 0 & 0 & -1 & 0 \\ A_{\beta\beta} & B_{\beta\beta} & F_\beta & G_\beta & 0 & 0 & 0 & -1 \\ H_\alpha & I_\alpha & J_\alpha & K_\alpha & 0 & 0 & 0 & 0 \\ H_\beta & I_\beta & J_\beta & K_\beta & 0 & 0 & 0 & 0 \end{bmatrix}$$

We reduce this matrix via row and column operations. Subtract the second and fourth rows from the first and third rows, respectively. Then, subtract the first row with the third row and clear the third row to get:

$$\text{(A.24)} \qquad DZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\alpha\beta} & B_{\alpha\beta} & F_\alpha & G_\alpha & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ A_{\beta\beta} & B_{\beta\beta} & F_\beta & G_\beta & 0 & 0 & 0 & -1 \\ H_\alpha & I_\alpha & J_\alpha & K_\alpha & 0 & 0 & 0 & 0 \\ H_\beta & I_\beta & J_\beta & K_\beta & 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtract the fourth row from the second row and the sixth row from the fifth row and then, clear the fourth row to obtain:

$$\text{(A.25)} \qquad DZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\alpha\beta} - A_{\beta\beta} & B_{\alpha\beta} - B_{\beta\beta} & m_{m1} & -m_{m1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ H_\alpha - H_\beta & I_\alpha - I_\beta & -m_{y1} & m_{y1} & 0 & 0 & 0 & 0 \\ H_\beta & I_\beta & J_\beta & K_\beta & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add the second and fourth columns to the first and third columns, respectively. Then, clear the last row to derive:

$$\text{(A.26)} \qquad DZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{\alpha\beta} - B_{\beta\beta} & 0 & -m_{m1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & I_\alpha - I_\beta & 0 & m_{y1} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following submatrix in DZ has rank 2:

$$\text{(A.27)} \qquad \begin{bmatrix} B_{\alpha\beta} - B_{\beta\beta} & -m_{m1} \\ I_{\alpha} - I_{\beta} & m_{y1} \end{bmatrix}$$

Therefore, the reduced Jacobian matrix is given by:

$$\text{(A.28)} \qquad DZ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank 5.

There is, however, a functional dependency between γ and ρ, since

$$\text{(A.29)} \qquad \frac{m_{1,t}^{s_t} - \bar{m}_1^{s_t}}{b_{1,t}^{s_t} - \bar{b}_1^{s_t}} = -\frac{\gamma\left(m - \bar{m}_1^{s_t}\right)}{\rho\left(m - \bar{m}_1^{s_t}\right)} = -\frac{\gamma}{\rho}$$

$$\Longleftrightarrow \gamma = -\rho\left(\frac{m_{1,t}^{s_t} - \bar{m}_1^{s_t}}{b_{1,t}^{s_t} - \bar{b}_1^{s_t}}\right)$$

which implies that we have at most five independent equations in five variables.

Thus, the IFT applies from the rank calculation and there exist recursive Markov Nash equilibria generated by linear forecast functions in a neighborhood of the deterministic steady states for the offer constrained OLG market game under sell-all strategy with sufficiently small shocks.

A.4 Proposition 3

We show that perfect consumption smoothing cannot be the outcome of the steady states in the deterministic OLG market game with $\delta = 1$ using the method of contradiction.

Thus, we start assuming that the perfectly smoothed consumptions are stationary allocations, i.e. $c_1 = c_2 = c_3$. Then, $b_1 = b_2 = b_3$ at the steady states by the consumption allocation rule under the sell-all strategy. The first-order conditions require $q_1 = q_2 = q_3$ for the equal lifetime consumptions to be individual optimal choices. The relationships among consumptions, bids and offers indicate $m_1 = m_2 = 0$ from the budget constraints which is contradictory to the money market clearing condition where $m_1 + m_2 = M > 0$. Therefore, the perfect consumption smoothing does not hold at the steady states for any endowment streams in the deterministic OLG economy with $\delta = 1$ if there are strategic interactions.

The steady-states consumptions rather vary over ages since agents consume more in a certain age depending on their income as follows. We first write the Euler equations at the stationary allocations under the sell-all strategy:

$$\text{(A.30)} \qquad u'\left(\frac{b_1}{B}\Omega\right)\frac{B_{-1}}{\Omega_{-1}} = u'\left(\frac{b_2}{B}\Omega\right)\frac{B_{-2}}{\Omega_{-2}} = u'\left(\frac{b_3}{B}\Omega\right)\frac{B_{-3}}{\Omega_{-3}}$$

where $\Omega = n(\omega_1 + \omega_2 + \omega_3)$ and $\Omega_{-i} = \Omega - \omega_i$ for $i \in \{1, 2, 3\}$.

Without the loss of generality, we normalize the total bid to be unity. Then, (A.30) becomes:

$$\frac{u'(b_1\Omega)(1-b_1)}{\Omega_{-1}} = \frac{u'(b_2\Omega)(1-b_2)}{\Omega_{-2}} = \frac{u'(b_3\Omega)(1-b_3)}{\Omega_{-3}} \tag{A.31}$$

$u'(x\Omega)(1-x)$ is strictly decreasing in x for $\forall x \in [0,1]$ under the standard assumptions on the utility function. We focus on the cases where households' bids are positive to consume normal goods.$\frac{1}{\Omega_{-i}}$ is proportional to the endowment of age i. Thus, b_i should be larger for ages with higher endowments to satisfy (A.31). This relationthip results in larger consumptions in ages with higher endowments.

Intuitively speaking, agents face lower marginal prices when offering more and thus, they transfer their wealth to ages with large endowments to purchase more goods cheaper. Therefore, the consumption stream parallels the income streams at the stationary equilibria under the strategic interactions.

A.5 Corollary 1

According to Proposition 3, consumptions are proportional to endowments at the stationary equilibria. Thus, when $\omega_2 > \omega_1 > \omega_3 = 0$, we obtain that $c_2 > c_1 > c_3$. We first consider the case that $c_2 \leq \frac{1}{3}(\omega_1 + \omega_2)$, then $c_1 < \frac{1}{3}(\omega_1 + \omega_2)$ and $c_3 < \frac{1}{3}(\omega_1 + \omega_2)$ because of the order of the lifetime consumptions. This case violates the good market clearing condition because $c_1 + c_2 + c_3 < \omega_1 + \omega_2$. Therefore, $c_2 > \frac{1}{3}(\omega_1 + \omega_2)$. Likewise, $c_3 < \frac{1}{3}(\omega_1 + \omega_2)$. If not, the good market clearing condition will be violated. Unlike c_2 and c_3, c_1 can be larger than, equal to or smaller than $\frac{1}{3}(\omega_1 + \omega_2)$ which depends on endowment ratios between the young and the middle-aged and utility functions.

To characterize the condition for c_1 to be lager than $\frac{1}{3}(\omega_1 + \omega_2)$, we consider the endowment structure with $\omega_1 = \omega_2 = \frac{\Omega}{2n}$. In this case, the first equality in (A.31) becomes $u'(b_1\Omega)(1-b_1) = u'(b_2\Omega)(1-b_2)$. To satisfy this equation, $b_1 = b_2$ since $u'(x\Omega)(1-x)$ is strictly decreasing in x. This result implies that $c_1 = c_2 > \frac{1}{3}(\omega_1 + \omega_2)$. By continuity, $c_1 > \frac{1}{3}(\omega_1 + \omega_2)$ if ω_1 is smaller than but close enough to $\frac{\Omega}{2n}$.

The endowment structure with $\omega_1 = \frac{\Omega}{3n}$ and $\omega_2 = \frac{2\Omega}{3n}$ transforms (A.31) to:

$$\frac{u'(b_1\Omega)(1-b_1)}{3n-1} = \frac{u'(b_2\Omega)(1-b_2)}{3n-2} = \frac{u'(b_3\Omega)(1-b_3)}{3n} \tag{A.32}$$

This equation indicates that $u'(b_1\Omega)(1-b_1) = \frac{u'(b_2\Omega)(1-b_2)+u'(b_3\Omega)(1-b_3)}{2}$. $u'(x\Omega)(1-x)$ is strictly convex in x from the assumption that $u'''(\cdot) > 0$. Then, by Jensen's inequality, $b_1 < \frac{b_2+b_3}{2}$. From this result, we know that $b_1 < \frac{B}{3n}$ and thus, $c_1 < \frac{1}{3}(\omega_1 + \omega_2)$. Hence, if ω_1 is larger than but sufficiently close to $\frac{\Omega}{3n}$, $c_1 < \frac{1}{3}(\omega_1 + \omega_2)$ by continuity.

B Algorithm

In this appendix, we explain how to compute the symmetric recursive Markov Nash equilibria in the three period SOLG models with the Shapley-Shubik market game. We adopt the projection method to approximate the equilibrium policy functions with high-degree Chebyshev polynomials. It is enough to interpolate the young's money demand function and the bidding functions for all ages because the money demands of the middle-aged are redundant with the young's ones from the money market clearing condition. We use the young's money holdings as a unique endogenous state variable. We apply a Newton-type nonlinear equation solver to find the coefficients of the policy functions satisfying the equilibrium conditions.

We summarize the algorithm in step by step as follows.

1. Define the type of polynomials to approximate the equilibrium policy functions.
   - In this paper, we use the Chebyshev polynomials of which domain is $[-1,1]$.
2. Set the degree of polynomials, N, for the unique endogenous state variable in the policy functions.
   - We use the same degree of polynomials for all approximated functions, $\{T_k(\cdot)\}_{k=0}^N$.
   - $\{\theta_k^{m_1,s}\}_{k=0}^N$ and $\left\{\theta_k^{b_i,s}\right\}_{k=0}^N$ are the coefficients of the policy functions for the young's money holding and the bidding in age i in state s.
   - The total number of coefficients is (# of policy functions) $\times$ (# of states) $\times$ $(N+1)$ = $4 \times 2 \times (N+1)$.
3. Generate $(N+1)$ nodes to apply the projection method.
   - We produce $(N+1)$ Chebyshev grids in $[-1,1]$.
4. Approximate the policy functions as follows.

(B.1)
$$\widehat{m_1^s} = \sum_{k=0}^{N} \theta_k^{m_1,s} T_k\left(\tilde{m}_{1,-1}\right)$$

(B.2)
$$\widehat{b_i^s} = \sum_{k=0}^{N} \theta_k^{b_i,s} T_k\left(\tilde{m}_{1,-1}\right) \text{ for } i \in \{1,2,3\}$$

   where $\tilde{m}_{1,-1}$ is the young's money holding in the previous period which is transformed into $[-1,1]$ using an appropriate interval for the unique endogenous state variable, $[m_{1,min}, m_{1,max}]$.
5. Solve for the state-dependent coefficients satisfying the equilibrium conditions on the grids from Step 3.
   - In this problem, the equilibrium conditions consist of two first-order conditions for the young and the middle-aged combined with the consumption allocation rule and two independent individual budget constraints. Note that the last budget constraint is linearly dependent on the other two.
   - We construct a system of non-linear equations by evaluating the policy functions on each grid and inserting them into the equilibrium conditions. Then, we use a Newton-type solver to find the state-dependent coefficients at once.

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