Valid Inequalities for Mixed-Integer Linear and Mixed-Integer Conic Programs

Sercan Yıldız

April 15, 2016

Tepper School of Business
Carnegie Mellon University
Pittsburgh, PA 15213

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Algorithms, Combinatorics, and Optimization.

Keywords: Integer programming, valid inequalities, cutting-planes, convexification, separation

Contents

List of Figures

Abstract

Mixed-integer programming provides a natural framework for modeling optimization problems which require discrete decisions. Valid inequalities, used as cutting-planes and cutting-surfaces in integer programming solvers, are an essential part of today's integer programming technology. They enable the solution of mixed-integer programs of greater scale and complexity by providing tighter mathematical formulations of the feasible solution set. This dissertation presents new structural results on general-purpose valid inequalities for mixed-integer linear and mixed-integer conic programs.

Cut-generating functions are a priori formulas for generating a cut from the data of a mixed-integer linear program. This concept has its roots in the work of Gomory and Johnson from the 1970s. It has received renewed attention in the past few years. Gomory and Johnson studied cut-generating functions for the corner relaxation, which is obtained by ignoring the nonnegativity constraints on the basic variables in a tableau formulation. We consider models where these constraints are not ignored. In our first contribution, we generalize a classical result of Gomory and Johnson characterizing minimal cut-generating functions in terms of subadditivity, symmetry, and periodicity. Our analysis also exposes shortcomings in the usual definition of minimality in our general setting. To remedy this, we consider stronger notions of minimality and show that these impose additional structure on cut-generating functions. A stronger notion than the minimality of a cut-generating function is its extremality. While extreme cut-generating functions produce powerful cutting-planes, their structure can be very complicated. Gomory and Johnson identified a "simple" class of extreme cut-generating functions for the corner relaxation of a one-row integer linear program by showing that continuous, piecewise linear, minimal cut-generating functions with only two distinct slope values are extreme. In our second contribution, we establish a similar result for a one-row problem which takes the nonnegativity constraint on the basic variable into account. In our third contribution, we consider a related model where only nonbasic continuous variables are present. Conforti, Cornuéjols, Daniilidis, Lemaréchal, and Malick recently showed that not all cutting-planes can be obtained from cut-generating functions in this framework. They also conjectured a natural condition under which cut-generating functions might be sufficient. In our third contri-

bution, we prove that this conjecture is true. This justifies the recent research interest in cut-generating functions for this model.

Despite the power of mixed-integer linear programming, many optimization problems of practical and theoretical interest cannot be modeled using a linear objective function and constraints alone. Next, we turn to a natural generalization of mixed-integer linear programming which allows nonlinear convex constraints: mixed-integer conic programming. Disjunctive inequalities, introduced by Balas in the context of mixed-integer linear programming in the 1970s, have been a principal ingredient to the practical success of integer programming in the last two decades. In order to extend our understanding of disjunctive inequalities to mixed-integer conic programming, we pursue a principled study of two-term disjunctions on conic sets. In our fourth contribution, we consider two-term disjunctions on a general regular cone. A result of Kılınç-Karzan indicates that conic minimal valid linear inequalities are all that is needed for a closed convex hull description of such sets. First we characterize the structure of conic minimal and tight valid linear inequalities for the disjunction. Then we develop structured nonlinear valid inequalities for the disjunction by grouping subsets of valid linear inequalities. We analyze the structure of these inequalities and identify conditions which guarantee that a single such inequality characterizes the closed convex hull of the disjunction. In our fifth and sixth contributions, we specialize our earlier results to the cases where the regular cone under consideration is a direct product of second order cones and nonnegative rays and where it is the positive semidefinite cone. These cases deserve attention because of their importance for mixed-integer second-order cone and mixed-integer semidefinite programming. We identify conditions under which our valid convex inequalities can be expressed in computationally tractable forms and present techniques to generate low-complexity relaxations when these conditions are not satisfied. In our final contribution, we provide closed convex hull descriptions for homogeneous two-term disjunctions on the second-order cone and general two-term disjunctions on affine cross-sections of the second-order cone, extending the aforementioned results in two directions. Our results yield strong convex disjunctive inequalities which can be used as cutting-surfaces in generic mixed-integer conic programming solvers.

Chapter 1

Introduction

1.1 Mixed-Integer Linear Programming

Mixed-integer linear programming is a natural framework for modeling optimization problems which require discrete decisions. In a mixed-integer linear program, we optimize a linear function of the decision variables over a set defined by linear equations, nonnegativity constraints, and integrality constraints on a subset of the decision variables. To be precise, a *mixed-integer linear program (MILP)* is a problem of the form

$$\begin{aligned} \text{minimize} \quad & d^\top x && (1.1a)\\ \text{subject to} \quad & Ax = b, && (1.1b)\\ & x \in \mathbb{R}^n_+, && (1.1c)\\ & x_j \in \mathbb{Z} \quad \forall j \in \mathbb{J}, && (1.1d) \end{aligned}$$

where A is an $m \times n$ rational matrix, d and b are rational vectors of appropriate dimensions, and $\mathbb{J} \subset \{1, \ldots, n\}$. The set of feasible solutions to (1.1) is

$$\mathbb{C}_I = \{x \in \mathbb{R}^n_+ : \quad Ax = b, \quad x_j \in \mathbb{Z}\ \forall j \in \mathbb{J}\}.$$

In this section we give a short overview of mixed-integer linear programming. For a more detailed introduction to the topic, the reader is referred to the excellent textbooks [46, 93, 97].

The modeling flexibility of mixed-integer linear programming allows many problems of practical and theoretical interest to be cast as mixed-integer linear programs. The real-world impact of mixed-integer linear programming can be seen in almost every sector of business from healthcare to energy, as well as in science and engineering. Although mixed-integer linear programming is NP-hard in general, the last two decades have seen a tremendous improvement in our ability to solve mixed-integer linear programs. State-of-the-art integer programming solvers such as CPLEX [1], Gurobi [2], and Xpress [4] can routinely handle problems of scale and complexity that was considered impossible in the 1990s. This improvement is a result of significant advances in our understanding of linear and mixed-integer linear programs, together with the availability of increased computing power [31]. Therefore, further theoretical study of mixed-integer linear programming has the potential to bring problems that remain challenging for today's technology within the power of computation in the future.

Arguably, the most successful approach to solving mixed-integer linear programs relies on a combination of two algorithmic ideas, *branch-and-bound* and *cutting-planes*. This approach, which is termed *branch-and-cut*, exploits the fact that linear programming is both theoretically and practically well-understood. To this end, one considers the natural *continuous relaxation* of (1.1) which is obtained after dropping the integrality constraints (1.1d) from the formulation (1.1):

$$\text{minimize} \quad d^\top x \tag{1.2a}$$
$$\text{subject to} \quad Ax = b, \tag{1.2b}$$
$$x \in \mathbb{R}^n_+. \tag{1.2c}$$

The problem (1.2) is a *linear program* and can be solved efficiently. Its set of feasible solutions $\mathbb{C} = \{x \in \mathbb{R}^n_+ : \ Ax = b\}$ is a *polyhedron*. With slight abuse of terminology, we also call $\mathbb{C}$ the continuous relaxation of $\mathbb{C}_I$. The problem (1.2) is indeed a relaxation of (1.1); its optimal value yields a lower bound on the optimal value of (1.1). Furthermore, if the optimal solution x^* to (1.2) satisfies the integrality constraints (1.1d), it is the optimal solution to (1.1). However, the optimal solution x^* is often fractional and does not satisfy the integrality constraints. In order to make progress towards finding an optimal solution to (1.1), it then becomes necessary to exclude the fractional solution x^* from consideration and work with tighter relaxations of (1.1). Branch-and-bound and cutting-planes represent two strategies towards achieving this outcome.

The branch-and-bound method prescribes a systematic tree search of the feasible solution set $\mathbb{C}_I$. The algorithm searches for the optimal solution to (1.1) as it successively divides $\mathbb{C}$ into smaller sets. At the root node of the search tree, the continuous relaxation (1.2) is solved and the optimal solution x^* is found. If x^* satisfies the integrality constraints (1.1d), the optimal solution to (1.1) has been found and the algorithm stops. Otherwise, $\mathbb{C}$ is split into polyhedral subsets $\mathbb{C}_1, \dots, \mathbb{C}_k$ whose union contains the set $\mathbb{C}_I$, but not the fractional solution x^*. The procedure is repeated in each of the subsets $\mathbb{C}_1, \dots, \mathbb{C}_k$. Figure 1.1 illustrates this branching step: The two sets, $\mathbb{C}_1$ and $\mathbb{C}_2$, are created by requiring that an integer-constrained variable, say x, takes values that are less than or equal k in $\mathbb{C}_1$ and greater than or equal to $k + 1$ in $\mathbb{C}_2$, for some integer k. The sets $\mathbb{C}_1$ and $\mathbb{C}_2$ are depicted in dark blue. The branch-and-bound method also takes advantage of information obtained from the linear programs $\min\{d^\top x : x \in \mathbb{C}_i\}$ to guide its search: Because the optimal value of the linear program $\min\{d^\top x : x \in \mathbb{C}_i\}$ provides a lower bound on that of $\min\{d^\top x : x \in \mathbb{C}_i, x_j \in \mathbb{Z}\ \forall j \in \mathbb{J}\}$, the algorithm discards a subset $\mathbb{C}_i$ if the optimal value of $\min\{d^\top x : x \in \mathbb{C}_i\}$ is too large.

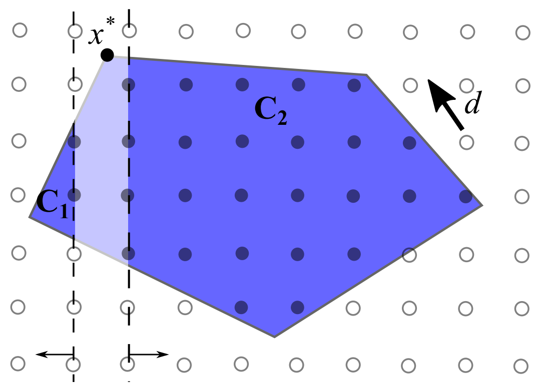


Figure 1.1: The branch-and-bound method for MILPs.

The cutting-plane method strives to strengthen the mathematical description of $\mathbb{C}$ with new linear inequalities that are satisfied by all feasible solutions in $\mathbb{C}_I$. Such an inequality is said to be a *valid inequality* for $\mathbb{C}_I$. In the cutting-plane method, first the continuous relaxation (1.2) is solved. If the optimal solution x^* to (1.2) satisfies the integrality constraints (1.1d), the optimal solution to (1.1) has been found. Otherwise, one has to find a linear inequality which is valid for $\mathbb{C}_I$ but strictly separates the fractional solution x^* from $\mathbb{C}_I$. Such a valid inequality is called a *cutting-plane*, or a *cut*. The addition of this cutting-plane to the description of $\mathbb{C}$ leads to a tighter approximation of $\mathbb{C}_I$, and the procedure is repeated. In Figure 1.2, the set $\mathbb{C}$ is depicted in dark blue, whereas the halfspace associated with a recently-added cutting-plane is depicted in light red. Note that this cutting-plane separates x^* from $\mathbb{C}_I$. The intersection of the blue and red regions corresponds to the new strengthened formulation.

Although a classical result in integer programming states that the mixed-integer linear program (1.1) can be solved after adding a finite number of cutting-planes to the continuous relaxation (1.2) and thus after a finite number of iterations of the cutting-plane method [87], it is commonly observed that algorithms that rely solely on the cutting-plane method do not perform well in practice. Combining cutting-planes and branch-and-bound in a branch-and-cut framework, on the other hand, can be highly effective. This approach has

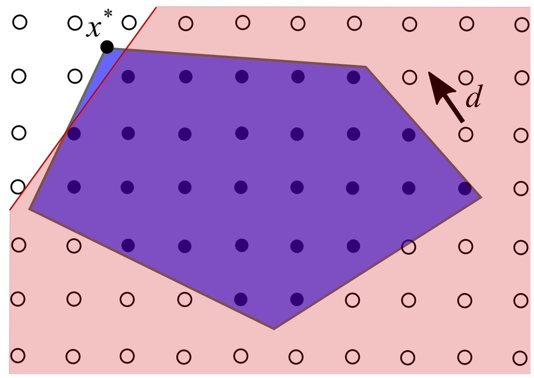


Figure 1.2: The cutting-plane method for MILPs.

been the principal solution method in mixed-integer linear programming computation since the 1990s and is used in today's leading integer programming solvers.

1.2 Mixed-Integer Conic Programming

A natural generalization of mixed-integer linear programming is mixed-integer conic programming. Let $\mathbb{E}$ be an n-dimensional Euclidean space which has the inner product $\langle\cdot,\cdot\rangle$. Any such space $(\mathbb{E}, \langle\cdot,\cdot\rangle)$ is isomorphic to $(\mathbb{R}^n, {}^\top)$; in order to keep the notation simple and similar to (1.1), we assume here that $\mathbb{E} = \mathbb{R}^n$ and $\langle\alpha, x\rangle = \alpha^\top x$. A *mixed-integer conic program (MICP)* is a problem of the form

$$\begin{aligned} \text{minimize} \quad & d^\top x && (1.3a) \\ \text{subject to} \quad & Ax = b, && (1.3b) \\ & x \in \mathbb{K}, && (1.3c) \\ & x_j \in \mathbb{Z} \quad \forall j \in \mathbb{J}, && (1.3d) \end{aligned}$$

where $\mathbb{K} \subset \mathbb{R}^n$ is a regular (closed, convex, full-dimensional, and pointed) cone, A is an $m \times n$ real matrix, d and b are real vectors of appropriate dimensions, and $\mathbb{J} \subset \{1, \dots, n\}$. Examples of regular cones include the nonnegative orthant $\mathbb{R}^k_+ = \Big\{x \in \mathbb{R}^k : \ x_j \geq 0 \ \forall j \in \{1, \dots, k\}\Big\}$, the second-order cone $\mathbb{L}^k = \Big\{x \in \mathbb{R}^k : \ \sqrt{x_1^2 + \ldots + x_{k-1}^2} \leq x_k\Big\}$, the positive semidefinite cone $\mathbb{S}^k_+ = \Big\{x \in \mathbb{R}^{k\times k} : \ x^\top = x, \ a^\top x\, a \geq 0 \ \forall a \in \mathbb{R}^k\Big\}$, and their direct products. Mixed-integer linear programming is the special case of mixed-integer conic programming where $\mathbb{K} = \mathbb{R}^n_+$. Other important special cases of mixed-integer conic programming include mixed-integer second-order cone programming, where $\mathbb{K}$ is a direct product of second-order cones and nonnegative rays, and mixed-integer semidefinite programming, where $\mathbb{K}$ is the positive semidefinite cone. The set of feasible solutions to (1.3) is

$$\mathbb{C}_I = \{x \in \mathbb{K} : \quad Ax = b, \quad x_j \in \mathbb{Z}\ \forall j \in \mathbb{J}\}.$$

The natural *continuous relaxation* of (1.3) is obtained after dropping the integrality constraints (1.3d):

$$\begin{aligned} &\text{minimize} && d^\top x && (1.4a)\\ &\text{subject to} && Ax = b, && (1.4b)\\ & && x \in \mathbb{K}. && (1.4c) \end{aligned}$$

The problem (1.4) is a *conic program.* It generalizes linear, second-order cone, and semidefinite programs and can be solved efficiently in these cases [28, 36]. The continuous relaxation of $\mathbb{C}_I$ is $\mathbb{C} = \{x \in \mathbb{K} : \ Ax = b\}$, an affine *cross-section* of the cone $\mathbb{K}$.

Despite the power of mixed-integer linear programming, many optimization problems of practical and theoretical interest cannot be modeled using a linear objective function and constraints alone. The possibility of using general conic constraints and integer variables allows mixed-integer conic programming significant representation power. Even without recourse to integer variables, second-order cone and semidefinite programs model a wide range of problems [7, 28, 36]. Considering additional discrete decisions in these models or explicitly requiring some of the existing variables to be integers leads to mixed-integer second-order cone and mixed-integer semidefinite programs. On the one hand, second-order cone and positive semidefinite cone constraints are used to capture inherent nonlinear relationships between the decision variables in application areas such as power distribution network design and control [72, 102], queuing system design [59], production scheduling [6], data clustering [37, 98], sparse learning [94], and least-squares estimation with integer inputs [67]. On the other hand, mixed-integer second-order cone and mixed-integer semidefinite programs arise as the robust or stochastic counterparts of mixed-integer linear programs in optimization under uncertainty. Some application areas in this context include capital budgeting [105], portfolio optimization [74, 86], telecommunications network design [68], supply chain network design [12], and truss topology design [109]. The surveys [26, 29] contain further examples of mixed-integer conic programming applications. In addition, it is well-known that semidefinite programming formulations provide strong convex relaxations for hard combinatorial optimization problems such as maximum cut and maximum stable set [85]. Reintroducing the integrality constraints into these relaxations yields exact mixed-integer conic programming formulations. Therefore, a good understanding of mixed-integer conic programming can also be particularly relevant to combinatorial optimization.

The potential of mixed-integer conic programming has compelled significant attention from researchers and practitioners in the last few years. Leading integer programming solvers such as CPLEX [1], Gurobi [2], MOSEK [3], and Xpress [4] have responded to this interest with new and expanded features for handling mixed-integer conic programs. However, the development of practical solution methods for mixed-integer conic programs has remained a challenge. Today's mixed-integer conic programming technology is based to a great extent on algorithms for solving general mixed-integer convex programs and employ a combination of two techniques: branch-and-bound and linear outer approximation. See [26] for a detailed account. The branch-and-bound method can be generalized from mixed-integer linear to mixed-integer conic programming in a straightforward fashion. At the root node of the branch-and-bound tree, the continuous relaxation (1.4) is solved and the optimal solution x^* is found. If x^* does not satisfy the integrality constraints (1.3d), the set $\mathbb{C}$ is split into smaller sets $\mathbb{C}_1, \ldots, \mathbb{C}_k$ and the algorithm continues its search at each subset. Figure 1.3 illustrates the procedure. Note that, as described, this method requires the solution of a conic program at every node of the search tree. In linearization-based methods, on the other hand, the mixed-integer conic program is reduced to a mixed-integer linear program. A linear outer approximation to $\mathbb{C}$ is created and maintained dynamically, and the resulting mixed-integer linear program is solved via branch-and-bound and cutting-planes. While these techniques have their advantages, the theory of valid inequalities for mixed-integer conic programs is relatively underdeveloped. In particular, generic branch-and-bound methods for mixed-integer conic programs are not equipped with powerful valid inequalities which can be used to strengthen the mathematical description of $\mathbb{C}$ in a branch-and-cut framework. This places today's technology for solving mixed-integer conic programs at a position where mixed-integer linear programming technology was more than two decades ago. On a related note, the inherent nonlinear structure of general mixed-integer conic programs exposes a possible shortcoming of the cutting-plane approach. It is no longer guaranteed that these problems can be solved to optimality after the addition of a finite number of *linear* inequalities. This raises a possible need and potential for *nonlinear* valid inequalities which can be represented in computationally tractable forms and used as *cutting-surfaces*. The development and practical implementation of such cutting-surfaces in mixed-integer conic programming solvers is a topic of active research.

1.3 Outline of the Dissertation

In the cutting-plane method to mixed-integer linear programming, we first solve the continuous relaxation of a problem. If the optimal solution to the continuous relaxation does not satisfy the integrality constraints, a cutting-plane which separates this fractional solution

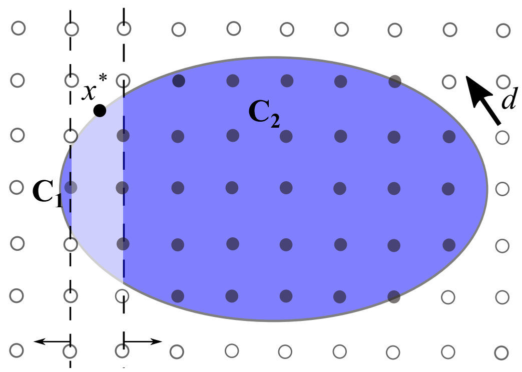


Figure 1.3: The branch-and-bound method for MICPs.

from the set of integer-feasible solutions is generated and added to the problem formulation. Consider the optimal simplex tableau of the continuous relaxation. Let $\{x_i\}_{i=1}^n$, $\{s_j\}_{j=1}^k$, and $\{y_j\}_{j=1}^m$ denote the basic, nonbasic continuous, and nonbasic integer variables in this simplex tableau, respectively. Then the tableau has the form

$$x = f + R_C s + R_I y, \tag{1.5a}$$
$$x \in \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}, \tag{1.5b}$$
$$s \in \mathbb{R}_+^k, \tag{1.5c}$$
$$y \in \mathbb{Z}_+^m, \tag{1.5d}$$

where $R_C = [r_C^1 \dots r_C^k]$ and $R_I = [r_I^1 \dots r_I^m]$ are real matrices of dimension $n \times k$ and $n \times m$ respectively and $f \in \mathbb{R}_+^n$. The optimal solution to the continuous relaxation is the basic solution associated with this simplex tableau, which is $x = f$, $s = 0$, $y = 0$ in our notation. If $f \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$, then this solution satisfies all integrality constraints. Otherwise, we would like to generate a cutting-plane that eliminates this fractional solution.

In Chapters 2, 3, and 4, we study this problem in a more general light. Let $\mathbb{S} \subset \mathbb{R}^n$ be a nonempty closed set and $f \in \mathbb{R}^n \setminus \mathbb{S}$. We consider the model

$$x = f + R_C s + R_I y, \tag{1.6a}$$
$$x \in \mathbb{S}, \tag{1.6b}$$
$$s \in \mathbb{R}_+^k, \tag{1.6c}$$
$$y \in \mathbb{Z}_+^m. \tag{1.6d}$$

The basic solution associated with this tableau, $x = f$, $s = 0$, $y = 0$, is still not feasible in this framework. For a better mathematical description of the set of feasible solutions, we would like to generate a cutting-plane which separates the infeasible basic solution from the set of feasible solutions. In particular, we would like to be able to generate a cutting-plane for every realization of the matrices R_I and R_C. This motivates the concept of "cut-generating functions": Consider $\mathbb{S}$ and f fixed. We say that functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$ form a *cut-generating function pair* (ψ, π) for (1.6) if the inequality $\sum_{j=1}^{k} \psi(r_C^j) s_j + \sum_{j=1}^{m} \pi(r_I^j) y_j \geq 1$ holds for all feasible solutions (x, s, y) to (1.6) for every choice of k, m, R_C, and R_I. Notice that this inequality cuts off the basic solution $x = f$, $s = 0$, $y = 0$. While even the claim that cut-generating functions exist may sound bold in the first place, such functions underlie the theory of cutting-planes in mixed-integer linear programming. Some of the most powerful general-purpose cutting-planes are obtained in this framework. Note that the nonnegativity constraints (1.6c) and (1.6d) on the nonbasic variables impose a natural hierarchy on cut-generating function pairs for (1.6). A cut-generating function pair (ψ, π) is said to be *minimal* if there does not exist a cut-generating function pair (ψ', π') distinct from (ψ, π) such that $\psi(r) \geq \psi'(r)$ and $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. Gomory and Johnson [63, 64] and Johnson [75] studied cut-generating function pairs for (1.6) when $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$. They characterized minimal cut-generating functions in terms of subadditivity, symmetry, and periodicity [63, 75]. Bachem, Johnson, and Schrader [13] presented a similar characterization for the case $\mathbb{S} = \{0\}$. The case $\mathbb{S} = \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$ is of particular interest because of its relation to (1.5) above. In Chapter 2 we generalize existing characterizations of minimal cut-generating functions to the case where $\mathbb{S} \subset \mathbb{R}^n$ is a nonempty closed set which does not contain f. Our analysis also exposes shortcomings in the usual definition of minimality for this general case. In response, we consider stronger notions of minimality and demonstrate how they impose additional structure on cut-generating functions under varying assumptions on the set $\mathbb{S}$. This chapter is based on joint work with Gérard Cornuéjols [106].

In Chapter 3 we consider the model (1.6) with only integer nonbasic variables:

$$x = f + R_I y, \tag{1.7a}$$
$$x \in \mathbb{S}, \tag{1.7b}$$
$$y \in \mathbb{Z}_+^m. \tag{1.7c}$$

A cut-generating function for (1.7) is defined as before: A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (1.7) if the inequality $\sum_{j=1}^{m} \pi(r_I^j) y_j \geq 1$ holds for all feasible solutions (x, y) to (1.7) for every choice of m and R_I. A cut-generating function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *minimal* if there does not exist a cut-generating function π' distinct from π such that $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. A stronger notion than the minimality of a cut-generating function is its extremality: A cut-generating function π is said to be *extreme* if any two cut-generating functions π_1, π_2 satisfying $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ must also satisfy $\pi = \pi_1 = \pi_2$. While extreme cut-generating functions produce powerful cutting-planes, their structure can be very complicated. In the case $\mathbb{S} = \mathbb{Z}$ and $f \in \mathbb{R} \setminus \mathbb{Z}$, Gomory and Johnson [64, 65] identified a "simple" class of extreme cut-generating functions for (1.7) by showing that continuous, piecewise linear, minimal cut-generating functions with only two distinct slope values are extreme. In Chapter 3, we establish a similar result for the case $\mathbb{S} = \mathbb{Z}_+$ and $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. This chapter is based on joint work with Gérard Cornuéjols [106].

In Chapter 4 we consider the model (1.6) with only continuous nonbasic variables:

$$x = f + R_C s, \tag{1.8a}$$
$$x \in \mathbb{S}, \tag{1.8b}$$
$$s \in \mathbb{R}^k_+. \tag{1.8c}$$

As before, a function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (1.8) if the inequality $\sum_{j=1}^k \psi(r_C^j) s_j \geq 1$ holds for all feasible solutions (x, s) to (1.8) for every choice of k and R_C. Conforti et al. [47] showed that cut-generating functions for (1.8) enjoy significant structure. However, they also gave an example showing that not all cutting-planes $c^\top s \geq 1$ can be obtained from cut-generating functions in the framework (1.8). They conjectured that cut-generating functions might be sufficient under the natural condition $\mathbb{S} - f \subset \operatorname{cone} R_C$, where $\operatorname{cone} R_C$ represents the cone generated by the columns of R_C. In Chapter 4, we prove that this conjecture is true. This justifies the recent research interest in cut-generating functions for (1.8). This chapter is based on joint work with Gérard Cornuéjols and Laurence Wolsey [51].

Cut-generating functions provide a means for separating the fractional optimal solution of the continuous relaxation of a mixed-integer linear program from the set of its feasible solutions. An alternative (and complementary) solution to the same problem comes from the disjunctive programming perspective of Balas [15]. Consider again the optimal simplex tableau (1.5). Suppose the optimal solution $x = f$, $s = 0$, $y = 0$ does not satisfy the integrality constraints (1.5b). Then there exists an integer basic variable, say x_1, whose current value f_1 is not an integer. Because any integer-feasible solution must satisfy either $x_1 \leq \lfloor f_1 \rfloor$ or $x_1 \geq \lceil f_1 \rceil$, the disjunction $x_1 \leq \lfloor f_1 \rfloor \vee x_1 \geq \lceil f_1 \rceil$ can be used to remove the fractional solution $x = f$, $s = 0$, $y = 0$ from the continuous relaxation while maintaining all feasible solutions to the mixed-integer linear program. More generally, the integrality constraints on the variables imply linear *two-term disjunctions* $c_1^\top x \geq c_{1,0} \vee c_2^\top x \geq c_{2,0}$ on the continuous relaxation. When the two halfspaces defined by $c_1^\top x \geq c_{1,0}$ and $c_2^\top x \geq c_{2,0}$ are opposing and disjoint, such two-term disjunctions are called *split disjunctions*. As an example, the disjunction $x_1 \leq \lfloor f_1 \rfloor \vee x_1 \geq \lceil f_1 \rceil$ mentioned above is a split disjunction. An inequality which is valid for a disjunction on the continuous relaxation of a mixed-integer linear program is called a *disjunctive inequality* [14].

In Chapters 5, 6, 7, and 8, we turn to mixed-integer conic programming. Disjunctive inequalities have been a principal ingredient in the practical success of mixed-integer linear programming in the last two decades. In order to extend our understanding of disjunctive inequalities to mixed-integer conic programming, we follow a principled study of two-term disjunctions on conic sets. In Chapter 5 we consider the disjunction $c_1^\top x \geq c_{1,0} \vee c_2^\top x \geq c_{2,0}$ on a general regular cone $\mathbb{K} \subset \mathbb{E}$. Associated with such a disjunction, we define the sets

$$\mathbb{C}_i = \{x \in \mathbb{K} : \, c_i^\top x \geq c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}. \tag{1.9}$$

Sets of the form $\mathbb{C}_1 \cup \mathbb{C}_2$ provide fundamental relaxations for mixed-integer conic programs. Convex inequalities that are valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ can be used as general-purpose cutting-surfaces in mixed-integer conic programming solvers. To derive the strongest such cutting-surfaces, we study the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. It is a well-known fact from convex analysis that the closed convex hull of any set can be described with valid linear inequalities alone. A result of Kılınç-Karzan [80] indicates, however, that *conic minimal* valid linear inequalities are all that is needed for a closed convex hull description of $\mathbb{C}_1 \cup \mathbb{C}_2$. In the first part of Chapter 5, we characterize the structure of conic minimal and tight valid linear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$. In the second part, we develop structured nonlinear valid inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$ by grouping subsets of valid linear inequalities through conic programming duality. This yields a family of convex valid inequalities which collectively describe the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ in the space of the original variables. We formulate the general form of these inequalities and analyze their structure in detail. Under certain conditions on the choice of disjunction, we can show that a single inequality from this family defines the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. These conditions are satisfied, for example, in the case of split disjunctions. This chapter is based on joint work with Fatma Kılınç-Karzan [83, 84, 108].

In Chapters 6 and 7, we specialize the results of Chapter 5 to the cases where $\mathbb{K}$ is a direct product of second-order cones and nonnegative rays and where $\mathbb{K}$ is the positive semidefinite cone, respectively. These cases deserve attention because of their importance for mixed-integer second-order cone and mixed-integer semidefinite programming. In Chapter 6 we develop closed-form expressions for the nonlinear inequalities of Chapter 5 in the case where $\mathbb{K}$ is a direct product of second-order cones and nonnegative rays. These inequalities can always be represented in second-order cone form in a lifted space with few additional variables. In the case where $\mathbb{K}$ is a single second-order cone, the additional variables can be eliminated if the disjunction satisfies a certain disjointness condition, yielding a valid second-order cone inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ in the space of the original variables. As a consequence of our results in Chapter 5, the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ can be described with a single convex inequality for certain disjunctions. In general, however, a complete description may require every inequality from our family of valid convex inequalities. In the case where $\mathbb{K}$ is a single second-order cone, we outline a procedure to reach explicit closed convex hull descriptions of $\mathbb{C}_1 \cup \mathbb{C}_2$. Our results on two-term disjunctions on a single second-order cone generalize related results on split disjunctions from the literature [8, 89]. Chapter 6 is based on joint work with Fatma Kılınç-Karzan [83, 84]. In Chapter 7 we develop closed-form expressions for the nonlinear inequalities of Chapter 5 in the case where $\mathbb{K}$ is the positive semidefinite cone. For a class of elementary disjunctions, we demonstrate that these inequalities can be expressed in a simple second-order conic form. For more general disjunctions, we present several techniques to generate low-complexity convex inequalities that are valid for $\mathbb{C}_1 \cup \mathbb{C}_2$. Chapter 6 is based on joint work with Fatma Kılınç-Karzan [108].

In Chapter 8 we consider homogeneous two-term disjunctions on the second-order cone and general two-term disjunctions on affine *cross-sections* of the second-order cone. First, we show that a convex inequality of the form developed in Chapter 6 defines the convex hull of homogeneous two-term disjunctions on the second-order cone. Second, we show that such an inequality can characterize the closed convex hull of two-term disjunctions on affine *cross-sections* of the second-order cone under certain conditions. These conditions are satisfied in particular by all two-term disjunctions on ellipsoids and paraboloids, a large class of two-term disjunctions on hyperboloids, and all split disjunctions on all cross-sections of the second-order cone. The inequalities can be represented in second-order cone form in the space of the original variables if the disjunctions satisfy certain disjointness conditions in either case. Our results generalize the related results on specific classes of two-term disjunctions on cross-sections of the second-order cone from the literature [27, 52, 89]. This chapter is based on joint work with Gérard Cornuéjols [107].

The remainder of this dissertation assumes a fundamental knowledge of optimization theory. Explicit references to specific results are provided as needed. The necessary background on integer programming, conic programming, and convex analysis can be found in the textbooks [46], [28], and [69, 96], respectively.

Chapter 2

Minimal Cut-Generating Functions for Integer Variables

Acknowledgments. This chapter is based on joint work with Gérard Cornuéjols [106].

2.1 Introduction

2.1.1 Motivation

An ongoing debate in integer linear programming centers on the value of general-purpose cuts (Gomory cuts are a famous example) versus facet-defining inequalities for special problem structures (for example, comb inequalities for the traveling salesman problem). Both have been successful in practice. In this chapter we focus on the former type of cuts, which are attractive for their wide applicability. Nowadays, state-of-the-art integer programming solvers routinely use several classes of general-purpose cuts. Recently, there has been a renewed interest in the *theory* of general-purpose cuts. This was sparked by a beautiful paper of Andersen, Louveaux, Weismantel, and Wolsey [9] on 2-row cuts which illuminated their connection to lattice-free convex sets. This line of research focused on cut coefficients for the continuous nonbasic variables in a tableau form, and lifting properties for the integer nonbasic variables [17, 21, 35, 44, 47, 53, 54]. Decades earlier, Gomory and Johnson [63, 64] and Johnson [75] had studied cut coefficients for the integer nonbasic variables directly. Although their characterization involves concepts that are not always easy to verify algorithmically (such as subadditivity), it provides a useful framework for the study of cutting-planes. Jeroslow [73], Blair [32], and Bachem, Johnson, and Schrader [13] extended the work of Gomory and Johnson on minimal cuts for the corner relaxation to general integer linear programs. In this chapter we pursue the study of general-purpose cuts in integer programming, further extending the framework introduced by Gomory and Johnson. Our focus is also on the cut coefficients of the integer variables.

Consider a pure integer linear program and the optimal simplex tableau of its linear programming relaxation. We select n rows of the tableau, corresponding to n basic variables $\{x_i\}_{i=1}^n$. Let $\{y_j\}_{j=1}^m$ denote the nonbasic variables. The tableau restricted to these n rows is of the form

$$x = f + \sum_{j=1}^{m} r^j y_j, \tag{2.1a}$$

$$x \in \mathbb{Z}_+^n, \tag{2.1b}$$

$$y_j \in \mathbb{Z}_+ \quad \forall j \in \{1, \dots, m\}, \tag{2.1c}$$

where $f \in \mathbb{R}_+^n$ and $r^j \in \mathbb{R}^n$ for all $j \in \{1, \dots, m\}$. We assume $f \notin \mathbb{Z}^n$; therefore, the basic solution $x = f$, $y = 0$ is not feasible. We would like to generate cutting-planes that cut off this infeasible solution.

A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (2.1) if the inequality $\sum_{j=1}^m \pi(r^j) y_j \geq 1$ holds for all feasible solutions (x, y) to (2.1) for any possible number m of nonbasic variables and any choice of nonbasic columns r^j. Gomory and Johnson [63, 64] and Johnson [75] characterized such functions for the corner relaxation of (2.1) which relaxes $x \in \mathbb{Z}_+^n$ to $x \in \mathbb{Z}^n$. They also introduced the *infinite group relaxation*

$$x = f + \sum_{r \in \mathbb{R}^n} r y_r, \tag{2.2a}$$

$$x \in \mathbb{Z}^n, \tag{2.2b}$$

$$y_r \in \mathbb{Z}_+ \quad \forall r \in \mathbb{R}^n, \tag{2.2c}$$

$$y \text{ has finite support}, \tag{2.2d}$$

as a master model for all corner relaxations. Here an infinite-dimensional vector is said to have *finite support* if it has a finite number of nonzero entries.

Let $\mathbb{S} \subset \mathbb{R}^n$ be any nonempty subset of the Euclidean space. Here we consider the following generalization of the Gomory-Johnson model:

$$x = f + \sum_{r \in \mathbb{R}^n} r y_r, \tag{2.3a}$$

$$x \in \mathbb{S}, \tag{2.3b}$$

$$y_r \in \mathbb{Z}_+ \quad \forall r \in \mathbb{R}^n, \tag{2.3c}$$

$$y \text{ has finite support}. \tag{2.3d}$$

This flexibility in the choice of $\mathbb{S} \subset \mathbb{R}^n$ makes (2.3) a relevant model for i) integer convex and conic programs, and ii) integer programs with complementarity constraints, as well as integer linear programs; see [47, Section 1.1]. The Gomory-Johnson model (2.2) is the special case of (2.3) where $\mathbb{S} = \mathbb{Z}^n$. The model of Bachem et al. [13] corresponds to the case $\mathbb{S} = \{0\}$. The case where $\mathbb{S} = \mathbb{Z}^n_+$, or more generally where $\mathbb{S} \subset \mathbb{R}^n$ is the set of integer points in a full-dimensional rational polyhedron, is of particular interest in integer linear programming due to its connection to (2.1) above. It is a main focus of this chapter. In the context of *mixed*-integer linear programming, the model (2.3) with continuous as well as integer variables is also interesting; we will discuss it in Section 2.3.4 (where we allow continuous basic variables) and Section 2.5 (where we also allow continuous nonbasic variables).

Note that (2.3) is nonempty since for any $\bar{x} \in \mathbb{S}$, the solution $x = \bar{x}$, $y_{\bar{x}-f} = 1$, and $y_r = 0$ for all $r \neq \bar{x} - f$ is feasible. In the remainder of the chapter, we assume that $f \in \mathbb{R}^n \setminus \mathbb{S}$. Therefore, the basic solution $x = f$, $y = 0$ is not a feasible solution of (2.3). We are interested in valid inequalities for (2.3) that cut off the above infeasible basic solution.

We can generalize the notion of cut-generating function as follows. A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for (2.3) if the inequality $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq 1$ holds for all feasible solutions (x, y) to (2.3). For example, the function that takes the value 1 for all $r \in \mathbb{R}^n$ is a cut-generating function because every feasible solution of (2.3) satisfies $y_r \geq 1$ for at least one $r \in \mathbb{R}^n$. When $\mathbb{S} = \mathbb{Z}^n_+$, we recover the earlier definition of a cut-generating function for (2.1).

A key feature that distinguishes the cut-generating functions for model (2.3) from those that were studied by Gomory and Johnson for model (2.2) is that they need not be nonnegative even if we assume continuity. In fact, they can take any real value, positive and negative, as the following examples illustrate.

Example 2.1. Consider the model (2.3) where $n = 1$, $0 < f < 1$, and $\mathbb{S} = \mathbb{Z}_+$. Cornuéjols, Kis, and Molinaro [50] showed that, for $0 < \alpha \leq 1$, the following family of functions $\pi^1_\alpha : \mathbb{R} \to \mathbb{R}$ are cut-generating functions:

$$\pi^1_\alpha(r) \;=\; \min\left\{\frac{r - \lfloor \alpha r \rfloor}{1-f},\; \frac{-r}{f} + \frac{\lceil \alpha r \rceil (1-\alpha f)}{\alpha f(1-f)}\right\}.$$

Note that when $\alpha = 1$, the function $\pi^1_1(r) = \min\left\{\frac{r-\lfloor r\rfloor}{1-f}, \frac{\lceil r\rceil - r}{f}\right\}$ is the well-known Gomory function. This function is periodic and takes its values in the interval $[0, 1]$. However, when $\alpha < 1$, this is not the case any more: The function π^1_α takes all real values between $-\infty$ and $+\infty$ and is not periodic in the usual sense. See Figure 2.1.

The next example is mostly of theoretical interest. It illustrates another property of model (2.3) that does not arise in the Gomory-Johnson model (2.2).

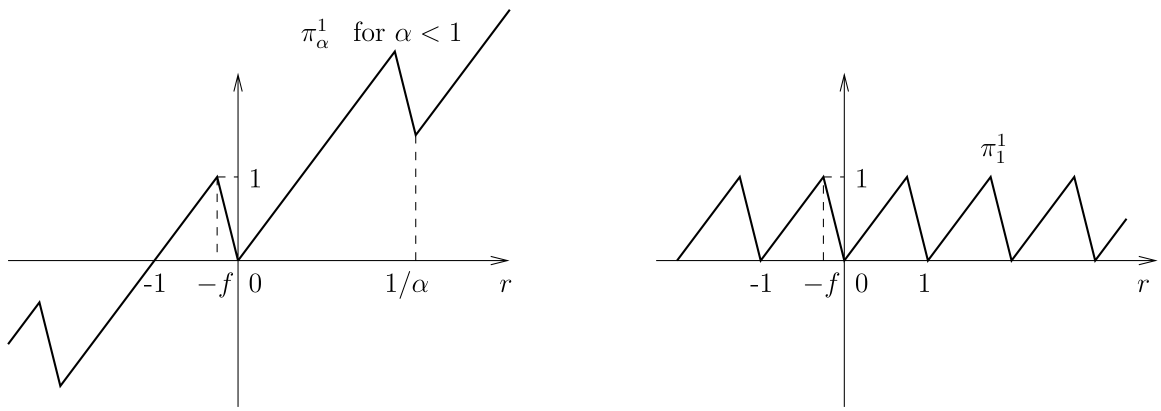


Figure 2.1: Two cut-generating functions: π^1_α for some $\alpha < 1$ and π^1_1.

Example 2.2. Consider the model (2.3) where $n = 1$, $f > 0$, and $\mathbb{S} = \{0\}$. In this case, the model (2.3) reduces to the constraints $\sum_{r\in\mathbb{R}} r y_r = -f$, $y_r \in \mathbb{Z}_+$ for $r \in \mathbb{R}$, and y has finite support. For any $\alpha \leq -\frac{1}{f} < 0$, the linear function $\pi^2_\alpha : \mathbb{R} \to \mathbb{R}$ defined as $\pi^2_\alpha(r) = \alpha r$ is a cut-generating function. This can be seen by observing that $\sum_{r\in\mathbb{R}} \pi^2_\alpha(r) y_r = \sum_{r\in\mathbb{R}} (\alpha r) y_r = \alpha \sum_{r\in\mathbb{R}} r y_r = -\alpha f \geq 1$ for any y feasible to (2.3).

2.1.2 Related Work

In this section we give a brief overview of some existing work. We comment on the connections between our results and other results from the literature further throughout the chapter.

Gomory and Johnson [63, 64] introduced the infinite group relaxation (2.2) as a master framework for research into general-purpose cutting-planes in integer linear programming. It has since then become a central problem in integer programming and a fertile ground for research. The reader is referred to the excellent surveys [22, 23, 45, 95] for extensive accounts of classical as well as recent results on the infinite group relaxation and its variants. In their seminal papers [63, 75], Gomory and Johnson investigated minimal cut-generating functions for (2.2); these are cut-generating functions π such that there does not exist a cut-generating function π' distinct from π which satisfies $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. Gomory and Johnson characterized minimal cut-generating functions for (2.2) in terms of subadditivity, periodicity with respect to $\mathbb{Z}^n$, and a certain symmetry condition. See Section 2.3.3 for a precise statement. Bachem et al. [13] provided a similar characterization for the model (2.3) in the special case $\mathbb{S} = \{0\}$.

In a parallel stream of literature, Jeroslow and Blair considered valid inequalities for an integer linear program with *fixed* data. In this framework, minimality of a valid inequality is defined for the particular problem instance under consideration, rather than for a master problem or a class of problems. Jeroslow [73] characterized minimal valid inequalities for integer linear programs with bounded feasible regions in terms of their value functions. Blair [32] extended this characterization to integer linear programs with rational data. Johnson [76] analyzed minimal valid inequalities for disjunctive sets. In all of these models, the set of feasible solutions is contained in the nonnegative orthant, and the minimality of a valid inequality is defined with respect to the nonnegative orthant as well. Recently, on a model for disjunctive conic programs, Kılınç-Karzan [80] generalized this notion broadly by defining and analyzing the minimality of a valid inequality with respect to an arbitrary regular cone which contains the feasible solution set. She also showed that these conic minimal inequalities describe the closed convex hull of the disjunctive conic set together with the cone constraint under a technical condition.

2.1.3 Notation and Terminology

Let $\mathbb{Z}_{++}$ be the set of strictly positive integers. For $k \in \mathbb{Z}_{++}$, we let $[k] = \{1, \ldots, k\}$. For $i \in [n]$, we let e^i denote the i-th standard unit vector in $\mathbb{R}^n$. We let $\operatorname{cl}\mathbb{V}$ and $\overline{\operatorname{conv}}\,\mathbb{V}$ represent the closure and closed convex hull of a set $\mathbb{V} \in \mathbb{R}^n$, respectively. We use $\operatorname{rec}\mathbb{V}$ and $\operatorname{lin}\mathbb{V}$ to refer to the recession cone and lineality space of a closed convex set $\mathbb{V} \subset \mathbb{R}^n$, respectively.

We say that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$; it is *symmetric* or satisfies the *symmetry condition* if $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^n$; it is *periodic with respect to* $\mathbb{Z}^n$ if $\pi(r) = \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n$; and it is *nondecreasing with respect to* $\mathbb{S} \subset \mathbb{R}^n$ if $\pi(r) \leq \pi(r + w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{S}$.

2.1.4 Outline of the Chapter

Minimal Cut-Generating Functions

Throughout the chapter, we consider the model (2.3) under the running assumption that $\mathbb{S} \neq \emptyset$. We say that a cut-generating function π' for (2.3) *dominates* another cut-generating function π if $\pi \geq \pi'$, that is, $\pi(r) \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. A cut-generating function π is *minimal* if there is no cut-generating function π' distinct from π that dominates π. When $n = 1$, $\mathbb{S} = \mathbb{Z}_+$, and $0 < f < 1$, the cut-generating functions π^1_α of Example 2.1 are minimal [50]. Later in Section 2.1.4, we will show that the linear cut-generating functions π^2_α of Example 2.2 are also minimal. The following theorem shows that minimal cut-generating functions for (2.3) indeed always exist when $\mathbb{S} \neq \emptyset$. This result also appears in a recent paper of Basu and Paat [17].

Theorem 2.1. *Every cut-generating function for* (2.3) *is dominated by a minimal cut-generating function.*

Proof. Let π be a cut-generating function for (2.3). Denote by Π the set of cut-generating functions π' that dominate π. Let $\{\pi_\ell\}_{\ell\in\mathbb{L}} \subset \Pi$ be a nonempty family of cut-generating functions such that for any pair $\ell', \ell'' \in \mathbb{L}$, we have $\pi_{\ell'} \leq \pi_{\ell''}$ or $\pi_{\ell'} \geq \pi_{\ell''}$. To prove the claim, it is enough to show according to Zorn's Lemma (see, e.g., [42]) that there exists a cut-generating function that is a lower bound on $\{\pi_\ell\}_{\ell\in\mathbb{L}}$.

Define the function $\bar{\pi} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ as $\bar{\pi}(r) = \inf_\ell\{\pi_\ell(r) : \ell \in \mathbb{L}\}$. Clearly, the function $\bar{\pi}$ is a lower bound on $\{\pi_\ell\}_{\ell\in\mathbb{L}}$. We show that it is a cut-generating function for (2.3). First we prove that $\bar{\pi}$ is finite everywhere. Choose $\bar{x} \in \mathbb{S}$. For any $\bar{r} \in \mathbb{R}^n$, let $\bar{y}$ be defined as $\bar{y}_{\bar{r}} = 1$, $\bar{y}_{\bar{x}-f-\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise. The solution $(\bar{x}, \bar{y})$ is feasible to (2.3). Then for any $\ell \in \mathbb{L}$, the cut-generating function π_ℓ satisfies $\sum_{r\in\mathbb{R}^n} \pi_\ell(r)\bar{y}_r = \pi_\ell(\bar{r}) + \pi_\ell(\bar{x} - f - \bar{r}) \geq 1$. Moreover, we have $\pi_\ell \leq \pi$ because $\pi_\ell \in \Pi$; hence,

$$\pi_\ell(\bar{r}) \geq 1 - \pi_\ell(\bar{x} - f - \bar{r}) \geq 1 - \pi(\bar{x} - f - \bar{r}).$$

Therefore, $\bar{\pi}(\bar{r}) \geq 1 - \pi(\bar{x} - f - \bar{r})$. This shows that $\bar{\pi}(r)$ is finite for all $r \in \mathbb{R}^n$. That is, $\bar{\pi} : \mathbb{R}^n \to \mathbb{R}$. Now consider any feasible solution (x, y) of (2.3). Note that $\{\pi_\ell\}_{\ell\in\mathbb{L}}$ is a totally ordered set, $\bar{\pi}$ is finite everywhere, and only a finite number of the terms y_r are nonzero. Combining these facts, we can write

$$\sum_{r\in\mathbb{R}^n} \bar{\pi}(r)y_r = \sum_{r\in\mathbb{R}^n} \inf_\ell \{\pi_\ell(r) : \ell \in \mathbb{L}\}\, y_r = \inf_\ell \left\{ \sum_{r\in\mathbb{R}^n} \pi_\ell(r)y_r : \ell \in \mathbb{L} \right\} \geq 1.$$

This proves that $\bar{\pi}$ is a cut-generating function. □

Theorem 2.1 shows that there always exists a minimal cut-generating function which separates the infeasible basic solution $x = f$, $y = 0$ from the feasible solutions to (2.3). Hence, when we search for a cut-generating function which will separate $x = f$, $y = 0$, we can restrict our attention to minimal cut-generating functions without any loss of generality.

When $\mathbb{S} = \mathbb{Z}^n$, cut-generating functions are traditionally assumed to be nonnegative. In this setting, Gomory and Johnson showed that a function $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is a minimal cut-generating function if and only if $\pi(0) = 0$, π is subadditive, symmetric, and periodic with respect to $\mathbb{Z}^n$ [45, 63, 75]. However, for general $\mathbb{S} \subset \mathbb{R}^n$, Examples 2.1 and 2.2 show that minimal cut-generating functions do not necessarily satisfy periodicity with respect to $\mathbb{Z}^n$, nor symmetry. We define a new condition, which we call the generalized symmetry condition, to replace symmetry and periodicity in the characterization of minimal cut-generating functions for (2.3). A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the *generalized* *symmetry condition* if

$$\pi(r) = \sup_{x,k} \left\{ \frac{1 - \pi(x - f - kr)}{k} : \ x \in \mathbb{S}, \, k \in \mathbb{Z}_{++} \right\} \quad \text{for all } r \in \mathbb{R}^n. \tag{2.4}$$

The functions π^1_α and π^2_α of Examples 2.1 and 2.2 satisfy the generalized symmetry condition. We briefly outline the proof in each case.

Example 2.1 continued. Consider the function π^1_α of Example 2.1. The inequality $\bar{k}\pi^1_\alpha(\bar{r}) + \pi^1_\alpha(\bar{x} - f - \bar{k}\bar{r}) \geq 1$ holds for any $\bar{r} \in \mathbb{R}$, $\bar{k} \in \mathbb{Z}_{++}$, and $\bar{x} \in \mathbb{Z}_+$ because π^1_α is a cut-generating function [50] and the solution $x = \bar{x}$, $y_{\bar{r}} = \bar{k}$, $y_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $y_r = 0$ otherwise is feasible to (2.3). Hence, $\pi^1_\alpha(r) \geq \frac{1}{k}(1 - \pi^1_\alpha(x - f - kr))$ for all $r \in \mathbb{R}$, $k \in \mathbb{Z}_{++}$, and $x \in \mathbb{Z}_+$. Furthermore, the graph of π^1_α is symmetric relative to the point $(-f/2, 1/2)$. In other words, the symmetry condition holds: $\pi^1_\alpha(r) = 1 - \pi^1_\alpha(-f - r)$ for all $r \in \mathbb{R}$. Therefore, for all $r \in \mathbb{R}$, we have

$$\pi^1_\alpha(r) = 1 - \pi^1_\alpha(-f - r) \leq \sup_{x,k} \left\{ \frac{1 - \pi^1_\alpha(x - f - kr)}{k} : x \in \mathbb{Z}_+, k \in \mathbb{Z}_{++} \right\} \leq \pi^1_\alpha(r).$$

This shows that π^1_α satisfies the generalized symmetry condition.

Example 2.2 continued. Consider the function π^2_α of Example 2.2. Because $\mathbb{S} = \{0\}$, the term x disappears from (2.4). Using $\alpha \leq \frac{-1}{f}$, for any $r \in \mathbb{R}$ we can write

$$\sup_{k \in \mathbb{Z}_{++}} \left\{ \frac{1 - \pi^2_\alpha(-f - kr)}{k} \right\} = \alpha r + \sup_{k \in \mathbb{Z}_{++}} \frac{1 + \alpha f}{k} = \alpha r = \pi^2_\alpha(r).$$

This shows that π^2_α satisfies the generalized symmetry condition.

Our main result about minimal cut-generating functions for (2.3) is the following theorem which holds for any nonempty $\mathbb{S} \subset \mathbb{R}^n$. This result will be proved in Section 2.2.

Theorem 2.2. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal cut-generating function for* (2.3) *if and only if $\pi(0) = 0$, π is subadditive and satisfies the generalized symmetry condition.*

Strengthening the Notion of Minimality

The notion of minimality that we defined above can be unsatisfactory for certain choices of $\mathbb{S} \subset \mathbb{R}^n$. We illustrate this in the next proposition and remark.

Proposition 2.3. *If a cut-generating function for* (2.3) *is linear, then it is minimal.*

Proof. Let π be a linear cut-generating function for (2.3). By Theorem 2.1, there exists a minimal cut-generating function π' such that $\pi' \leq \pi$. By Theorem 2.2, π' is subadditive and $\pi'(0) = 0$. For any $r \in \mathbb{R}^n$, the inequality $\pi' \leq \pi$ implies $\pi(r) + \pi(-r) \geq \pi'(r) + \pi'(-r) \geq \pi'(0) = 0 = \pi(r) + \pi(-r)$ where the last equality follows from the linearity of π. Hence, $\pi' = \pi$. □

Linear cut-generating functions are closely related to linear inequalities that strictly separate f from $\mathbb{S}$. To see this, let $\alpha \in \mathbb{R}^n$, and consider a linear function $\pi : \mathbb{R}^n \to \mathbb{R}$ such that $\pi(r) = \alpha^\top r$. For any (x, y) feasible to (2.3), we have $\sum_{r\in\mathbb{R}^n} \pi(r)y_r = \sum_{r\in\mathbb{R}^n} \alpha^\top r y_r = \alpha^\top(x - f)$. Thus, π is a cut-generating function for (2.3) if and only if $\alpha^\top(x - f) \geq 1$ is valid for $\mathbb{S}$.

Remark 2.4. *For a minimal cut-generating function π, it is possible that the inequality $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq 1$ is implied by an inequality $\sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq 1$ arising from some other cut-generating function π'. Indeed, for $n = 1$, $f > 0$, and $\mathbb{S} = \{0\}$, consider again the cut-generating functions π^2_α of Example 2.2 with $\alpha \leq -\frac{1}{f}$. These are minimal by Proposition 2.3. However, the inequalities $|\alpha| f \sum_{r\in\mathbb{R}} \frac{-r}{f} y_r \geq 1$ generated from π^2_α for $\alpha < -\frac{1}{f}$ are implied by the inequality $\sum_{r\in\mathbb{R}} \frac{-r}{f} y_r \geq 1$ generated for $\alpha = -\frac{1}{f}$.*

This shortfall in the traditional definition of minimality was also noted in [80, Example 7]. Thus, it makes sense to define a stronger notion of minimality as follows: A cut-generating function π' for (2.3) *implies* another cut-generating function π *via scaling* if there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$. Note that when the function π' is nonnegative, this notion is identical to the notion of domination introduced earlier; however, the two notions are distinct when π' can take negative values. A cut-generating function π is *restricted minimal* if there is no cut-generating function π' distinct from π that implies π via scaling. This notion was the one used by Jeroslow [73], Blair [32], and Bachem et al. [13]; they just called it minimality. In this chapter we call it restricted minimality to distinguish it from the notion of minimality introduced in Section 2.1.4. The next proposition shows that restricted minimal cut-generating functions are the minimal cut-generating functions which enjoy an additional "tightness" property.

Proposition 2.5. *A cut-generating function π for* (2.3) *is restricted minimal if and only if it is minimal and* $\inf_x\{\pi(x - f) : x \in \mathbb{S}\} = 1$.

The proof of this proposition will be presented at the end of Section 2.2.

The next proposition shows that there always exists a restricted minimal cut-generating function which separates the infeasible basic solution $x = f$, $y = 0$ from the feasible solutions to (2.3). As a corollary, we obtain that restricted minimal cut-generating functions always exist.

Proposition 2.6. *Every cut-generating function for* (2.3) *is implied via scaling by a restricted minimal cut-generating function.*

Proof. Let π be a cut-generating function. Let $\mu = \inf_{x,y}\{\sum_{r\in\mathbb{R}^n}\pi(r)y_r : (x,y) \text{ satisfies } (2.3)\}$; note that $\mu \geq 1$. Define $\pi' = \frac{\pi}{\mu}$. The function π' is also a cut-generating function, and it satisfies $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n}\pi'(r)y_r : (x,y) \text{ satisfies } (2.3)\} = 1$. By Theorem 2.1, there exists a minimal cut-generating function π^* that dominates π'. The function π^* implies π via scaling since $\mu\pi^* \leq \mu\pi' = \pi$. We claim that π^* is restricted minimal. First note that $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n}\pi^*(r)y_r : (x,y) \text{ satisfies } (2.3)\} = 1$. Now consider $\beta \geq 1$ and a cut-generating function π^{**} such that $\pi^* \geq \beta\pi^{**}$. We must have $\beta = 1$ since $\inf_{x,y}\{\sum_{r\in\mathbb{R}^n}\pi^{**}(r)y_r : (x,y) \text{ satisfies } (2.3)\} \geq 1$. Then because π^* is minimal, we get $\pi^{**} = \pi^*$. This proves the claim. □

In the case $\mathbb{S} = \{0\}$, Bachem et al. [13] showed that restricted minimal cut-generating functions satisfy the symmetry condition. This can be generalized as in the next theorem.

Theorem 2.7. *Let* $\mathbb{K} \subset \mathbb{R}^n$ *be a closed convex cone and* $\mathbb{S} = \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. *Let* $\pi : \mathbb{R}^n \to \mathbb{R}$. *The function* π *is a restricted minimal cut-generating function for* (2.3) *if and only if* $\pi(0) = 0$, π *is subadditive, nondecreasing with respect to* $\mathbb{S} \subset \mathbb{R}^n$, *and satisfies the symmetry condition.*

This theorem will be proved in Section 2.3.

The notion of minimality can be strengthened even further if we take into consideration the linear inequalities that are valid for $\mathbb{S} \subset \mathbb{R}^n$. Let $\alpha^\top(x-f) \geq \alpha_0$ be a valid inequality for $\mathbb{S}$. Because $f + \sum_{r\in\mathbb{R}^n} ry_r = x \in \mathbb{S}$ for any (x,y) feasible to (2.3), such a valid inequality can be translated to the space of the nonbasic variables y as $\sum_{r\in\mathbb{R}^n}\alpha^\top ry_r \geq \alpha_0$. We say that a cut-generating function π' for (2.3) *implies* another cut-generating function π for (2.3) if there exists a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{S}$ and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \alpha^\top r + \beta\pi'(r)$ for all $r \in \mathbb{R}^n$. This definition makes sense because if $\sum_{r\in\mathbb{R}^n}\pi'(r)y_r \geq 1$ is a valid inequality for (2.3), then $\sum_{r\in\mathbb{R}^n}\pi(r)y_r \geq \sum_{r\in\mathbb{R}^n}\alpha^\top ry_r + \beta\sum_{r\in\mathbb{R}^n}\pi'(r)y_r \geq \alpha_0 + \beta \geq 1$ is also valid for (2.3). When the closed convex hull of $\mathbb{S} \subset \mathbb{R}^n$ is equal to the whole of $\mathbb{R}^n$, the only inequalities that are valid for $\mathbb{S}$ are those that have $\alpha = 0$ and $\alpha_0 \leq 0$; in this case, a cut-generating function may imply another only via scaling. However, the two notions may be different when $\overline{\text{conv}}(S) \subsetneq \mathbb{R}^n$. We say that a cut-generating function π is *strongly minimal* if there does not exist a cut-generating function π' distinct from π that implies π. Note that strongly minimal cut-generating functions are restricted minimal. Indeed, if π is a cut-generating function that is not restricted minimal, there exists a cut-generating function $\pi' \neq \pi$ and $\beta \geq 1$ such that $\pi \geq \beta\pi'$; but then π' implies π by taking $\alpha = 0$ and $\alpha_0 = 0$ which shows that π is not strongly minimal. For a fixed integer programming instance, the three definitions of minimality that we explore in this chapter can be seen from the perspective of [80]. We comment on this connection in the appendix to this chapter. In the setting of infinite relaxations, our results demonstrate how strengthening the notion of minimality imposes additional structure on cut-generating functions for (2.3). See also [80, Remark 7] for a related discussion.

In Section 2.4.1, we prove the following theorem about strongly minimal cut-generating functions for (2.3) when $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$.

Theorem 2.8. *Let* $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ *and* $f \in \mathbb{R}^n_+ \setminus \mathbb{S}$. *Let* $\pi : \mathbb{R}^n \to \mathbb{R}$. *The function* π *is a strongly minimal cut-generating function for* (2.3) *if and only if* $\pi(0) = 0$, $\pi(-e^i) = 0$ *for all* $i \in [p]$ *and* $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ *for all* $i \in [n] \setminus [p]$, π *is subadditive and satisfies the symmetry condition.*

In Section 2.4.2, we give an example showing that strongly minimal cut-generating functions do not always exist. On the other hand, when $\mathbb{S} \subset \mathbb{R}^n$ is a full-dimensional polyhedron, we can show that there always exists a strongly minimal cut-generating function which separates the infeasible basic solution $x = f$, $y = 0$ from the feasible solutions to (2.3). As a corollary, this shows that strongly minimal cut-generating functions always exist in this case.

Theorem 2.9. *Suppose the closed convex hull of* $\mathbb{S} \subset \mathbb{R}^n$ *is a full-dimensional polyhedron. Let* $f \in \overline{\operatorname{conv}}\,\mathbb{S}$. *Then every cut-generating function for* (2.3) *is implied by a strongly minimal cut-generating function.*

The proof will be given in Section 2.4.2.

Section 2.5 extends some of the earlier results to a mixed-integer model where nonbasic continuous and nonbasic integer variables are both present.

2.2 Characterization of Minimal Cut-Generating Functions

In this section, we characterize minimal cut-generating functions for (2.3) under the basic assumption that $\mathbb{S} \neq \emptyset$. In the next three lemmas, we state necessary conditions that are satisfied by all minimal cut-generating functions.

Lemma 2.10. *If* π *is a minimal cut-generating function for* (2.3), *then* $\pi(0) = 0$.

Proof. Suppose $\pi(0) < 0$, and let $(\bar{x}, \bar{y})$ be a feasible solution of (2.3). Then there exists some $\bar{k} \in \mathbb{Z}_{++}$ such that $\pi(0)\bar{k} < 1 - \sum_{r \in \mathbb{R}^n \setminus \{0\}} \pi(r)\bar{y}_r$ since the right-hand side of the inequality is a constant. Define $\tilde{y}$ as $\tilde{y}_0 = \bar{k}$ and $\tilde{y}_r = \bar{y}_r$ for all $r \neq 0$. Note that $(\bar{x}, \tilde{y})$

is a feasible solution of (2.3). This contradicts the assumption that π is a cut-generating function since $\sum_{r\in\mathbb{R}^n}\pi(r)\tilde{y}_r < 1$. Thus, $\pi(0)\geq 0$.

Let $(\bar{x},\bar{y})$ be a feasible solution of (2.3), and consider $\tilde{y}$ defined as $\tilde{y}_0 = 0$ and $\tilde{y}_r = \bar{y}_r$ for all $r\neq 0$. Then $(\bar{x},\tilde{y})$ is a feasible solution of (2.3). Now define the function π' as $\pi'(0) = 0$ and $\pi'(r) = \pi(r)$ for all $r\neq 0$. Observe that $\sum_{r\in\mathbb{R}^n}\pi'(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n}\pi(r)\tilde{y}_r \geq 1$ where the inequality follows because π is a cut-generating function. This implies that π' is also a cut-generating function for (2.3). Since π is minimal and $\pi'\leq\pi$, we must have $\pi = \pi'$ and $\pi(0) = 0$. □

The proof of the next lemma is similar to the one presented by Gomory and Johnson [63] for the case $\mathbb{S} = \mathbb{Z}$ and Johnson [75] for the case $\mathbb{S} = \mathbb{Z}^n$. It is included here for the sake of completeness.

Lemma 2.11. *If π is a minimal cut-generating function for* (2.3)*, then π is subadditive.*

Proof. Let $r^1, r^2\in\mathbb{R}^n$. We need to show $\pi(r^1)+\pi(r^2)\geq\pi(r^1+r^2)$. This inequality holds when $r^1 = 0$ or $r^2 = 0$ by Lemma 2.10.

Assume now that $r^1\neq 0$ and $r^2\neq 0$. Define the function π' as $\pi'(r^1+r^2) = \pi(r^1)+\pi(r^2)$ and $\pi'(r) = \pi(r)$ for $r\neq r^1+r^2$. We show that π' is a cut-generating function. Since π is minimal, it then follows that $\pi(r^1+r^2)\leq\pi'(r^1+r^2) = \pi(r^1)+\pi(r^2)$.

Consider any feasible solution $(\bar{x},\bar{y})$ to (2.3). Define $\tilde{y}$ as $\tilde{y}_{r^1} = \bar{y}_{r^1}+\bar{y}_{r^1+r^2}$, $\tilde{y}_{r^2} = \bar{y}_{r^2}+\bar{y}_{r^1+r^2}$, $\tilde{y}_{r^1+r^2} = 0$, and $\tilde{y}_r = \bar{y}_r$ otherwise. Note that $\tilde{y}$ is well-defined since $r^1\neq 0$ and $r^2\neq 0$. It is easy to verify that $\tilde{y}$ has finite support, $\tilde{y}_r\in\mathbb{Z}_+$ for all $r\in\mathbb{R}^n$, and $\sum_{r\in\mathbb{R}^n} r\tilde{y}_r = \sum_{r\in\mathbb{R}^n} r\bar{y}_r$. These together show that $(\bar{x},\tilde{y})$ is a feasible solution of (2.3). Furthermore, we have $\sum_{r\in\mathbb{R}^n}\pi'(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n}\pi(r)\tilde{y}_r$ which is greater than or equal to 1 since π is a cut-generating function. This proves that π' is a cut-generating function. □

The next lemma shows that all minimal cut-generating functions satisfy the generalized symmetry condition (2.4). A similar argument appears in the proof of the main result of [13] for the case $\mathbb{S} = \{0\}$.

Lemma 2.12. *If π is a minimal cut-generating function for* (2.3)*, then it satisfies the generalized symmetry condition.*

Proof. Let $\bar{r}\in\mathbb{R}^n$. For any $\bar{x}\in\mathbb{S}$ and $\bar{k}\in\mathbb{Z}_{++}$, define $\bar{y}$ as $\bar{y}_{\bar{r}} = \bar{k}$, $\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise. Since $(\bar{x},\bar{y})$ is feasible to (2.3) and π is a cut-generating function for (2.3), we have $\pi(\bar{r})\geq\frac{1}{\bar{k}}(1-\pi(\bar{x}-f-\bar{k}\bar{r}))$. Then the definition of supremum implies $\pi(\bar{r})\geq\sup_{x,k}\left\{\frac{1}{k}(1-\pi(x-f-k\bar{r})) : x\in\mathbb{S}, k\in\mathbb{Z}_{++}\right\}$. Note that the value on the right-hand side is bounded from above since π is a real-valued function and the left-hand side is finite.

Let the function $\rho : \mathbb{R}^n \to \mathbb{R}$ be defined as $\rho(r) = \sup_{x,k} \left\{ \frac{1}{k}(1 - \pi(x - f - kr)) : x \in \mathbb{S}, k \in \mathbb{Z}_{++} \right\}$. Note that $\pi \geq \rho$ by the first part. Now suppose π does not satisfy the generalized symmetry condition. Then there exists $\tilde{r} \in \mathbb{R}^n$ such that $\pi(\tilde{r}) > \rho(\tilde{r})$. Define the function π' as $\pi'(\tilde{r}) = \rho(\tilde{r})$ and $\pi'(r) = \pi(r)$ for all $r \neq \tilde{r}$. Consider any feasible solution $(\tilde{x}, \tilde{y})$ to (2.3). If $\tilde{y}_{\tilde{r}} = 0$, we get $\sum_{r \in \mathbb{R}^n} \pi'(r)\tilde{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$. Otherwise, $\tilde{y}_{\tilde{r}} \geq 1$ and we have $\pi'(\tilde{r})\tilde{y}_{\tilde{r}} + \sum_{r \in \mathbb{R}^n \setminus \{\tilde{r}\}} \pi'(r)\tilde{y}_r \geq 1 - \pi(\tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}) + \sum_{r \in \mathbb{R}^n \setminus \{\tilde{r}\}} \pi(r)\tilde{y}_r \geq 1$ where we use $\pi'(\tilde{r}) = \rho(\tilde{r}) \geq \frac{1}{\tilde{y}_{\tilde{r}}}(1 - \pi(\tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}))$ to obtain the first inequality and the subadditivity of π and $\sum_{r \in \mathbb{R}^n \setminus \{\tilde{r}\}} r\tilde{y}_r = \tilde{x} - f - \tilde{y}_{\tilde{r}}\tilde{r}$ to obtain the second inequality. Thus, π' is a cut-generating function for (2.3). Since $\pi' \leq \pi$ and $\pi'(\tilde{r}) = \rho(\tilde{r}) < \pi(\tilde{r})$, this contradicts the minimality of π. □

We now prove Theorem 2.2 stated in the introduction.

Theorem 2.2. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal cut-generating function for* (2.3) *if and only if $\pi(0) = 0$, π is subadditive and satisfies the generalized symmetry condition.*

Proof. The necessity of these conditions has been proven in Lemmas 2.10, 2.11, and 2.12. We now prove their sufficiency.

Assume that $\pi(0) = 0$, π is subadditive and satisfies the generalized symmetry condition. Since $\pi(0) = 0$, the generalized symmetry condition implies $\pi(\bar{x} - f) \geq 1$ for all $\bar{x} \in \mathbb{S}$ by taking $r = 0$, $x = \bar{x}$, and $k = 1$ in (2.4). We first show that π is a cut-generating function for (2.3). To see this, note that any feasible solution $(\bar{x}, \bar{y})$ for (2.3) satisfies $\sum_{r \in \mathbb{R}^n} r\bar{y}_r = \bar{x} - f$, and using the subadditivity of π, we can write $\sum_{r \in \mathbb{R}^n} \pi(r)\bar{y}_r \geq \pi(\sum_{r \in \mathbb{R}^n} r\bar{y}_r) = \pi(\bar{x} - f) \geq 1$.

If π is not minimal, then by Theorem 2.1, there exists a minimal cut-generating function π' such that $\pi' \leq \pi$ and $\pi'(\bar{r}) < \pi(\bar{r})$ for some $\bar{r} \in \mathbb{R}^n$. Let $\epsilon = \pi(\bar{r}) - \pi'(\bar{r})$. Because π satisfies the generalized symmetry condition, there exists $\bar{x} \in \mathbb{S}$ and $\bar{k} \in \mathbb{Z}_{++}$ such that $\pi(\bar{r}) - \frac{\epsilon}{2} \leq \frac{1}{\bar{k}}(1 - \pi(\bar{x} - f - \bar{k}\bar{r}))$. Rearranging the terms and using $\pi' \leq \pi$ and $\pi(\bar{r}) - \pi'(\bar{r}) = \epsilon$, we obtain

$$1 \geq \bar{k}\left(\pi(\bar{r}) - \frac{\epsilon}{2}\right) + \pi\left(\bar{x} - f - \bar{k}\bar{r}\right) \geq \bar{k}\left(\pi'(\bar{r}) + \frac{\epsilon}{2}\right) + \pi'\left(\bar{x} - f - \bar{k}\bar{r}\right)$$

which implies $\bar{k}\pi'(\bar{r}) + \pi'(\bar{x} - f - \bar{k}\bar{r}) < 1$. This contradicts the hypothesis that π' is a cut-generating function because the solution $x = \bar{x}$, $\bar{y}_{\bar{r}} = \bar{k}$, $\bar{y}_{\bar{x}-f-\bar{k}\bar{r}} = 1$, and $\bar{y}_r = 0$ otherwise is feasible to (2.3). □

Next we state two properties of subadditive functions that will be used later in the chapter. The first lemma below shows that if the supremum is achieved in the generalized symmetry condition, it must be achieved for $k = 1$.

Lemma 2.13. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function that satisfies the generalized symmetry condition. Suppose $r \in \mathbb{R}^n$ is a point for which the supremum in* (2.4) *is achieved. Then the supremum is achieved when $k = 1$, that is, $\pi(r) = 1 - \pi(x - f - r)$ for some $x \in \mathbb{S}$.*

Proof. Consider a vector $r \in \mathbb{R}^n$ for which the supremum in (2.4) is achieved. Namely, there exists $x \in \mathbb{S}$ and $k \in \mathbb{Z}_{++}$ such that $\pi(r) = \frac{1}{k}(1 - \pi(x - f - kr))$. This equation can be rewritten as

$$k\pi(r) + \pi(x - f - kr) = 1. \tag{2.5}$$

We also have

$$k\pi(r) + \pi(x - f - kr) = \pi(r) + (k-1)\pi(r) + \pi(x - f - kr) \geq \pi(r) + \pi(x - f - r) \geq 1$$

where the first inequality follows from the subadditivity of π and the second from $\pi(r) \geq 1 - \pi(x - f - r)$ by the generalized symmetry condition. Using (2.5), we see that equality holds throughout. In particular, $\pi(r) + \pi(x - f - r) = 1$. Thus, the supremum in (2.4) is achieved when $k = 1$. □

For a subadditive function $\pi : \mathbb{R}^n \to \mathbb{R}$, we have $\pi(r) \geq \frac{\pi(kr)}{k}$ for all $r \in \mathbb{R}^n$ and $k \in \mathbb{Z}_{++}$. Hence, $\pi(r) \geq \sup_{k \in \mathbb{Z}_{++}} \frac{\pi(kr)}{k}$. In fact, we have $\pi(r) = \sup_{k \in \mathbb{Z}_{++}} \frac{\pi(kr)}{k}$ because equality holds for $k = 1$. When $\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k}$ for some $r \in \mathbb{R}^n$, a result of Bachem et al. [13] shows that π is actually linear in $k \in \mathbb{Z}_{++}$.

Lemma 2.14 (Bachem, Johnson, and Schrader [13])**.** *If a subadditive function $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies $\pi(r) = \limsup_{k \in \mathbb{Z}_{++}, k \to \infty} \frac{\pi(kr)}{k}$ for some $r \in \mathbb{R}^n$, then $\pi(kr) = k\pi(r)$ for all $k \in \mathbb{Z}_{++}$.*

We close this section with a proof of Proposition 2.5 which was stated in the introduction.

Proposition 2.5. *A cut-generating function π for* (2.3) *is restricted minimal if and only if it is minimal and* $\inf_x\{\pi(x - f) : x \in \mathbb{S}\} = 1$.

Proof. If π is a cut-generating function, we have $\pi(\bar{x} - f) \geq 1$ for any $\bar{x} \in \mathbb{S}$. To see this, note that the solution $x = \bar{x}$, $y_{\bar{x}-f} = 1$, and $y_r = 0$ for all $r \neq \bar{x} - f$ is feasible to (2.3) and the inequality $\sum_{r \in \mathbb{R}^n} \pi(r)y_r \geq 1$ reduces to $\pi(\bar{x} - f) \geq 1$.

To prove the "only if" part, let π be a restricted minimal cut-generating function. Then there does not exist any cut-generating function $\pi' \neq \pi$ that implies π via scaling by $\beta \geq 1$. By taking $\beta = 1$, we note that no cut-generating function $\pi' \neq \pi$ dominates π. Thus, π is minimal. Let $\nu = \inf_x\{\pi(x - f) : x \in \mathbb{S}\}$. By the above observation, we have $\nu \geq 1$. Suppose $\nu > 1$, and let $\pi' = \frac{\pi}{\nu}$. For any feasible solution (x, y) to (2.3), we have $\sum_{r\in\mathbb{R}^n}\pi'(r)y_r = \frac{1}{\nu}\sum_{r\in\mathbb{R}^n}\pi(r)y_r \geq \frac{1}{\nu}\pi(\sum_{r\in\mathbb{R}^n} ry_r) = \frac{1}{\nu}\pi(x-f) \geq 1$ where the first inequality follows from the subadditivity of π (Theorem 2.2) and the second from the definition of ν. Therefore, π' is a cut-generating function. Since π' is distinct from π and implies π via scaling, this contradicts the hypothesis that π is restricted minimal. Therefore, $\nu = \inf_x\{\pi(x-f) : x \in \mathbb{S}\} = 1$.

For the converse, let π be a minimal cut-generating function such that $\inf_x\{\pi(x-f) : x \in \mathbb{S}\} = 1$. Suppose π is not restricted minimal. Then there exists a cut-generating function $\pi' \neq \pi$ that implies π via scaling. That is, there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$. Because π is minimal, we must have $\beta > 1$, but then $\inf_x\{\pi'(x-f) : x \in \mathbb{S}\} = \frac{1}{\beta}\inf_x\{\pi(x-f) : x \in \mathbb{S}\} < 1$. This implies that there exists $x \in \mathbb{S}$ such that $\pi'(x-f) < 1$, contradicting the choice of π' as a cut-generating function. □

2.3 Specializing the Set $\mathbb{S}$

In this section, we turn our attention to sets $\mathbb{S} \subset \mathbb{R}^n$ that arise in the context of integer programming. The majority of the results in this section consider $\mathbb{S} = \mathbb{C}\cap(\mathbb{Z}^p\times\mathbb{R}^{n-p})$ where $\mathbb{C} \subset \mathbb{R}^n$ is a closed convex set and p is an integer between 0 and n. The case $p = n$ and $\mathbb{C} = \mathbb{R}^n_+$ is of particular interest since it corresponds to the pure integer linear programming case. At the other extreme, when $p = 0$ and $\mathbb{C}$ is a closed convex cone, we recover the infinite relaxation of a mixed-integer conic programming model studied by Morán, Dey, and Vielma [91]. In their model, Morán, Dey, and Vielma presented an extension of the duality theory to mixed-integer conic programs and showed that subadditive functions that are nondecreasing with respect to $\mathbb{C}$ can generate all valid inequalities under a technical condition.

2.3.1 The Case $\mathbb{S} = \mathbb{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for a Convex Set $\mathbb{C}$

We first show that when $\mathbb{S} \subset \mathbb{R}^n$ is the set of mixed-integer points in a closed convex set, a function that satisfies the generalized symmetry condition is monotone in a certain sense. Let $\mathbb{K}$ be a closed convex cone and $\mathbb{L}$ be a linear subspace in $\mathbb{R}^n$. Recall that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *nondecreasing with respect to* $\mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ if $\pi(r) \leq \pi(r+w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. We say that the function π is *periodic with respect to* $\mathbb{L} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ if $\pi(r) = \pi(r+w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{L} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Note that when $\mathbb{L} = \mathbb{R}^n$ and $p = n$, this definition of periodicity reduces to the earlier definition of periodicity with respect to $\mathbb{Z}^n$.

Proposition 2.15. *Let $\mathbb{C} \subset \mathbb{R}^n$ be a closed convex set, $\mathbb{S} = \mathbb{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, and $f \in \mathbb{R}^n$. If $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies the generalized symmetry condition, then it is nondecreasing with* *respect to* $\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$*. In particular, it is periodic with respect to* $\operatorname{lin}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.

Proof. Suppose π satisfies the generalized symmetry condition. Then for any $r \in \mathbb{R}^n$ and $\epsilon > 0$, there exist $x^\epsilon \in \mathbb{S}$ and $k^\epsilon \in \mathbb{Z}_{++}$ such that $\frac{1}{k^\epsilon}(1 - \pi(x^\epsilon - f - k^\epsilon r)) > \pi(r) - \epsilon$. Let $w \in \operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Observing that $x^\epsilon + k^\epsilon w \in \mathbb{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}) = \mathbb{S}$, condition (2.4) implies

$$\pi(r+w) \geq \frac{1}{k^\epsilon}\left(1 - \pi\left((x^\epsilon + k^\epsilon w) - f - k^\epsilon(r+w)\right)\right) = \frac{1}{k^\epsilon}(1 - \pi(x^\epsilon - f - k^\epsilon r)) > \pi(r) - \epsilon.$$

Taking limits of both sides as $\epsilon \downarrow 0$, we get $\pi(r+w) \geq \pi(r)$. The second statement follows from the observation that $w, -w \in \operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ if $w \in \operatorname{lin}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. In this case, repeating the same argument with both w and $-w$ gives us the equality necessary to establish the periodicity of π. □

Proposition 2.16. *Let* $\mathbb{C} \subset \mathbb{R}^n$ *be a closed convex set,* $\mathbb{S} = \mathbb{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$*, and* $f \in \mathbb{R}^n$*. Let* $\mathbb{X} \subset \mathbb{S}$ *be such that* $\mathbb{S} = \mathbb{X} + (\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$*. The function* $\pi : \mathbb{R}^n \to \mathbb{R}$ *satisfies the generalized symmetry condition if and only if it is nondecreasing with respect to* $\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ *and satisfies the condition*

$$\pi(r) = \sup_{x,k} \left\{ \frac{1 - \pi(x - f - kr)}{k} : x \in \mathbb{X}, k \in \mathbb{Z}_{++} \right\} \quad \text{for all } r \in \mathbb{R}^n. \tag{2.6}$$

Proof. Suppose π satisfies the generalized symmetry condition. By Proposition 2.15, π is nondecreasing with respect to $\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Let $r \in \mathbb{R}^n$ and $\epsilon > 0$. For any $x \in \mathbb{X}$ and $k \in \mathbb{Z}_{++}$, we have $k\pi(r) + \pi(x - f - kr) \geq 1$. Because π satisfies the generalized symmetry condition, there exist $x^\epsilon \in \mathbb{S}$ and $k^\epsilon \in \mathbb{Z}_{++}$ such that $k^\epsilon \pi(r) + \pi(x^\epsilon - f - k^\epsilon r) < 1 + k^\epsilon \epsilon$. Let $\bar{x} \in \mathbb{X}$ be such that $x^\epsilon \in \bar{x} + (\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$. Because π is nondecreasing with respect to $\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, we get $k^\epsilon \pi(r) + \pi(\bar{x} - f - k^\epsilon r) \leq k^\epsilon \pi(r) + \pi(x^\epsilon - f - k^\epsilon r) < 1 + k^\epsilon \epsilon$. This shows that π satisfies (2.6).

To prove the converse, suppose π is nondecreasing with respect to $\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ and satisfies (2.6). Let $r \in \mathbb{R}^n$ and $\epsilon > 0$. For any $x \in \mathbb{S}$ and $k \in \mathbb{Z}_{++}$, there exists $\bar{x} \in \mathbb{X}$ such that $x \in \bar{x} + (\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$ and we have $k\pi(r) + \pi(x - f - kr) \geq k\pi(r) + \pi(\bar{x} - f - kr) \geq 1$. Furthermore, there exist $x^\epsilon \in \mathbb{X} \subset \mathbb{S}$ and $k^\epsilon \in \mathbb{Z}_{++}$ such that $\pi(r) - \epsilon < \frac{1}{k^\epsilon}(1 - \pi(x^\epsilon - f - k^\epsilon r))$. This shows that π satisfies the generalized symmetry condition. □

When the set $\mathbb{X}$ in the statement of Proposition 2.16 can be chosen finite, condition (2.6) further implies that

$$\forall r \in \mathbb{R}^n \quad \exists x^r \in \mathbb{X} \quad \text{such that } \pi(r) = \sup_k \left\{ \frac{1 - \pi(x^r - f - kr)}{k} : k \in \mathbb{Z}_{++} \right\}. \tag{2.7}$$

A finite set $\mathbb{X}$ satisfying the hypothesis of Proposition 2.16 exists for two choices of unbounded sets $\mathbb{S} \subset \mathbb{R}^n$ which are important in integer programming. When $\mathbb{S} \subset \mathbb{R}^n$ is the set of pure integer points in a rational (possibly unbounded) polyhedron, the existence of such a finite set $\mathbb{X}$ follows from Meyer's Theorem and its proof [87]. When $\mathbb{S} \subset \mathbb{R}^n$ is the set of mixed-integer points in a closed convex cone $\mathbb{K}$, one can simply let $\mathbb{X} = \{0\}$. Then (2.6) can be stated as

$$\pi(r) = \sup_k \left\{ \frac{1 - \pi(-f - kr)}{k} : k \in \mathbb{Z}_{++} \right\} \quad \text{for all } r \in \mathbb{R}^n. \tag{2.8}$$

In general, (2.8) is a weaker requirement than symmetry on subadditive functions. However, the next proposition shows that (2.8) implies symmetry if the supremum is achieved for all $r \in \mathbb{R}^n$.

Proposition 2.17. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function.*

i. Let $\mathbb{X} \subset \mathbb{S}$ be a finite set, and suppose π satisfies (2.7). Fix $r \in \mathbb{R}^n$, and choose $x^r \in \mathbb{X}$ as in (2.7). The supremum in (2.7) is attained if and only if $\pi(r) + \pi(x^r - f - r) = 1$.

ii. Suppose π satisfies (2.8). Fix $r \in \mathbb{R}^n$. The supremum in (2.8) is attained if and only if $\pi(r) + \pi(-f - r) = 1$. Furthermore, the supremum in (2.8) is attained for all $r \in \mathbb{R}^n$ if and only if π satisfies the symmetry condition.

Proof. We prove statement (i) first. Fix $r \in \mathbb{R}^n$, and choose $x^r \in \mathbb{X}$ as in (2.7). Suppose the supremum on the right-hand side of (2.7) is attained. Let $k^* \in \mathbb{Z}_{++}$ be such that $\frac{1}{k^*}(1 - \pi(x^r - f - k^* r)) \geq \frac{1}{k}(1 - \pi(x^r - f - kr))$ for all $k \in \mathbb{Z}_{++}$. Because π satisfies (2.7), we have $\pi(r) \geq 1 - \pi(x^r - f - r)$ and $\pi(r) = \frac{1}{k^*}(1 - \pi(x^r - f - k^* r))$. Using the subadditivity of π, we can write

$$1 = k^*\pi(r) + \pi(x^r - f - k^* r) = \pi(r) + (k^* - 1)\pi(r) + \pi(x^r - f - k^* r) \geq \pi(r) + \pi(x^r - f - r) \geq 1.$$

This shows $\pi(r) + \pi(x^r - f - r) = 1$. To prove the converse, suppose $\pi(r) + \pi(x^r - f - r) = 1$. Then $1 - \pi(x^r - f - r) = \pi(r) = \sup_k\{\frac{1}{k}(1 - \pi(x^r - f - kr)) : k \in \mathbb{Z}_{++}\}$ which shows that the supremum is attained for $k = 1$. This concludes the proof of statement (i).

Statement (ii) follows from statement (i) by noting that (2.8) is equivalent to (2.7) with $\mathbb{X} = \{0\}$. In this case, $x^r \in \mathbb{X}$ in (2.7) is necessarily equal to zero for any $r \in \mathbb{R}^n$. Let $r \in \mathbb{R}^n$. By statement (i), the supremum in (2.8) is attained if and only if $\pi(r) + \pi(-f - r) = 1$. If the supremum is attained for all $r \in \mathbb{R}^n$, then $\pi(r) + \pi(-f - r) = 1$ for all $r \in \mathbb{R}^n$, which is the symmetry condition on π. □

Proposition 2.18. *Let $\mathbb{X} \subset \mathbb{S}$ be a finite set, and let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function such that $\pi(0) = 0$ and π satisfies (2.7). Fix $r \in \mathbb{R}^n$, and choose $x^r \in \mathbb{X}$ as in (2.7). If the* *supremum in* (2.7) *is not attained, then*

$$\pi(r) = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{\pi(kr)}{k} = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}.$$

Furthermore, $\pi(kr) = k\pi(r)$ *for all* $k \in \mathbb{Z}_{++}$.

Proof. Fix $r \in \mathbb{R}^n$, and choose $x^r \in \mathbb{X}$ as in (2.7). Suppose the supremum in (2.7) is not attained. Since π satisfies (2.7), $\pi(r) \geq \frac{1}{k}(1 - \pi(x^r - f - kr))$ for all $k \in \mathbb{Z}_{++}$. It follows that $\pi(r) \geq \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{1}{k}(1-\pi(x^r - f - kr))$. Let $\epsilon = \pi(r) - \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{1}{k}(1-\pi(x^r - f - kr))$, and suppose $\epsilon > 0$. By the definition of limit supremum, there exists $k_0 \in \mathbb{Z}_{++}$ such that $\pi(r) - \frac{\epsilon}{2} \geq \frac{1}{k}(1 - \pi(x^r - f - kr))$ for all $k \geq k_0$. It follows that the supremum in (2.7) must be attained for some $k < k_0$, a contradiction. Therefore, $\epsilon = 0$. Using $\pi(0) = 0$ and the subadditivity of π, we can write

$$\begin{aligned}\pi(r) &= \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{1-\pi(x^r - f - kr)}{k} = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(x^r - f - kr)}{k} \\ &\leq \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr) + \pi(-x^r + f)}{k} = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k} \\ &\leq \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{\pi(kr)}{k} \leq \pi(r).\end{aligned}$$

In particular, $\pi(r) = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{\pi(kr)}{k} = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}$. It follows from Lemma 2.14 that $\pi(kr) = k\pi(r)$ for all $k \in \mathbb{Z}_{++}$. □

When the set $\mathbb{X}$ in the statement of Proposition 2.16 is finite, we can obtain a simplified version of (2.6) in which the double supremum over x and k is decoupled through Propositions 2.17 and 2.18.

Corollary 2.19. *Let* $\mathbb{C} \subset \mathbb{R}^n$ *be a closed convex set,* $\mathbb{S} = \mathbb{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$*, and* $f \in \mathbb{R}^n$*. Let* $\mathbb{X} \subset \mathbb{S}$ *be a finite set such that* $\mathbb{S} = \mathbb{X} + (\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}))$*. Let* $\pi : \mathbb{R}^n \to \mathbb{R}$ *be a subadditive function such that* $\pi(0) = 0$*. The function* π *satisfies the generalized symmetry condition if and only if it is nondecreasing with respect to* $\operatorname{rec}(\mathbb{C}) \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ *and satisfies the condition*

$$\pi(r) = \max\left\{\max_{x\in\mathbb{X}}\{1 - \pi(x - f - r)\},\ \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}\right\} \quad \text{for all } r \in \mathbb{R}^n. \tag{2.9}$$

Proof. By Proposition 2.16, it will be enough to show that π satisfies (2.6) if and only if it satisfies (2.9). Suppose π satisfies (2.6). Fix $r \in \mathbb{R}^n$. By (2.6), we have $\pi(r) \geq$

$\max_{x\in\mathbb{X}}\{1-\pi(x-f-r)\}$. By the subadditivity of π, $\pi(r) \geq \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{\pi(kr)}{k} \geq \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}$. The "only if" part then follows from Propositions 2.17 and 2.18 by observing that $\mathbb{X}$ is finite and π satisfies (2.7). To prove the "if" part, suppose π satisfies (2.9). Fix $r \in \mathbb{R}^n$. Observe that (2.9) implies $\pi(r) \geq 1-\pi(x-f-r)$ for all $x \in \mathbb{X}$. By the subadditivity of π, $k\pi(r)+\pi(x-f-kr) \geq \pi(r)+\pi(x-f-r) \geq 1$ for all $x \in \mathbb{X}$ and $k \in \mathbb{Z}_{++}$. In particular, $\pi(r) \geq \sup_{x,k}\left\{\frac{1}{k}(1-\pi(x-f-kr)) : x \in \mathbb{X}, k \in \mathbb{Z}_{++}\right\}$. If there exists $x^r \in \mathbb{X}$ such that $\pi(r) = 1-\pi(x^r-f-r)$, then (2.7) holds for that x^r. If $\pi(r) = \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}$, then (2.7) holds for any $x \in \mathbb{X}$ since

$$\begin{aligned}\limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{1-\pi(x-f-kr)}{k} &\geq \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{1-\pi(x-f)-\pi(-kr)}{k}\\ &= \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k} = \pi(r).\end{aligned}$$

In either case, condition (2.6) is satisfied. □

2.3.2 The Case $\mathbb{S} = \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for a Convex Cone $\mathbb{K}$

In this section, we consider the case where $\mathbb{S} \subset \mathbb{R}^n$ is the set of mixed-integer points in a closed convex cone $\mathbb{K}$. The following theorem recapitulates the results of Theorem 2.2 and Proposition 2.16 for this case.

Theorem 2.20. *Let $\mathbb{K} \subset \mathbb{R}^n$ be a closed convex cone and $\mathbb{S} = \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a minimal cut-generating function for* (2.3) *if and only if $\pi(0) = 0$, π is subadditive, nondecreasing with respect to $\mathbb{S}$, and satisfies* (2.8).

When $\mathbb{S} = \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ for a closed convex cone $\mathbb{K}$, we can choose $\mathbb{X} = \{0\}$ in Corollary 2.19. Then (2.8) in the statement of Theorem 2.20 can be replaced without any loss of generality with (2.9) which now reads $\pi(r) = \max\Big\{1-\pi(-f-r),\ \limsup_{k\in\mathbb{Z}_{++},k\to\infty} \frac{-\pi(-kr)}{k}\Big\}$ for all $r \in \mathbb{R}^n$. This condition simplifies further to just $\pi(r) = 1-\pi(-f-r)$, the symmetry condition, when we consider *restricted* minimal cut-generating functions. This will be proved next in Theorem 2.7, which was already stated in the introduction. Theorem 2.7 generalizes to $\mathbb{S} = \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ a result of Bachem et al. [13] for the case $\mathbb{S} = \{0\}$.

Theorem 2.7. *Let $\mathbb{K} \subset \mathbb{R}^n$ be a closed convex cone and $\mathbb{S} = \mathbb{K} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a restricted minimal cut-generating function for* (2.3) *if and only if $\pi(0) = 0$, π is subadditive, nondecreasing with respect to $\mathbb{S}$, and satisfies the symmetry condition.*

Proof. We first prove the "if" part. Assume $\pi(0) = 0$, π is subadditive, nondecreasing with respect to $\mathbb{S}$, and satisfies the symmetry condition. Since condition (2.8) is a weaker requirement than symmetry, it follows from Theorem 2.20 that π is a minimal cut-generating function. Because π is nondecreasing with respect to $\mathbb{S}$, we have $\pi(x - f) \geq \pi(-f)$ for all $x \in \mathbb{S}$. Furthermore, by taking $r = -f$, the symmetry condition implies $\pi(-f) = 1$. It follows that $\min\{\pi(x - f) : x \in \mathbb{S}\} = \pi(-f) = 1$. Then by Proposition 2.5, π is restricted minimal.

We now prove the "only if" part. Assume that π is a restricted minimal cut-generating function. By Proposition 2.5, π is a minimal cut-generating function and satisfies $\inf_x\{\pi(x - f) : x \in \mathbb{S}\} = 1$. Since π is minimal, Theorem 2.20 implies that $\pi(0) = 0$, π is subadditive, nondecreasing with respect to $\mathbb{S}$, and satisfies (2.8). Because π is nondecreasing with respect to $\mathbb{S} \subset \mathbb{R}^n$, we have $\pi(-f) = \inf_x\{\pi(x - f) : x \in \mathbb{S}\} = 1$. Now suppose that there exists $\bar{r} \in \mathbb{R}^n$ such that $\pi(\bar{r}) > 1 - \pi(-f - \bar{r})$. Letting $\mathbb{X} = \{0\}$ and using Proposition 2.17(i), we see that the supremum in (2.8) is not attained. By Proposition 2.18, $\pi(k\bar{r}) = k\pi(\bar{r})$ for all $k \in \mathbb{Z}_{++}$. By the subadditivity of π, $\pi(-f + k(f + \bar{r})) + (k - 1)\pi(-f) \geq \pi(k\bar{r}) = k\pi(\bar{r})$ for all $k \in \mathbb{Z}_{++}$. Rearranging terms and using $\pi(-f) = 1$, we get $k(1 - \pi(\bar{r})) \geq 1 - \pi(-f + k(f + \bar{r}))$. Thus, $1 - \pi(\bar{r}) \geq \frac{1}{k}(1 - \pi(-f + k(f + \bar{r})))$ for all $k \in \mathbb{Z}_{++}$. This implies

$$1 - \pi(\bar{r}) \geq \sup_k \left\{ \frac{1 - \pi(-f - k(-f - \bar{r}))}{k} : k \in \mathbb{Z}_{++} \right\} = \pi(-f - \bar{r})$$

where the equality follows from (2.8). This contradicts the hypothesis that $\pi(\bar{r}) > 1 - \pi(-f - \bar{r})$. □

Let $\mathbb{K}_1, \mathbb{K}_2 \in \mathbb{R}^n$ be two closed convex cones such that $\mathbb{K}_2 \subset \mathbb{K}_1$. Because $\mathbb{K}_2 \subset \mathbb{K}_1$, every cut-generating function for (2.3) when $\mathbb{S} = \mathbb{K}_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is a cut-generating function for (2.3) when $\mathbb{S} = \mathbb{K}_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. However, it is rather surprising that every *restricted minimal* cut-generating function for (2.3) when $\mathbb{S} = \mathbb{K}_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is also a restricted minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{K}_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. A similar statement is also true for minimal cut-generating functions. We show this in the next proposition.

Proposition 2.21. *Let $\mathbb{K}_1, \mathbb{K}_2 \in \mathbb{R}^n$ be two closed convex cones such that $\mathbb{K}_2 \subset \mathbb{K}_1$. If π is a (restricted) minimal cut-generating function for* (2.3) *when $\mathbb{S} = \mathbb{K}_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, then π is also a (restricted) minimal cut-generating function for* (2.3) *when $\mathbb{S} = \mathbb{K}_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.*

Proof. We prove the statement for the case of restricted minimality only. A similar claim on minimal cut-generating functions follows by using Theorem 2.20 instead of Theorem 2.7.

Assume π is a restricted minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{K}_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. By Theorem 2.7, $\pi(0) = 0$, π is subadditive, nondecreasing with respect to $\mathbb{K}_1 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, and satisfies the symmetry condition. Because $\mathbb{K}_2 \subset \mathbb{K}_1$, π is also nondecreasing with respect to $\mathbb{K}_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. Therefore, again by Theorem 2.7, π is a restricted minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{K}_2 \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$. □

In particular, Proposition 2.21 implies that a (restricted) minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$ is still (restricted) minimal for (2.3) when $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, and a (restricted) minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ is still (restricted) minimal for (2.3) when $\mathbb{S} = \{0\}$. We focus on the cases $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$ and $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ in the next two sections.

2.3.3 The Case $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$

Gomory and Johnson [63] and Johnson [75] characterized minimal cut-generating functions for (2.2) in terms of subadditivity, symmetry, and periodicity with respect to $\mathbb{Z}^n$. In this section, we relate our Theorems 2.7 and 2.20 to their results.

For the model (2.2), Theorem 2.20 states that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function if and only if $\pi(0) = 0$, π is subadditive, periodic with respect to $\mathbb{Z}^n$, and satisfies (2.8). For the same model, Theorem 2.7 shows that π is restricted minimal if and only if it satisfies the symmetry condition as well as the conditions for minimality above. In the context of model (2.2), cut-generating functions are conventionally required to be nonnegative; therefore, the minimal ones take values in the interval $[0, 1]$ only. (See [45, 63, 75].) While the above implications of Theorems 2.7 and 2.20 hold without this additional assumption, the notions of minimality and restricted minimality coincide for nonnegative cut-generating functions for (2.2). To see this, note that any nonnegative minimal cut-generating function π for (2.2) satisfies $\pi(-f) \geq 1$ because $0 \in \mathbb{S}$ and $\pi(-f) \leq 1$ because it takes values in $[0, 1]$ only. The periodicity of π with respect to $\mathbb{Z}^n$ then implies $\min_x\{\pi(x-f) : x \in \mathbb{Z}^n\} = \pi(-f) = 1$. It follows from Proposition 2.5 that any nonnegative minimal cut-generating function for (2.2) is in fact restricted minimal. Hence, by taking $\mathbb{K} = \mathbb{R}^n$ and $p = n$ in the statement of Theorem 2.7, we can recover the well-known result of Gomory and Johnson on nonnegative minimal cut-generating functions for (2.2).

Theorem 2.22 (Gomory and Johnson [63], Johnson [75])**.** *Let* $\pi : \mathbb{R}^n \to \mathbb{R}_+$. *The function* π *is a minimal cut-generating function for* (2.2) *if and only if* $\pi(0) = 0$, π *is subadditive, symmetric, and periodic with respect to* $\mathbb{Z}^n$.

Note that when $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$, a minimal cut-generating function $\pi : \mathbb{R}^n \to \mathbb{R}$ for (2.3) has to be periodic with respect to $\mathbb{Z}^p \times \mathbb{R}^{n-p}$ by Theorem 2.20. In particular, the value of π cannot depend on the last $n - p$ entries of its argument. This shows a simple bijection between minimal cut-generating functions for $\mathbb{S} = \mathbb{Z}^p$ and those for $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$: Let $\text{proj}_{\mathbb{R}^p} : \mathbb{R}^n \to \mathbb{R}^p$ denote the orthogonal projection onto the first p coordinates. The function $\pi' : \mathbb{R}^p \to \mathbb{R}$ is a minimal cut-generating function for $\mathbb{S} = \mathbb{Z}^p$ if and only if $\pi = \pi' \circ \text{proj}_{\mathbb{R}^p}$ is a minimal cut-generating function for $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$. Using the same arguments, one can also show that such a bijection exists between restricted minimal cut-generating functions for $\mathbb{S} = \mathbb{Z}^p$ and those for $\mathbb{S} = \mathbb{Z}^p \times \mathbb{R}^{n-p}$.

2.3.4 The Case $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$

In this section, we focus on the case where $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ which is of particular importance in integer linear programming. We simplify the statement of Theorems 2.7 and 2.20 for this special case exploiting the fact that $\mathbb{R}^n_+$ has the finite generating set $\{e^i\}_{i=1}^n$. However, we first prove a simple lemma.

Lemma 2.23. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function. For any $\alpha > 0$ and $r \in \mathbb{R}^n$, $\frac{\pi(\alpha r)}{\alpha} \leq \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$.*

Proof. Consider $\epsilon = \frac{\alpha}{k}$ for $k \in \mathbb{Z}_{++}$. We have $k\pi(\frac{\alpha}{k} r) \geq \pi(\alpha r)$ by the subadditivity of π. Thus, $\frac{\pi(\alpha r)}{\alpha} \leq \frac{\pi(\frac{\alpha}{k} r)}{\frac{\alpha}{k}}$. Letting $k \to +\infty$, this implies $\frac{\pi(\alpha r)}{\alpha} \leq \limsup_{\epsilon \to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$. □

Proposition 2.24. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ be a subadditive function such that $\pi(0) = 0$. The function π is nondecreasing with respect to $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ if and only if $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$.*

Proof. Suppose π is nondecreasing with respect to $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. Because $\pi(0) = 0$, π has to have $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$. Now suppose $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$. For any $w \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, using the subadditivity of π and Lemma 2.23 with $\alpha = w_i$ for $i \in [n] \setminus [p]$, we can write

$$\pi(-w) \leq \sum_{i=1}^n \pi(-w_i e^i) \leq \sum_{i=1}^p w_i \pi(-e^i) + \sum_{i=p+1}^n w_i \limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0.$$

Thus, for any $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, $\pi(r + w) \geq \pi(r) - \pi(-w) \geq \pi(r)$. This shows that π is nondecreasing with respect to $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. □

Theorem 2.20 and Proposition 2.24 thus show the following: A function $\pi : \mathbb{R}^n \to \mathbb{R}$ is a minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ if and only if $\pi(0) = 0$, $\pi(-e^i) \leq 0$ for all $i \in [p]$ and $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$, π is subadditive and satisfies (2.8). Similarly, Theorem 2.7 and Proposition 2.24 show the following.

Theorem 2.25. *Let* $\mathbb{S} = \mathbb{Z}_+^p \times \mathbb{R}_+^{n-p}$ *and* $\pi : \mathbb{R}^n \to \mathbb{R}$. *The function* π *is a restricted minimal cut-generating function for* (2.3) *if and only if* $\pi(0) = 0$, $\pi(-e^i) \leq 0$ *for all* $i \in [p]$ *and* $\limsup_{\epsilon \to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ *for all* $i \in [n] \setminus [p]$, π *is subadditive and satisfies the symmetry condition.*

2.4 Strongly Minimal Cut-Generating Functions

The following example illustrates the distinction between restricted minimal and strongly minimal cut-generating functions.

Example 2.3. Consider the model (2.3) where $n = 1$, $0 < f < 1$, and $\mathbb{S} = \mathbb{Z}_+$. The Gomory function $\pi_1^1(r) = \min\left\{\frac{r-\lfloor r\rfloor}{1-f}, \frac{\lceil r\rceil - r}{f}\right\}$ is a cut-generating function in this setting [62]. For any $\alpha \geq 0$, we define perturbations of the Gomory function as $\pi_\alpha^3(r) = \alpha r + (1+\alpha f)\pi_1^1(r)$. One can easily verify that $\pi_\alpha^3(0) = 0$ and $\pi_\alpha^3(-1) = -\alpha \leq 0$. Furthermore, π_α^3 is symmetric and subadditive since π_1^1 is. By Theorem 2.25, π_α^3 is a restricted minimal cut-generating function. However, for $\alpha > 0$, π_α^3 is not strongly minimal because it is implied by the Gomory function π_1^1.

When $f \notin \overline{\operatorname{conv}}\,\mathbb{S}$, any valid inequality that strictly separates f from $\mathbb{S}$ can be used to cut off the infeasible solution $x = f$, $y = 0$. Therefore, when we analyze strongly minimal cut-generating functions, our focus will be on the case $f \in \overline{\operatorname{conv}}\,\mathbb{S}$.

Lemma 2.26. *Suppose* $f \in \overline{\operatorname{conv}}\,\mathbb{S}$. *Let* π *be a (restricted) minimal cut-generating function for* (2.3). *Any cut-generating function for* (2.3) *that implies* π *is also (restricted) minimal.*

Proof. We will prove the claim for the case of restricted minimality only. The proof for minimality is similar.

Let π be a restricted minimal cut-generating function for (2.3). Let π' be a cut-generating function that implies π. Then there exist a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{S}$ and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \beta\pi'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\operatorname{conv}}\,\mathbb{S}$, the inequality $\alpha^\top(x-f) \geq \alpha_0$ is also valid for $x = f$. Hence, $\alpha_0 \leq 0$, and $\beta \geq 1$. We claim that π' is restricted minimal.

Let $\bar\pi'$ be a restricted minimal cut-generating function that implies π' via scaling. Such a function $\bar\pi'$ always exists by Proposition 2.6. Then there exists $\nu \geq 1$ such that $\pi' \geq \nu\bar\pi'$. By Proposition 2.5 and Theorem 2.2, $\bar\pi'$ is subadditive. We first show that $\bar\pi : \mathbb{R}^n \to \mathbb{R}$, defined as $\bar\pi(r) = \beta\bar\pi'(r) + \frac{\alpha^\top r}{\nu}$, is also a cut-generating function. Indeed, for any feasible solution (x, y) to (2.3), we can use the validity of $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{S}$ and the subadditivity of $\bar\pi'$ to write

$$\sum_{r\in\mathbb{R}^n}\bar\pi(r)y_r=\sum_{r\in\mathbb{R}^n}\frac{\alpha^\top r}{\nu}y_r+\beta\sum_{r\in\mathbb{R}^n}\bar\pi'(r)y_r\geq\frac{\alpha^\top(x-f)}{\nu}+\beta\bar\pi'(x-f)\geq\frac{\alpha_0}{\nu}+\beta\geq\alpha_0+\beta\geq 1.$$

Therefore, $\bar\pi$ is a cut-generating function. Because $\nu\geq 1$, so is $\nu\bar\pi$. Furthermore, for all $r\in\mathbb{R}^n$, we have

$$\nu\bar\pi(r)=\alpha^\top r+\beta\nu\bar\pi'(r)\leq\alpha^\top r+\beta\pi'(r)\leq\pi(r). \tag{2.10}$$

Since π is a restricted minimal cut-generating function, $\nu\bar\pi=\bar\pi=\pi$, $\nu=1$, and equality holds throughout (2.10). In particular, the first inequality in (2.10) is tight. Using this, $\nu=1$, and $\beta\geq 1$, we get $\bar\pi'=\pi'$. This proves that π' is restricted minimal. ☐

The next proposition characterizes strongly minimal cut-generating functions as a certain subset of restricted minimal cut-generating functions.

Proposition 2.27. *Suppose $\mathbb{S}\subset\mathbb{R}^n$ is full-dimensional and $f\in\overline{\operatorname{conv}}\,\mathbb{S}$. Let $\pi:\mathbb{R}^n\to\mathbb{R}$. The function π is a strongly minimal cut-generating function for* (2.3) *if and only if it is a restricted minimal cut-generating function for* (2.3) *and for any valid inequality $\alpha^\top(x-f)\geq\alpha_0$ for $\mathbb{S}$ such that $\alpha\neq 0$, there exists $x^*\in\mathbb{S}$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0}<1$.*

Proof. We first prove the "only if" part of the statement. Let π be a strongly minimal cut-generating function for (2.3). It follows by setting $\alpha=0$ and $\alpha_0=0$ in the definition of strong minimality that π is restricted minimal. In particular, it is subadditive by Theorem 2.2 and Proposition 2.5. Suppose there exists a valid inequality $\alpha^\top(x-f)\geq\alpha_0$ for $\mathbb{S}$ such that $\alpha\neq 0$ and $\frac{\pi(x-f)-\alpha^\top(x-f)}{1-\alpha_0}\geq 1$ for all $x\in\mathbb{S}$. Because $f\in\overline{\operatorname{conv}}\,\mathbb{S}$, we must have $\alpha_0\leq 0$. Define the function $\pi':\mathbb{R}^n\to\mathbb{R}$ by letting $\pi'(r)=\frac{\pi(r)-\alpha^\top r}{1-\alpha_0}$. We claim that π' is a cut-generating function. To see this, first note that π' is subadditive because π is. Also, $\pi'(x-f)\geq 1$ for all $x\in\mathbb{S}$ by our hypothesis. Then for any feasible solution (x,y) to (2.3), we can write $\sum_{r\in\mathbb{R}^n}\pi'(r)y_r\geq\pi'(\sum_{r\in\mathbb{R}^n}ry_r)=\pi'(x-f)\geq 1$. Thus, π' is indeed a cut-generating function for (2.3). Furthermore, it is not difficult to show that π' is distinct from π. Consider $\bar x\in\mathbb{S}$ such that $\alpha^\top(\bar x-f)>\alpha_0$; such a point exists because $\mathbb{S}$ is full-dimensional. Because π is a cut-generating function, $\pi(\bar x-f)\geq 1$. Then $\pi'(\bar x-f)=\frac{\pi(\bar x-f)-\alpha^\top(\bar x-f)}{1-\alpha_0}<\pi(\bar x-f)$ because $\alpha^\top(\bar x-f)>\alpha_0\geq\alpha_0\pi(\bar x-f)$. Finally, note that π' implies π since $\pi(r)\geq(1-\alpha_0)\pi'(r)+\alpha^\top r$ for all $r\in\mathbb{R}^n$. Because π' is distinct from π, this contradicts the strong minimality of π.

Now we prove the "if" part. Let π be a restricted minimal cut-generating function for (2.3). Suppose that for any valid inequality $\alpha^\top(x-f)\geq\alpha_0$ for $\mathbb{S}$ such that $\alpha\neq 0$, there exists $x^*\in\mathbb{S}$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0}<1$. Let π' be a cut-generating function that implies π. Then there exists a valid inequality $\mu^\top(x-f)\geq\mu_0$ and $\nu\geq 0$ for $\mathbb{S}$ such that

$\mu_0 + \nu \geq 1$ and $\pi(r) \geq \nu\pi'(r) + \mu^\top r$ for all $r \in \mathbb{R}^n$. Note that $\mu_0 \leq 0$ because $f \in \overline{\text{conv}}\,\mathbb{S}$. We will show $\pi' = \pi$, proving that π is strongly minimal. First suppose $\mu \neq 0$. Then by our hypothesis, there exists $x^* \in \mathbb{S}$ such that $1 > \frac{\pi(x^*-f)-\mu^\top(x^*-f)}{1-\mu_0} \geq \frac{\nu\pi'(x^*-f)}{1-\mu_0}$. Rearranging the terms, we get $\pi'(x^* - f) < \frac{1-\mu_0}{\nu} \leq 1$. This contradicts the fact that π' is a cut-generating function because the solution $x = x^*$, $y_{x^*-f} = 1$, and $y_r = 0$ otherwise is feasible to (2.3). Hence, we can assume $\mu = 0$. Then we actually have $\pi \geq \nu\pi'$ for some $\nu \geq 1$. Because π is restricted minimal, it must be that $\pi' = \pi$. □

2.4.1 Strongly Minimal Cut-Generating Functions for $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$

The main result of this section is Theorem 2.8 which was already stated in the introduction.

Theorem 2.8. *Let $\mathbb{S} = \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ and $f \in \mathbb{R}^n_+ \setminus \mathbb{S}$. Let $\pi : \mathbb{R}^n \to \mathbb{R}$. The function π is a strongly minimal cut-generating function for (2.3) if and only if $\pi(0) = 0$, $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$, π is subadditive and satisfies the symmetry condition.*

Proof. Let π be a restricted minimal cut-generating function. By Theorem 2.25 and Proposition 2.27, it will be enough to show that $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$ if and only if, for any valid inequality $\alpha^\top(x - f) \geq \alpha_0$ for $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\alpha \neq 0$, there exists $x^* \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0} < 1$.

We first prove the "if" part of the statement above. Because π is restricted minimal, Theorem 2.25 implies that $\pi(-e^i) \leq 0$ for all $i \in [p]$, $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n]\setminus[p]$, π is subadditive and symmetric. The symmetry condition implies in particular that $\pi(-f) = 1$. Suppose in addition that for any valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ with $\alpha \neq 0$, there exists $x^* \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\frac{\pi(x^*-f)-\alpha^\top(x^*-f)}{1-\alpha_0} < 1$. Let $\alpha \in \mathbb{R}^n$ be such that $\alpha_i = -\pi(-e^i)$ for all $i \in [p]$ and $\alpha_i = -\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon}$ for all $i \in [n] \setminus [p]$. Note that α is well-defined since π is subadditive and $\pi(-e^i) \leq \limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = -\alpha_i \leq 0$ for all $i \in [n] \setminus [p]$ by Lemma 2.23. Now consider the inequality $\alpha^\top(x - f) \geq -\alpha^\top f$ which is valid for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ because $\alpha \in \mathbb{R}^n_+$. Note that for any $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, we can write

$$\begin{aligned}
\pi(x - f) - \alpha^\top x &= \pi(x - f) + \sum_{i=1}^{p} \pi(-e^i)x_i + \sum_{i=p+1}^{n} \limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} x_i \\
&\geq \pi(x - f) + \sum_{i=1}^{p} \pi(-e^i)x_i + \sum_{i=p+1}^{n} \pi(-e^i x_i) \geq \pi(-f) = 1
\end{aligned}$$

by using Lemma 2.23 and the subadditivity of π to obtain the first and second inequality, respectively. Because $\alpha, f \in \mathbb{R}^n_+$ and $\pi(x-f) - \alpha^\top x \geq 1$ for any $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, we have $\frac{\pi(x-f)-\alpha^\top(x-f)}{1+\alpha^\top f} \geq 1$ for any $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. Then by our hypothesis, we must have $\alpha = 0$.

We now prove the "only if" part. Via Theorem 2.25, the restricted minimality of π implies that $\pi(-e^i) \leq 0$ for all $i \in [p]$, $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} \leq 0$ for all $i \in [n] \setminus [p]$, and π is subadditive. Suppose in addition that $\pi(-e^i) = 0$ for all $i \in [p]$ and $\limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = 0$ for all $i \in [n] \setminus [p]$. Let $\alpha^\top(x-f) \geq \alpha_0$ be a valid inequality for $\mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ such that $\frac{\pi(x-f)-\alpha^\top(x-f)}{1-\alpha_0} \geq 1$ for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$. We are going to show $\alpha = 0$. First observe that because the inequality $\alpha^\top(x-f) \geq \alpha_0$ is valid for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$, we must have $\alpha \in \mathbb{R}^n_+$ and $\alpha_0 \leq 0$. Define the function $\pi' : \mathbb{R}^n \to \mathbb{R}$ by letting $\pi'(r) = \frac{\pi(r)-\alpha^\top r}{1-\alpha_0}$. Then π' is subadditive because π is. Furthermore, $\pi'(x-f) \geq 1$ for all $x \in \mathbb{Z}^p_+ \times \mathbb{R}^{n-p}_+$ by our choice of the inequality $\alpha^\top(x-f) \geq \alpha_0$. These two observations imply that π' is a cut-generating function because for any solution (x,y) feasible to (2.3), we have $\sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq \pi(x-f) \geq 1$. Furthermore, π' implies π by definition. It follows from Lemma 2.26 that π' is also restricted minimal. Then by Theorem 2.25, $0 \geq \pi'(-e^i) = \frac{\pi(-e^i)+\alpha_i}{1-\alpha_0} = \frac{\alpha_i}{1-\alpha_0}$ for all $i \in [p]$ and

$$0 \geq \limsup_{\epsilon\to 0^+} \frac{\pi'(-\epsilon e^i)}{\epsilon} = \frac{\alpha_i}{1-\alpha_0} + \frac{1}{1-\alpha_0} \limsup_{\epsilon\to 0^+} \frac{\pi(-\epsilon e^i)}{\epsilon} = \frac{\alpha_i}{1-\alpha_0}$$

for all $i \in [n] \setminus [p]$. Together with $\alpha \in \mathbb{R}^n_+$ and $\alpha_0 \leq 0$, this implies $\alpha = 0$. □

Example 2.4. Theorem 2.8 implies in particular that the cut-generating functions π^1_α of Example 2.1 are strongly minimal. On the other hand, none of the minimal cut-generating functions π^2_α of Example 2.2 are strongly minimal. Indeed, for $f > 0$, the inequality $\alpha(x-f) \geq 1$ is valid for $\mathbb{S} = \{0\}$ when $\alpha \leq -\frac{1}{f}$. Therefore, setting $\alpha_0 = 1$ and $\beta = 0$ in the definition of implication shows that π^2_α is implied by the trivial cut-generating function π_0 that takes the value 1 for all $r \in \mathbb{R}$. Note that π_0 is not minimal since it does not satisfy Lemma 2.10.

2.4.2 Existence of Strongly Minimal Cut-Generating Functions

We observe that Theorem 2.8 is stated for a rather special set $\mathbb{S} \subset \mathbb{R}^n$. One issue is the existence of strongly minimal cut-generating functions for a general set $\mathbb{S} \subset \mathbb{R}^n$. In particular, in Example 2.2, no strongly minimal cut-generating function exists despite the existence of minimal and restricted minimal cut-generating functions. We show this in the next proposition.

Proposition 2.28. *No strongly minimal cut-generating function exists for the model of Example 2.2.*

Proof. Let π be a cut-generating function for the model of Example 2.2. We will show that π cannot be strongly minimal. That is, we will show that there exists a cut-generating function $\pi' \neq \pi$ such that π' implies π.

Let $0 < \beta < 1$, $\alpha_0 = 1 - \beta > 0$, and $\alpha = \frac{\alpha_0}{-f} < 0$. Note that $\alpha(x - f) \geq \alpha_0$ is valid for $\mathbb{S} = \{0\}$. Define the function π' by letting $\pi'(r) = \frac{\pi(r) - \alpha r}{\beta}$. Suppose $\pi' = \pi$. This implies $\pi(r) = \frac{r}{-f}$ for $r \in \mathbb{R}$, but we showed in Example 2.4 that such a linear function is implied by the trivial cut-generating function π_0. Therefore, π cannot be strongly minimal in this case. Hence, we may assume $\pi' \neq \pi$. We will show that π' is a cut-generating function. Since π' implies π, this will prove that π is not strongly minimal. For every feasible solution (x, y), we have $\sum_{r \in \mathbb{R}} \pi(r) y_r \geq 1$ and $\sum_{r \in \mathbb{R}} r y_r = -f$. By the definition of π', we can write $\sum_{r \in \mathbb{R}} \pi'(r) y_r = \frac{1}{\beta}(\sum_{r \in \mathbb{R}} \pi(r) y_r - \alpha \sum_{r \in \mathbb{R}} r y_r) \geq \frac{1}{\beta}(1 + \alpha f) = \frac{1}{\beta}(1 - \alpha_0) = 1$. Thus, π' is a cut-generating function. □

Next we prove Theorem 2.9 stated in the introduction.

Theorem 2.9. *Suppose the closed convex hull of $\mathbb{S} \subset \mathbb{R}^n$ is a full-dimensional polyhedron. Let $f \in \overline{\operatorname{conv}}\, \mathbb{S}$. Then every cut-generating function for (2.3) is implied by a strongly minimal cut-generating function.*

Proof. Let π be a cut-generating function for (2.3). By Proposition 2.6, there exists a restricted minimal cut-generating function π^0 that implies π via scaling. By Proposition 2.5 and Theorem 2.2, π^0 is subadditive. Furthermore, $\pi^0(x - f) \geq 1$ for all $x \in \mathbb{S}$. Consider an explicit description of the closed convex hull of $\mathbb{S} \subset \mathbb{R}^n$ with t linear inequalities: $\overline{\operatorname{conv}}(S) = \{x \in \mathbb{R}^n : {\alpha^i}^\top (x - f) \geq \alpha_0^i \ \forall i \in [t]\}$. Note that $\alpha_0^i \leq 0$ for all $i \in [t]$ because $f \in \overline{\operatorname{conv}}\, \mathbb{S}$. Let $\lambda_0^* = 0$. We define a finite sequence of functions $\{\pi^i\}_{i=1}^t$ iteratively as follows:

A. Given π^{i-1}, let λ_i^* be the largest value λ_i that satisfies $\frac{\pi^{i-1}(x-f) - \lambda_i {\alpha^i}^\top (x-f)}{1 - \lambda_i \alpha_0^i} \geq 1$ for all $x \in \mathbb{S}$.
B. Define the function π^i by letting $\pi^i(r) = \frac{\pi^{i-1}(r) - \lambda_i^* {\alpha^i}^\top r}{1 - \lambda_i^* \alpha_0^i}$.

Claim 1. *For all $i \in \{0, \ldots, t\}$, $\lambda_i^* \geq 0$ and π^i is a restricted minimal cut-generating function.*

We prove the claim by induction. The claim holds for $i = 0$. Assume that it holds for $i = j-1$ where $j \in [t]$. Note that λ_j^* is well-defined because the closed convex hull of $\mathbb{S} \subset \mathbb{R}^n$ is full-dimensional and there exists $x^j \in \mathbb{S}$ such that ${\alpha^j}^\top (x^j - f) > \alpha_0^j$. Furthermore, $\lambda_j^* \geq 0$ because $\pi^{j-1}(x - f) \geq 1$ for all $x \in \mathbb{S}$. The function π^j is a subadditive cut-generating function because it satisfies $\pi^j(x-f) \geq 1$ for all $x \in \mathbb{S}$ and π^{j-1} is subadditive by Proposition 2.5 and Theorem 2.2. Moreover, π^j is restricted minimal by Lemma 2.26 because it implies π^{j-1} by definition and π^{j-1} is restricted minimal. This concludes the proof of Claim 1.

Claim 2. For all $i \in [t]$ and $x \in \mathbb{S}$, $\pi^i(x-f) \leq \pi^{i-1}(x-f)$.

Indeed, for all $i \in [t]$ and $x \in \mathbb{S}$, we can write

$$\pi^i(x-f) = \frac{\pi^{i-1}(x-f) - \lambda_i^* {\alpha^i}^\top (x-f)}{1-\lambda_i^*\alpha_0^i} \leq \frac{\pi^{i-1}(x-f) - \lambda_i^*\alpha_0^i}{1-\lambda_i^*\alpha_0^i} \leq \pi^{i-1}(x-f).$$

The first inequality above follows from the validity of ${\alpha^i}^\top(x-f) \geq \alpha_0^i$ for $\mathbb{S}$, the second inequality follows from $\alpha_0^i \leq 0$ and the fact that $\pi^{i-1}(x-f) \geq 1$ for all $x \in \mathbb{S}$. This concludes the proof of Claim 2.

Claim 3. For all $i \in [t]$ and $\lambda > 0$, there exists $x \in \mathbb{S}$ such that $\frac{\pi^i(x-f) - \lambda {\alpha^i}^\top(x-f)}{1-\lambda\alpha_0^i} < 1$.

To see this, fix $i \in [t]$ and suppose that the claim is not true. Then there exists $\lambda > 0$ such that

$$\begin{aligned} 1 \leq \frac{\pi^i(x-f) - \lambda {\alpha^i}^\top(x-f)}{1-\lambda\alpha_0^i} &= \frac{\frac{\pi^{i-1}(x-f)-\lambda_i^*{\alpha^i}^\top(x-f)}{1-\lambda_i^*\alpha_0^i} - \lambda {\alpha^i}^\top(x-f)}{1-\lambda\alpha_0^i} \\ &= \frac{\pi^{i-1}(x-f) - (\lambda_i^* + \lambda(1-\lambda_i^*\alpha_0^i)){\alpha^i}^\top(x-f)}{1-(\lambda_i^*+\lambda(1-\lambda_i^*\alpha_0^i))\alpha_0^i} \end{aligned}$$

for all $x \in \mathbb{S}$. Because $\lambda(1-\lambda_i^*\alpha_0^i) > 0$, we get $\lambda_i^* + \lambda(1-\lambda_i^*\alpha_0^i) > \lambda_i^*$ which contradicts the maximality of λ_i^*. This concludes the proof of Claim 3.

Claim 4. For all $i \in [t]$ and $\lambda \in \mathbb{R}_+^i \setminus \{0\}$, there exists $x \in \mathbb{S}$ such that $\frac{\pi^i(x-f) - \sum_{\ell=1}^i \lambda_\ell {\alpha^\ell}^\top(x-f)}{1-\sum_{\ell=1}^i \lambda_\ell\alpha_0^\ell} < 1$.

We have already proved this for $i = 1$ in Claim 3. Assume now that the claim holds for $i = j-1 \in [t-1]$. Let $\lambda \in \mathbb{R}_+^j \setminus \{0\}$. If $\lambda_j = 0$, we can write

$$\frac{\pi^j(x-f) - \sum_{\ell=1}^j \lambda_\ell {\alpha^\ell}^\top(x-f)}{1-\sum_{\ell=1}^j \lambda_\ell\alpha_0^\ell} \leq \frac{\pi^{j-1}(x-f) - \sum_{\ell=1}^{j-1} \lambda_\ell {\alpha^\ell}^\top(x-f)}{1-\sum_{\ell=1}^{j-1} \lambda_\ell\alpha_0^\ell} < 1.$$

Here we have used Claim 2 to obtain the first inequality and the induction hypothesis to obtain the second inequality. If $\lambda_j > 0$, we get

$$\frac{\pi^j(x-f) - \sum_{\ell=1}^j \lambda_\ell {\alpha^\ell}^\top(x-f)}{1-\sum_{\ell=1}^j \lambda_\ell\alpha_0^\ell} \leq \frac{\pi^j(x-f) - \sum_{\ell=1}^{j-1} \lambda_\ell\alpha_0^\ell - \lambda_j {\alpha^j}^\top(x-f)}{1-\sum_{\ell=1}^{j-1} \lambda_\ell\alpha_0^\ell - \lambda_j\alpha_0^j} < 1$$

by using Claim 3 to obtain the second inequality. This concludes the proof of Claim 4.

By Claim 1, π^t is a restricted minimal cut-generating function. Furthermore, π^t implies π^0. By Proposition 2.27, to prove that π^t is strongly minimal, it is enough to show that for any valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{S}$ such that $\alpha \neq 0$, there exists $x \in \mathbb{S}$ such that $\frac{\pi(x-f)-\alpha^\top(x-f)}{1-\alpha_0} < 1$. Let $\alpha^\top(x-f) \geq \alpha_0$ be any valid inequality for $\mathbb{S}$ such that $\alpha \neq 0$. We have $\alpha_0 \leq 0$ because $f \in \overline{\text{conv}}\,\mathbb{S}$. By Farkas' Lemma, there exists $\lambda \in \mathbb{R}^t_+ \setminus \{0\}$ such that $\alpha = \sum_{\ell=1}^t \lambda_\ell \alpha^\ell$ and $\sum_{\ell=1}^t \lambda_\ell \alpha_0^\ell \geq \alpha_0$. By Claim 4 above, there exists $x \in \mathbb{S}$ such that $\pi^t(x-f) - \sum_{\ell=1}^t \lambda_\ell \alpha^\ell < 1 - \sum_{\ell=1}^t \lambda_\ell \alpha_0^\ell \leq 1 - \alpha_0$. Proposition 2.27 now implies that π^t is strongly minimal. □

2.5 Minimal Cut-Generating Functions for Mixed-Integer Programs

We now turn to mixed-integer linear programming. As before, it is convenient to work with an infinite model:

$$x = f + \sum_{r \in \mathbb{R}^n} r s_r + \sum_{r \in \mathbb{R}^n} r y_r, \tag{2.11a}$$

$$x \in \mathbb{S}, \tag{2.11b}$$

$$s_r \in \mathbb{R}_+ \quad \forall r \in \mathbb{R}^n, \tag{2.11c}$$

$$y_r \in \mathbb{Z}_+ \quad \forall r \in \mathbb{R}^n, \tag{2.11d}$$

$$s, y \text{ have finite support.} \tag{2.11e}$$

As earlier, we assume that $\emptyset \neq \mathbb{S} \subset \mathbb{R}^n$. We will also need to assume that $f \in \mathbb{R}^n$ is not in the closure of $\mathbb{S}$, that is, $f \notin \text{cl}\,\mathbb{S}$.

Two functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$ are said to form a *cut-generating function pair* if the inequality $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \sum_{r\in\mathbb{R}^n} \pi(r)y_r \geq 1$ holds for every feasible solution (x, s, y) of (2.11). Cut-generating function pairs can be used to generate cutting-planes in mixed-integer linear programming by simply restricting the above inequality to the vectors r that appear as nonbasic columns.

Note that the assumption $f \notin \text{cl}\,\mathbb{S}$ is needed for the existence of ψ in cut-generating function pairs (ψ, π). Suppose for example that $\mathbb{S} = \mathbb{R} \setminus \{f\}$. Let $\bar{r} \in \mathbb{R} \setminus \{0\}$ and $\epsilon > 0$. Then the solution $x = f + \epsilon\bar{r}$, $y = 0$, $s_{\bar{r}} = \epsilon$, and $s_r = 0$ for all $r \neq \bar{r}$ is feasible to (2.11). Therefore, in any cut-generating function pair (ψ, π) for (2.11), the function $\psi : \mathbb{R} \to \mathbb{R}$ would have to satisfy $\sum_{r\in\mathbb{R}} \pi(r)y_r + \sum_{r\in\mathbb{R}} \psi(r)s_r = \epsilon\psi(\bar{r}) \geq 1$. This, however, implies $\psi(\bar{r}) \geq \frac{1}{\epsilon}$ for all $\epsilon > 0$, contradicting $\psi(\bar{r}) \in \mathbb{R}$.

The definitions of minimality, restricted minimality, and strong minimality extend readily to cut-generating function pairs for the model (2.11). A cut-generating function pair (ψ', π') for (2.11) *dominates* another cut-generating function pair (ψ, π) if $\psi \geq \psi'$ and $\pi \geq \pi'$, *implies* (ψ, π) *via scaling* if there exists $\beta \geq 1$ such that $\psi \geq \beta\psi'$ and $\pi \geq \beta\pi'$, and *implies* (ψ, π) if there exists $\beta \geq 0$ and a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ for $\mathbb{S}$ such that $\alpha_0 + \beta \geq 1$, $\psi(r) \geq \beta\psi'(r) + \alpha^\top r$, and $\pi(r) \geq \beta\pi'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. A cut-generating function pair (ψ, π) is *minimal* (resp. *restricted minimal*, *strongly minimal*) if it is not *dominated* (resp. *implied via scaling*, *implied*) by a cut-generating function pair other than itself. As for the model (2.3), strongly minimal cut-generating function pairs for (2.11) are restricted minimal, and restricted minimal cut-generating function pairs for (2.11) are minimal.

The following theorem extends Theorem 2.1, Proposition 2.6, and Theorem 2.9 to the model (2.11). The proof of each claim is similar to the proof of its aforementioned counterpart for the model (2.3) and is therefore omitted.

Theorem 2.29.

i. Every cut-generating function pair for (2.11) *is dominated by a minimal cut-generating function pair.*

ii. Every cut-generating function pair for (2.11) *is implied via scaling by a restricted minimal cut-generating function pair.*

iii. Suppose the closed convex hull of $\mathbb{S} \subset \mathbb{R}^n$ *is a full-dimensional polyhedron. Let* $f \in \overline{\operatorname{conv}}\,\mathbb{S}$*. Then every cut-generating function pair for* (2.11) *is implied by a strongly minimal cut-generating function pair.*

Next we state two simple lemmas which will be used in the proof of Theorem 2.32. We omit a complete proof of Lemma 2.30. Its first claim follows from the observation that for any cut-generating function pair (ψ, π), the related pair (ψ, π') where π' is the pointwise minimum of ψ and π is a cut-generating function pair that dominates (ψ, π). Its second claim has a similar proof to that of Lemma 2.11. The reader is referred to [45] for the proof of Lemma 2.30 in the case $\mathbb{S} = \mathbb{Z}^n$, which remains valid for general $\mathbb{S} \subset \mathbb{R}^n$.

Lemma 2.30. *Let* (ψ, π) *be a minimal cut-generating function pair for* (2.11)*. Then*

i. $\pi \leq \psi$*,*

ii. ψ *is sublinear, that is, subadditive and positively homogeneous.*

Lemma 2.31. *Let* $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$*. If* π *is a cut-generating function for* (2.3)*,* ψ *is sublinear, and* $\psi \geq \pi$*, then* (ψ, π) *is a cut-generating function pair for* (2.11)*.*

Proof. Let $(\bar{x}, \bar{s}, \bar{y})$ be a feasible solution of (2.11), and let $\bar{r} = \sum_{r \in \mathbb{R}^n} r\bar{s}_r$. Note that $(\bar{x}, \tilde{y})$, where $\tilde{y}_{\bar{r}} = \bar{y}_{\bar{r}} + 1$ and $\tilde{y}_r = \bar{y}_r$ for $r \neq \bar{r}$, is a feasible solution to (2.3). Then $\pi(\bar{r}) + \sum_{r \in \mathbb{R}^n} \pi(r)\bar{y}_r = \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$ because π is a cut-generating function for (2.3).

Using the sublinearity of ψ and $\psi \geq \pi$, we can write $\sum_{r\in\mathbb{R}^n} \psi(r)\bar{s}_r + \sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r \geq \psi(\bar{r}) + \sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r \geq \pi(\bar{r}) + \sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r \geq 1$. This shows that (ψ, π) is a cut-generating function pair for (2.11). □

Gomory and Johnson [63] characterized minimal cut-generating function pairs for (2.11) when $\mathbb{S} = \mathbb{Z}$. Johnson [75] generalized this result as follows: Consider $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$. The pair (ψ, π) is a minimal cut-generating function pair for (2.11) when $\mathbb{S} = \mathbb{Z}^n$ if and only if π is a minimal cut-generating function for (2.3) when $\mathbb{S} = \mathbb{Z}^n$ and ψ satisfies

$$\psi(r) = \limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \quad \text{for all } r \in \mathbb{R}^n. \tag{2.12}$$

In the next result, we give similar characterizations of minimal, restricted minimal, and strongly minimal cut-generating function pairs for (2.11). Our proof follows the proofs in [45, 75] for similar results on minimal cut-generating function pairs in the case $\mathbb{S} = \mathbb{Z}^n$.

Theorem 2.32. *Let $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$.*

i. The pair (ψ, π) is a (restricted) minimal cut-generating function pair for (2.11) *if and only if π is a (restricted) minimal cut-generating function for* (2.3) *and ψ satisfies* (2.12)*.*

ii. Suppose $\mathbb{S} \subset \mathbb{R}^n$ is full-dimensional and $f \in \overline{\operatorname{conv}}\,\mathbb{S}$. The pair (ψ, π) is a strongly minimal cut-generating function pair for (2.11) *if and only if π is a strongly minimal cut-generating function for* (2.3) *and ψ satisfies* (2.12)*.*

Proof. We will prove the statement (ii) only. The proof of statement (i) is similar.

We first prove the "only if" part. Suppose (ψ, π) is a strongly minimal cut-generating function pair for (2.11). Because (ψ, π) is minimal, we have that $\psi \geq \pi$ and ψ is sublinear by Lemma 2.30. Furthermore, π is a cut-generating function for (2.3) since for any feasible solution $(\bar{x}, \bar{y})$ to (2.3), there exists a feasible solution $(\bar{x}, \bar{s}, \bar{y})$ to (2.11) such that $\bar{s}_r = 0$ for all $r \in \mathbb{R}^n$, and $\sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r = \sum_{r\in\mathbb{R}^n} \psi(r)\bar{s}_r + \sum_{r\in\mathbb{R}^n} \pi(r)\bar{y}_r \geq 1$. We claim that π is a strongly minimal cut-generating function for (2.3). Suppose not. Then there exists a cut-generating function $\pi' \neq \pi$, a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ for $\mathbb{S}$, and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \beta\pi'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\operatorname{conv}}\,\mathbb{S}$, $\alpha_0 \leq 0$ and $\beta \geq 1$. Define the function $\psi' : \mathbb{R}^n \to \mathbb{R}$ by letting $\psi'(r) = \frac{\psi(r) - \alpha^\top r}{\beta}$. The pair (ψ', π') is a cut-generating function pair for (2.11). To see this, first note that ψ' is sublinear because ψ is. Furthermore, $\psi' \geq \pi'$ because $\psi'(r) = \frac{\psi(r) - \alpha^\top r}{\beta} \geq \frac{\pi(r) - \alpha^\top r}{\beta} \geq \pi'(r)$ for all $r \in \mathbb{R}^n$. It then follows from Lemma 2.31 that (ψ', π') is a cut-generating function pair. Because $\pi' \neq \pi$ and (ψ', π') implies (ψ, π), this contradicts the strong minimality of (ψ, π). Thus, π is a strongly minimal cut-generating function for (2.3). In particular, π is minimal, and subadditive by Theorem 2.2.

Define the function $\psi'' : \mathbb{R}^n \to \mathbb{R}$ by letting $\psi''(r) = \limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$. We first show that ψ'' is well-defined, that is, it is finite everywhere, and that $\psi'' \leq \psi$. By Lemma 2.30, $\pi \leq \psi$ and ψ is sublinear. Thus, for all $\epsilon > 0$ and $r \in \mathbb{R}^n$, we have

$$-\psi(-r) = \frac{-\psi(-\epsilon r)}{\epsilon} \leq \frac{-\pi(-\epsilon r)}{\epsilon} \leq \frac{\pi(\epsilon r)}{\epsilon} \leq \frac{\psi(\epsilon r)}{\epsilon} = \psi(r).$$

The second inequality above holds because $\pi(r) + \pi(-r) \geq \pi(0) = 0$ for all $r \in \mathbb{R}^n$ by the subadditivity of π. This implies

$$-\psi(-r) \leq \psi''(r) = \limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon} \leq \psi(r),$$

which proves both claims since ψ is real-valued.

It is easy to verify from the definition of ψ'' that it is sublinear. Furthermore, $\pi \leq \psi''$ by Lemma 2.23. It then follows from Lemma 2.31 that (ψ'', π) is a cut-generating function pair for (2.11). Because the cut-generating function pair (ψ, π) is minimal and $\psi'' \leq \psi$, we get $\psi = \psi''$, proving that ψ satisfies (2.12).

We now prove the "if" part. Suppose π is a strongly minimal cut-generating function for (2.3) and ψ satisfies (2.12). Note that ψ is sublinear by definition and $\psi \geq \pi$ by Lemma 2.23. It follows from Lemma 2.31 that (ψ, π) is a cut-generating function pair for (2.11). Let (ψ', π') be a cut-generating function pair that implies (ψ, π). We will show $\psi' = \psi$ and $\pi' = \pi$, proving that (ψ, π) is strongly minimal. Let (ψ'', π'') be a minimal cut-generating function pair that dominates (ψ', π'). By the choice of (ψ', π'), there exist a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\psi(r) \geq \beta\psi'(r) + \alpha^\top r \geq \beta\psi''(r) + \alpha^\top r$, $\pi(r) \geq \beta\pi'(r) + \alpha^\top r \geq \beta\pi''(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Furthermore, $\alpha_0 \leq 0$ and $\beta \geq 1$ because $f \in \overline{\operatorname{conv}}\,\mathbb{S}$. By the "only if" part of statement (i), π'' is a minimal cut-generating function for (2.3) and $\psi''(r) = \limsup_{\epsilon\to 0^+} \frac{\pi''(\epsilon r)}{\epsilon}$ for all $r \in \mathbb{R}^n$. The function π'' implies π. By the strong minimality of π, we have $\pi'' = \pi$. Then $\pi(r) \geq \beta\pi(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Let $\bar{x} \in \mathbb{S}$ be such that $\alpha^\top(\bar{x} - f) > \alpha_0$; such a point exists because $\mathbb{S} \subset \mathbb{R}^n$ is full-dimensional. If $\beta > 1$, then $\pi(\bar{x} - f) \leq \frac{-\alpha^\top(\bar{x}-f)}{\beta-1} < \frac{-\alpha_0}{\beta-1} \leq 1$ which contradicts the fact that π is a cut-generating function. Hence, we can assume $\beta = 1$. Then $\alpha^\top r \leq 0$ for all $r \in \mathbb{R}^n$; therefore, $\alpha = 0$. Using $\alpha = 0$ and $\beta = 1$, we get $\pi = \pi'' \leq \pi' \leq \pi$ and $\psi'' \leq \psi' \leq \psi$. Finally, note that $\psi''(r) = \limsup_{\epsilon\to 0^+} \frac{\pi''(\epsilon r)}{\epsilon} = \limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon} = \psi(r)$ for all $r \in \mathbb{R}^n$. This shows $\psi'' = \psi' = \psi$ and concludes the proof. □

Example 2.5. Let $n = 1$, $\mathbb{S} = \mathbb{Z}_+$, and $0 < f < 1$. We consider the classical Gomory function $\psi(r) = \max\left\{\frac{-r}{f}, \frac{r}{1-f}\right\}$ for the continuous nonbasic variables. In the spirit of [53], the *trivial lifting* of ψ can be defined as

$$\pi^5(r) = \inf_{x\in\mathbb{Z}_+} \{\psi(r + x)\}.$$

Note that π^5 coincides with the Gomory function $\pi_1^1(r) = \min\left\{\frac{r-\lfloor r\rfloor}{1-f}, \frac{\lceil r\rceil - r}{f}\right\}$ of Example 2.1 on the negative points and with ψ on the nonnegative points. Using standard techniques, one can verify that (ψ, π^5) is a cut-generating function pair for (2.11). Nevertheless, (ψ, π^5) is not a *minimal* pair. To prove this, it is enough by Theorem 2.32 and Proposition 2.16 to show that π^5 does not satisfy (2.8) and hence is not a minimal cut-generating function for (2.3). Indeed, note that $\pi^5(1) = \frac{1}{1-f}$, whereas $\pi^5(-f-k) = 1$ for all $k \in \mathbb{Z}_{++}$. Therefore, $\pi^5(1) = \frac{1}{1-f} \neq 0 = \sup\left\{\frac{1}{k}(1 - \pi^5(-f-k)) : k \in \mathbb{Z}_{++}\right\}$ which violates (2.8).

2.6 Appendix

In this appendix, we consider the integer program

$$x = f + \sum_{j=1}^{m} r^j y_j, \tag{2.13a}$$

$$x \in \mathbb{S}, \tag{2.13b}$$

$$y_j \in \mathbb{Z}_+ \quad \forall j \in [m], \tag{2.13c}$$

where $\mathbb{S} \subset \mathbb{R}^n$ is a nonempty set and $f \notin \mathbb{S}$. Let us say that an inequality ${\pi'}^\top y \geq 1$ valid for (2.13) implies another valid inequality $\pi^\top y \geq 1$ if there exists an inequality $\alpha^\top(x - f) \geq \alpha_0$ valid for $\mathbb{S}$ and $\beta \geq 0$ such that $\pi \geq \beta\pi' + R^\top\alpha$ and $\alpha_0 + \beta \geq 1$. We show how this notion can be seen as dominance with respect to a cone in a lifted space [80]. Let us define

$$\mathbb{K} = \left\{ \begin{pmatrix} t \\ y \end{pmatrix} \in \mathbb{R}^{m+1}_+ : \quad \begin{pmatrix} t \\ tf + \sum_{j=1}^m r^j y_j \end{pmatrix} \in \operatorname{cone}\begin{pmatrix} 1 \\ \mathbb{S} \end{pmatrix} \right\}.$$

Then a point (x, y) satisfies (2.13) if and only if $(1, x, y)$ satisfies

$$x = ft + \sum_{j=1}^{m} r^j y_j, \tag{2.14a}$$

$$x \in \mathbb{S}, \tag{2.14b}$$

$$y_j \in \mathbb{Z}_+ \quad \forall j \in [m], \tag{2.14c}$$

$$\begin{pmatrix} t \\ y \end{pmatrix} \in \mathbb{K}, \tag{2.14d}$$

$$t = 1. \tag{2.14e}$$

The system (2.14) is an exact reformulation of (2.13). Therefore, $\pi^\top y \geq 1$ is valid for (2.13) if and only if $\pi^\top y \geq t$ is valid for (2.14). Furthermore, an inequality ${\pi'}^\top y \geq 1$ valid for (2.13) implies another valid inequality $\pi^\top y \geq 1$ if and only if $\beta{\pi'}^\top y \geq \beta t$ dominates $\pi^\top y \geq t$ with respect to $\mathbb{K}$ for some $\beta \geq 0$.

Chapter 3

Extreme Cut-Generating Functions for the One-Row Problem

Acknowledgments. This chapter is based on joint work with Gérard Cornuéjols [106].

3.1 Introduction

3.1.1 Motivation

Let $\mathbb{S} \subset \mathbb{R}^n$ be a nonempty subset of the Euclidean space. In this chapter, as in Chapter 2, we consider the infinite relaxation

$$x = f + \sum_{r \in \mathbb{R}^n} r y_r, \tag{3.1a}$$

$$x \in \mathbb{S}, \tag{3.1b}$$

$$y_r \in \mathbb{Z}_+ \quad \forall r \in \mathbb{R}^n, \tag{3.1c}$$

$$y \text{ has finite support.} \tag{3.1d}$$

The model (3.1) generalizes Gomory and Johnson's infinite group relaxation [63, 64, 75], which corresponds to the case $\mathbb{S} = \mathbb{Z}^n$, and a model studied by Bachem, Johnson, and Schrader [13], which corresponds to the case $\mathbb{S} = \{0\}$. The reader is referred to Section 2.1 for a related discussion. In Chapter 2 we characterized minimal cut-generating functions for (3.1) under different notions of minimality and assumptions on the structure of $\mathbb{S}$. A yet stronger notion than the minimality of a cut-generating function is its extremality: A cut-generating function π is said to be *extreme* if any two cut-generating functions π_1, π_2 satisfying $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ must also satisfy $\pi = \pi_1 = \pi_2$. In this chapter we investigate extreme cut-generating functions for (3.1). We focus on the one-row problem where $n = 1$.

The structure of extreme cut-generating functions can be very complicated. Constructing extreme cut-generating functions for (3.1), or even verifying that a given cut-generating function is extreme for (3.1), often requires ad hoc techniques. For the case $\mathbb{S} = \mathbb{Z}$, Gomory and Johnson [64, 65] established the Two-Slope Theorem which identifies an interesting class of "simple" extreme cut-generating functions. We state this result next. Recall that, when $\mathbb{S} = \mathbb{Z}$, cut-generating functions must be nonnegative over the rationals, and they are usually assumed to be nonnegative on the whole real line.

Assumption 3.1. *When $\mathbb{S} = \mathbb{Z}$, all cut-generating functions π satisfy $\pi \geq 0$, that is, $\pi(r) \geq 0$ for all $r \in \mathbb{R}$.*

Let $\mathbb{I} \subset \mathbb{R}$ be a compact interval of the real line. We say that a function $\pi : \mathbb{I} \to \mathbb{R}$ is *piecewise linear* if there are finitely many values $\min \mathbb{I} = r_0 < r_1 < \ldots < r_t = \max \mathbb{I}$ such that $\pi(r) = a_j r + b_j$ for some $a_j, b_j \in \mathbb{R}$ at each one of the open intervals (r_{j-1}, r_j). The piecewise linear function π is continuous if and only if $\pi(r_0) = a_1 r_0 + b_1$, $\pi(r_t) = a_t r_t + b_t$, and $\pi(r_j) = a_j r_j + b_j = a_{j+1} r_j + b_{j+1}$ for $j \in \{1, \ldots, t-1\}$.

Theorem 3.1 (Gomory-Johnson Two-Slope Theorem [64, 65])**.** *Let $\mathbb{S} = \mathbb{Z}$ and $f \in \mathbb{R} \setminus \mathbb{Z}$. Suppose Assumption 3.1 holds. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a minimal cut-generating function for* (3.1)*. If the restriction of π to the interval* $[0, 1]$ *is a continuous piecewise linear function with only two slopes, then π is extreme.*

Despite their simplicity, two-slope cut-generating functions produce powerful cutting-planes. Gomory mixed-integer inequalities [62], which are among the most effective cutting-planes in mixed-integer linear programming [31], are generated by two-slope functions. Motivated by the success of two-slope cut-generating functions in the case $\mathbb{S} = \mathbb{Z}$, in this chapter we prove a similar result for the case $\mathbb{S} = \mathbb{Z}_+$.

It follows from the definition of extremality that extreme cut-generating functions are minimal [63, 75]. In Section 3.2 we show that extreme cut-generating functions for (3.1) must in fact be strongly minimal. In Section 3.3 we prove a Two-Slope Theorem for extreme cut-generating functions for (3.1) when $\mathbb{S} = \mathbb{Z}_+$, in the spirit of the Gomory-Johnson Two-Slope Theorem for $\mathbb{S} = \mathbb{Z}$. A similar extension of the Two-Slope Theorem has recently appeared in [104].

3.1.2 Notation and Terminology

Let $\mathbb{Q}$ and $\mathbb{Z}_{++}$ be the set of rational numbers and strictly positive integers, respectively. For $k \in \mathbb{Z}_{++}$, we let $[k] = \{1, \ldots, k\}$. We let $\overline{\text{conv}}\, \mathbb{V}$ represent the closed convex hull of a set $\mathbb{V} \in \mathbb{R}^n$.

We define the minimality, restricted minimality, and strong minimality of a cut-generating function as in Chapter 2. A cut-generating function π' for (3.1) *dominates* another cut-generating function π if $\pi \geq \pi'$, *implies π via scaling* if there exists $\beta \geq 1$ such that $\pi \geq \beta\pi'$, and *implies* π if there exists $\beta \geq 0$ and a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{S}$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \beta\pi'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. A cut-generating function π is *minimal* (resp. *restricted minimal*, *strongly minimal*) if it is not *dominated* (resp. *implied via scaling*, *implied*) by a cut-generating function other than itself. We say that a function $\pi : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$; it is *symmetric* or satisfies the *symmetry condition* if $\pi(r) + \pi(-f-r) = 1$ for all $r \in \mathbb{R}^n$; and it is *nondecreasing with respect to* $\mathbb{S} \subset \mathbb{R}^n$ if $\pi(r) \leq \pi(r+w)$ for all $r \in \mathbb{R}^n$ and $w \in \mathbb{S}$.

3.2 Two Results for General $\mathbb{S}$

The results of this section hold for general nonempty $\mathbb{S} \subset \mathbb{R}^n$. We also assume $f \in \overline{\operatorname{conv}}\,\mathbb{S}$ since otherwise any linear inequality which strictly separates f from $\mathbb{S}$ can be used to cut off the infeasible solution $x = f$, $y = 0$. The following result shows that extreme cut-generating functions must in fact be strongly minimal; see Gomory and Johnson [63], Johnson [75], and Kılınç-Karzan [80] for analogous results.

Lemma 3.2. *Suppose $f \in \overline{\operatorname{conv}}\,\mathbb{S}$. Any extreme cut-generating function for* (3.1) *is strongly minimal.*

Proof. We prove the contrapositive, namely, any cut-generating function that is not strongly minimal cannot be extreme. Let π be a cut-generating function for (3.1) that is not strongly minimal. Then there exist a cut-generating function $\pi' \neq \pi$, a valid inequality $\alpha^\top(x-f) \geq \alpha_0$ for $\mathbb{S}$, and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi(r) \geq \alpha^\top r + \beta\pi'(r)$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\operatorname{conv}}\,\mathbb{S}$, we must have $\alpha_0 \leq 0$, and $\beta \geq 1$. We divide the rest of the proof into two cases. In each case, we exhibit cut-generating functions π_1, π_2 that are distinct from π and satisfy $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$.
Case i: $\alpha_0 + \beta > 1$. Let $\delta > 0$ be such that $\alpha_0 + \beta = 1 + \delta$. Let π_1 and π_2 be defined as $\pi_1 = \frac{1}{1+\delta}\pi$ and $\pi_2 = \frac{1+2\delta}{1+\delta}\pi$. It is easy to check that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Furthermore, π_1 and π_2 are distinct from π since for any $x \in \mathbb{S}$, $\pi_1(x-f) \neq \pi(x-f)$ and $\pi_2(x-f) \neq \pi(x-f)$. We show that π_1 and π_2 are indeed cut-generating functions. Let (x,y) be a feasible solution to (3.1) so that $f + \sum_{r\in\mathbb{R}^n} r y_r = x \in \mathbb{S}$. Then $\sum_{r\in\mathbb{R}^n} \pi_1(r) y_r \geq \frac{1}{1+\delta}(\sum_{r\in\mathbb{R}^n} \alpha^\top r y_r + \beta \sum_{r\in\mathbb{R}^n} \pi'(r) y_r) \geq \frac{1}{1+\delta}(\alpha^\top(x-f) + \beta) \geq \frac{\alpha_0+\beta}{1+\delta} = 1$. Similarly, $\sum_{r\in\mathbb{R}^n} \pi_2(r) y_r = \frac{1+2\delta}{1+\delta} \sum_{r\in\mathbb{R}^n} \pi(r) y_r \geq \frac{1+2\delta}{1+\delta} > 1$. Thus, π_1 and π_2 are cut-generating functions.
Case ii: $\alpha_0 + \beta = 1$. Let π_1 and π_2 be defined as $\pi_1 = \pi'$ and $\pi_2 = \pi + (\pi - \pi')$. It is again easy to see that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. The function π_1 is a cut-generating function that is distinct from π by hypothesis. Furthermore, π_2 is distinct from π because π_1 is distinct

from π. We show that π_2 is a cut-generating function. Note that $\alpha_0+\beta = 1$; hence, $\pi_2(r) = \pi(r)+(\pi(r)-(\alpha_0+\beta)\pi'(r)) = \pi(r)+((\pi(r)-\beta\pi'(r))-\alpha_0\pi'(r)) \geq \pi(r)+(\alpha^\top r-\alpha_0\pi'(r))$ for all $r \in \mathbb{R}^n$. For any feasible solution (x, y) to (3.1), we can write $\sum_{r\in\mathbb{R}^n} \pi_2(r)y_r \geq \sum_{r\in\mathbb{R}^n} \pi(r)y_r+\sum_{r\in\mathbb{R}^n} \alpha^\top r y_r-\alpha_0 \sum_{r\in\mathbb{R}^n} \pi'(r)y_r \geq \sum_{r\in\mathbb{R}^n} \pi(r)y_r+\alpha^\top(x-f)-\alpha_0 \geq 1$ where the second inequality is obtained by using $\alpha_0 \leq 0$. Thus, π_2 is a cut-generating function. □

Recall that any minimal cut-generating function π for (3.1) is subadditive by Theorem 2.2. Thus, $\pi(r^1) + \pi(r^2) \geq \pi(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$. We denote by $\mathbb{E}(\pi)$ the set of all pairs (r^1, r^2) for which this inequality is satisfied at equality.

Lemma 3.3. *Suppose $\mathbb{S}$ is full-dimensional and $f \in \overline{\operatorname{conv}}\,\mathbb{S}$. Let π be a strongly minimal cut-generating function for* (3.1). *Suppose there exist cut-generating functions π_1 and π_2 such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. Then π_1 and π_2 are strongly minimal cut-generating functions and $\mathbb{E}(\pi) \subset \mathbb{E}(\pi_1) \cap \mathbb{E}(\pi_2)$.*

Proof. We first prove that π_1 and π_2 are strongly minimal cut-generating functions. Suppose π_1 is not strongly minimal. Then there exists a cut-generating function $\pi_1' \neq \pi_1$, a valid inequality $\alpha^\top(x - f) \geq \alpha_0$ for $\mathbb{S}$, and $\beta \geq 0$ such that $\alpha_0 + \beta \geq 1$ and $\pi_1(r) \geq \beta\pi_1'(r) + \alpha^\top r$ for all $r \in \mathbb{R}^n$. Because $f \in \overline{\operatorname{conv}}\,\mathbb{S}$, α_0 and $\beta \geq 1$. Define the function $\pi' : \mathbb{R}^n \to \mathbb{R}$ as $\pi' = \frac{\beta}{\beta+1}\pi_1' + \frac{1}{\beta+1}\pi_2$. The function π' is a cut-generating function because it is a convex combination of two cut-generating functions. Furthermore, $\pi(r) = \frac{1}{2}\pi_1(r) + \frac{1}{2}\pi_2(r) \geq \frac{\beta}{2}\pi_1'(r) + \frac{1}{2}\pi_2(r) + \frac{1}{2}\alpha^\top r = \frac{\beta+1}{2}\pi_1'(r) + \frac{1}{2}\alpha^\top r$ for all $r \in \mathbb{R}^n$. Because the linear inequality $\frac{1}{2}\alpha^\top(x - f) \geq \frac{\alpha_0}{2}$ is valid for $\mathbb{S}$, $\frac{\beta+1}{2} \geq 0$, and $\frac{\beta+1}{2} + \frac{\alpha_0}{2} \geq 1$, the function π' implies π. If $\alpha = 0$ and $\beta = 1$, then $\pi' = \frac{1}{2}\pi_1' + \frac{1}{2}\pi_2$ and $\pi' \neq \pi$ because $\pi_1' \neq \pi_1$. If $\alpha = 0$ and $\beta > 1$, then $\pi \geq \frac{\beta+1}{2}\pi'$. For any $x \in \mathbb{S}$, we have $\pi(x-f) > \pi'(x-f)$ because π' is a cut-generating function and $\pi'(x - f) \geq 1$. If $\alpha \neq 0$, then there exists $\bar{x} \in \mathbb{S}$ such that $\alpha^\top(\bar{x} - f) > \alpha_0$. Such a point $\bar{x}$ exists because $\mathbb{S}$ is full-dimensional. Then we can write $\pi(\bar{x}-f) \geq \frac{\beta+1}{2}\pi'(\bar{x}-f)+\frac{1}{2}\alpha^\top(\bar{x}-f) > \frac{\beta+1}{2}\pi'(\bar{x}-f)+\frac{\alpha_0}{2} \geq \pi'(\bar{x}-f)+\frac{\alpha_0+\beta-1}{2} \geq \pi'(\bar{x}-f)$ by using $\pi'(\bar{x}-f) \geq 1$ and $\alpha_0+\beta \geq 1$ to obtain the third and fourth inequality, respectively. In all three cases, $\pi' \neq \pi$ which contradicts the strong minimality of π.

Now let $(r^1, r^2) \in \mathbb{E}(\pi)$. Because π_1 and π_2 are minimal cut-generating functions, they are subadditive by Theorem 2.2. Then

$$\begin{aligned}\pi(r^1 + r^2) = \pi(r^1) + \pi(r^2) &= \frac{1}{2}(\pi_1(r^1) + \pi_1(r^2)) + \frac{1}{2}(\pi_2(r^1) + \pi_2(r^2))\\ &\geq \frac{1}{2}\pi_1(r^1 + r^2) + \frac{1}{2}\pi_2(r^1 + r^2) = \pi(r^1 + r^2).\end{aligned}$$

This shows that the inequality above must in fact be satisfied as an equality and $\pi_j(r^1) +$

$\pi_j(r^2) = \pi_j(r^1 + r^2)$ for $j \in [2]$. Equivalently, $(r^1, r^2) \in \mathbb{E}(\pi_1) \cap \mathbb{E}(\pi_2)$. Hence, $\mathbb{E}(\pi) \subset \mathbb{E}(\pi_1) \cap \mathbb{E}(\pi_2)$. □

3.3 The One-Row Problem for $\mathbb{S} = \mathbb{Z}_+$

The main purpose of this section is to prove a Two-Slope Theorem for extreme cut-generating functions for (3.1) when $\mathbb{S} = \mathbb{Z}_+$, in the spirit of the Gomory-Johnson Two-Slope Theorem for $\mathbb{S} = \mathbb{Z}$. We assume that $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$.

When $\mathbb{S} = \mathbb{Z}_+$, any cut-generating function for (3.1) must take nonnegative values at nonnegative rationals because minimal cut-generating functions are subadditive and take nonnegative values at nonnegative integers. In the remainder, we restrict our attention to cut-generating functions for (3.1) that take nonnegative values at *all* nonnegative points. This is satisfied in particular by cut-generating functions that are left or right-continuous on the nonnegative halfline. Therefore, we make the following assumption.

Assumption 3.2. *When $\mathbb{S} = \mathbb{Z}_+$, all cut-generating functions π satisfy $\pi(r) \geq 0$ for all $r \geq 0$.*

This assumption means, in particular, that a cut-generating function π is extreme if and only if it cannot be written as $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ where π_1 and π_2 are distinct cut-generating functions satisfying Assumption 3.2. We now state the main result of this section.

Theorem 3.4 (Two-Slope Theorem)**.** *Let $\mathbb{S} = \mathbb{Z}_+$ and $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. Suppose Assumption 3.2 holds. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a strongly minimal cut-generating function for* (3.1)*. If the restriction of π to any compact interval is a continuous piecewise linear function with at most two slopes, then π is extreme.*

Theorem 3.4 implies, for example, that the cut-generating functions π^1_α of Example 2.1 are extreme.

The proof of Theorem 3.4 will go through two lemmas.

Lemma 3.5. *Let $\mathbb{S} = \mathbb{Z}_+$ and $f \in \mathbb{R}_+ \setminus \mathbb{Z}_+$. Let $\pi : \mathbb{R} \to \mathbb{R}$ be a minimal cut-generating function for* (3.1)*. If the restriction of π to any compact interval is a continuous piecewise linear function, then there exist $0 < \epsilon \leq \min\{f - \lfloor f \rfloor, \lceil f \rceil - f\}$ and $s^- < 0 < s^+$ such that $\pi(r) = s^- r$ for $r \in [-\epsilon, 0]$ and $\pi(r) = s^+ r$ for $r \in [0, \epsilon]$.*

Proof. Suppose π is a minimal cut-generating function for (3.1). By Theorem 2.20, $\pi(0) = 0$ and π is subadditive. Together with $\pi(0) = 0$, the continuity and piecewise linearity of π imply that there exist $0 < \epsilon \leq \min\{f - \lfloor f \rfloor, \lceil f \rceil - f\}$ and $s^-, s^+ \in \mathbb{R}$ such that $\pi(r) = s^- r$ for $r \in [-\epsilon, 0]$ and $\pi(r) = s^+ r$ for $r \in [0, \epsilon]$. Because π is a cut-generating function for (3.1), it must satisfy $\pi\Big(\lfloor f \rfloor - f\Big) \geq 1$ and $\pi\Big(\lceil f \rceil - f\Big) \geq 1$. The subadditivity of π then implies $k\pi\Big(\frac{\lfloor f\rfloor - f}{k}\Big) \geq \pi\Big(\lfloor f\rfloor - f\Big) \geq 1$ and $k\pi\Big(\frac{\lceil f\rceil - f}{k}\Big) \geq \pi\Big(\lceil f\rceil - f\Big) \geq 1$ for all $k \in \mathbb{Z}_{++}$. For k large enough, $\frac{\lfloor f\rfloor - f}{k} \in [-\epsilon, 0]$ and $\frac{\lceil f\rceil - f}{k} \in [0, \epsilon]$. This proves $s^- < 0 < s^+$. □

A fundamental tool in the proof of Theorem 3.4 will be the Interval Lemma, as was already the case in the proof of Gomory and Johnson's Two-Slope Theorem [64, 65]. The Interval Lemma has numerous variants (see, for example, Aczél [5], Kannappan [78], Dey et al. [55], and Basu et al. [20]). Below we give another variant which is well-suited to our needs in proving Theorem 3.4 because it only assumes a function that is bounded from below on a finite interval. This condition is known to be equivalent to the classical continuity assumption in the literature on Cauchy's additive equation; see Kannappan [78, Theorem 1.2]. We include a proof of our Interval Lemma here for the sake of completeness. Our proof follows the approach of [20]. Interval lemmas are usually stated in terms of a single function, but they can also be worded using three functions; this variant is known as Pexider's additive equation (see, for example, Aczél [5] or Basu, Hildebrand, and Köppe [22]). We state and prove our lemma in this more general form.

Lemma 3.6 (Interval Lemma). *Let $a_1 < a_2$ and $b_1 < b_2$. Consider the intervals $\mathbb{A} = [a_1, a_2]$, $\mathbb{B} = [b_1, b_2]$, and $\mathbb{A} + \mathbb{B} = [a_1 + b_1, a_2 + b_2]$. Let $f : \mathbb{A} \to \mathbb{R}$, $g : \mathbb{B} \to \mathbb{R}$, and $h : \mathbb{A} + \mathbb{B} \to \mathbb{R}$. Assume that f is bounded from below on $\mathbb{A}$. If $f(a) + g(b) = h(a + b)$ for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$, then f, g, and h are affine functions with identical slopes in the intervals $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{A} + \mathbb{B}$, respectively.*

Proof. The lemma will follow from several claims about the functions f, g, h.

Claim 1. Let $a \in \mathbb{A}$, and let $b \in \mathbb{B}$, $\epsilon > 0$ be such that $b + \epsilon \in \mathbb{B}$. For all $k \in \mathbb{Z}_{++}$ such that $a + k\epsilon \in \mathbb{A}$, we have $f(a + k\epsilon) - f(a) = k[g(b + \epsilon) - g(b)]$.

For $\ell \in [k]$, we have $f(a + \ell\epsilon) + g(b) = h(a + b + \ell\epsilon) = f(a + (\ell - 1)\epsilon) + g(b + \epsilon)$ by the hypothesis of the lemma. This implies $f(a + \ell\epsilon) - f(a + (\ell - 1)\epsilon) = g(b + \epsilon) - g(b)$ for $\ell \in [k]$. Summing all k equations, we obtain $f(a + k\epsilon) - f(a) = k[g(b + \epsilon) - g(b)]$. This concludes the proof of Claim 1.

Let $\bar{a}, \bar{a}' \in \mathbb{A}$ be such that $\bar{a}' - \bar{a} \in \mathbb{Q}$ and $\bar{a}' > \bar{a}$. Define $c = \frac{f(\bar{a}') - f(\bar{a})}{\bar{a}' - \bar{a}}$.

Claim 2. For all $a, a' \in \mathbb{A}$ such that $a' - a \in \mathbb{Q}$, we have $f(a') - f(a) = c(a' - a)$.

Assume without any loss of generality that $a' > a$. Choose a positive rational ϵ such that $\bar{a}' - \bar{a} = \bar{p}\epsilon$ for some integer $\bar{p}$, $a' - a = p\epsilon$ for some integer p, and $b_1 + \epsilon \in \mathbb{B}$. From Claim 1, we have

$$f(\bar{a}') - f(\bar{a}) = \bar{p}[g(b_1 + \epsilon) - g(b_1)] \text{ and } f(a') - f(a) = p[g(b_1 + \epsilon) - g(b_1)].$$

Dividing the first equality by $\bar{a}' - \bar{a} = \bar{p}\epsilon$ and the second by $a' - a = p\epsilon$, we obtain

$$\frac{f(a') - f(a)}{a' - a} = \frac{g(b_1 + \epsilon) - g(b_1)}{\epsilon} = \frac{f(\bar{a}') - f(\bar{a})}{\bar{a}' - \bar{a}} = c.$$

Thus, $f(a') - f(a) = c(a' - a)$. This concludes the proof of Claim 2.

Claim 3. *For all* $a \in \mathbb{A}$, $f(a) = f(a_1) + c(a - a_1)$.

Let $\delta(x) = f(x) - cx$. We show that $\delta(a) = \delta(a_1)$ for all $a \in \mathbb{A}$ to prove the claim. Because f is bounded from below on $\mathbb{A}$, δ is bounded from below on $\mathbb{A}$ as well. Let M be a number such that $\delta(a) \geq M$ for all $a \in \mathbb{A}$.

Suppose for a contradiction that there exists some $a^* \in \mathbb{A}$ such that $\delta(a^*) \neq \delta(a_1)$. The lower bound on δ implies $\delta(a_1), \delta(a^*) \geq M$. Let $D = \max\{\delta(a_1), \delta(a^*)\}$. Let $N \in \mathbb{Z}_{++}$ be such that $N|\delta(a^*) - \delta(a_1)| > D - M$. By Claim 2, $\delta(a_1) = \delta(a)$ and $\delta(a^*) = \delta(a')$ for all $a, a' \in \mathbb{A}$ such that $a_1 - a$ and $a^* - a'$ are rational. If $\delta(a^*) < \delta(a_1)$, choose $\bar{a}, \bar{a}' \in \mathbb{A}$ such that $\bar{a} < \bar{a}'$, $\delta(a_1) = \delta(\bar{a})$, $\delta(a^*) = \delta(\bar{a}')$, $\bar{a} + N(\bar{a}' - \bar{a}) \in \mathbb{A}$, and $b_1 + (\bar{a}' - \bar{a}) \in \mathbb{B}$. Otherwise, choose $\bar{a}, \bar{a}' \in \mathbb{A}$ such that $\bar{a} < \bar{a}'$, $\delta(a_1) = \delta(\bar{a}')$, $\delta(a^*) = \delta(\bar{a})$, $\bar{a} + N(\bar{a}' - \bar{a}) \in \mathbb{A}$, and $b_1 + (\bar{a}' - \bar{a}) \in \mathbb{B}$. In either case we have $\bar{a} < \bar{a}'$ and $\delta(\bar{a}) > \delta(\bar{a}')$. Furthermore, the choices of $\bar{a}$, $\bar{a}'$, and N imply

$$N[\delta(\bar{a}') - \delta(\bar{a})] = -N\,|\delta(\bar{a}') - \delta(\bar{a})| = -N\,|\delta(a^*) - \delta(a_1)| < M - D.$$

Let $\epsilon = \bar{a}' - \bar{a}$. By Claim 1,

$$\delta(\bar{a} + N\epsilon) - \delta(\bar{a}) = N[\delta(b_1 + \epsilon) - \delta(b_1)] = N[\delta(\bar{a} + \epsilon) - \delta(\bar{a})] = N[\delta(\bar{a}') - \delta(\bar{a})].$$

Combining this with the previous inequality, we obtain

$$\delta(\bar{a} + N\epsilon) - \delta(\bar{a}) = N[\delta(\bar{a}') - \delta(\bar{a})] < M - D.$$

Because $\delta(\bar{a}) \leq \max\{\delta(a_1), \delta(a^*)\} = D$, this yields $\delta(\bar{a} + N\epsilon) < M - D + \delta(\bar{a}) < M$ which contradicts the choice of M. This concludes the proof of Claim 3.

Claim 4. *For all* $b \in \mathbb{B}$, $g(b) = g(b_1) + c(b - b_1)$.

Let k be the smallest positive integer such that $k(a_2 - a_1) \geq b - b_1$, and let $\epsilon = b - (b_1 + (k-1)(a_2 - a_1))$. For all $\ell \in [k-1]$, we have $g(b_1 + \ell(a_2 - a_1)) - g(b_1 + (\ell-1)(a_2 - a_1)) = f(a_1 + (a_2 - a_1)) - f(a_1) = c(a_2 - a_1)$ by Claim 1. Similarly, $g(b) - g(b_1 + (k-1)(a_2 - a_1)) = g(b_1 + (k-1)(a_2 - a_1) + \epsilon) - g(b_1 + (k-1)(a_2 - a_1)) = f(a_1 + \epsilon) - f(a_1) = c\epsilon$ by Claim 1. Summing all k equations, we obtain $g(b) - g(b_1) = c\epsilon + c(k-1)(a_2 - a_1) = c(b - b_1)$. This concludes the proof of Claim 4.

Finally, let $w \in \mathbb{A} + \mathbb{B}$, and let $a \in \mathbb{A}$, $b \in \mathbb{B}$ be such that $w = a + b$. By the hypothesis of the lemma and by Claims 3 and 4, we have $h(w) = f(a) + g(b) = f(a_1) + c(a - a_1) + g(b_1) + c(b - b_1) = h(a_1 + b_1) + c(w - (a_1 + b_1))$. □

We are now ready to prove Theorem 3.4. Our proof follows the proof outline of the original Gomory-Johnson Two-Slope Theorem for $\mathbb{S} = \mathbb{Z}$ [65].

Proof of Theorem 3.4. Let $\mathbb{I}$ be a compact interval of the real line containing $\Big[\lfloor -f\rfloor, 1\Big]$. By Lemma 3.5, there exist $0 < \epsilon \leq \min\{f - \lfloor f\rfloor, \lceil f\rceil - f\}$ and $s^- < 0 < s^+$ such that $\pi(r) = s^- r$ for $r \in [-\epsilon, 0]$ and $\pi(r) = s^+ r$ for $r \in [0, \epsilon]$. Thus, s^- and s^+ are the two slopes of π. Assume without any loss of generality that the slopes of π are distinct in the consecutive intervals delimited by the points $\min \mathbb{I} = r_{-q} < \ldots < r_{-1} < r_0 = 0 < r_1 < \ldots < r_t = \max \mathbb{I}$. It follows that π has slope s^+ in interval $[r_i, r_{i+1}]$ if i is even and slope s^- if i is odd.

Consider cut-generating functions π_1, π_2 such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$. By Lemma 3.3, π_1 and π_2 are strongly minimal cut-generating functions. By Theorem 2.8, π, π_1, and π_2 are symmetric and satisfy $\pi(0) = \pi_1(0) = \pi_2(0) = 0$ and $\pi(-1) = \pi_1(-1) = \pi_2(-1) = 0$. The symmetry condition implies in particular that $\pi(-f) = \pi_1(-f) = \pi_2(-f) = 1$.

We are going to obtain the theorem as a consequence of several claims.

Claim 1. In intervals $[r_i, r_{i+1}]$ with i even, π_1 and π_2 are affine functions with positive slopes s_1^+ and s_2^+, respectively.

Let $i \in \{-q, \ldots, t-1\}$ even. Let $0 < \epsilon \leq r_1$ be such that $r_i + \epsilon < r_{i+1}$. Define $\mathbb{A} = [0, \epsilon]$, $\mathbb{B} = [r_i, r_{i+1} - \epsilon]$. Then $\mathbb{A} + \mathbb{B} = [r_i, r_{i+1}]$. Note that the slope of π is s^+ in all three intervals and $\pi(a) + \pi(b) = \pi(a+b)$ for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$. By Lemma 3.3, $\pi_1(a) + \pi_1(b) = \pi_1(a+b)$ and $\pi_2(a) + \pi_2(b) = \pi_2(a+b)$ for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$. Consider either $j \in \{1, 2\}$. The function π_j is a cut-generating function, so $\pi_j(a) \geq 0$ for all $a \in \mathbb{A}$ by Assumption 3.2. Lemma 3.6 implies that π_j is an affine function with common slope s_j^+ in all three intervals $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{A} + \mathbb{B}$. Because π_j is a minimal cut-generating function, it is subadditive and satisfies $k\pi_j\Big(\frac{\lceil f\rceil - f}{k}\Big) \geq \pi_j\Big(\lceil f\rceil - f\Big) \geq 1$ for all $k \in \mathbb{Z}_{++}$. Choosing k large enough ensures $\frac{\lceil f\rceil - f}{k} \in \mathbb{A}$ and $k\pi_j\Big(\frac{\lceil f\rceil - f}{k}\Big) = s_j^+\Big(\lceil f\rceil - f\Big) \geq 1$. This shows $s_j^+ > 0$ and concludes the proof of Claim 1.

Claim 2. In intervals $[r_i, r_{i+1}]$ with i odd, π_1 and π_2 are affine functions with negative slopes s_1^- and s_2^-, respectively.

The proof of the claim is similar to the proof of Claim 1. One only needs to choose the intervals $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{A} + \mathbb{B}$ slightly more carefully while using Lemma 3.6. Let $i \in \{-q, \ldots, t-1\}$ odd. Let $0 < \epsilon \leq -r_{-1}$ be such that $r_i + \epsilon < r_{i+1}$ and $\epsilon \leq r_1$. Define $\mathbb{A} = [-\epsilon, 0]$, $\mathbb{B} = [r_i + \epsilon, r_{i+1}]$. Then $\mathbb{A} + \mathbb{B} = [r_i, r_{i+1}]$. Consider either $j \in \{1, 2\}$. Because π_j is a minimal cut-generating function, it is subadditive and satisfies $\pi_j(a) \geq -\pi_j(-a) = s_j^+ a$ for all $a \in \mathbb{A}$. Thus, π_j is minorized by a linear function and bounded from below on $\mathbb{A}$. Now using Lemmas 3.3 and 3.6, we see that π_j is an affine function with common slope s_j^- in all three intervals $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{A} + \mathbb{B}$. The negativity of s_j^- then follows from this, the subadditivity of π_j, $\pi_j(0) = 0$, and $\pi_j\Big(\lfloor f\rfloor - f\Big) \geq 1$. This concludes the proof of Claim 2.

Claims 1 and 2 show that π_1 and π_2 are continuous functions whose restrictions to the interval $\mathbb{I}$ are piecewise linear functions with two slopes.

Claim 3. $s^+ = s_1^+ = s_2^+$, $s^- = s_1^- = s_2^-$.

Define L^+_{-1} and L^+_{-f} as the sum of the lengths of intervals with positive slope contained in $[-1, 0]$ and $[-f, 0]$, respectively. Define L^-_{-1} and L^-_{-f} as the sum of the lengths of intervals with negative slope contained in $[-1, 0]$ and $[-f, 0]$, respectively. Note that $L^+_{-f}, L^-_{-f}, L^+_{-1}, L^-_{-1}$ are all nonnegative, $L^+_{-1} + L^-_{-1} = 1$, and $L^+_{-f} + L^-_{-f} = f$. Since $\pi(0) = \pi_1(0) = \pi_2(0) = 0$, $\pi(-f) = \pi_1(-f) = \pi_2(-f) = 1$, and $\pi(-1) = \pi_1(-1) = \pi_2(-1) = 0$, the vectors (s^+, s^-), (s_1^+, s_1^-), (s_2^+, s_2^-) all satisfy the system

$$\begin{aligned} L^+_{-1}\sigma^+ + L^-_{-1}\sigma^- &= 0, \\ L^+_{-f}\sigma^+ + L^-_{-f}\sigma^- &= -1. \end{aligned}$$

Note that $(L^+_{-1}, L^-_{-1}) \neq 0$ because $L^+_{-1} + L^-_{-1} = 1$. Suppose the constraint matrix of the system above is singular. Then the vector (L^+_{-f}, L^-_{-f}) must be a multiple λ of (L^+_{-1}, L^-_{-1}). However, this is impossible because the system has a solution (s^+, s^-) and the right-hand sides of the two equations would have to satisfy $0\lambda = -1$. Therefore, the constraint matrix is nonsingular and the system must have a unique solution. This implies $s^+ = s_1^+ = s_2^+$ and $s^- = s_1^- = s_2^-$, concluding the proof of Claim 3.

The functions π, π_1, and π_2 are continuous piecewise linear functions which have the same slope in each interval $[r_i, r_{i+1}]$ of $\mathbb{I}$. Therefore, $\pi(r) = \pi_1(r) = \pi_2(r)$ for all $r \in \mathbb{I}$. Because $\mathbb{I}$ can be chosen to be any compact interval that contains $\big[\lfloor -f \rfloor, 1\big]$, we have $\pi = \pi_1 = \pi_2$. □

Example 3.1. In Theorem 3.4, the cut-generating function π is assumed to be "strongly minimal". This assumption cannot be weakened to "minimal" or "restricted minimal" as illustrated by the following example. For $0 < f < 1$ and $\alpha \geq 1$, consider the function $\pi^4_\alpha : \mathbb{R} \to \mathbb{R}$ defined by

$$\pi^4_\alpha(r) = \begin{cases} \frac{\alpha r}{1-f}, & \text{for } r \geq 0, \\ \frac{-r}{f}, & \text{for } -f < r < 0, \\ 1 + \frac{\alpha(r+f)}{1-f}, & \text{for } r \leq -f. \end{cases}$$

The above function π^4_α is a continuous piecewise linear function with only two slopes (see Figure 3.1). Furthermore, $\frac{\alpha r}{1-f} \leq \pi^4_\alpha(r) \leq 1 + \frac{\alpha(r+f)}{1-f}$ for all $r \in \mathbb{R}$. We claim that

i. π^4_α is a restricted minimal cut-generating function for (3.1),
ii. π^4_α is neither strongly minimal nor extreme when $\alpha > 1$.

As a consequence of Theorem 2.25, to prove (i), it suffices to show that $\pi^4_\alpha(0) = 0$, $\pi^4_\alpha(-1) \leq 0$, π^4_α is subadditive and symmetric. The first two properties are straightforward to verify. We prove that π^4_α is subadditive, that is, $\pi^4_\alpha(r^1) + \pi^4_\alpha(r^2) \geq \pi^4_\alpha(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}$. We may assume $r^1 \leq r^2$.

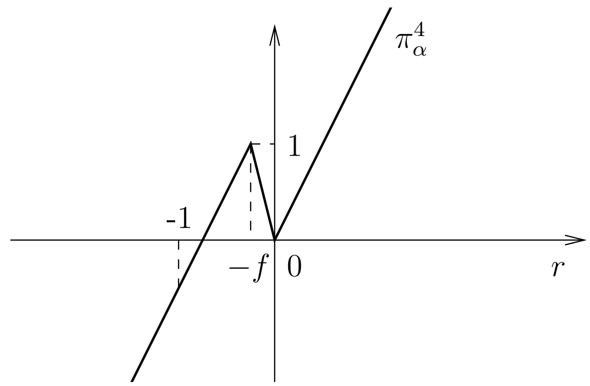


Figure 3.1: The restricted minimal cut-generating function π^4_α has only two slopes but is not extreme.

- If $r^1 \leq -f$, then $\pi^4_\alpha(r^1) + \pi^4_\alpha(r^2) \geq 1 + \frac{\alpha(r^1+f)}{1-f} + \frac{\alpha r^2}{1-f} = 1 + \frac{\alpha(r^1+r^2+f)}{1-f} \geq \pi^4_\alpha(r^1 + r^2)$.
- If $r^1 > -f$ and $r^1+r^2 < 0$, then $\pi^4_\alpha(r^1)+\pi^4_\alpha(r^2) \geq \frac{-r^1}{f} + \frac{-r^2}{f} = \frac{-(r^1+r^2)}{f} \geq \pi^4_\alpha(r^1+r^2)$.
- If $r^1 + r^2 \geq 0$, then $\pi^4_\alpha(r^1) + \pi^4_\alpha(r^2) \geq \frac{\alpha r^1}{1-f} + \frac{\alpha r^2}{1-f} = \pi^4_\alpha(r^1 + r^2)$.

Thus, π^4_α is subadditive. Furthermore, π^4_α is symmetric since the point $(-f/2, 1/2)$ is a point of symmetry in the graph of the function.

To prove (ii), note that for any $\alpha > 1$, $\pi^4_\alpha(-1) < 0$. It follows from Theorem 2.8 that π^4_α is not strongly minimal and from Lemma 3.2 that π^4_α is not extreme. Indeed, for any $\alpha > 1$, π^4_α can be written as $\pi^4_\alpha = \frac{1}{2}\pi^4_{\alpha-\epsilon} + \frac{1}{2}\pi^4_{\alpha+\epsilon}$, where both functions $\pi^4_{\alpha-\epsilon}$ and $\pi^4_{\alpha+\epsilon}$ are restricted minimal cut-generating functions if we choose $0 < \epsilon \leq \alpha - 1$.

Finally, we observe that when $\alpha = 1$, the conditions of Theorem 2.8 are satisfied. This implies that π^4_α is strongly minimal in this case and therefore extreme by Theorem 3.4.

Chapter 4

Sufficiency of Cut-Generating Functions

Acknowledgments. This chapter is based on joint work with Gérard Cornuéjols and Laurence Wolsey [51].

4.1 Introduction

4.1.1 Motivation

Let $\mathbb{S}' \subset \mathbb{R}^n$ be a nonempty closed set such that $0 \notin \mathbb{S}'$. In this chapter, we consider the model

$$\mathbb{X} = \mathbb{X}(R, \mathbb{S}') = \{s \in \mathbb{R}^k_+ : \quad Rs \in \mathbb{S}'\}, \tag{4.1}$$

where $R = [r^1, \dots, r^k]$ is a real $n \times k$ matrix. The model (4.1) has been studied in [47, 76, 80]. It arises in integer programming when studying Gomory's corner relaxation [63, 66] or the relaxation proposed by Andersen, Louveaux, Weismantel, and Wolsey [9]. It also arises in other optimization problems such as complementarity problems [77]. As in Chapters 2 and 3, in the framework (4.1) the goal is to generate inequalities that are valid for $\mathbb{X}$ but not for the origin. Such cutting planes are well-defined [47, Lemma 2.1] and can be written as

$$c^\top s \geq 1. \tag{4.2}$$

Let $\mathbb{S}' \subset \mathbb{R}^n$ be a given nonempty closed set such that $0 \notin \mathbb{S}'$. The set $\mathbb{S}'$ is assumed to be fixed in this paragraph. A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is a *cut-generating function* for $\mathbb{X}(R, \mathbb{S}')$ if it produces the coefficients $c_j = \rho(r^j)$ of a cut (4.2) valid for $\mathbb{X}(R, \mathbb{S}')$ for any choice of k and $R = [r^1, \dots, r^k]$. Conforti et al. [47] show that cut-generating functions enjoy significant structure, generalizing earlier work in integer programming [18, 54]. For instance, the minimal ones are sublinear and are closely related to $\mathbb{S}'$-free neighborhoods of the origin. We say that a closed, convex set is $\mathbb{S}'$*-free* if it contains no point of $\mathbb{S}'$ in its interior. For any minimal cut-generating function ρ, there exists a closed, convex, $\mathbb{S}'$-free set $\mathbb{V} \subset \mathbb{R}^n$ such that $0 \in \operatorname{int} \mathbb{V}$ and $\mathbb{V} = \{r \in \mathbb{R}^n : \rho(r) \leq 1\}$. A cut (4.2) with coefficients $c_j = \rho(r^j)$ is called an $\mathbb{S}'$*-intersection cut* in this chapter.

Now assume that both $\mathbb{S}'$ and R are fixed. Noting $\mathbb{X} \subset \mathbb{R}^k_+$, we say that a cutting plane $c^\top s \geq 1$ *dominates* $b^\top s \geq 1$ if $c_j \leq b_j$ for all $j \in \{1, \ldots, k\}$. A natural question is whether every cut (4.2) that is valid for $\mathbb{X}$ is dominated by an $\mathbb{S}'$-intersection cut. Conforti et al. provide an affirmative response to this question under the condition that $\operatorname{cone} R = \mathbb{R}^n$; see [47, Theorem 6.3]. However, they also give an example which demonstrates that it is not always the case. This example has the peculiarity that $\mathbb{S}'$ contains points that cannot be obtained as Rs for any $s \in \mathbb{R}^k_+$. Conforti et al. [47] propose the following open problem: Assuming $\mathbb{S}' \subset \operatorname{cone} R$, is it true that every cut (4.2) that is valid for $\mathbb{X}(R, \mathbb{S}')$ is dominated by an $\mathbb{S}'$-intersection cut? The main theorem of this chapter shows that this is indeed the case.

Theorem 4.1. *Let* $\mathbb{X}(R, \mathbb{S}') \subset \mathbb{R}^k$ *be a nonempty set defined as in* (4.1)*. Suppose* $\mathbb{S}' \subset \operatorname{cone} R$*. Then any valid inequality* $c^\top s \geq 1$ *separating the origin from* $\mathbb{X}(R, \mathbb{S}')$ *is dominated by an* $\mathbb{S}'$*-intersection cut.*

Earlier, for the case $n = 2$, Cornuéjols and Margot [49] showed that every valid cut (4.2) for $\mathbb{X}(R, \mathbb{S}')$ is dominated by an $\mathbb{S}'$-intersection cut for all choices of R when $\mathbb{S}' = b + \mathbb{Z}^n$ for some $b \in \mathbb{R}^n \backslash \mathbb{Z}^n$; see [49, Theorem 3.1]. Zambelli [110] generalized this result to arbitrary n. Conforti et al. [43] showed that a similar statement is true for Gomory's corner polyhedron. We note that any valid cut (4.2) must have $c \in \mathbb{R}^k_+$ in all of these settings because the recession cone of the closed convex hull of $\mathbb{X}(R, \mathbb{S}')$ equals the nonnegative orthant. Dey and Wolsey [54] extended these results to the case where $\mathbb{S}' = \mathbb{P} \cap (b + \mathbb{Z}^n)$ for some $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and a rational polyhedron $\mathbb{P} \subset \mathbb{R}^n$; see [54, Proposition 3.7]. Our Theorem 4.1 further extends them to the case where $\mathbb{S}' \subset \mathbb{R}^n$ is an arbitrary nonempty closed set such that $0 \notin \mathbb{S}'$. More recently, Theorem 4.1 has been generalized in [81, 82]. These papers build upon the earlier results of [76, 80] on minimal inequalities for disjunctive sets.

The remainder of the chapter is organized as follows: In Section 4.2 we prove Theorem 4.1. Section 4.3 elaborates on the geometric intuition behind the proof and illustrates its construction with an example.

4.1.2 Notation and Terminology

For a positive integer ℓ, we let $[\ell] = \{1, \ldots, \ell\}$. For $j \in [k]$, we let $e^j \in \mathbb{R}^k$ denote the j-th standard unit vector. We let $\operatorname{conv} \mathbb{V}$, $\operatorname{cone} \mathbb{V}$, and $\operatorname{span} \mathbb{V}$ represent the convex hull, conical hull, and linear span of a set $\mathbb{V} \subset \mathbb{R}^n$, respectively. We use $\operatorname{lin} \mathbb{V}$ and $\operatorname{rec} \mathbb{V}$ to refer to the lineality space and recession cone of a closed convex set $\mathbb{V} \subset \mathbb{R}^n$, respectively. The *polar cone* of $\mathbb{V} \subset \mathbb{R}^n$ is $\mathbb{V}^\circ = \{r \in \mathbb{R}^n : r^\top x \leq 0 \; \forall x \in \mathbb{V}\}$. The *dual cone* of $\mathbb{V} \subset \mathbb{R}^n$ is $\mathbb{V}^* = -\mathbb{V}^\circ$.

A function $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *positively homogeneous* if $\rho(\lambda x) = \lambda \rho(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$, and *subadditive* if $\rho(x_1) + \rho(x_2) \geq \rho(x_1 + x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$. Moreover, ρ is *sublinear* if it is both positively homogeneous and subadditive. Sublinear functions are known to be convex. For a nonempty set $\mathbb{V} \subset \mathbb{R}^n$, the *support function* of $\mathbb{V}$ is the function $\sigma_{\mathbb{V}} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as $\sigma_{\mathbb{V}}(r) = \sup_{x \in \mathbb{V}} r^\top x$. It is not difficult to show that $\sigma_{\mathbb{V}} = \sigma_{\operatorname{conv} \mathbb{V}}$. Support functions of nonempty sets are sublinear. For an in-depth treatment of sublinearity and support functions, the reader is referred to [69, Chapter C]. Given a closed, convex neighborhood $\mathbb{V} \subset \mathbb{R}^n$ of the origin, a *representation of* $\mathbb{V}$ is any sublinear function $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{V} = \{r \in \mathbb{R}^n : \rho(r) \leq 1\}$. Minkowski's gauge function is a representation of $\mathbb{V}$, but there can be other representations when $\mathbb{V}$ is unbounded. $\mathbb{S}'$-intersection cuts are generated by representations of closed, convex, $\mathbb{S}'$-free neighborhoods of the origin.

4.2 Proof of Theorem 4.1

Our proof of Theorem 4.1 will use several lemmas. Throughout this section we assume that $\mathbb{X} \neq \emptyset$ and $c^\top s \geq 1$ is a valid inequality separating the origin from $\mathbb{X}$.

Lemma 4.2. *Let $\mathbb{X} \subset \mathbb{R}^k$ be a nonempty set defined as in* (4.1)*. Consider a valid inequality $c^\top s \geq 1$ for $\mathbb{X}$. If $u \in \mathbb{R}^k_+$ and $Ru = 0$, then $c^\top u \geq 0$. Equivalently, $c \in \mathbb{R}^k_+ + \operatorname{Im} R^\top$.*

Proof. Let $\bar{s} \in \mathbb{X}$. Note that $R(\bar{s} + tu) = R\bar{s} \in \mathbb{S}'$ and $\bar{s} + tu \geq 0$ for all $t \geq 0$. By the validity of c, we have $c^\top(\bar{s} + tu) \geq 1$ for all $t \geq 0$. Observing $tc^\top u \geq 1 - c^\top \bar{s}$ and letting $t \to +\infty$ implies $c^\top u \geq 0$ as desired. Because u is an arbitrary vector in $\mathbb{R}^k_+ \cap \operatorname{Ker} R$, we can write $c \in (\mathbb{R}^k_+ \cap \operatorname{Ker} R)^*$. The equality $(\mathbb{R}^k_+ \cap \operatorname{Ker} R)^* = \mathbb{R}^k_+ + \operatorname{Im} R^\top$ follows from the facts $(\mathbb{R}^k_+)^* = \mathbb{R}^k_+$, $(\operatorname{Ker} R)^* = \operatorname{Im} R^\top$, and $\mathbb{R}^k_+ + \operatorname{Im} R^\top$ is closed (see [96, Cor. 16.4.2]). $\square$

Given the valid inequality $c^\top s \geq 1$, we now construct a sublinear function $h_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that produces a valid inequality $\sum_{j=1}^k h_c(r^j) s_j \geq 1$ which dominates $c^\top s \geq 1$.

Let

$$h_c(r) = \min \quad c^\top s \tag{4.3a}$$
$$Rs = r, \tag{4.3b}$$
$$s \geq 0. \tag{4.3c}$$

The function h_c is used here in analogy with the proof of Theorem 1 in [110]; see also Lemma 3.1 in [19] and Theorem 2.3 in [47]. This function was also studied in its primal form in [32, 73] and in its dual form in [76, 80] because of its connection with minimal valid inequalities for the set $\mathbb{X}$ corresponding to a fixed matrix R.

Remark 4.3. *Suppose the hypotheses of Lemma 4.2 are satisfied. Let $h_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined as in* (4.3)*.*

i. $h_c(r^j) \leq c_j$ *for all* $j \in [k]$*.*

ii. $h_c(\overline{r}) \geq 1$ *for all* $\overline{r} \in \mathbb{S}'$*.*

Proof. The first claim follows directly from the observation that $e^j \in \mathbb{R}^k$ is feasible to the linear program (4.3) associated with $r = r^j$. To prove the second claim, let $\overline{r} \in \mathbb{S}'$. If the linear program (4.3) associated with $r = \overline{r}$ is infeasible, $h_c(\overline{r}) = +\infty \geq 1$. Otherwise, any feasible solution $\bar{s}$ to this linear program satisfies $\bar{s} \in \mathbb{X}$ and $c^\top \bar{s} \geq 1$ by the validity of $c^\top s \geq 1$. Hence, $h_c(\overline{r}) \geq 1$. □

Lemma 4.4. *Let $\mathbb{X} \subset \mathbb{R}^k$ be a nonempty set defined as in* (4.1)*. Consider a valid inequality $c^\top s \geq 1$ for $\mathbb{X}$. Let $h_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined as in* (4.3)*.*

i. $h_c = \sigma_{\mathbb{P}}$ *where* $\mathbb{P} = \{y \in \mathbb{R}^n : R^\top y \leq c\}$*.*

ii. The function h_c is piecewise-linear and sublinear. Furthermore, it is finite on $\operatorname{cone} R$*.*

Proof. The dual of (4.3) is

$$\begin{aligned} \max \quad & r^\top y \\ & R^\top y \leq c. \end{aligned} \tag{4.4}$$

By Lemma 4.2, $c = c' + c''$ where $c' \in \mathbb{R}^k_+$ and $c'' \in \operatorname{Im} R^\top$. Because $c'' \in \operatorname{Im} R^\top$, there exists $y'' \in \mathbb{R}^n$ such that $R^\top y'' = c'' \leq c$. Hence, $y'' \in \mathbb{P}$ which shows that the dual linear program (4.4) is always feasible and strong duality holds. This shows that $h_c = \sigma_{\mathbb{P}}$ and h_c is indeed a sublinear function.

The linear program (4.3) is feasible if and only if $r \in \operatorname{cone} R$. Hence, $h_c(r) < +\infty$ for $r \in \operatorname{cone} R$ and $h_c(r) = +\infty$ for $r \in \mathbb{R}^n \setminus \operatorname{cone} R$. The conclusion that h_c is finite on $\operatorname{cone} R$ follows from $h_c = \sigma_{\mathbb{P}} > -\infty$. We now show that h_c is piecewise-linear. Let $\overline{r} \in \operatorname{cone} R$. Let $\mathbb{W}$ be a finite set of points for which $\mathbb{P} = \operatorname{conv} \mathbb{W} + \operatorname{rec} \mathbb{P}$. Observe that $\operatorname{rec} \mathbb{P} = (\operatorname{cone} R)^\circ$

and $\bar{r}^\top u \leq 0$ for all $u \in \operatorname{rec}\mathbb{P}$. Thus, $\bar{r}^\top(w+u) \leq \bar{r}^\top w$ for all $w \in \operatorname{conv}\mathbb{W}$ and $u \in \operatorname{rec}\mathbb{P}$, which implies

$$\sigma_{\mathbb{P}}(\bar{r}) = \sup_{p\in\mathbb{P}} \bar{r}^\top p \leq \sigma_{\operatorname{conv}\mathbb{W}}(\bar{r}) = \sup_{w\in\operatorname{conv}\mathbb{W}} \bar{r}^\top w = \sigma_{\mathbb{W}}(\bar{r}).$$

Since $\mathbb{W} \subset \mathbb{P}$ implies $\sigma_{\mathbb{W}} \leq \sigma_{\mathbb{P}}$, we have $\sigma_{\mathbb{P}}(\bar{r}) = \sigma_{\mathbb{W}}(\bar{r})$. Therefore, $h_c(\bar{r}) = \sigma_{\mathbb{P}}(\bar{r}) = \sigma_{\mathbb{W}}(\bar{r}) = \max_{w\in\mathbb{W}} \bar{r}^\top w$ where the last equality follows from the finiteness of $\mathbb{W}$. This and the fact that $\operatorname{cone} R$ is polyhedral imply that h_c is piecewise-linear. □

Lemma 4.4 implies in particular that $h_c(0) = 0$.

Proposition 4.5. *Theorem 4.1 holds when* $\operatorname{cone} R = \mathbb{R}^n$.

Proof. In this case h_c is finite everywhere. Let $\mathbb{V}_c = \{r \in \mathbb{R}^n : h_c(r) \leq 1\}$. The set $\mathbb{V}_c$ is a closed, convex neighborhood of the origin because h_c is sublinear and finite everywhere, and $h_c(0) = 0$. Because the Slater condition is satisfied with $h_c(0) = 0$, we have $\operatorname{int}\mathbb{V}_c = \{r \in \mathbb{R}^n : h_c(r) < 1\}$ (see, e.g., [69, Prop. D.1.3.3]). Then $\mathbb{V}_c$ is also $\mathbb{S}'$-free since $h_c(\bar{r}) \geq 1$ for all $\bar{r} \in \mathbb{S}'$ by Remark 4.3(ii). The function h_c is a cut-generating function because it represents the closed, convex, $\mathbb{S}'$-free neighborhood of the origin $\mathbb{V}_c$ by definition, and $\sum_{j=1}^k h_c(r^j)s_j \geq 1$ is an $\mathbb{S}'$-intersection cut that can be obtained from $\mathbb{V}_c$. By Remark 4.3(i), $h_c(r^j) \leq c_j$ for all $j \in [k]$. This shows that the $\mathbb{S}'$-intersection cut $\sum_{j=1}^k h_c(r^j)s_j \geq 1$ dominates $c^\top s \geq 1$. □

We now consider the case where $\operatorname{cone} R \subsetneq \mathbb{R}^n$. We want to extend the definition of h_c to the whole of $\mathbb{R}^n$ and show that this extension is a cut-generating function. We will first construct a function h'_c such that i) h'_c is finite everywhere on $\operatorname{span} R$, ii) h'_c coincides with h_c on $\operatorname{cone} R$. If $\operatorname{rank}(R) < n$, we will further extend h'_c to the whole of $\mathbb{R}^n$ by letting $h'_c(r) = h'_c(r')$ for all $r \in \mathbb{R}^n$, $r' \in \operatorname{span} R$, $r'' \in (\operatorname{span} R)^\perp$ such that $r = r' + r''$. Our proof of Theorem 4.1 will show that this procedure yields a function h'_c that is the desired extension of h_c.

Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$ where $\operatorname{ri}(\cdot)$ denotes the relative interior. Note that this guarantees $\operatorname{cone}(R \cup \{r_0\}) = \operatorname{span} R$ since there exist $\epsilon > 0$ and $d = \operatorname{rank}(R)$ linearly independent vectors $a_1, \ldots, a_d \in \operatorname{span} R$ such that $-r_0 \pm \epsilon a_i \in \operatorname{cone} R$ for all $i \in [d]$ which implies $\pm a_i \in \operatorname{cone}(R \cup \{r_0\})$. Now we define c_0 as

$$c_0 = \sup_{\substack{r\,\in\,\operatorname{cone} R\\ \alpha > 0}} \frac{h_c(r) - h_c(r - \alpha r_0)}{\alpha}. \tag{4.5}$$

Lemma 4.6. *Let $\mathbb{X} \subset \mathbb{R}^k$ be a nonempty set defined as in* (4.1)*. Consider a valid inequality $c^\top s \geq 1$ for $\mathbb{X}$. Let $h_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined as in* (4.3)*. The value c_0, defined as in* (4.5)*, is finite.*

Proof. Any pair $\overline{r} \in \operatorname{cone} R$ and $\overline{\alpha} > 0$ yields a lower bound on c_0: Our choice of r_0 ensures $\overline{r} - \overline{\alpha} r_0 \in \operatorname{cone} R$ and $c_0 \geq \frac{h_c(\overline{r}) - h_c(\overline{r} - \overline{\alpha} r_0)}{\overline{\alpha}}$. To get an upper bound on c_0, consider any $\tilde{r} \in \operatorname{cone} R$ and $\tilde{\alpha} \geq 0$. Observe that $\tilde{r} - \tilde{\alpha} r_0 \in \operatorname{cone} R$. By Lemma 4.4, $h_c(\tilde{r} - \tilde{\alpha} r_0) = \sigma_{\mathbb{P}}(\tilde{r} - \tilde{\alpha} r_0)$ where $\mathbb{P} = \{y \in \mathbb{R}^n : R^\top y \leq c\}$. Let $\mathbb{W}$ be a finite set of points for which $\mathbb{P} = \operatorname{conv} \mathbb{W} + \operatorname{rec} \mathbb{P}$. Because $\operatorname{rec} \mathbb{P} = (\operatorname{cone} R)^\circ$, we have $(\tilde{r} - \tilde{\alpha} r_0)^\top u \leq 0$ for all $u \in \operatorname{rec} \mathbb{P}$. This implies $\sigma_{\mathbb{P}}(\tilde{r} - \tilde{\alpha} r_0) = \sigma_{\mathbb{W}}(\tilde{r} - \tilde{\alpha} r_0)$, and we can write

$$c_0 = \sup_{\substack{r \in \operatorname{cone} R \\ \alpha > 0}} \frac{\sigma_{\mathbb{W}}(r) - \sigma_{\mathbb{W}}(r - \alpha r_0)}{\alpha} \leq \sup_{\substack{r \in \operatorname{cone} R \\ \alpha > 0}} \frac{\sigma_{\mathbb{W}}(\alpha r_0)}{\alpha} = \sigma_{\mathbb{W}}(r_0),$$

where we have used the sublinearity of $\sigma_{\mathbb{W}}$ in the inequality and the second equality. The conclusion follows now from the fact that $\mathbb{W}$ is a finite set. □

Remark 4.7. *Suppose the hypotheses of Lemma 4.6 are satisfied. If we scale r_0 by a positive scalar λ, then c_0 is scaled by λ as well.*

Proof. This follows from $\frac{h_c(r) - h_c(r - \alpha\lambda r_0))}{\alpha} = \lambda \frac{h_c(r/\lambda) - h_c(r/\lambda - \alpha r_0))}{\alpha}$ (positive homogeneity of h_c) and the fact that $r \in \operatorname{cone} R$ if and only if $r/\lambda \in \operatorname{cone} R$. □

Proposition 4.8. *Let $\mathbb{X}(R, \mathbb{S}') \subset \mathbb{R}^k$ be a nonempty set defined as in (4.1). Consider a valid inequality $c^\top s \geq 1$ for $\mathbb{X}(R, \mathbb{S}')$. Let $h_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined as in (4.3). Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$, and let c_0 be defined as in (4.5). Then $c_0 s_0 + c^\top s \geq 1$ is a valid inequality for $\mathbb{X}([r_0, R], \mathbb{S}')$.*

Proof. Let $(\bar{s}_0, \bar{s}) \in \mathbb{X}([r_0, R], \mathbb{S}')$ and $\overline{r} = r_0 \bar{s}_0 + R\bar{s} \in \mathbb{S}'$. Then

$$c_0 \bar{s}_0 + c^\top \bar{s} \geq c_0 \bar{s}_0 + \sum_{j=1}^k h_c(r^j) \bar{s}_j \geq c_0 \bar{s}_0 + h_c(R\bar{s}) = c_0 \bar{s}_0 + h_c(\overline{r} - \bar{s}_0 r_0),$$

where the first inequality follows from Remark 4.3(i) and the second from the sublinearity of h_c. Using the definition of c_0 and applying Remark 4.3(ii), we conclude $c_0 \bar{s}_0 + c^\top \bar{s} \geq c_0 \bar{s}_0 + h_c(\overline{r} - \bar{s}_0 r_0) \geq h_c(\overline{r}) \geq 1$. □

We define the function h'_c on $\operatorname{span} R$ by

$$\begin{aligned} h'_c(r) = \min \quad & c_0 s_0 + c^\top s \\ & r_0 s_0 + Rs = r, \\ & s_0 \geq 0,\ s \geq 0. \end{aligned} \tag{4.6}$$

The function h'_c is real-valued, piecewise-linear, and sublinear on $\operatorname{span} R$ as a consequence of Lemma 4.4 applied to the matrix $[r_0, R]$ and the inequality $c_0 s_0 + c^\top s \geq 1$ which is valid for $\mathbb{X}([r_0, R], \mathbb{S}')$ by Proposition 4.8.

Lemma 4.9. *Let* $\mathbb{X} \subset \mathbb{R}^k$ *be a nonempty set defined as in* (4.1)*. Consider a valid inequality* $c^\top s \geq 1$ *for* $\mathbb{X}$*. Let* $h_c, h'_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be defined as in* (4.3) *and* (4.6)*, respectively. The function* h'_c *coincides with* h_c *on* $\operatorname{cone} R$*.*

Proof. It is clear from the definitions (4.3) and (4.6) that $h'_c \leq h_c$ on $\operatorname{span} R$. Let $\bar{r} \in \operatorname{cone} R$ and suppose $h'_c(\bar{r}) < h_c(\bar{r})$. Then there exists $(\bar{s}_0, \bar{s})$ satisfying $r_0\bar{s}_0 + R\bar{s} = \bar{r}$, $\bar{s} \geq 0$, $\bar{s}_0 > 0$, and $c_0\bar{s}_0 + c^\top\bar{s} < h_c(\bar{r})$. Rearranging the terms and using Remark 4.3(i), we obtain

$$c_0 < \frac{h_c(\bar{r}) - c^\top \bar{s}}{\bar{s}_0} \leq \frac{h_c(\bar{r}) - \sum_{j=1}^k h_c(r^j)\bar{s}_j}{\bar{s}_0}.$$

Finally, the sublinearity of h_c and the observation that $R\bar{s} = \bar{r} - r_0\bar{s}_0$ give

$$c_0 < \frac{h_c(\bar{r}) - \sum_{j=1}^k h_c(r^j)\bar{s}_j}{\bar{s}_0} \leq \frac{h_c(\bar{r}) - h_c(R\bar{s})}{\bar{s}_0} = \frac{h_c(\bar{r}) - h_c(\bar{r} - r_0\bar{s}_0)}{\bar{s}_0}.$$

This contradicts the definition of c_0 and proves the claim. □

Lemma 4.9 and Remark 4.3 yield the following corollary.

Corollary 4.10. *Let* $\mathbb{X} \subset \mathbb{R}^k$ *be a nonempty set defined as in* (4.1)*. Consider a valid inequality* $c^\top s \geq 1$ *for* $\mathbb{X}$*. Let* $h'_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be defined as in* (4.6)*.*

i. $h'_c(r^j) \leq c_j$ *for all* $j \in [k]$*.*
ii. Suppose $\mathbb{S}' \subset \operatorname{cone} R$*. Then* $h'_c(\bar{r}) \geq 1$ *for all* $\bar{r} \in \mathbb{S}'$*.*

If $\operatorname{rank}(R) < n$, we extend the function h'_c defined in (4.6) to the whole of $\mathbb{R}^n$ by letting

$$h'_c(r) = h'_c(r') \text{ for all } r \in \mathbb{R}^n,\ r' \in \operatorname{span} R,\ r'' \in (\operatorname{span} R)^\perp \text{ such that } r = r' + r''. \tag{4.7}$$

Note that this extension preserves the sublinearity of h'_c.

Proof of Theorem 4.1. Let h'_c be defined as in (4.6) and (4.7), and let $\mathbb{V}'_c = \{r \in \mathbb{R}^n : h'_c(r) \leq 1\}$. Observe that $\mathbb{V}'_c$ is a closed, convex neighborhood of the origin because h'_c is sublinear and finite everywhere, and $h'_c(0) = 0$. Furthermore, $\operatorname{int} \mathbb{V}'_c = \{r \in \mathbb{R}^n : h'_c(r) < 1\}$ by the Slater property $h'_c(0) = 0$. This implies that $\mathbb{V}'_c$ is also $\mathbb{S}'$-free since $h'_c(\bar{r}) \geq 1$ for all $\bar{r} \in \mathbb{S}'$ by Corollary 4.10(ii). The function h'_c is a cut-generating function because it represents $\mathbb{V}'_c$, and $\sum_{j=1}^k h'_c(r^j)s_j \geq 1$ is an $\mathbb{S}'$-intersection cut. By Corollary 4.10(i), $h'_c(r^j) \leq c_j$ for all $j \in [k]$. This shows that the $\mathbb{S}'$-intersection cut $\sum_{j=1}^k h'_c(r^j)s_j \geq 1$ dominates $c^\top s \geq 1$. □

4.3 Constructing the $\mathbb{S}'$-Free Convex Neighborhood of the Origin

Here we give a geometric interpretation for the proof of Theorem 4.1 and explicitly describe the $\mathbb{S}'$-free neighborhood of the origin $\mathbb{V}'_c = \{r \in \mathbb{R}^n : h'_c(r) \le 1\}$ in terms of the vectors $r^1, \ldots, r^k$.

As in Section 4.2, we let $c^\top s \ge 1$ be a valid inequality separating the origin from $\mathbb{X}$. Assume without any loss of generality that the vectors $r^1, \ldots, r^k$ have been normalized so that $c_j \in \{0, \pm 1\}$ for all $j \in [k]$. Define the sets $\mathbb{J}_+ = \{j \in [k] : c_j = +1\}$, $\mathbb{J}_- = \{j \in [k] : c_j = -1\}$, and $\mathbb{J}_0 = \{j \in [k] : c_j = 0\}$. Let $\mathbb{C} = \operatorname{conv}(\{0\} \cup \{r^j : j \in \mathbb{J}_+\})$ and $\mathbb{K} = \operatorname{cone}(\{r^j : j \in \mathbb{J}_0 \cup \mathbb{J}_-\} \cup \{r^j + r^i : j \in \mathbb{J}_+, i \in \mathbb{J}_-\})$. Let $\mathbb{A} = \mathbb{C} + \mathbb{K}$. Defining h_c as in (4.3), one can show $\mathbb{A} = \{r \in \mathbb{R}^n : h_c(r) \le 1\}$.

When $\operatorname{cone} R \ne \mathbb{R}^n$, the origin lies on the boundary of $\mathbb{A}$. This happens in the example of Figure 4.1. In the proof of Theorem 4.1, we overcame the difficulty occurring when $\operatorname{cone} R \ne \mathbb{R}^n$ by extending h_c into a function h'_c which is defined on the whole of $\mathbb{R}^n$ and coincides with h_c on $\operatorname{cone} R$. The geometric counterpart is to extend the set $\mathbb{A}$ into a set $\mathbb{A}'$ that contains the origin in its interior. Let $r_0 \in -\operatorname{ri}(\operatorname{cone} R)$ and let c_0 be as defined in (4.5). When $c_0 \ne 0$, scale r_0 so that $c_0 \in \{\pm 1\}$ (this is possible by Remark 4.7). Introduce r_0 into the relevant subset of $[k]$ according to the sign of c_0: If $c_0 = +1$, let $\mathbb{J}'_+ = \mathbb{J}_+ \cup \{0\}$, $\mathbb{J}'_0 = \mathbb{J}_0$, and $\mathbb{J}'_- = \mathbb{J}_-$; if $c_0 = 0$, let $\mathbb{J}'_+ = \mathbb{J}_+$, $\mathbb{J}'_0 = \mathbb{J}_0 \cup \{0\}$, and $\mathbb{J}'_- = \mathbb{J}_-$; and if $c_0 = -1$, let $\mathbb{J}'_+ = \mathbb{J}_+$, $\mathbb{J}'_0 = \mathbb{J}_0$, and $\mathbb{J}'_- = \mathbb{J}_- \cup \{0\}$. Finally, let $\mathbb{C}' = \operatorname{conv}(\{0\} \cup \{r^j : j \in \mathbb{J}'_+\})$, $\mathbb{K}' = \operatorname{cone}(\{r^j : j \in \mathbb{J}'_0 \cup \mathbb{J}'_-\} \cup \{r^j + r^i : j \in \mathbb{J}'_+, i \in \mathbb{J}'_-\})$, and

$$\mathbb{A}' = \mathbb{C}' + \mathbb{K}' + (\operatorname{span} R)^\perp. \tag{4.8}$$

The example below illustrates this procedure for the cases $c_0 = +1$ and $c_0 = -1$.

Example 4.1. Let $R = [r^1, r^2, r^3]$ be a 2×3 real matrix where $r^1 = (1, 3)$, $r^2 = (1.5, 1.5)$, and $r^3 = (2, -1)$. Let $c \in \mathbb{R}^3$ where $c_1 = c_2 = +1$ and $c_3 = -1$. The shaded region in Figure 4.1 is the set $\mathbb{A}$. In Figure 4.2 we add the vector $r_0 = (-5, -1)$ to the collection of vectors $\{r^1, r^2, r^3\}$. The new vector r_0 has $c_0 = +1$. Its addition expands $\mathbb{A}$ to the set $\mathbb{A}'$ that is depicted. In Figure 4.3 we add the vector $r_0 = (-4, -5)$ with $c_0 = -1$ to the original collection and again obtain $\mathbb{A}'$.

The following proposition shows that the function h'_c defined in (4.6) and (4.7) represents the set $\mathbb{A}'$ defined in (4.8) above.

Proposition 4.11. *Let $\mathbb{X} \subset \mathbb{R}^k$ be a nonempty set defined as in* (4.1)*. Consider a valid inequality $c^\top s \ge 1$ for $\mathbb{X}$. Let $h'_c : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be defined as in* (4.6) *and* (4.7)*. Let $\mathbb{A}' \subset \mathbb{R}^n$ be defined as in* (4.8)*. Then $\mathbb{A}' = \{r \in \mathbb{R}^n : h'_c(r) \le 1\}$.*

Proof. Let $\mathbb{V}'_c = \{r \in \mathbb{R}^n : h'_c(r) \leq 1\}$. Note that $\mathbb{V}'_c$ is convex by the sublinearity of h'_c. We have $h'_c(r^j) \leq c_j = 1$ for all $j \in \mathbb{J}'_+$, $h'_c(r^j) \leq c_j \leq 0$ for all $j \in \mathbb{J}'_0 \cup \mathbb{J}'_-$, and $h'_c(r^j + r^i) \leq h'_c(r^j) + h'_c(r^i) \leq c_j + c_i = 0$ for all $j \in \mathbb{J}'_+$ and $i \in \mathbb{J}'_-$. Moreover, $h'_c(r) = h'_c(r + r')$ for all $r \in \mathbb{R}^n$ and $r' \in (\operatorname{span} R)^\perp$ by the definition of h'_c. Hence, $\mathbb{C}' \subset \mathbb{V}'_c$, $\mathbb{K}' \subset \operatorname{rec} \mathbb{V}'_c$, and $(\operatorname{span} R)^\perp \subset \operatorname{lin} \mathbb{V}'_c$, which together give us $\mathbb{A}' = \mathbb{C}' + \mathbb{K}' + (\operatorname{span} R)^\perp \subset \mathbb{V}'_c$.

To prove the converse, let $\bar{r} \in \mathbb{R}^n$ be such that $h'_c(\bar{r}) \leq 1$. We need to show $\bar{r} \in \mathbb{A}'$. We consider two distinct cases: $h'_c(\bar{r}) \leq 0$ and $0 < h'_c(\bar{r}) \leq 1$. First, let us suppose $h'_c(\bar{r}) \leq 0$. Then the definition of h'_c implies that there exist $(\bar{s}_0, \bar{s}) \in \mathbb{R} \times \mathbb{R}^k$ and $\bar{r}' \in (\operatorname{span} R)^\perp$ such that $(\bar{s}_0, \bar{s}) \geq 0$, $\sum_{j \in \mathbb{J}'_+} \bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i \leq 0$, and $r_0 \bar{s}_0 + R\bar{s} = \bar{r} - \bar{r}'$. Consider the cone $\mathbb{F} = \{(\bar{s}_0, \bar{s}) \geq 0 : \sum_{j \in \mathbb{J}'_+} \bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i \leq 0\}$ defined by the first two sets of inequalities. The extreme rays of $\mathbb{F}$ have all their components equal to 0 except for one or two components. Therefore, it is easy to verify by inspection that $\mathbb{F}$ is generated by the rays $\{e^j : j \in \mathbb{J}'_0 \cup \mathbb{J}'_-\} \cup \{e^j + e^i : j \in \mathbb{J}'_+, i \in \mathbb{J}'_-\}$. This shows $\bar{r} \in \mathbb{K}' + (\operatorname{span} R)^\perp \subset \mathbb{A}'$. Now suppose $0 < h'_c(\bar{r}) \leq 1$. Then there exist $(\bar{s}_0, \bar{s}) \in \mathbb{R} \times \mathbb{R}^k$ and $\bar{r}' \in (\operatorname{span} R)^\perp$ such that $(\bar{s}_0, \bar{s}) \geq 0$, $0 < \sum_{j \in \mathbb{J}'_+} \bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i \leq 1$, and $r_0 \bar{s}_0 + R\bar{s} = \bar{r} - \bar{r}'$. Define $\bar{s}_i^j = \bar{s}_i \frac{\bar{s}_j}{\sum_{j \in \mathbb{J}'_+} \bar{s}_j}$ for all $i \in \mathbb{J}'_-$ and $j \in \mathbb{J}'_+$. These values are well-defined since $0 \leq \sum_{i \in \mathbb{J}'_-} \bar{s}_i < \sum_{j \in \mathbb{J}'_+} \bar{s}_j$. Observe that $\sum_{j \in \mathbb{J}'_+} \bar{s}_i^j = \bar{s}_i$ and $r_0 \bar{s}_0 + R\bar{s} = \sum_{j \in \mathbb{J}'_+} (\bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i^j) r^j + \sum_{i \in \mathbb{J}'_-} \sum_{j \in \mathbb{J}'_+} \bar{s}_i^j (r^i + r^j) + \sum_{j \in \mathbb{J}'_0} \bar{s}_j r_j$. We have $\sum_{j \in \mathbb{J}'_+} (\bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i^j) = \sum_{j \in \mathbb{J}'_+} \bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i \leq 1$ together with $\bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i^j > 0$ which is true for all $j \in \mathbb{J}'_+$ because $\sum_{i \in \mathbb{J}'_-} \bar{s}_i^j = \bar{s}_j \frac{\sum_{i \in \mathbb{J}'_-} \bar{s}_i}{\sum_{j \in \mathbb{J}'_+} \bar{s}_j} < \bar{s}_j$. Hence, $\sum_{j \in \mathbb{J}'_+} (\bar{s}_j - \sum_{i \in \mathbb{J}'_-} \bar{s}_i^j) r^j \in \mathbb{C}'$. Moreover, $\sum_{i \in \mathbb{J}'_-} \sum_{j \in \mathbb{J}'_+} \bar{s}_i^j (r^i + r^j) + \sum_{j \in \mathbb{J}'_0} \bar{s}_j r^j \in \mathbb{K}'$. These yield $\bar{r} \in \mathbb{C}' + \mathbb{K}' + (\operatorname{span} R)^\perp = \mathbb{A}'$. $\square$

As a consequence, the set $\mathbb{A}'$ can be used to generate an $\mathbb{S}'$-intersection cut that dominates $c^\top s \geq 1$. Indeed, the proof of Theorem 4.1 shows that $\mathbb{V}'_c = \{r \in \mathbb{R}^n : h'_c(r) \leq 1\}$ is a closed, convex, $\mathbb{S}'$-free neighborhood of the origin. Proposition 4.11 shows that $\mathbb{A}' = \mathbb{V}'_c$. Therefore, $\sum_{j=1}^k h'_c(r^j) s_j \geq 1$ is an $\mathbb{S}'$-intersection cut obtained from $\mathbb{A}'$.

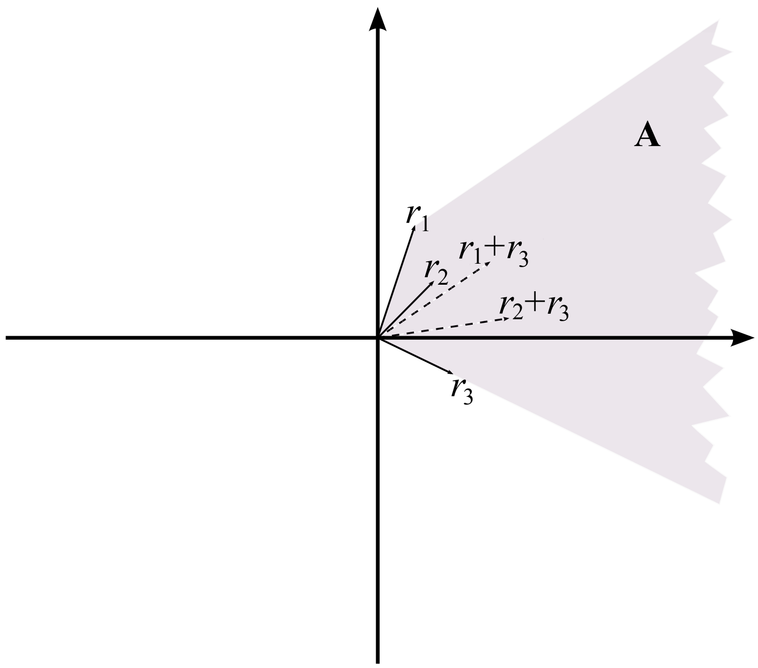


Figure 4.1: The set $\mathbb{A}$ for Example 4.1.

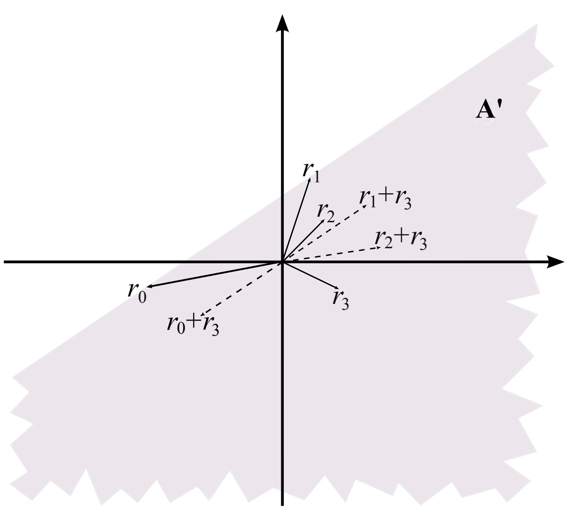


Figure 4.2: The set $\mathbb{A}$ is expanded to $\mathbb{A}'$ after the addition of $r_0 = (-5, -1)$.

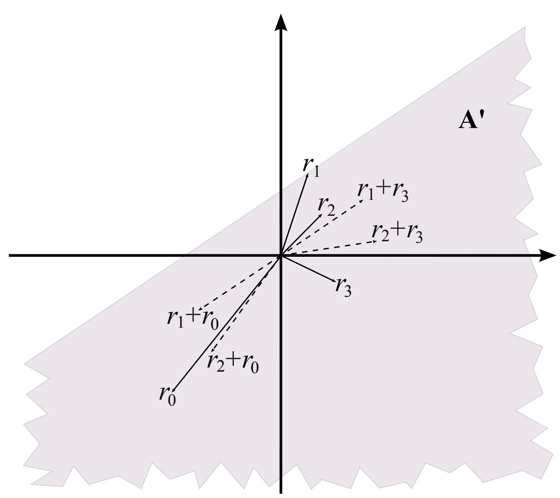


Figure 4.3: The set $\mathbb{A}$ is expanded to $\mathbb{A}'$ after the addition of $r_0 = (-4, -5)$.

Chapter 5

Two-Term Disjunctions on Regular Cones

Acknowledgments. This chapter is based on joint work with Fatma Kılınç-Karzan [84, 108]. A preliminary version of [84] appeared in [83].

5.1 Introduction

5.1.1 Motivation

Let $\mathbb{E}$ be a finite-dimensional Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$. Let $\mathbb{K} \subset \mathbb{E}$ be a regular (full-dimensional, closed, convex, and pointed) cone. In Chapters 5, 6, 7, and 8, we consider non-convex sets which result from the application of a linear two-term disjunction on a set of the form

$$\mathbb{C} = \{x \in \mathbb{K} : \quad \mathcal{A}x = b\}, \tag{5.1}$$

where $\mathcal{A} : \mathbb{E} \to \mathbb{R}^m$ is a linear map and $b \in \mathbb{R}^m$. Specifically, we consider a disjunction $\langle c_1, x \rangle \geq c_{1,0} \vee \langle c_2, x \rangle \geq c_{2,0}$ on $\mathbb{C}$. In reference to the disjunction, we define the sets

$$\mathbb{C}_i = \{x \in \mathbb{C} : \langle c_i, x \rangle \geq c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}. \tag{5.2}$$

The purpose of this chapter is to analyze the structure of the closed convex hull of the disjunctive conic set $\mathbb{C}_1 \cup \mathbb{C}_2$ under minimal assumptions on the structure of $\mathbb{K}$. We provide linear and nonlinear valid inequalities which explicitly describe this closed convex hull in the space of the original variables. We also develop various techniques for constructing low-complexity convex relaxations of $\mathbb{C}_1 \cup \mathbb{C}_2$ in the same space.

Sets of the form $\mathbb{C}_1 \cup \mathbb{C}_2$ are at the core of convex optimization based solution methods to conic programs with integrality requirements on the variables and other types of non-convex constraints. In the context of mixed-integer conic programs (MICPs), integrality conditions are naturally relaxed into disjunctions satisfied by all feasible solutions; convex inequalities that are valid for the resulting non-convex sets can then be added to the problem formulation to obtain a tighter description of the integer hull. Such inequalities are known as *disjunctive inequalities* [14]. We comment further on the use of disjunctive inequalities in the solution of MICPs in Section 5.1.2. In addition, two-term disjunctions are closely related to non-convex sets defined by rank-two quadratics of the form

$$\mathbb{X} = \{x \in \mathbb{E} : (c_{1,0} - \langle c_1, x\rangle)(c_{2,0} - \langle c_2, x\rangle) \le 0\}.$$

For instance, given that there does not exist any point $x \in \mathbb{K}$ which satisfies both $\langle c_1, x\rangle \ge c_{1,0}$ and $\langle c_2, x\rangle \ge c_{2,0}$ strictly, a two-term disjunction on $\mathbb{K}$ can be represented using the set $\mathbb{X}$: $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K} \cap \mathbb{X}$. We explore this relationship further in Section 5.2.2.

A conic program is the problem of optimizing a linear function over the intersection of a regular cone with an affine subspace. An MICP is a conic program where some decision variables are constrained to take integer values. In the special case where the regular cone which underlies the problem is a nonnegative orthant, MICPs reduce to mixed-integer linear programs (MILPs). The combined representation power of integer variables and conic constraints makes MICPs an attractive framework for modeling optimization problems which require discrete decisions. Following the development of stable and efficient algorithms for solving second-order cone programs and semidefinite programs, MICPs with second-order cone and positive semidefinite cone constraints have received significant attention in the recent years. These problems find applications in optimization under uncertainty as well as in engineering design and statistical learning. The reader is referred to Section 1.2 for a discussion of the applications of MICPs. Motivated by these applications, in Chapters 6, 7, and 8 we place special emphasis on the cases where $\mathbb{K}$ is the nonnegative orthant, the second-order cone, the positive semidefinite cone, or one of their direct products.

5.1.2 Related Work

Disjunctive inequalities, introduced in the early 1970s in the context of MILPs [14], are a main ingredient of today's successful integer programming technology. In their most general form, disjunctive inequalities are inequalities that are valid for disjunctions on a convex relaxation of an integer program. Despite their simplicity, the most powerful disjunctions in integer programming are *split disjunctions* of the form $\langle c_1, x\rangle \ge c_{1,0} \vee \langle c_2, x\rangle \ge c_{2,0}$ where the inequalities $\langle c_1, x\rangle \ge c_{1,0}$ and $\langle c_2, x\rangle \ge c_{2,0}$ define opposing, disjoint halfspaces. Disjunctive inequalities derived using split disjunctions are called *split inequali-*

ties [48]. Some of the most well-known families of cutting-planes for MILPs are split inequalities: Chvátal-Gomory inequalities [41, 61], Gomory mixed-integer inequalities [62], mixed-integer rounding inequalities [92], lift-and-project inequalities [16]... More general two-term disjunctions are used for complementarity problems [77, 101] and integer programs with non-convex quadratic constraints [24, 39]. When $\mathbb{K}$ is the nonnegative orthant, Bonami et al. [34] characterized the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ with a finite set of linear inequalities. There has been a lot of recent interest in extending the theory of disjunctive inequalities from the setting of MILPs to that of MICPs. Stubbs and Mehrotra [99, 100] generalized lift-and-project inequalities to mixed-integer convex programs with 0-1 variables. Çezik and Iyengar [40] investigated Chvátal-Gomory inequalities for pure-integer conic programs and lift-and-project inequalities for mixed-integer conic programs with 0-1 variables. Kılınç, Linderoth, and Luedtke [79] and Bonami [33] suggested improved methods for generating lift-and-project inequalities for mixed-integer convex programs. Atamtürk and Narayanan [11] presented a method to lift a valid conic inequality for a low-dimensional restriction of a mixed-integer conic set into a valid conic inequality for the actual mixed-integer set.

The set $\mathbb{C}_1 \cup \mathbb{C}_2$ exemplifies the simplest form of a disjunctive conic set, as defined by Kılınç-Karzan [80]. In this model, the underlying cone $\mathbb{K}$ of the disjunctive conic set determines a hierarchy for valid linear inequalities in terms of their dominance relations. Kılınç-Karzan studied valid linear inequalities which are minimal with respect to this hierarchy and showed that these inequalities generate the associated closed convex hulls under a mild technical condition which is also satisfied in our setup. Bienstock and Michalka [30] studied the characterization and separation of linear inequalities which are valid for the epigraph of a convex, differentiable function restricted to a non-convex domain. While a regular cone, which provides the base convex set for our disjunctions in Chapters 5, 6, and 7, can be seen as the epigraph of a convex function, this function is not differentiable. On the other hand, certain cross-sections of the second-order cone, which we consider in Chapter 8, correspond to epigraphs of convex, differentiable functions. Nevertheless, we note that in both cases two-term disjunctions on the *domain* of these functions are more limited than those we analyze.

As a special class of MICPs, mixed-integer second-order cone programs (MISOCPs) have received particular attention. Atamtürk and Narayanan [10] extended mixed-integer rounding inequalities to MISOCPs. See also [103] for a generalization of this to mixed-integer p-order cone programs. Drewes [56] analyzed Chvátal-Gomory and lift-and-project inequalities for MISOCPs. Drewes and Pokutta [57, 58] devised a lift-and-project cutting-plane framework for MISOCPs with a special structure. In the last few years, several authors investigated the problem of representing the closed convex convex hull of a two-term disjunction on the second-order cone or one of its affine cross-sections in the space of the original variables with closed-form convex inequalities. Dadush et al. [52] and Andersen and Jensen [8] derived split inequalities for ellipsoids and the second-order cone, respectively. Modaresi et al. extended this work on split disjunctions to essentially all cross-sections of the second-order cone in [89] and studied their relationship with extended formulations and conic mixed-integer rounding inequalities in [90]. Belotti et al. [25] studied the families of quadratic surfaces that have fixed intersections with two given hyperplanes and showed that these families can be described by a single parameter. Building on this, in [27] they identified a procedure for constructing two-term disjunctive inequalities under the assumptions that $\mathbb{C}_1 \cap \mathbb{C}_2 = \emptyset$ and the sets $\{x \in \mathbb{C} : \langle c_1, x\rangle = c_{1,0}\}$ and $\{x \in \mathbb{C} : \langle c_2, x\rangle = c_{2,0}\}$ are bounded.

Recently, results about two-term disjunctions on the second-order cone and its cross-sections have been extended to intersections of the second-order cone or its affine cross-sections with a single homogeneous quadratic [38, 88]. To the best of our knowledge, none of the papers from the existing literature provide closed convex hull characterizations of two-term disjunctions on the positive semidefinite cone in the space of the original variables.

5.1.3 Notation and Terminology

In this chapter, we consider a finite-dimensional Euclidean space $\mathbb{E}$ equipped with the inner product $\langle\cdot,\cdot\rangle$. If $\mathbb{E}$ is a direct product $\mathbb{E} = \prod_{j=1}^{p} \mathbb{E}^j$ of lower-dimensional Euclidean spaces $\mathbb{E}^j$, we define $\langle\cdot,\cdot\rangle$ as the sum of individual inner products $\langle\cdot,\cdot\rangle_j$ on $\mathbb{E}^j$. We assume that $\mathbb{R}^n$ is equipped with the inner product $\langle\alpha, x\rangle = \alpha^\top x$. The (standard) Euclidean norm $\|\cdot\| : \mathbb{E} \to \mathbb{R}$ on $\mathbb{E}$ is defined as $\|x\| = \sqrt{\langle x, x\rangle}$. For any positive integer k, we let $[k] = \{1, \ldots, k\}$. For $i \in [n]$, we let e^i be the i-th unit vector in $\mathbb{R}^n$.

Throughout the chapter, we consider a regular cone $\mathbb{K} \subset \mathbb{E}$. In the case where $\mathbb{E} = \prod_{j=1}^{p} \mathbb{E}^j$, if each $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone for each $j \in [p]$, then the direct product $\mathbb{K} = \prod_{j=1}^{p} \mathbb{K}^j$ is also a regular cone in $\mathbb{E}$. The *dual cone* of $\mathbb{V} \subset \mathbb{E}$ is $\mathbb{V}^* = \{\alpha \in \mathbb{E} : \langle x, \alpha\rangle \geq 0\ \forall x \in \mathbb{V}\}$. The dual cone $\mathbb{K}^*$ of a regular cone $\mathbb{K}$ is also regular, and the dual of $\mathbb{K}^*$ is $\mathbb{K}$ itself. If $\mathbb{K} = \prod_{j=1}^{p} \mathbb{K}^j$, then $\mathbb{K}^* = \prod_{j=1}^{p} (\mathbb{K}^j)^*$. Given a set $\mathbb{V} \subset \mathbb{E}$, we let $\operatorname{conv} \mathbb{V}$, $\overline{\operatorname{conv}}\, \mathbb{V}$, $\operatorname{int} \mathbb{V}$, and $\operatorname{bd} \mathbb{V}$ denote the convex hull, closed convex hull, topological interior, and boundary of $\mathbb{V}$, respectively. We use $\operatorname{rec} \mathbb{V}$ to refer to the recession cone of a closed convex set $\mathbb{V}$.

5.1.4 Outline of the Chapter

The remainder of this chapter is organized as follows: In Section 5.2 we introduce the basic elements of our study. In Section 5.2.1 we define the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ and identify the setup for our analysis with Conditions 5.1 and 5.2. Condition 5.1 is a natural assumption for this study, whereas Condition 5.2 is only needed in results which provide complete closed convex hull characterizations of $\mathbb{C}_1 \cup \mathbb{C}_2$. We discuss the pathologies that arise in the absence of Condition 5.2 in Section 5.3.3. In Section 5.2.2, we establish a connection between two-term disjunctions on $\mathbb{C}$ and non-convex sets $\mathbb{C} \cap \mathbb{X}$ defined by simple quadratics; we show that this connection carries over to convex hulls of these sets as well.

In Section 5.3 we consider two-term disjunctions on a general regular cone $\mathbb{K}$. It is a well-known fact from convex analysis that the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ can be described with linear inequalities alone. However, the set of linear inequalities that are valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ is typically very large, and only a small subset of these are needed in a description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ besides the cone constraint $x \in \mathbb{K}$. In Section 5.3.1, for a two-term disjunction on a regular cone $\mathbb{K}$, we identify and characterize the structure of a subset of strong valid linear inequalities which, along with the cone constraint $x \in \mathbb{K}$, are sufficient to describe the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. These inequalities are tight on $\mathbb{C}_1 \cup \mathbb{C}_2$ and $\mathbb{K}$-minimal in the sense defined in [80]. We term such linear inequalities "undominated" in this chapter. In Section 5.3.2 we identify and study certain cases where the characterization of undominated valid linear inequalities can be refined further.

In Section 5.4 we develop structured nonlinear valid inequalities for the sets under consideration through conic programming duality. In Section 5.4.1, we consider two-term disjunctions on a regular cone $\mathbb{K}$. We formulate the general form of a family of convex inequalities that are valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ and explore their structure in detail. The refined linear inequality characterization of Section 5.3.2 guarantees that a single convex inequality from this family characterizes the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ under certain conditions. In Section 5.4.2, using the connection established in Section 5.2.2 between two-term disjunctions and non-convex sets defined by rank-two quadratics, we develop convex valid inequalities and closed convex hull descriptions for sets of the form $\mathbb{K} \cap \mathbb{X}$. In Section 5.4.3 we show how the results of Section 5.4.1 can be strengthened when $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy a certain disjointness condition.

We note that our results on disjunctions on regular cones easily extend to disjunctions on homogeneous cross-sections of regular cones if we work in the linear subspace which defines the cross-section.

5.2 Preliminaries

5.2.1 Two-Term Disjunctions on Convex Sets

Let $\mathbb{C} \subset \mathbb{E}$ be defined as in (5.1). In this section we start our analysis of the set $\mathbb{C}_1 \cup \mathbb{C}_2$ and its closed convex hull, where $\mathbb{C}_1$ and $\mathbb{C}_2$ are defined as in (5.2). We first describe some conditions which will simplify our analysis of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$.

The inequalities $\langle c_1, x\rangle \geq c_{1,0}$ and $\langle c_2, x\rangle \geq c_{2,0}$ can always be scaled so that their right-hand sides are 0 or ± 1. Therefore, from now on we assume $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ for convenience. Furthermore, $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{C}_2$ when $\mathbb{C}_1 \subset \mathbb{C}_2$, and $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{C}_1$ when $\mathbb{C}_1 \supset \mathbb{C}_2$. In both cases, the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ has an immediate description. In the remainder we assume $\mathbb{C}_1 \not\subset \mathbb{C}_2$ and $\mathbb{C}_1 \not\supset \mathbb{C}_2$.

Condition 5.1. $\mathbb{C}_1 \not\subset \mathbb{C}_2$ *and* $\mathbb{C}_1 \not\supset \mathbb{C}_2$.

In particular, Condition 5.1 implies $\mathbb{C}_1, \mathbb{C}_2 \neq \emptyset$ and $\mathbb{C}_1, \mathbb{C}_2 \neq \mathbb{C}$. Condition 5.1 has a simple implication which we state next. The lemma extends ideas from Balas [15] to disjunctions on more general convex sets.

Lemma 5.1. *Let* $\mathbb{C} \subset \mathbb{E}$ *be a convex set. Consider* $\mathbb{C}_i = \{x \in \mathbb{C} : \langle c_i, x\rangle \geq c_{i,0}\}$ *for* $i \in \{1, 2\}$. *Suppose* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *satisfy Condition 5.1.*

i. The set $\mathbb{C}_1 \cup \mathbb{C}_2$ *is not convex unless* $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{C}$,

ii. If $\mathbb{C}$ *is closed and pointed, then* $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \text{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+)$ *where* $\mathbb{C}_1^+ = \mathbb{C}_1 + \text{rec}\, \mathbb{C}_2$ *and* $\mathbb{C}_2^+ = \mathbb{C}_2 + \text{rec}\, \mathbb{C}_1$.

Proof. To prove (i), suppose $\mathbb{C}_1 \cup \mathbb{C}_2 \subsetneq \mathbb{C}$ and pick $x_0 \in \mathbb{C} \setminus (\mathbb{C}_1 \cup \mathbb{C}_2)$. Also, pick $x_1 \in \mathbb{C}_1 \setminus \mathbb{C}_2$ and $x_2 \in \mathbb{C}_2 \setminus \mathbb{C}_1$. Let x' be the point on the line segment between x_0 and x_1 such that $\langle c_1, x'\rangle = c_{1,0}$. Similarly, let x'' be the point between x_0 and x_2 such that $\langle c_2, x''\rangle = c_{2,0}$. Note that $x' \notin \mathbb{C}_2$ and $x'' \notin \mathbb{C}_1$ by the convexity of $\mathbb{C} \setminus \mathbb{C}_1$ and $\mathbb{C} \setminus \mathbb{C}_2$. Then a point that is a strict convex combination of x' and x'' is in the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ but not in $\mathbb{C}_1 \cup \mathbb{C}_2$.

Now we prove (ii). Corollary 9.1.2 in [96] implies $\mathbb{C}_1^+$ and $\mathbb{C}_2^+$ are closed and $\text{rec}\, \mathbb{C}_1^+ = \text{rec}\, \mathbb{C}_2^+ = \text{rec}\, \mathbb{C}_1 + \text{rec}\, \mathbb{C}_2$ because $\mathbb{C}$ is pointed. The inclusions $\mathbb{C}_1 \subset \mathbb{C}_1^+$ and $\mathbb{C}_2 \subset \mathbb{C}_2^+$ imply that $\text{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) \subset \text{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+)$. Furthermore, the convex hull of $\mathbb{C}_1^+ \cup \mathbb{C}_2^+$ is closed by Corollary 9.8.1 in [96] since $\mathbb{C}_1^+$ and $\mathbb{C}_2^+$ have the same recession cone. Hence, $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) \subset \text{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+)$. We claim $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \text{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+)$. Let $x^+ \in \text{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+)$. Then there exist $u_1 \in \mathbb{C}_1$, $v_2 \in \text{rec}\, \mathbb{C}_2$, $u_2 \in \mathbb{C}_2$, and $v_1 \in \text{rec}\, \mathbb{C}_1$ such that $x^+ \in \text{conv}\{u_1 + v_2, u_2 + v_1\}$. To prove the claim, it is enough to show that $u_1 + v_2, u_2 + v_1 \in \overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$. Consider the point $u_1 + v_2$ and the sequence

$$\left\{\left(1 - \frac{1}{k}\right) u_1 + \frac{1}{k}\left(u_2 + k v_2\right)\right\}_{k=1}^{\infty}.$$

For any $k > 0$, we have $u_1 \in \mathbb{C}_1$ and $u_2 + kv_2 \in \mathbb{C}_2$. Therefore, the sequence above is contained in the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. Furthermore, it converges to $u_1 + v_2$ as $k \to \infty$ which implies $u_1 + v_2 \in \overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$. A similar argument shows $u_2 + v_1 \in \overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$ and proves the claim. □

We also use the following additional technical condition in some of our results.

Condition 5.2. *$\mathbb{C}_1$ and $\mathbb{C}_2$ are strictly feasible. That is, the sets $\mathbb{C}_1 \cap \operatorname{int} \mathbb{K}$ and $\mathbb{C}_2 \cap \operatorname{int} \mathbb{K}$ are nonempty.*

Throughout the chapter, we are interested in sets $\mathbb{C}_1$ and $\mathbb{C}_2$ which are defined as in (5.2). If $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Conditions 5.1 and 5.2, we say that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the *basic disjunctive setup*.

5.2.2 Intersection of a Convex Set with Non-Convex Rank-Two Quadratics

In this section we consider the set $\mathbb{C} \cap \mathbb{X}$ where

$$\mathbb{X} = \{x \in \mathbb{E} :\ (c_{1,0} - \langle c_1, x\rangle)(c_{2,0} - \langle c_2, x\rangle) \le 0\} \tag{5.3}$$

is a non-convex set defined by a rank-two quadratic inequality. As in Section 5.2.1, we can assume without any loss of generality that $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$. Under a disjointness assumption, the two-term disjunction $\langle c_1, x\rangle \ge c_{1,0} \vee \langle c_2, x\rangle \ge c_{2,0}$ on $\mathbb{C}$ can be written as the intersection of $\mathbb{C}$ with the non-convex set $\mathbb{X}$. We discuss this connection further in Section 5.4.3.

Note that $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2$ where

$$\begin{aligned}\mathbb{X}_1 &= \{x \in \mathbb{E} :\ \langle c_1, x\rangle \ge c_{1,0},\ \langle c_2, x\rangle \le c_{2,0}\},\\ \mathbb{X}_2 &= \{x \in \mathbb{E} :\ \langle c_1, x\rangle \le c_{1,0},\ \langle c_2, x\rangle \ge c_{2,0}\}.\end{aligned}$$

Associated with $\mathbb{X}, \mathbb{C} \subset \mathbb{E}$, we define the sets $\mathbb{C}_i^+, \mathbb{C}_i^- \subset \mathbb{E}$ where

$$\mathbb{C}_i^+ = \{x \in \mathbb{C} : \langle c_i, x\rangle \ge c_{i,0}\}, \quad \mathbb{C}_i^- = \{x \in \mathbb{C} : \langle c_i, x\rangle \le c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}. \tag{5.4}$$

Then $\mathbb{C} \cap \mathbb{X}_1 = \mathbb{C}_1^+ \cap \mathbb{C}_2^-$ and $\mathbb{C} \cap \mathbb{X}_2 = \mathbb{C}_1^- \cap \mathbb{C}_2^+$. Furthermore, $\mathbb{C} \cap \mathbb{X}$ equals the intersection of $\mathbb{C}_1^+ \cup \mathbb{C}_2^+$ and $\mathbb{C}_1^- \cup \mathbb{C}_2^-$. In Proposition 5.2 below, we show that the *convex hull* of $\mathbb{C} \cap \mathbb{X}$ equals the intersection of the convex hulls of $\mathbb{C}_1^+ \cup \mathbb{C}_2^+$ and $\mathbb{C}_1^- \cup \mathbb{C}_2^-$.

Proposition 5.2. *Let $\mathbb{C} \subset \mathbb{E}$ be a convex set. Let $\mathbb{X} \subset \mathbb{E}$ and $\mathbb{C}_i^+, \mathbb{C}_i^- \subset \mathbb{E}$ be defined as in* (5.3) *and* (5.4)*, respectively.*

i. $\mathrm{conv}(\mathbb{C} \cap \mathbb{X}) = \mathrm{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$.

ii. If $\mathbb{C}$ is closed, then $\overline{\mathrm{conv}}(\mathbb{C} \cap \mathbb{X}) = \overline{\mathrm{conv}}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \overline{\mathrm{conv}}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$.

Proof. First we prove (i). Because $\mathbb{C} \cap \mathbb{X} = (\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap (\mathbb{C}_1^- \cup \mathbb{C}_2^-)$, we immediately have $\mathrm{conv}(\mathbb{C} \cap \mathbb{X}) \subset \mathrm{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$. If $\mathrm{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-) = \emptyset$, then we have equality throughout. Let $x \in \mathrm{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$. We will show $x \in \mathrm{conv}(\mathbb{C} \cap \mathbb{X})$. If $x \in \mathbb{X}$, then we are done, because $\mathrm{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-) \subset \mathbb{C}$. Hence, we assume $x \notin \mathbb{X}$. Then $x \in \mathbb{T}^+ \cup \mathbb{T}^-$ where $\mathbb{T}^+ = \{x \in \mathbb{E} : \langle c_1, x\rangle > c_{1,0},\ \langle c_2, x\rangle > c_{2,0}\}$ and $\mathbb{T}^- = \{x \in \mathbb{E} : \langle c_1, x\rangle < c_{2,0},\ \langle c_2, x\rangle < c_{2,0}\}$.

Consider the case where $x \in \mathbb{T}^+$. The case for $x \in \mathbb{T}^-$ is similar. Because $x \in \mathbb{T}^+$, we have $\langle c_1, x\rangle > c_{1,0}$ and $\langle c_2, x\rangle > c_{2,0}$. Because $x \in \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$, there exists $x_1, x_2 \in \mathbb{C}_1^- \cup \mathbb{C}_2^-$ such that $x \in \mathrm{conv}\{x_1, x_2\}$. We claim $x_1, x_2 \in \mathbb{X}$. Suppose not. Then $x_1 \in \mathbb{T}^-$ or $x_2 \in \mathbb{T}^-$. In the first case, x_1 satisfies $\langle c_1, x_1\rangle < c_{1,0}$ and $\langle c_2, x_1\rangle < c_{2,0}$, whereas $x_2 \in \mathbb{C}_1^- \cup \mathbb{C}_2^-$ implies that x_2 satisfies at least one of $\langle c_1, x_2\rangle \le c_{1,0}$ or $\langle c_2, x_2\rangle \le c_{2,0}$. This contradicts $x \in \mathbb{T}^+$. The case where $x_2 \in \mathbb{T}^-$ is analogous and leads to the same conclusion.

Now we prove (ii). The inclusion $\overline{\mathrm{conv}}(\mathbb{C} \cap \mathbb{X}) \subset \overline{\mathrm{conv}}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \overline{\mathrm{conv}}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$ follows from statement (i). As in the proof of statement (i), we can assume $\overline{\mathrm{conv}}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \overline{\mathrm{conv}}(\mathbb{C}_1^- \cup \mathbb{C}_2^-) \neq \emptyset$. Let $x \in \overline{\mathrm{conv}}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \overline{\mathrm{conv}}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$. We will show $x \in \overline{\mathrm{conv}}(\mathbb{C} \cap \mathbb{X})$. Because $x \in \mathbb{C}$, it is enough to consider $x \notin \mathbb{X}$. Suppose $x \in \mathbb{T}^+$. Because $x \in \overline{\mathrm{conv}}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$, there exists a sequence $\{u^i\}_{i=1}^\infty \subset \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$ which converges to x. The subsequence $\{u^i\}_{i=1}^\infty \cap \mathbb{T}^+$ is infinite, contained in $\mathrm{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \cap \mathrm{conv}(\mathbb{C}_1^- \cup \mathbb{C}_2^-)$, and also converges to x. By statement (i), this subsequence is also contained in $\mathrm{conv}(\mathbb{C} \cap \mathbb{X})$. Therefore, $x \in \overline{\mathrm{conv}}(\mathbb{C} \cap \mathbb{X})$. □

5.3 Properties of Valid Linear Inequalities for Disjunctions on Regular Cones

In the rest of this chapter, we consider the case where the description of $\mathbb{C}$ contains an empty set of linear equations. In other words, we let $\mathbb{C} = \mathbb{K}$. With this, the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ take the form

$$\mathbb{C}_i = \{x \in \mathbb{K} : \langle c_i, x\rangle \ge c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}. \tag{5.5}$$

The main purpose of this section is to characterize the structure of undominated valid linear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$. As before, we assume that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Condition 5.1 and each inequality $\langle c_i, x\rangle \ge c_{i,0}$ has been scaled so that $c_{i,0} \in \{0, \pm 1\}$. For some results, we also require $\mathbb{C}_1$ and $\mathbb{C}_2$ to satisfy Condition 5.2. When this is the case, we say that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup.

Under Condition 5.2, the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ always have nonempty interior. Note that the set $\mathbb{C}_i$ is always strictly feasible when it is nonempty and $c_{i,0} \in \{\pm 1\}$. Therefore, we need Condition 5.2 to supplement Condition 5.1 only when $c_{1,0} = 0$ or $c_{2,0} = 0$. We note that Condition 5.2 is primarily needed for sufficiency results, that is, explicit closed convex hull characterizations, and even in the absence of Condition 5.2, our techniques yield convex valid inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$. We evaluate the necessity of Condition 5.2 for our sufficiency results with an example in Section 5.3.3.

The next lemma obtains a natural consequence of Condition 5.1 through conic duality.

Lemma 5.3. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose Condition 5.1 holds. Then the following system of inequalities in the variable β_1 is inconsistent:*

$$\beta_1 \geq 0, \quad \beta_1 c_{1,0} \geq c_{2,0}, \quad c_2 - \beta_1 c_1 \in \mathbb{K}^*. \tag{5.6}$$

Similarly, the following system of inequalities in the variable β_2 is inconsistent:

$$\beta_2 \geq 0, \quad \beta_2 c_{2,0} \geq c_{1,0}, \quad c_1 - \beta_2 c_2 \in \mathbb{K}^*. \tag{5.7}$$

Proof. Suppose there exists β_1^* satisfying (5.6). For all $x \in \mathbb{K}$, this implies $\langle c_2 - \beta_1^* c_1, x\rangle \geq 0 \geq c_{2,0} - \beta_1^* c_{1,0}$. Then any point $x \in \mathbb{C}_1$ satisfies $\beta_1^* \langle c_1, x\rangle \geq \beta_1^* c_{1,0}$ and therefore, $\langle c_2, x\rangle \geq c_{2,0}$. Hence, $\mathbb{C}_1 \subset \mathbb{C}_2$ which contradicts Condition 5.1. The proof for the inconsistency of (5.7) is similar. □

5.3.1 Undominated Valid Linear Inequalities

It is well-known that the closed convex hull of any set can be described with valid linear inequalities alone (see, e.g., [69, Theorem 4.2.3]). In this section, using the special structure of the set $\mathbb{C}_1 \cup \mathbb{C}_2$, we show that a subset of strong valid linear inequalities is sufficient for a description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. Besides being smaller in size, this subset of linear inequalities also has a particular structure which is instrumental in the derivation of the nonlinear valid inequalities in Chapter 6.

A valid linear inequality $\langle \mu, x\rangle \geq \mu_0$ for a nonempty set $\mathbb{S} \subset \mathbb{K}$ is said to be *tight* if $\inf_x\{\langle \mu, x\rangle : x \in \mathbb{S}\} = \mu_0$ and *strongly tight* if there exists $x^* \in \mathbb{S}$ such that $\langle \mu, x^*\rangle = \mu_0$. A valid linear inequality $\langle \nu, x\rangle \geq \nu_0$ for a strictly feasible set $\mathbb{S} \subset \mathbb{K}$ is said to *dominate* another valid linear inequality $\langle \mu, x\rangle \geq \mu_0$ if there exists $\beta > 0$ such that $(\mu - \beta\nu, \mu_0 - \beta\nu_0) \in (\mathbb{K}^* \times -\mathbb{R}_+) \setminus \{(0,0)\}$. A valid linear inequality $\langle \mu, x\rangle \geq \mu_0$ is said to be *undominated* if there does not exist another valid linear inequality $\langle \nu, x\rangle \geq \nu_0$ such that $(\mu - \nu, \mu_0 - \nu_0) \in (\mathbb{K}^* \times -\mathbb{R}_+) \setminus \{(0,0)\}$. This notion of domination is closely tied with the conic minimality definition of [80] which says that a valid linear inequality $\langle \mu, x\rangle \geq \mu_0$ is minimal with respect to the cone $\mathbb{K}$, or $\mathbb{K}$-*minimal*, if there does not exist another valid linear inequality $\langle \nu, x\rangle \geq \nu_0$ such that $(\mu-\nu, \mu_0-\nu_0) \in (\mathbb{K}^* \setminus \{0\}) \times -\mathbb{R}_+$. In particular, a valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ is undominated in the sense considered here if and only if it is $\mathbb{K}$-minimal and tight on $\mathbb{C}_1 \cup \mathbb{C}_2$. Kılınç-Karzan [80] defines and studies $\mathbb{K}$-minimal inequalities for disjunctive conic sets of the form $\{x \in \mathbb{K} : \mathcal{A}x \in \mathbb{S}\}$ where $\mathbb{S} \subset \mathbb{R}^m$ is an arbitrary nonempty set, $\mathcal{A} : \mathbb{E} \to \mathbb{R}^m$ is a linear map, and $\mathbb{K} \subset \mathbb{E}$ is a regular cone. Our set $\mathbb{C}_1 \cup \mathbb{C}_2$ can be represented in this form as

$$\left\{x \in \mathbb{K} : \begin{pmatrix}\langle c_1, x\rangle \\ \langle c_2, x\rangle\end{pmatrix} \in \begin{pmatrix}c_{1,0}+\mathbb{R}_+ \\ \mathbb{R}\end{pmatrix} \bigcup \begin{pmatrix}\mathbb{R} \\ c_{2,0}+\mathbb{R}_+\end{pmatrix}\right\}.$$

Because $\mathbb{C}_1 \cup \mathbb{C}_2$ is full-dimensional under Condition 5.2, [80, Proposition 2] implies that the extreme rays of the convex cone of valid linear inequalities

$$\{(\mu, \mu_0) \in \mathbb{E} \times \mathbb{R} : \quad \langle \mu, x\rangle \geq \mu_0 \quad \forall x \in \mathbb{C}_1 \cup \mathbb{C}_2\}$$

are either $\mathbb{K}$-minimal, or they are implied by the cone constraint $x \in \mathbb{K}$. It is also not difficult to show that these extreme rays have to be tight valid linear inequalities. Hence, undominated valid linear inequalities can produce an outer description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$, together with the cone constraint $x \in \mathbb{K}$.

Because $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Condition 5.2, the strong duality theorem of conic programming implies that an inequality $\langle \mu, x\rangle \geq \mu_0$ is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ if and only if there exist $\alpha_1, \alpha_2, \beta_1$, and β_2 such that $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies

$$\begin{gathered}\mu = \alpha_1 + \beta_1 c_1, \quad \mu = \alpha_2 + \beta_2 c_2, \\ \beta_1 c_{1,0} \geq \mu_0, \qquad \beta_2 c_{2,0} \geq \mu_0, \\ \alpha_1 \in \mathbb{K}^*, \beta_1 \in \mathbb{R}_+, \quad \alpha_2 \in \mathbb{K}^*, \beta_2 \in \mathbb{R}_+.\end{gathered} \tag{5.8}$$

Consider $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ which satisfies (5.8). If $\mu_0 < \beta_1 c_{1,0}$ and $\mu_0 < \beta_2 c_{2,0}$, the inequality $\langle \mu, x\rangle \geq \mu_0$ is not tight on $\mathbb{C}_1 \cup \mathbb{C}_2$. Any such inequality is dominated by $\langle \mu, x\rangle \geq \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$ which has a larger right-hand side. Furthermore, when $\beta_1 = 0$ or $\beta_2 = 0$, the inequality $\langle \mu, x\rangle \geq \mu_0$ is implied by the cone constraint $x \in \mathbb{K}$. Therefore, any inequality $\langle \mu, x\rangle \geq \mu_0$ which is valid for and tight on $\mathbb{C}_1 \cup \mathbb{C}_2$ and not implied by $x \in \mathbb{K}$ is characterized by a tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ that satisfies

$$\begin{gathered}\mu = \alpha_1 + \beta_1 c_1, \quad \mu = \alpha_2 + \beta_2 c_2, \\ \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\} = \mu_0, \\ \alpha_1 \in \mathbb{K}^*, \beta_1 \in \mathbb{R}_+ \setminus \{0\}, \quad \alpha_2 \in \mathbb{K}^*, \beta_2 \in \mathbb{R}_+ \setminus \{0\}.\end{gathered} \tag{5.9}$$

In Proposition 5.5 below, we show that this system can be strengthened significantly when we consider *undominated* valid linear inequalities. We first prove a simple lemma.

Lemma 5.4. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Let $r \in \mathbb{E}$.*

i. There exist $\alpha_1, \alpha_2 \in \mathbb{K}^*$ such that $\alpha_1 - \alpha_2 = r$.

ii. Consider $\alpha_1, \alpha_2 \in \mathbb{K}^*$ such that $\alpha_1 - \alpha_2 = r$. Suppose $r \notin \pm \operatorname{int} \mathbb{K}^*$. Then there exist $\alpha_1', \alpha_2' \in \operatorname{bd} \mathbb{K}^*$ such that $\alpha_1' - \alpha_2' = r$, $\alpha_1 - \alpha_1' \in \mathbb{K}^*$, and $\alpha_2 - \alpha_2' \in \mathbb{K}^*$.

Proof. First we prove statement (i). The dual cone $\mathbb{K}^*$ is also a regular cone. Let $e \in \operatorname{int} \mathbb{K}^*$. Then there exists $\epsilon > 0$ such that $e + \mathbb{B}(\epsilon) \subset \mathbb{K}^*$ where $\mathbb{B}(\epsilon) = \{x \in \mathbb{E} : \|x\| \leq \epsilon\}$. Let $r \in \mathbb{E}$. Then $\frac{\epsilon}{\|r\|} r \in \mathbb{B}(\epsilon)$. Hence, $e + \frac{\epsilon}{\|r\|} r \in \mathbb{K}^*$. After scaling, we obtain $\frac{\|r\|}{\epsilon} e + r \in \mathbb{K}^*$, which implies that r can be written as the difference of some point in $\mathbb{K}^*$ and $\frac{\|r\|}{\epsilon} e$.

If $r \in \operatorname{bd} \mathbb{K}^*$, let $\alpha_1' = r$ and $\alpha_2' = 0$. If $r \in -\operatorname{bd} \mathbb{K}^*$, let $\alpha_1' = 0$ and $\alpha_2' = r$. In either case, α_1' and α_2' satisfy the claims of the lemma. Now consider the case $r \notin \pm \mathbb{K}^*$. The rays α_1 and α_2 must be distinct and nonzero. Let $\epsilon_1 \geq 0$ be such that $\alpha_2 - \epsilon_1 \alpha_1 \in \operatorname{bd} \mathbb{K}^*$. Let $\epsilon_2 \geq 0$ be such that $(1 - \epsilon_1)\alpha_1 - \epsilon_2(\alpha_2 - \epsilon_1 \alpha_1) \in \operatorname{bd} \mathbb{K}^*$. Here the scalars ϵ_1 and ϵ_2 are well-defined because $\mathbb{K}^*$ is pointed. The points $\alpha_1' = (1 - \epsilon_1)\alpha_1 - \epsilon_2(\alpha_2 - \epsilon_1 \alpha_1)$ and $\alpha_2' = (1 - \epsilon_2)(\alpha_2 - \epsilon_1 \alpha_1)$ satisfy the claims of the lemma. ☐

Proposition 5.5. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then, up to positive scaling, any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ has the form $\langle \mu, x \rangle \geq \mu_0$ with $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying*

$$\begin{gathered} \mu = \alpha_1 + \beta_1 c_1, \quad \mu = \alpha_2 + \beta_2 c_2, \\ \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\} = \mu_0, \\ \alpha_1 \in \operatorname{bd} \mathbb{K}^*, \beta_1 \in \mathbb{R}_+ \setminus \{0\}, \quad \alpha_2 \in \operatorname{bd} \mathbb{K}^*, \beta_2 \in \mathbb{R}_+ \setminus \{0\}. \end{gathered} \tag{5.10}$$

Proof. Let $\langle \nu, x \rangle \geq \nu_0$ be a valid inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$. Then there exist $\alpha_1, \alpha_2, \beta_1$, and β_2 such that $(\nu, \nu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies (5.8). If $(\nu, \nu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ does not satisfy (5.9), then it is dominated. Hence, we can assume without any loss of generality that $(\nu, \nu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies (5.9). Let $r = \beta_2 c_2 - \beta_1 c_1$. If $r \notin \pm \operatorname{int} \mathbb{K}^*$, then the inequality $\langle \nu, x \rangle \geq \nu_0$ is either equivalent to or dominated by the inequality $\langle \mu, x \rangle \geq \nu_0$ where $\mu = \alpha_1' + \beta_1 c_1 = \alpha_2' + \beta_2 c_2$ for α_1' and α_2' chosen as in Lemma 5.4(ii). In the remainder of the proof, we consider the case $r \in \pm \operatorname{int} \mathbb{K}^*$. We will show that the inequality $\langle \nu, x \rangle \geq \nu_0$ is either equivalent to or dominated by an inequality $\langle \mu, x \rangle \geq \nu_0$ which satisfies (5.10).

Suppose $r \in \operatorname{int} \mathbb{K}^*$; the analysis for the case $r \in -\operatorname{int} \mathbb{K}^*$ is similar. By Lemma 5.3 and taking $\beta_1, \beta_2 > 0$ into account, we conclude i) $\beta_2 c_{2,0} > \beta_1 c_{1,0}$, and ii) $\alpha_1 = \alpha_2 + r \in \operatorname{int} \mathbb{K}^*$. Statement (i) further implies $\nu_0 = \beta_1 c_{1,0}$. There are two cases that we need to consider: $\alpha_2 \neq 0$ and $\alpha_2 = 0$.

First suppose $\alpha_2 \neq 0$. Let $\alpha_1' = r$, $\alpha_2' = 0$, and $\mu = \nu - \alpha_2$. Then the inequality $\langle \mu, x \rangle \geq \nu_0$ is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ because $(\mu, \nu_0, \alpha_1', \alpha_2', \beta_1, \beta_2)$ satisfies (5.8). Furthermore, $\langle \mu, x \rangle \geq \nu_0$ dominates $\langle \nu, x \rangle \geq \nu_0$ since $\nu - \mu = \alpha_2 \in \mathbb{K}^* \setminus \{0\}$.

Now suppose $\alpha_2 = 0$. Then $\alpha_1 = r \in \operatorname{int} \mathbb{K}^*$. If $\nu_0 > 0$, then $c_{2,0} > 0$ as well. In this case, we must have $c_2 \notin -\mathbb{K}^*$; otherwise, Condition 5.2 is violated. Let $\epsilon > 0$ be such that $\alpha_1' = \alpha_1 - \epsilon c_2 \in \operatorname{bd} \mathbb{K}^*$. Here the scalar ϵ is well-defined because $c_2 \notin -\mathbb{K}^*$. We define $\beta_2' = \beta_2 - \epsilon$ and $\mu = \nu - \epsilon c_2$. If $\nu_0 < 0$, we can assume $c_2 \notin \mathbb{K}^*$; otherwise, the inequality $\langle \nu, x \rangle \geq \nu_0$ is implied by the cone constraint $x \in \mathbb{K}$. Let $\epsilon > 0$ be such that $\alpha_1' = \alpha_1 + \epsilon c_2 \in \operatorname{bd} \mathbb{K}^*$. The scalar ϵ is well-defined because $c_2 \notin \mathbb{K}^*$. We define $\beta_2' = \beta_2 + \epsilon'$ and $\mu = \nu + \epsilon' c_2$. In either case, Lemma 5.3 shows that $\beta_2' c_{2,0} > \beta_1 c_{1,0}$ since $\beta_2' c_2 - \beta_1 c_1 = \alpha_1' \in \mathbb{K}^*$. Furthermore, the inequality $\langle \mu, x \rangle \geq \nu_0$ is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ because $(\mu, \nu_0, \alpha_1', \alpha_2, \beta_1, \beta_2')$ satisfies (5.10). It dominates $\langle \nu, x \rangle \geq \nu_0$ because $\mu = \frac{\beta_2'}{\beta_2} \nu$ which satisfies $\beta_2' < \beta_2$ when $\nu_0 > 0$ and $\beta_2' > \beta_2$ when $\nu_0 < 0$. In the case $\nu_0 = 0$, we can assume $c_2 \notin \pm \mathbb{K}^*$; otherwise, either Condition 5.2 is violated or $\langle \nu, x \rangle \geq \nu_0$ is implied by the cone constraint $x \in \mathbb{K}$. Let $\epsilon > 0$ be such that $\alpha_1' = \alpha_1 + \epsilon c_2 \in \operatorname{bd} \mathbb{K}^*$. We define $\beta_2' = \beta_2 + \epsilon$ and $\mu = \nu + \epsilon c_2$. Then the inequality $\langle \mu, x \rangle \geq \nu_0$ is a positive multiple of $\langle \nu, x \rangle \geq \nu_0$. Furthermore, $(\mu, \nu_0, \alpha_1', \alpha_2, \beta_1, \beta_2')$ satisfies (5.10). □

Any tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying (5.10) must also satisfy $r = \beta_2 c_2 - \beta_1 c_1 \notin \pm \operatorname{int} \mathbb{K}^*$ since having $r \in \pm \operatorname{int} \mathbb{K}^*$ contradicts either $\alpha_1 = \alpha_2 + r \in \operatorname{bd} \mathbb{K}^*$ or $\alpha_2 = \alpha_1 - r \in \operatorname{bd} \mathbb{K}^*$. For ease of exposition in the remainder of this section, let us define the scalar $\mu_0(\beta_1, \beta_2) = \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$. Let us also define the sets

$$\mathbb{B} = \{(\beta_1, \beta_2) \in \mathbb{R}^2 : \quad \beta_1, \beta_2 > 0, \quad \beta_2 c_2 - \beta_1 c_1 \notin \pm \operatorname{int} \mathbb{K}^*\}, \tag{5.11}$$

$$\mathbb{M}'(\beta_1, \beta_2) = \{\mu \in \mathbb{E} : \quad \exists\, \alpha_1, \alpha_2 \in \operatorname{bd} \mathbb{K}^*, \quad \mu = \alpha_1 + \beta_1 c_1 = \alpha_2 + \beta_2 c_2\}. \tag{5.12}$$

Proposition 5.5 implies the following result.

Corollary 5.6. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is*

$$\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \{x \in \mathbb{K} : \quad \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2) \quad \forall \mu \in \mathbb{M}'(\beta_1, \beta_2),\ (\beta_1, \beta_2) \in \mathbb{B}\}\,.$$

The system (5.10) is homogeneous in the tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$. Therefore, in an undominated valid inequality $\langle \mu, x \rangle \geq \mu_0$, we can assume without any loss of generality that the whole tuple has been scaled by a positive real number so that $\beta_1 = 1$ or $\beta_2 = 1$.

Proposition 5.7. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then, up to positive scaling, any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ has the form $\langle \mu, x \rangle \geq \mu_0$ with*

$(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ *satisfying one of the following systems:*

$$(i) \quad \begin{aligned} &\mu = \alpha_1 + \beta_1 c_1, \\ &\mu = \alpha_2 + c_2, \\ &\beta_1 c_{1,0} \geq c_{2,0} = \mu_0, \\ &\alpha_1, \alpha_2 \in \operatorname{bd} \mathbb{K}^*, \\ &\beta_1 \in \mathbb{R}_+ \setminus \{0\}, \beta_2 = 1, \end{aligned} \qquad (ii) \quad \begin{aligned} &\mu = \alpha_1 + c_1, \\ &\mu = \alpha_2 + \beta_2 c_2, \\ &\beta_2 c_{2,0} \geq c_{1,0} = \mu_0, \\ &\alpha_1, \alpha_2 \in \operatorname{bd} \mathbb{K}^*, \\ &\beta_2 \in \mathbb{R}_+ \setminus \{0\}, \beta_1 = 1. \end{aligned} \tag{5.13}$$

Keeping $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ in mind, observe that the first of the two systems in (5.13) is infeasible when $c_{2,0} > c_{1,0}$ and the second is infeasible when $c_{1,0} > c_{2,0}$. Therefore, in these cases it is enough to consider only one of these systems. When $c_{1,0} = c_{2,0}$ however, one may need valid linear inequalities that are associated with either of the two systems in (5.13) for a description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. Still, for this case Proposition 5.7 implies that any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ can be written in the form $\langle \mu, x \rangle \geq \mu_0$ where $\mu_0 = c_{1,0} = c_{2,0}$.

Proposition 5.7 can be used to strengthen the statement of Corollary 5.6 as follows. Let $r = c_2 - \beta_1 c_1$. First, note that any tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying the first system in (5.13) must also satisfy $r \notin \pm \operatorname{int} \mathbb{K}^*$ since having $r \in \pm \operatorname{int} \mathbb{K}^*$ contradicts either $\alpha_1 = \alpha_2 + r \in \operatorname{bd} \mathbb{K}^*$ or $\alpha_2 = \alpha_1 - r \in \operatorname{bd} \mathbb{K}^*$. Analogously, any tuple $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying the second system in (5.13) must also satisfy $c_1 - \beta_2 c_2 \notin \pm \operatorname{int} \mathbb{K}^*$. Let us define the sets

$$\mathbb{B}_1 = \{\beta_1 > 0 : \quad \beta_1 c_{1,0} \geq c_{2,0}, \quad c_2 - \beta_1 c_1 \notin \pm \operatorname{int} \mathbb{K}^*\}, \tag{5.14a}$$

$$\mathbb{B}_2 = \{\beta_2 > 0 : \quad \beta_2 c_{2,0} \geq c_{1,0}, \quad \beta_2 c_2 - c_1 \notin \pm \operatorname{int} \mathbb{K}^*\}. \tag{5.14b}$$

Now Proposition 5.7 implies the following result.

Corollary 5.8. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is*

$$\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{K} : \begin{array}{ll} \langle \mu, x \rangle \geq c_{2,0} & \forall \mu \in \mathbb{M}'(\beta_1, 1),\ \beta_1 \in \mathbb{B}_1, \\ \langle \mu, x \rangle \geq c_{1,0} & \forall \mu \in \mathbb{M}'(1, \beta_2),\ \beta_2 \in \mathbb{B}_2 \end{array} \right\}.$$

5.3.2 When Does a Single (β_1, β_2) Pair Suffice?

In this section we continue to study the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. The main result of this section is Theorem 5.9, which shows that under certain conditions the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ has a simpler outer description than the one given in Corollary 5.8.

Theorem 5.9. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5). *Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Let $\mu_0 = \min\{c_{1,0}, c_{2,0}\}$. Suppose one of the conditions below holds:*

i. The points $c_1, c_2 \in \mathbb{E}$ satisfy $c_1 \in \mathbb{K}^*$ or $c_2 \in \mathbb{K}^*$.

ii. The convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed and $c_{1,0} = c_{2,0} \in \{\pm 1\}$.

Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is

$$\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \{x \in \mathbb{K} : \quad \langle \mu, x \rangle \geq \mu_0 \quad \forall \mu \in \mathbb{M}'(1,1)\}.$$

Theorem 5.9 is a consequence of several lemmas, which refine the results of Section 5.3.1 on the structure of undominated valid linear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$. These lemmas are the subject of the next two sections.

The Recession Cones of $\mathbb{C}_1$ and $\mathbb{C}_2$

The lemma below shows that the statement of Proposition 5.7 can be strengthened substantially when $c_1 \in \mathbb{K}^*$ or $c_2 \in \mathbb{K}^*$. Note that $c_i \in \mathbb{K}^*$ implies $\operatorname{rec} \mathbb{C}_i = \mathbb{K}$ in either case.

Lemma 5.10. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5). *Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Suppose $c_1 \in \mathbb{K}^*$ or $c_2 \in \mathbb{K}^*$. Let $\mu_0 = \min\{c_{1,0}, c_{2,0}\}$. Then, up to positive scaling, any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ has the form $\langle \mu, x \rangle \geq \mu_0$ where $\mu \in \mathbb{M}'(1,1)$.*

Proof. Having $c_i \in \mathbb{K}^*$ implies $\operatorname{rec} \mathbb{C}_i = \mathbb{K}$. If $c_{1,0} \leq 0$ or $c_{2,0} \leq 0$, then Lemma 5.1 shows $\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{K}$. In this case all valid inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$ are implied by the cone constraint $x \in \mathbb{K}$, and the claim holds trivially because there are no undominated valid inequalities. Thus, we only need to consider the case $c_{1,0} = c_{2,0} = 1$.

Assume without any loss of generality that $c_2 \in \mathbb{K}^*$. Consider an undominated valid inequality $\langle \nu, x \rangle \geq \nu_0$ for $\mathbb{C}_1 \cup \mathbb{C}_2$. Up to positive scaling, it satisfies the conditions of Proposition 5.7. Hence, i) $\nu_0 = c_{1,0} = c_{2,0} = 1$, and ii) there exist $\alpha_1, \alpha_2, \beta_1$, and β_2 such that $(\nu, 1, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies one of the two systems in (5.13). In particular, this implies $\nu = \alpha_1 + \beta_1 c_1 = \alpha_2 + \beta_2 c_2 \in \mathbb{K}^*$ and $\min\{\beta_1, \beta_2\} = 1$. Let $r = \beta_2 c_2 - \beta_1 c_1$. By Lemma 5.3, we also have $r \notin \pm\mathbb{K}^*$. We will show that the inequality $\langle \nu, x \rangle \geq 1$ is dominated if $\nu \notin \mathbb{M}'(1,1)$. If $\beta_1 = \beta_2 = 1$, then $\nu \in \mathbb{M}'(1,1)$. We divide the rest of the proof into the following two cases: $\beta_1 > \beta_2$ and $\beta_1 < \beta_2$.

First suppose $\beta_1 > \beta_2$. Then $\beta_2 = 1$ and $\nu = \alpha_1 + \beta_1 c_1 = \alpha_2 + c_2$. Having $\alpha_2 = 0$ contradicts $r \notin \pm\mathbb{K}^*$; therefore, we assume $\alpha_2 \neq 0$. Let ϵ' be such that $0 < \epsilon' \leq \frac{\beta_1 - 1}{\beta_1}$. We define $\alpha_1' = (1-\epsilon')\alpha_1 + \epsilon' c_2$, $\beta_1' = (1-\epsilon')\beta_1$, $\alpha_2' = (1-\epsilon')\alpha_2$, and $\mu = \nu - \epsilon' \alpha_2$.

The inequality $\langle \mu, x\rangle \geq 1$ is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ because $(\mu, 1, \alpha_1', \alpha_2', \beta_1', 1)$ satisfies (5.8). Furthermore, it dominates $\langle \nu, x\rangle \geq 1$ since $\nu - \mu = \epsilon' \alpha_2 \in \mathbb{K}^* \setminus \{0\}$.

Now suppose $\beta_2 > \beta_1 = 1$. Observe that the tuple $(\nu, 1, \alpha_1, \alpha_2 + (\beta_2 - 1)c_2, 1, 1)$ also satisfies (5.8). Having $\alpha_1 = 0$ contradicts $r \notin \pm\mathbb{K}^*$; therefore, we assume $\alpha_1 \neq 0$. In the case $\alpha_2 + (\beta_2 - 1)c_2 \in \operatorname{int} \mathbb{K}^*$, we can find a valid inequality that dominates $\langle \nu, x\rangle \geq 1$ by subtracting a positive multiple of α_1 from μ as in the proof of Proposition 5.5. Otherwise, $\alpha_2 + (\beta_2 - 1)c_2 \in \operatorname{bd} \mathbb{K}^*$ which implies that $\nu \in \mathbb{M}'(1,1)$ since $\nu = \alpha_1 + c_1 = (\alpha_2 + (\beta_2 - 1)c_2) + c_2$. □

The Topology of the Convex Hull

When $c_{1,0} = c_{2,0} \in \{\pm 1\}$, the characterization of Proposition 5.7 can be strengthened similarly for the family of undominated valid linear inequalities which are tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$.

Lemma 5.11. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Suppose $\mu_0 = c_{1,0} = c_{2,0} \in \{\pm 1\}$. Then, up to positive scaling, any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ which is tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$ has the form $\langle \mu, x\rangle \geq \mu_0$ where $\mu \in \mathbb{M}'(1,1)$.*

Proof. Let $\langle \mu, x\rangle \geq \mu_0$ be an undominated valid inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ that is tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$. Using Proposition 5.7, we can assume that $\mu_0 = c_{1,0} = c_{2,0}$ and there exist $\alpha_1, \alpha_2, \beta_1$, and β_2 such that $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies one of the two systems in (5.13). In particular, either $\beta_1 = 1$ and $\beta_2\mu_0 \geq \mu_0$, or $\beta_2 = 1$ and $\beta_1\mu_0 \geq \mu_0$. In any case, $\min\{\beta_1\mu_0, \beta_2\mu_0\} = \mu_0$. We will show $\beta_1 = \beta_2 = 1$.

Consider the following pair of minimization problems

$$\inf_x\{\langle \mu, x\rangle : \ x \in \mathbb{C}_1\} \quad \text{and} \quad \inf_x\{\langle \mu, x\rangle : \ x \in \mathbb{C}_2\},$$

and their duals

$$\sup_{\delta,\gamma}\{\delta\mu_0 : \ \mu = \gamma + \delta c_1, \gamma \in \mathbb{K}^*, \delta \geq 0\} \ \text{ and } \ \sup_{\delta,\gamma}\{\delta\mu_0 : \ \mu = \gamma + \delta c_2, \gamma \in \mathbb{K}^*, \delta \geq 0\}.$$

The pairs (α_1, β_1) and (α_2, β_2) are feasible solutions to the first and second dual problems, respectively. Because the inequality $\langle \mu, x\rangle \geq \mu_0$ is tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$, the optimal values of both minimization problems are μ_0. Then we must have $\beta_1\mu_0 \leq \mu_0$ and $\beta_2\mu_0 \leq \mu_0$ by duality. This implies $\beta_1\mu_0 = \beta_2\mu_0 = \mu_0$ and $\beta_1 = \beta_2 = 1$. □

Next, we identify an important case where the family of tight inequalities specified in Lemma 5.11 is rich enough to describe the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ completely.

Proposition 5.12. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in (5.5). Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Suppose the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed. Then undominated valid linear inequalities which are strongly tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$ are sufficient to describe the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$, together with the cone constraint $x \in \mathbb{K}$.*

Proof. Suppose the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed. When $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{K}$, no new inequalities are needed for a description of the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$; hence, the claim holds trivially. Therefore, assume $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) \subsetneq \mathbb{K}$. We prove that given $u \in \mathbb{K} \setminus \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$, there exists an undominated valid inequality which separates u from the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ and which is strongly tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$.

Let $v \in \operatorname{int}(\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)) \setminus (\mathbb{C}_1 \cup \mathbb{C}_2)$. Note that such a point exists since otherwise, we have $\operatorname{int}(\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)) \subset \mathbb{C}_1 \cup \mathbb{C}_2$, which implies $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) \subset \mathbb{C}_1 \cup \mathbb{C}_2$ from the closedness of $\mathbb{C}_1 \cup \mathbb{C}_2$. By Lemma 5.1, this is possible only if $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{K}$ which we have already ruled out. Let $0 < \lambda < 1$ be such that $w = (1-\lambda)u + \lambda v \in \operatorname{bd}(\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2))$. Then $w \in \mathbb{K} \setminus (\mathbb{C}_1 \cup \mathbb{C}_2)$ by the convexity of $\mathbb{K} \setminus (\mathbb{C}_1 \cup \mathbb{C}_2) = \{x \in \mathbb{K} : \langle c_1, x\rangle < c_{1,0},\ \langle c_2, x\rangle < c_{2,0}\}$. Because $w \in \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$, there exist $x_1 \in \mathbb{C}_1$, $x_2 \in \mathbb{C}_2$, and $0 < \kappa < 1$ such that $w = \kappa x_1 + (1-\kappa)x_2$. Furthermore, by Corollary 5.6, the fact that $w \in \operatorname{bd}(\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2))$ implies that there exists an undominated valid inequality $\langle \mu, x\rangle \geq \mu_0$ for $\mathbb{C}_1 \cup \mathbb{C}_2$ such that $\langle \mu, w\rangle = \mu_0$. Because $\langle \mu, w\rangle = \kappa\langle \mu, x_1\rangle + (1-\kappa)\langle \mu, x_2\rangle = \mu_0$, $\langle \mu, x_1\rangle \geq \mu_0$, and $\langle \mu, x_2\rangle \geq \mu_0$, it must be the case that $\langle \mu, x_1\rangle = \langle \mu, x_2\rangle = \mu_0$. Thus, the inequality $\langle \mu, x\rangle \geq \mu_0$ is strongly tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$. The only thing that remains is to show that $\langle \mu, x\rangle \geq \mu_0$ separates u from the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. To see this, observe that $u = \frac{1}{1-\lambda}(w - \lambda v)$ and that $\langle \mu, v\rangle > \mu_0$ since $v \in \operatorname{int}(\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2))$. Hence, we conclude $\langle \mu, u\rangle = \frac{1}{1-\lambda}(\langle \mu, w\rangle - \lambda\langle \mu, v\rangle) < \mu_0$. □

We now give the proof of Theorem 5.9, which we stated at the beginning of this section.

Proof of Theorem 5.9. Consider an inequality $\langle \mu, x\rangle \geq \mu_0$ where $\mu \in \mathbb{M}'(1,1)$ and $\mu_0 = \min\{c_{1,0}, c_{2,0}\}$. This inequality is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ because there exist $\alpha_1, \alpha_2 \in \mathbb{K}^*$ such that the tuple $(\mu, \mu_0, \alpha_1, \alpha_2, 1, 1)$ satisfies (5.8). Furthermore, Lemmas 5.10 and 5.11 and Proposition 5.12 show that all undominated valid linear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$ have this form. The result follows. □

Proposition 5.12 demonstrates the close relationship between the closedness of the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ and the sufficiency of valid linear inequalities which are tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$. This motivates us to understand better the cases where the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed.

The convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is always closed when $c_{1,0} = c_{2,0} = 0$ (see, e.g., [96, Corollary 9.1.3]) or when $\mathbb{C}_1$ and $\mathbb{C}_2$ are defined by a split disjunction (see Dadush et al. [52, Lemma 2.3]). In Proposition 5.13 below, we generalize the result of Dadush et al.: We give a sufficient condition for the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ to be closed and show that this condition is almost necessary. In Corollary 5.14, we show that the sufficient condition of Proposition 5.13 can be rewritten in a more specialized form using conic duality when the base set is the regular cone $\mathbb{K}$.

Proposition 5.13. *Let $\mathbb{C} \subset \mathbb{E}$ be a closed, convex, and pointed set. Let $\mathbb{C}_i = \{x \in \mathbb{C} : \langle c_i, x\rangle \geq c_{i,0}\}$ for $i \in \{1,2\}$. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Condition 5.1. If*

$$\begin{aligned} &\{r \in \operatorname{rec}\mathbb{C} :\ \langle c_2, r\rangle = 0\} \subset \{r \in \operatorname{rec}\mathbb{C} :\ \langle c_1, r\rangle \geq 0\} \ \textit{and} \\ &\{r \in \operatorname{rec}\mathbb{C} :\ \langle c_1, r\rangle = 0\} \subset \{r \in \operatorname{rec}\mathbb{C} :\ \langle c_2, r\rangle \geq 0\}, \end{aligned} \tag{5.15}$$

then the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed. Conversely, if

i. there exists $r^* \in \operatorname{rec}\mathbb{C}$ such that $\langle c_1, r^*\rangle < 0 = \langle c_2, r^*\rangle$ and the problem $\inf_x\{\langle c_2, x\rangle : x \in \mathbb{C}_1\}$ is solvable, or

ii. there exists $r^* \in \operatorname{rec}\mathbb{C}$ such that $\langle c_2, r^*\rangle < 0 = \langle c_1, r^*\rangle$ and the problem $\inf_x\{\langle c_1, x\rangle : x \in \mathbb{C}_2\}$ is solvable,

then the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is not closed.

Proof. Let $\mathbb{C}_1^+ = \mathbb{C}_1 + \operatorname{rec}\mathbb{C}_2$ and $\mathbb{C}_2^+ = \mathbb{C}_2 + \operatorname{rec}\mathbb{C}_1$. We have $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) \subset \overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+)$ by Lemma 5.1. We will show $\operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \subset \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$ to prove that the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed when (5.15) is satisfied. Let $x^+ \in \mathbb{C}_1^+$. Then there exist $u_1 \in \mathbb{C}_1$ and $v_2 \in \operatorname{rec}(\mathbb{C}_2)$ such that $x^+ = u_1 + v_2$. If $\langle c_2, v_2\rangle > 0$, then there exists $\epsilon \geq 1$ such that $x^+ + \epsilon v_2 \in \mathbb{C}_2$ and we have $x^+ \in \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$. Otherwise, $\langle c_2, v_2\rangle = 0$, and by the hypothesis, $\langle c_1, v_2\rangle \geq 0$. This implies $x^+ \in \mathbb{C}_1$, and thus $\mathbb{C}_1^+ \subset \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$. Through a similar argument, one can show $\mathbb{C}_2^+ \subset \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$. Hence, $\mathbb{C}_1^+ \cup \mathbb{C}_2^+ \subset \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$. Taking the convex hull of both sides yields $\operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) \subset \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$.

For the converse, suppose condition (i) holds, and let $x^* \in \mathbb{C}_1$ be such that $\langle c_2, x^*\rangle \leq \langle c_2, x\rangle$ for all $x \in \mathbb{C}_1$. Note that $\langle c_2, x^*\rangle < c_{2,0}$ since otherwise, $\mathbb{C}_1 \subset \mathbb{C}_2$. Pick $\delta > 0$ such that $x' = x^* + \delta r^* \notin \mathbb{C}_1$. Then $x' \notin \mathbb{C}_2$ too because $\langle c_2, x'\rangle = \langle c_2, x^*\rangle < c_{2,0}$. For any $0 < \lambda < 1$, $x_1 \in \mathbb{C}_1$, and $x_2 \in \mathbb{C}_2$, we can write $\langle c_2, \lambda x_1 + (1-\lambda)x_2\rangle \geq \lambda\langle c_2, x^*\rangle + (1-\lambda)c_{2,0} > \langle c_2, x'\rangle$. Hence, $x' \notin \operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$. On the other hand, $x' \in \mathbb{C}_1^+ \subset \operatorname{conv}(\mathbb{C}_1^+ \cup \mathbb{C}_2^+) = \overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$ where the last equality follows from Lemma 5.1. □

Corollary 5.14 shows that the sufficient condition of Proposition 5.13 can be rewritten in a more specialized form using conic duality when the base set is the regular cone $\mathbb{K}$.

Corollary 5.14. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5). *Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. If there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $c_1 - \beta_2 c_2 \in \mathbb{K}^*$ and $c_2 - \beta_1 c_1 \in \mathbb{K}^*$, then the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed.*

Proof. Suppose there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that $c_1 - \beta_2 c_2 \in \mathbb{K}^*$ and $c_2 - \beta_1 c_1 \in \mathbb{K}^*$. Consider the following minimization problem

$$\inf_u \{\langle c_1, u\rangle : \ \langle c_2, u\rangle = 0, u \in \mathbb{K}\}$$

and its dual

$$\sup_\delta \{0 : \ c_1 - \delta c_2 \in \mathbb{K}^*\}.$$

Because β_2 is a feasible solution to the dual problem, we have $\langle c_1, u\rangle \geq 0$ for all $u \in \mathbb{K}$ such that $\langle c_2, u\rangle = 0$. Similarly, one can use the existence of β_1 to show that the second part of (5.15) holds too. Then by Proposition 5.13, the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed. □

Lemma 5.11 allows us to simplify the characterization (5.10) of undominated valid linear inequalities which are tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$ in the case $c_{1,0} = c_{2,0} \in \{\pm 1\}$. The next proposition shows the necessity of the condition $c_{1,0} = c_{2,0}$ in the statement of this lemma. Unfortunately, when $c_{1,0} \neq c_{2,0}$, undominated valid linear inequalities are tight on exactly one of the two sets $\mathbb{C}_1$ and $\mathbb{C}_2$.

Proposition 5.15. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. If $c_{1,0} > c_{2,0}$, then any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ is tight on $\mathbb{C}_2$ but not on $\mathbb{C}_1$.*

Proof. Every undominated valid inequality has to be tight on either $\mathbb{C}_1$ or $\mathbb{C}_2$; otherwise, we can just increase the right-hand side to obtain a dominating valid inequality. By Proposition 5.7, undominated valid inequalities are of the form $\langle \mu, x\rangle \geq \mu_0$ where $(\mu, \mu_0, \alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfies the first system in (5.13). In particular, we have $\beta_1 > 0$, $\beta_1 c_{1,0} \geq c_{2,0}$, and $\mu_0 = c_{2,0}$. Now consider the following minimization problem

$$\inf_u \{\langle \mu, u\rangle : \ u \in \mathbb{C}_1\}$$

and its dual

$$\sup_\delta \{\delta c_{1,0} : \ \mu - \delta c_1 \in \mathbb{K}^*, \ \delta \geq 0\}.$$

Note that β_1 is a feasible solution to the dual problem. The set $\mathbb{C}_1$ is strictly feasible by Condition 5.2, so strong duality applies to this pair of conic optimization problems. The dual problem admits an optimal solution δ^* which satisfies $\delta^* \geq \beta_1 > 0$ because $c_{1,0} \geq 0$. Then

$$\text{sign}\{\delta^* c_{1,0}\} = \text{sign}\{c_{1,0}\} = c_{1,0} > c_{2,0} = \mu_0.$$

Hence, the inequality $\langle \mu, x\rangle \geq \mu_0$ cannot be tight on $\mathbb{C}_1$. □

This result, when combined with Proposition 5.12, yields the following corollary.

Corollary 5.16. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (5.5)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Suppose $c_{1,0} > c_{2,0}$. If $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) \neq \mathbb{K}$, then the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is not closed.*

Proof. Suppose the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed, and let $x \in \mathbb{K} \setminus \text{conv}(\mathbb{C}_1 \cup \mathbb{C}_2)$. By Proposition 5.12, there exists an undominated valid linear inequality which cuts off x from the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ and is tight on both $\mathbb{C}_1$ and $\mathbb{C}_2$. This contradicts Proposition 5.15. □

5.3.3 Revisiting Condition 5.2

Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in (5.5). When $\mathbb{C}_i$ is nonempty and $c_{i,0} \in \{\pm 1\}$, it is not difficult to show that $\mathbb{C}_i$ has to be strictly feasible. Therefore, Condition 5.2 is not needed when, for instance, $\mathbb{C}_1$ and $\mathbb{C}_2$ are nonempty sets defined by a split disjunction which excludes the origin. Indeed, the only situation where Condition 5.2 may be needed in addition to Condition 5.1 occurs when $c_{1,0} = 0$ or $c_{2,0} = 0$. Note that in such a case, linear inequalities that satisfy system (5.8) (or (5.10)) are still valid for the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$; they may just not be sufficient to define it completely. We next give an example which shows that Condition 5.2 is necessary to establish the sufficiency of the linear inequalities that satisfy (5.8) (or (5.10)) when $c_{1,0} = c_{2,0} = 0$.

Let $\mathbb{E} = \mathbb{R}^3$ and $\mathbb{K} = \mathbb{L}^3$. Consider the disjunction $x_1 - x_3 \geq 0 \vee -x_1 - x_3 \geq 0$ ($c_1 = e^1 - e^3$, $c_2 = -e^1 - e^3$, $c_{1,0} = c_{2,0} = 0$) on $\mathbb{L}^3$. Note that $c_1, c_2 \in -\text{bd}\,\mathbb{L}^3$, and $\mathbb{C}_1$ and $\mathbb{C}_2$ are the rays generated by $e^1 + e^3$ and $-e^1 + e^3$, respectively. Therefore, $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \{x \in \mathbb{L}^3 : x_2 = 0\}$ and $x_2 \geq 0$ is a valid inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$. However, letting $\mu = e^2$ in (5.8), we see that any α_1 which satisfies $\mu = \alpha_1 + \beta_1 c_1$ for some $\beta_1 \in \mathbb{R}$ cannot be in $\mathbb{L}^3$ because $\alpha_1 = -\beta_1 e^1 + e^2 + \beta_1 e^3 \notin \mathbb{L}^3$.

5.4 Nonlinear Inequalities with Special Structure

Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. In this section we continue to study the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$, where $\mathbb{C}_1$ and $\mathbb{C}_2$ are defined as in (5.5). Consider a pair $(\beta_1, \beta_2) \in \mathbb{B}$. For this pair, we let

$$\mu_0(\beta_1, \beta_2) = \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}.$$

Let us also define the set

$$\mathbb{M}(\beta_1, \beta_2) = \{\mu \in \mathbb{E} : \quad \exists\, \alpha_1, \alpha_2 \in \mathbb{K}^*, \quad \mu = \alpha_1 + \beta_1 c_1 = \alpha_2 + \beta_2 c_2\}.$$

Recall from (5.8) that, for $\mu \in \mathbb{M}(\beta_1, \beta_2)$, an inequality $\langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2)$ is always valid for $\mathbb{C}_1 \cup \mathbb{C}_2$, regardless of whether or not $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. When, however, $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup, Corollary 5.6 has the following simple consequence.

Remark 5.17. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in (5.5). Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is given by*

$$\begin{aligned}\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) &= \{x \in \mathbb{K} : \quad \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2) \quad \forall \mu \in \mathbb{M}'(\beta_1, \beta_2), \quad (\beta_1, \beta_2) \in \mathbb{B}\} \\ &= \{x \in \mathbb{K} : \quad \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2) \quad \forall \mu \in \mathbb{M}(\beta_1, \beta_2), \quad (\beta_1, \beta_2) \in \mathbb{B}\}.\end{aligned}$$

In this section, for given $(\beta_1, \beta_2) \in \mathbb{B}$, we develop structured valid nonlinear inequalities for $\mathbb{C}_1 \cup \mathbb{C}_2$ by grouping the linear inequalities $\langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2)$ associated with all $\mu \in \mathbb{M}(\beta_1, \beta_2)$. Let $(\beta_1, \beta_2) \in \mathbb{B}$. A point $x \in \mathbb{E}$ satisfies $\langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2)$ for all $\mathbb{M}(\beta_1, \beta_2)$ if and only if it satisfies

$$\inf_{\mu \in \mathbb{M}(\beta_1, \beta_2)} \langle \mu, x \rangle \geq \mu_0(\beta_1, \beta_2). \tag{5.16}$$

Theorem 5.9 demonstrates that there are important cases where an inequality of the form (5.16) associated with a single pair $(\beta_1, \beta_2) \in \mathbb{B}$ provides a complete description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. However, in general (5.16) is a convex inequality which is derived from a relaxation $\langle \beta_1 c_1, x \rangle \geq \mu_0(\beta_1, \beta_2) \vee \langle \beta_2 c_2, x \rangle \geq \mu_0(\beta_1, \beta_2)$ of the original disjunction on the cone $\mathbb{K}$. Somewhat contrary to intuition, inequalities (5.16) obtained from such weaker disjunctions are sometimes necessary for a complete description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. With this understanding, from now on we consider $(\beta_1, \beta_2) \in \mathbb{B}$ fixed. Letting $d_i = \beta_i c_i$ for $i \in \{1, 2\}$ and suppressing the arguments of $\mathbb{M}(\beta_1, \beta_2)$ and $\mu_0(\beta_1, \beta_2)$, we concentrate our analysis on the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$ where

$$\mathbb{D}_i = \{x \in \mathbb{K} : \quad \langle d_i, x \rangle \geq \mu_0\} \quad \text{for} \quad i \in \{1, 2\}. \tag{5.17}$$

Given $\mathbb{C}_1$ and $\mathbb{C}_2$ which satisfy the basic disjunctive setup, the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ always satisfy Condition 5.2 because $\mathbb{D}_1 \supset \mathbb{C}_1$ and $\mathbb{D}_2 \supset \mathbb{C}_2$. However, they may violate Condition 5.1. When this is the case, the convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$ is equal to one of $\mathbb{D}_1$ or $\mathbb{D}_2$. Therefore, we are primarily interested in cases where $\mathbb{D}_1$ and $\mathbb{D}_2$ also satisfy Condition 5.1. By Lemma 5.3, this can happen only if $d_2 - d_1 \notin \pm\mathbb{K}^*$. Therefore, while studying convex relaxations for $\mathbb{D}_1 \cup \mathbb{D}_2$ in subsequent sections, we sometimes state our results under the assumption that $r = d_2 - d_1 \notin \pm\mathbb{K}^*$.

In Sections 5.4.1 and 5.4.3, we study the general form of (5.16) under various assumptions on the structure of $\mathbb{D}_1$ and $\mathbb{D}_2$.

5.4.1 Inequalities for Two-Term Disjunctions

In this section we consider sets $\mathbb{D}_1$ and $\mathbb{D}_2$ which are defined as in (5.17). Let $\mathbb{M} = \{\mu \in \mathbb{E} : \exists \alpha_1, \alpha_2 \in \mathbb{K}^*,\ \mu = \alpha_1 + d_1 = \alpha_2 + d_2\}$. As discussed in Section 5.2.1, any point $x \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies

$$\inf_{\mu \in \mathbb{M}} \langle \mu, x \rangle \geq \mu_0, \tag{5.18}$$

regardless of whether or not $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the basic disjunctive setup. Furthermore, whenever $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Theorem 5.9, an inequality of the form (5.18) describes the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$. Our main purpose here is to investigate the general form of this inequality under minimal assumptions on the structure of $\mathbb{K}$. This generality will enable us to establish results about disjunctions on direct products of second-order cones and the positive semidefinite cone in Chapters 6 and 7.

Throughout this section, we denote $r = d_2 - d_1 \in \mathbb{E}$. We start with a simple observation which yields an alternate representation of the disjunction $\langle d_1, x \rangle \geq \mu_0 \vee \langle d_2, x \rangle \geq \mu_0$.

Remark 5.18. *A point $x \in \mathbb{E}$ satisfies the disjunction $\langle d_1, x \rangle \geq \mu_0 \vee \langle d_2, x \rangle \geq \mu_0$ if and only if it satisfies*

$$|\langle r, x \rangle| \geq 2\mu_0 - \langle d_1 + d_2, x \rangle. \tag{5.19}$$

The next proposition states (5.18) in an alternate form.

Proposition 5.19. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. A point $x \in \mathbb{E}$ satisfies* (5.18) *if and only if it satisfies*

$$f_{\mathbb{K},r}(x) \geq 2\mu_0 - \langle d_1 + d_2, x \rangle \tag{5.20}$$

where $f_{\mathbb{K},r} : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$f_{\mathbb{K},r}(x) = \inf_{\alpha_1, \alpha_2} \{\langle \alpha_1 + \alpha_2, x \rangle : \quad \alpha_1 - \alpha_2 = r, \quad \alpha_1, \alpha_2 \in \mathbb{K}^*\} \tag{5.21}$$

$$= \max_{\rho} \{\langle r, \rho \rangle : \quad x - \rho \in \mathbb{K}, \quad x + \rho \in \mathbb{K}\}. \tag{5.22}$$

Proof. Consider (5.18). Note that

$$\begin{aligned}\inf_{\mu} \{\langle \mu, x \rangle : \mu \in \mathbb{M}\} &= \inf_{\mu, \alpha_1, \alpha_2} \{\langle \mu, x \rangle : \quad \mu = \alpha_1 + d_1, \quad \mu = \alpha_2 + d_2, \quad \alpha_1, \alpha_2 \in \mathbb{K}^*\} \\ &= \frac{1}{2}\langle d_1 + d_2, x \rangle + \frac{1}{2} \inf_{\alpha_1, \alpha_2} \left\{ \langle \alpha_1 + \alpha_2, x \rangle : \quad \begin{array}{c} \alpha_1 - \alpha_2 = r, \\ \alpha_1, \alpha_2 \in \mathbb{K}^* \end{array} \right\} \\ &= \frac{1}{2}\langle d_1 + d_2, x \rangle + \frac{1}{2} f_{\mathbb{K},r}(x).\end{aligned}$$

Therefore, (5.18) is equivalent to (5.20). Lemma 5.4(i) shows that there always exist $\widehat{\alpha}_1, \widehat{\alpha}_2 \in \mathbb{K}^*$ such that $\widehat{\alpha}_1 - \widehat{\alpha}_2 = r$. Hence, (5.21) is always feasible. Indeed, this minimization problem is strictly feasible because, for any $e \in \operatorname{int} \mathbb{K}^*$, we have $\widehat{\alpha}_1 + e, \widehat{\alpha}_2 + e \in \operatorname{int} \mathbb{K}^*$

and $(\widehat{\alpha}_1 + e) - (\widehat{\alpha}_2 + e) = r$. Therefore, the strong duality theorem of conic programming applies, and the dual problem (5.22) is solvable whenever the optimal value of (5.21) is bounded from below. □

Next, we make a series of immediate observations on the function $f_{\mathbb{K},r}(x)$.

Remark 5.20. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $r \in \mathbb{E}$.*

i. As a function of x, $-f_{\mathbb{K},r}(x)$ is the support function of a nonempty set (see (5.21)). Therefore, it is closed and sublinear. Furthermore, the value of $-f_{\mathbb{K},r}(x)$ is finite if and only if $x \in \mathbb{K}$.

ii. The function $f_{\mathbb{K},r}(x)$ satisfies $f_{\mathbb{K},r}(x) \geq |\langle r, x\rangle|$ for any $x \in \mathbb{K}$. If x is an extreme ray of $\mathbb{K}$, then $f_{\mathbb{K},r}(x) = |\langle r, x\rangle|$.

Proof. We only prove statement (ii). Let $x \in \mathbb{K}$. Both x and $-x$ are feasible solutions to (5.22). Therefore, $f_{\mathbb{K},r}(x) \geq |\langle r, x\rangle|$. Now suppose x is an extreme ray of $\mathbb{K}$. Let $\rho \in \mathbb{E}$ be any feasible solution to (5.22). We show $\rho \in \operatorname{conv}\{x, -x\}$. First, note that $\frac{1}{2}(x - \rho) + \frac{1}{2}(x + \rho) = x$. Because x is an extreme ray of K, there must exist $\lambda_1, \lambda_2 \geq 0$ such that $x - \rho = \lambda_1 x$ and $x + \rho = \lambda_2 x$. It follows that $\rho = (1 - \lambda_1)x = (\lambda_2 - 1)x$ and $\lambda_1 + \lambda_2 = 2$, which completes the proof of the claim. □

Remark 5.20(i) immediately implies the convexity of the inequality (5.20) because its right-hand side is a linear function of x.

Recall from Remark 5.18 that (5.19) provides an exact representation of the disjunction $\langle d_1, x\rangle \geq \mu_0 \vee \langle d_2, x\rangle \geq \mu_0$. Remark 5.20 shows that $f_{\mathbb{K},r}(x)$ is a concave function of x which satisfies $f_{\mathbb{K},r}(x) \geq |\langle r, x\rangle|$ for any $x \in \mathbb{K}$. Replacing the term $|\langle r, x\rangle|$ on the left-hand side of (5.19) with any such function would define a convex relaxation of $\mathbb{D}_1 \cup \mathbb{D}_2$ inside the cone $\mathbb{K}$. However, $f_{\mathbb{K},r}(x)$ is a "tight" concave overestimator of the function $x \mapsto |\langle r, x\rangle| : \mathbb{E} \to \mathbb{R}$ over $\mathbb{K}$: It satisfies $f_{\mathbb{K},r}(x) = |\langle r, x\rangle|$ whenever x is an extreme ray of $\mathbb{K}$. This indicates that an extreme ray $x \in \mathbb{K}$ satisfies (5.20) if and only if $x \in \mathbb{D}_1 \cup \mathbb{D}_2$. Furthermore, if the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Theorem 5.9, the inequality (5.20) defines the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$.

Remark 5.21. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $x \in \mathbb{K}$.*

i. As a function of r, $f_{\mathbb{K},r}(x)$ is the support function of a bounded set which contains the origin (see (5.22)). Therefore, it is nonnegative, finite-valued, and sublinear.

ii. As a function of r, $f_{\mathbb{K},r}(x)$ is symmetric with respect to the origin, that is, $f_{\mathbb{K},r}(x) = f_{\mathbb{K},-r}(x)$ for any $r \in \mathbb{E}$.

Remark 5.22. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Let $x \in \mathbb{K}$.*

i. If $r \in \mathbb{K}^*$, then $f_{\mathbb{K},r}(x) = \langle r, x\rangle$; if $-r \in \mathbb{K}^*$, then $f_{\mathbb{K},r}(x) = \langle -r, x\rangle$. Thus, $f_{\mathbb{K},r}(x) = |\langle r, x\rangle|$ if $r \in \pm\mathbb{K}^*$.

ii. If $r \notin \pm\mathbb{K}^*$, then $f_{\mathbb{K},r}(x) = f'_{\mathbb{K},r}(x)$ where $f'_{\mathbb{K},r} : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$f'_{\mathbb{K},r}(x) = \inf_{\alpha_1,\alpha_2} \{\langle \alpha_1 + \alpha_2, x\rangle : \ \alpha_1 - \alpha_2 = r, \ \alpha_1, \alpha_2 \in \operatorname{bd} \mathbb{K}^*\}. \tag{5.23}$$

Proof. We only prove statement (*ii*). The inequality $f_{\mathbb{K},r}(x) \le f'_{\mathbb{K},r}(x)$ follows from the observation that the feasible solution set of the minimization problem (5.23) is a restriction of the feasible solution set of the minimization problem (5.21). The inequality $f_{\mathbb{K},r}(x) \ge f'_{\mathbb{K},r}(x)$ follows from Lemma 5.4(ii) and the hypothesis $x \in \mathbb{K}$. □

Remark 5.22(ii) shows that, when $r \notin \pm\mathbb{K}^*$, the variables α_1, α_2 in the minimization problem (5.21) can be restricted to the boundary of the cone $\mathbb{K}^*$ without changing the optimal value of the problem. Note that this conclusion parallels the necessary conditions for undominated valid linear inequalities obtained in Proposition 5.5.

We can use Proposition 5.19 together with Remarks 5.20(i) and 5.21(i) to build simple convex inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$ as follows.

Remark 5.23. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $r \in \mathbb{E}$. For any $r_1, \ldots, r_\ell \in \mathbb{E}$ such that $r = \sum_{i=1}^{\ell} r_i$, we have $\sum_{i=1}^{\ell} f_{\mathbb{K},r_i}(x) \ge f_{\mathbb{K},r}(x)$. Therefore, the inequality $\sum_{i=1}^{\ell} f_{\mathbb{K},r_i}(x) \ge 2\mu_0 - \langle d_1 + d_2, x\rangle$ is a relaxation of* (5.20). *Furthermore, note from Remark 5.20(i) that each function $f_{\mathbb{K},r_i}(x)$ is a concave function of x; hence, the resulting inequality is convex.*

Remark 5.23 suggests a general procedure for developing convex inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$ which might have nicer structural properties than (5.20). Furthermore, it allows great flexibility in the choice of the decomposition $r = \sum_{i=1}^{\ell} r_i$. For certain choices of $r_1, \ldots, r_\ell \in \mathbb{E}$, the relaxation suggested in Remark 5.23 has the interpretation of relaxing the underlying disjunction. We comment more on this interpretation in Section 7.2.4. Next we consider an immediate application of the procedure outlined in Remark 5.23 which gives valid linear inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$ as a consequence of Remark 5.22(i).

Remark 5.24. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Fix $r \in \mathbb{E}$. By Lemma 5.4, there exists $r_+, r_- \in \mathbb{K}^*$ such that $r = r_+ - r_-$. Remark 5.21(i) shows that $f_{\mathbb{K},r}(x) \le f_{\mathbb{K},r_+}(x) + f_{\mathbb{K},-r_-}(x) = f_{\mathbb{K},r_+}(x) + f_{\mathbb{K},r_-}(x)$. Moreover, because $r_+, r_- \in \mathbb{K}^*$, Remark 5.22(i) implies $f_{\mathbb{K},r_+}(x) = \langle r_+, x\rangle$ and $f_{\mathbb{K},r_-}(x) = \langle r_-, x\rangle$. Finally, using Proposition 5.19, we conclude that any $x \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies the linear inequality*

$$\langle r_+ + r_-, x\rangle \ge 2\mu_0 - \langle d_1 + d_2, x\rangle. \tag{5.24}$$

Note that any possible choice of $r_+, r_- \in \mathbb{K}^*$ satisfying $r = r_+ - r_-$ leads to a different inequality of the form (5.24). *Given a two-term disjunction and a point $x \in \mathbb{K}$ that is desired to be cut off, we can select the best possible inequality of the form* (5.24) *via a conic optimization problem.*

Remark 5.25. *Let $\mathbb{K} \subset \mathbb{E}$ and $\mathbf{K} \subset \mathbb{E}$ be regular cones such that $\mathbf{K} \supset \mathbb{K}$. Then $\mathbf{K}^* \subset \mathbb{K}^*$, and for any $x, r \in \mathbb{E}$, we have $f_{\mathbf{K},r}(x) \geq f_{\mathbb{K},r}(x)$.*

The monotonicity result from Remark 5.25 can be useful when one would like to develop structured convex relaxations of $\mathbb{D}_1 \cup \mathbb{D}_2$ by replacing $\mathbb{K}$ with a regular cone $\mathbf{K} \supset \mathbb{K}$ such that an expression for $f_{\mathbf{K},r}(x)$ is readily available.

Remark 5.26. *Let $\mathbb{E} = \prod_{j=1}^p \mathbb{E}^j$ be a direct product of finite-dimensional Euclidean spaces. Suppose $\mathbb{K} = \prod_{j=1}^p \mathbb{K}^j$ and each $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone. Then*

$$f_{\mathbb{K},r}(x) = \sum_{j=1}^{p} \inf_{\alpha_1^j, \alpha_2^j} \left\{ \langle \alpha_1^j + \alpha_2^j, x^j \rangle_j : \ \alpha_1^j - \alpha_2^j = r^j, \ \alpha_1^j, \alpha_2^j \in (\mathbb{K}^j)^* \right\} = \sum_{j=1}^{p} f_{\mathbb{K}^j, r^j}(x^j).$$

Under the hypotheses of Remark 5.26, let us define the following sets with respect to $r = (r^1, \ldots, r^p) \in \mathbb{E}$:

$$\mathbb{P}^+ = \left\{ j \in [p] \colon -r^j \in (\mathbb{K}^j)^* \right\}, \ \mathbb{P}^- = \left\{ j \in [p] \colon r^j \in (\mathbb{K}^j)^* \right\}, \ \mathbb{P}^\circ = \left\{ j \in [p] \colon r^j \notin \pm(\mathbb{K}^j)^* \right\}. \quad (5.25)$$

Next we state a consequence of Proposition 5.19 and Remarks 5.22(i) and 5.26.

Proposition 5.27. *Let $\mathbb{E} = \prod_{j=1}^p \mathbb{E}^j$ be a direct product of finite-dimensional Euclidean spaces. Suppose $\mathbb{K} = \prod_{j=1}^p \mathbb{K}^j$ and each $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone. Define the sets $\mathbb{P}^+$, $\mathbb{P}^-$, and $\mathbb{P}^\circ$ as in* (5.25).

i. A point $x \in \mathbb{K}$ satisfies (5.20) *if and only if it satisfies*

$$\sum_{j \in \mathbb{P}^\circ} f_{\mathbb{K}^j, r^j}(x^j) + \sum_{j \in \mathbb{P}^\circ} \langle d_1^j + d_2^j, x^j \rangle_j + 2 \sum_{j \in \mathbb{P}^+} \langle d_1^j, x^j \rangle_j + 2 \sum_{j \in \mathbb{P}^-} \langle d_2^j, x^j \rangle_j \geq 2\mu_0. \quad (5.26)$$

ii. A point $x \in \mathbb{K}$ satisfies (5.26) *if and only if there exist $z^j \in \mathbb{R}$, $j \in [p]$, such that*

$$f_{\mathbb{K}^j, r^j}(x^j) \geq |2z^j - \langle d_1^j + d_2^j, x^j \rangle| \quad \forall j \in [p], \quad (5.27a)$$

$$\sum_{j=1}^{p} z^j \geq \mu_0. \quad (5.27b)$$

Furthermore, for each $j \in [p]$, (5.27a) *is equivalent to*

$$\left[f_{\mathbb{K}^j, r^j}(x^j) \right]^2 - \langle r^j, x^j \rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j \rangle_j)(z^j - \langle d_2^j, x^j \rangle_j). \quad (5.28)$$

Proof. Statement (i) follows directly from Proposition 5.19 and Remarks 5.22(i) and 5.26. Fix $x \in \mathbb{K}$. The "if" part of statement (ii) is clear. To show the "only if" part, let $\bar{z}^j = \frac{1}{2}(f_{\mathbb{K}^j, r^j}(x^j) + \langle d_1^j + d_2^j, x^j \rangle_j)$ for each $j \in [p]$. Recall from Remark 5.21(i) that each

$f_{\mathbb{K}^j,r^j}(x^j)$ is finite and nonnegative. Then $2\bar{z}^j - \langle d_1^j + d_2^j, x^j\rangle_j = f_{\mathbb{K}^j,r^j}(x^j) \geq 0$. Hence, $(\bar{z}^1, \ldots, \bar{z}^p)$ satisfies (5.27).

To finish the proof, we show that (5.27a) is equivalent to $[f_{\mathbb{K}^j,r^j}(x^j)]^2 - \langle r^j, x^j\rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j\rangle_j)(z^j - \langle d_2^j, x^j\rangle_j)$ for any $z^j \in \mathbb{R}$. The nonnegativity of $f_{\mathbb{K}^j,r^j}(x^j)$ implies

$$\begin{aligned} f_{\mathbb{K}^j,r^j}(x^j) \geq |2z^j - \langle d_1^j + d_2^j, x^j\rangle_j| \quad &\Leftrightarrow \quad \left[f_{\mathbb{K}^j,r^j}(x^j)\right]^2 \geq (2z^j - \langle d_1^j + d_2^j, x^j\rangle_j)^2 \\ &\Leftrightarrow \quad \left[f_{\mathbb{K}^j,r^j}(x^j)\right]^2 - \langle r^j, x^j\rangle_j^2 \geq 4(z^j - \langle d_1^j, x^j\rangle_j)(z^j - \langle d_2^j, x^j\rangle_j). \end{aligned}$$

□

Remark 5.28. *Under the hypotheses of Proposition 5.27, Remark 5.22(i) shows that* $f_{\mathbb{K}^j,r^j}(x^j) = |\langle r^j, x^j\rangle|$ *for* $j \in \mathbb{P}^+ \cup \mathbb{P}^-$. *Therefore,* (5.27a) *simplifies to* $\langle d_1^j, x^j\rangle \geq z^j \geq \langle d_2^j, x^j\rangle$ *for* $j \in \mathbb{P}^+$ *and to* $\langle d_2^j, x^j\rangle \geq z^j \geq \langle d_1^j, x^j\rangle$ *for* $j \in \mathbb{P}^-$. *Hence, the auxiliary variables* z^j, $j \in \mathbb{P}^+ \cup \mathbb{P}^-$, *can be eliminated from* (5.27) *after setting them equal to their corresponding upper bounds.*

The next remark recovers a well-known result about disjunctions on the nonnegative orthant, as a consequence of Remark 5.28.

Remark 5.29. *Let* $\mathbb{E} = \mathbb{R}^p$ *and* $\mathbb{K} = \mathbb{R}^p_+$. *Note that* $\mathbb{R}^p_+$ *is a decomposable cone: It can be seen as a direct product* $\prod_{j=1}^p \mathbb{K}^j$ *where* $\mathbb{K}^j = \mathbb{R}_+$ *for all* $j \in [p]$. *Then Remark 5.22(i), together with the fact that* $r^j \in \pm\mathbb{R}_+$ *for all* $j \in [p]$, *implies* $f_{\mathbb{R}^p_+,r}(x) = \sum_{j=1}^p |r^j x^j| = \sum_{j=1}^p |r^j| x^j$ *for all* $x \in \mathbb{R}^p_+$. *Proposition 5.19 shows that the inequality* $\sum_{j=1}^p |r^j| x^j \geq 2\mu_0 - \langle d_1 + d_2, x\rangle$ *is valid for* $\mathbb{D}_1 \cup \mathbb{D}_2$. *This inequality can be further simplified into*

$$\sum_{j=1}^p \max\left\{d_1^j, d_2^j\right\} x^j \geq \mu_0.$$

5.4.2 Inequalities for Intersections with Rank-Two Non-Convex Quadratics

In this section, we consider sets of the form $\mathbb{K} \cap \mathbb{F}$ where $\mathbb{K} \subset \mathbb{E}$ is a regular cone and $\mathbb{F} \subset \mathbb{E}$ is a non-convex set defined by a rank-two quadratic inequality:

$$\mathbb{F} = \{x \in \mathbb{E} : \ (\mu_0 - \langle d_1, x\rangle)(\mu_0 - \langle d_2, x\rangle) \leq 0\}. \tag{5.29}$$

We will show how the results of Sections 5.2.2 and 5.4.1 can be combined to build convex relaxations and convex hull descriptions for $\mathbb{K} \cap \mathbb{F}$.

As in the previous section, we denote $r = d_2 - d_1 \in \mathbb{E}$. We start with a simple observation on an alternate representation of $\mathbb{F}$, which parallels Remark 5.18.

Remark 5.30. *A point $x \in \mathbb{E}$ satisfies $(\mu_0 - \langle d_1, x\rangle)(\mu_0 - \langle d_2, x\rangle) \leq 0$ if and only if it satisfies*

$$|\langle r, x\rangle| \geq |2\mu_0 - \langle d_1 + d_2, x\rangle| \tag{5.30}$$

The following result is a consequence of Remark 5.21(ii) and Propositions 5.2 and 5.19.

Proposition 5.31. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{F} \subset \mathbb{E}$ defined as in* (5.29)*. Let $\mathbb{D}_i^+ = \{x \in \mathbb{K} : \langle d_i, x\rangle \geq \mu_0\}$ and $\mathbb{D}_i^- = \{x \in \mathbb{K} : \langle d_i, x\rangle \leq \mu_0\}$ for $i \in \{1, 2\}$.*

i. Any point $x \in \mathbb{K} \cap \mathbb{F}$ satisfies

$$f_{\mathbb{K},r}(x) \geq |2\mu_0 - \langle d_1 + d_2, x\rangle|. \tag{5.31}$$

ii. Suppose $\overline{\text{conv}}(\mathbb{D}_1^+ \cup \mathbb{D}_2^+) = \mathbb{K}$ or the sets $\mathbb{D}_1^+$ and $\mathbb{D}_2^+$ satisfy the conditions of Theorem 5.9. Suppose also that $\overline{\text{conv}}(\mathbb{D}_1^- \cup \mathbb{D}_2^-) = \mathbb{K}$ or the sets $\mathbb{D}_1^-$ and $\mathbb{D}_2^-$ satisfy the conditions of Theorem 5.9. Then

$$\overline{\text{conv}}(\mathbb{K} \cap \mathbb{F}) = \{x \in \mathbb{K} :\ f_{\mathbb{K},r}(x) \geq |2\mu_0 - \langle d_1 + d_2, x\rangle|\}. \tag{5.32}$$

Proof. Note that $\mathbb{K} \cap \mathbb{F} = (\mathbb{D}_1^+ \cup \mathbb{D}_2^+) \cap (\mathbb{D}_1^- \cup \mathbb{D}_2^-)$. Using Proposition 5.19 for $\mathbb{D}_1^+ \cup \mathbb{D}_2^+$ and $\mathbb{D}_1^- \cup \mathbb{D}_2^-$ shows that the inequalities $f_{\mathbb{K},r}(x) \geq 2\mu_0 - \langle d_1 + d_2, x\rangle$ and $f_{\mathbb{K},-r}(x) \geq -2\mu_0 + \langle d_1 + d_2, x\rangle$ are both valid for $\mathbb{K} \cap \mathbb{F}$. By Remark 5.21(ii), $f_{\mathbb{K},-r}(x) = f_{\mathbb{K},r}(x)$ for any $r \in \mathbb{E}$ and $x \in \mathbb{K}$. Therefore, the two inequalities together are equivalent to (5.31). Under the hypotheses of statement (*ii*), we have

$$\begin{aligned}\overline{\text{conv}}(\mathbb{D}_1^+ \cup \mathbb{D}_2^+) &= \{x \in \mathbb{K} :\ f_{\mathbb{K},r}(x) \geq 2\mu_0 - \langle d_1 + d_2, x\rangle\} \text{ and} \\ \overline{\text{conv}}(\mathbb{D}_1^- \cup \mathbb{D}_2^-) &= \{x \in \mathbb{K} :\ f_{\mathbb{K},-r}(x) \geq -2\mu_0 + \langle d_1 + d_2, x\rangle\}.\end{aligned}$$

Then Proposition 5.2 shows (5.32). □

The next proposition shows that the linear inequality in (5.27) can be replaced with a linear equality when we consider the intersection of $\mathbb{K}$ with a rank-two non-convex quadratic instead of a two-term disjunction.

Proposition 5.32. *Let $\mathbb{E} = \prod_{j=1}^p \mathbb{E}^j$ be a direct product of finite-dimensional Euclidean spaces. Suppose $\mathbb{K} = \prod_{j=1}^p \mathbb{K}^j$ and each $\mathbb{K}^j \subset \mathbb{E}^j$ is a regular cone. A point $x \in \mathbb{K}$ satisfies* (5.31) *if and only if there exist $z^j \in \mathbb{R}$, $j \in [p]$, such that* (5.27a) *(or, equivalently* (5.28)*) holds together with $\sum_{j=1}^p z^j = \mu_0$.*

Proof. Fix $x \in \mathbb{K}$. The "if" part follows from the triangle inequality. To show the "only if" part, recall from Proposition 5.27(ii) that x satisfies $f_{\mathbb{K}^j,r^j}(x^j) \geq 2\mu_0 - \langle d_1^j + d_2^j, x^j\rangle$ if and only if there exist $t_1^j \in \mathbb{R}$, $j \in [p]$, such that

$$f_{\mathbb{K}^j,r^j}(x^j) \geq |2t_1^j - \langle d_1^j + d_2^j, x^j \rangle| \quad \forall j \in [p], \tag{5.33a}$$

$$\sum_{j=1}^{p} t_1^j \geq \mu_0. \tag{5.33b}$$

Furthermore, x satisfies $f_{\mathbb{K}^j,r^j}(x^j) \geq -2\mu_0 + \langle d_1^j + d_2^j, x^j \rangle$ if and only if there exist $t_2^j \in \mathbb{R}$, $j \in [p]$, such that

$$f_{\mathbb{K}^j,r^j}(x^j) \geq |-2t_2^j + \langle d_1^j + d_2^j, x^j \rangle| \quad \forall j \in [p], \tag{5.34a}$$

$$-\sum_{j=1}^{p} t_2^j \geq -\mu_0. \tag{5.34b}$$

Let $0 \leq \delta \leq 1$ such that $\delta \sum_{j=1}^{p} t_1^j - (1-\delta) \sum_{j=1}^{p} t_2^j = \mu_0$. For all $j \in [p]$, we also define $z^j = \delta t_1^j - (1-\delta) t_2^j$. Then $\sum_{j=1}^{p} z^j = \mu_0$. For any $j \in [p]$, combining (5.33a) and (5.34a) with weights δ and $1-\delta$, we have

$$\begin{aligned} f_{\mathbb{K}^j,r^j}(x^j) &\geq \delta|2t_1^j - \langle d_1^j + d_2^j, x^j \rangle| + (1-\delta)|-2t_2^j + \langle d_1^j + d_2^j, x^j \rangle| \\ &= \delta|2t_1^j - \langle d_1^j + d_2^j, x^j \rangle| + (1-\delta)|2t_2^j - \langle d_1^j + d_2^j, x^j \rangle| \\ &\geq |2z^j - \langle d_1^j + d_2^j, x^j \rangle|, \end{aligned}$$

where the second inequality holds because the function $z \mapsto |2z - \langle d_1^j + d_2^j, x^j \rangle| : \mathbb{R} \to \mathbb{R}$ is convex. This completes the proof of the first part. Finally, we note that the equivalence of (5.27a) to $\left[f_{\mathbb{K}^j,r^j}(x^j)\right]^2 - \langle r^j, x^j \rangle_j^2 \geq 4\left(z^j - \langle d_1^j, x^j \rangle_j\right)\left(z^j - \langle d_2^j, x^j \rangle_j\right)$ can be shown as in the proof of Proposition 5.27. ☐

We close this section by presenting a result which complements the relation between convex hulls of non-convex quadratic sets of form $\mathbb{K} \cap \mathbb{F}$ and the associated disjunctions given in Proposition 5.31. In particular, we show that given a structured and explicit characterization of the closed convex hull of $\mathbb{F} \cap \mathbb{K}$, we can obtain a convex hull characterization of $\mathbb{D}_1 \cup \mathbb{D}_2$ even when the disjointness assumption is violated.

Proposition 5.33. *Let $\mathbb{K} \subset \mathbb{E}$ be a regular cone. Consider $\mathbb{D}_1, \mathbb{D}_2 \subset \mathbb{E}$ defined as in* (5.17) *and $\mathbb{F} \subset \mathbb{E}$ defined as in* (5.29). *Let $g(x) : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ be an upper semi-continuous, concave function such that $g(x) \geq 0$ for any $x \in \mathbb{K}$ and $\mathbb{K} \cap \mathbb{F} \subset \{x \in \mathbb{K} : g(x) \geq |2\mu_0 - \langle d_1 + d_2, x \rangle|\}$.*

i. Any point $x \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies the convex inequality

$$g(x) \geq 2\mu_0 - \langle d_1 + d_2, x \rangle. \tag{5.35}$$

ii. *If* $\overline{\text{conv}}(\mathbb{K} \cap \mathbb{F}) = \{x \in \mathbb{K} : \ g(x) \geq |2\mu_0 - \langle d_1 + d_2, x\rangle|\}$, *then*

$$\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = \{x \in \mathbb{K} : \ g(x) \geq 2\mu_0 - \langle d_1 + d_2, x\rangle\}. \tag{5.36}$$

Proof. Note that $\mathbb{D}_1 \cup \mathbb{D}_2 = (\mathbb{K} \cap \mathbb{F}) \cup (\mathbb{D}_1 \cap \mathbb{D}_2)$. Our hypotheses ensure that any $x \in \mathbb{K} \cap \mathbb{F}$ satisfies (5.35). Moreover, for any $x \in \mathbb{D}_1 \cap \mathbb{D}_2$, we have $0 \geq 2\mu_0 - \langle d_1 + d_2, x\rangle$. Then (5.35) is valid for $\mathbb{D}_1 \cap \mathbb{D}_2$ because $g(x)$ is nonnegative for $x \in \mathbb{K}$.

Statement (i), together with the concavity of $g(x)$, shows that (5.35) is valid for the convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$. The continuity of $g(x)$ implies the validity of (5.35) for the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$. If $\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = \mathbb{K}$, then (5.35) is redundant. Suppose $\overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) \neq \mathbb{K}$. Assume for contradiction that there exists $\bar{x} \in \mathbb{K}$ satisfying (5.35) but $\bar{x} \notin \overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2)$. Then $\bar{x} \notin \overline{\text{conv}}(\mathbb{K} \cap \mathbb{F})$ as well; thus $g(\bar{x}) < |2\mu_0 - \langle d_1 + d_2, \bar{x}\rangle|$. Combining this with (5.35), we arrive at

$$|2\mu_0 - \langle d_1 + d_2, \bar{x}\rangle| > g(\bar{x}) \geq 2\mu_0 - \langle d_1 + d_2, \bar{x}\rangle,$$

which implies $0 > 2\mu_0 - \langle d_1 + d_2, \bar{x}\rangle$. Then at least one of $0 > \mu_0 - \langle d_1, \bar{x}\rangle$ or $0 > \mu_0 - \langle d_2, \bar{x}\rangle$ must hold. Hence, $\bar{x} \in \mathbb{D}_1 \cup \mathbb{D}_2$, contradicting the assumption $\bar{x} \notin \overline{\text{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2)$. This proves the relation stated in (5.36). □

5.4.3 Inequalities for Disjoint Two-Term Disjunctions

As in Section 5.4.1, we consider sets $\mathbb{D}_1$ and $\mathbb{D}_2$ defined as in (5.17). In this section, we assume $\{x \in \mathbb{K} : \ \langle d_1, x\rangle > \mu_0, \ \langle d_2, x\rangle > \mu_0\} = \emptyset$. Whenever this is the case, we say that $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the *disjointness condition.* Such sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are naturally associated with rank-two quadratic constraints: In particular, under the disjointness condition, $\mathbb{D}_1 \cup \mathbb{D}_2 = \mathbb{K} \cap \mathbb{F}$ where $\mathbb{F}$ is defined as in (5.29). Therefore, we can immediately use the results of Section 5.4.2 in this case. Specifically, we have the following result.

Corollary 5.34. *Let* $\mathbb{K} \subset \mathbb{E}$ *be a regular cone. Consider* $\mathbb{D}_1$ *and* $\mathbb{D}_2$ *defined as in* (5.17).

i. *Let* $x \in \mathbb{K}$ *be such that* $\langle d_1, x\rangle \leq \mu_0 \vee \langle d_2, x\rangle \leq \mu_0$. *Then* x *satisfies* (5.20) *if and only if it satisfies* (5.31).

ii. *Suppose* $\mathbb{D}_1$ *and* $\mathbb{D}_2$ *satisfy the disjointness condition. Then a point* $x \in \mathbb{K}$ *satisfies* (5.20) *if and only if it satisfies* (5.31).

Proof. We first prove statement (i). Let $x \in \mathbb{K}$ be such that $\langle d_1, x\rangle \leq \mu_0 \vee \langle d_2, x\rangle \leq \mu_0$. Then x satisfies the inequality $f_{\mathbb{K},-r}(x) \geq -2\mu_0 + \langle d_1 + d_2, x\rangle$. Recall from Remark 5.21(ii) that $f_{\mathbb{K},-r}(x) = f_{\mathbb{K},r}(x)$ for any $r \in \mathbb{E}$. Hence, x satisfies (5.20) if and only if it satisfies (5.31).

Under the disjointness condition, any point $x \in \mathbb{K}$ satisfies the disjunction $\langle d_1, x\rangle \leq \mu_0 \vee \langle d_2, x\rangle \leq \mu_0$. The result follows from statement (i). □

5.5 Conclusion

In this chapter we have considered two-term disjunctions on a regular cone $\mathbb{K}$ and intersections of a regular cone $\mathbb{K}$ with rank-two non-convex quadratics. These sets provide fundamental non-convex relaxations for conic programs with integrality requirements and other types of non-convex constraints. We have characterized the structure of undominated valid linear inequalities for two-term disjunctions on $\mathbb{K}$ and developed a general theory for constructing closed convex hull descriptions and low-complexity relaxations of such sets in the space of the original variables. These relaxations can be used to strengthen the natural continuous relaxations of MICPs.

In Chapters 6 and 7, we consider the cases where $\mathbb{K}$ is a direct product of second-order cones and nonnegative rays and where $\mathbb{K}$ is the positive semidefinite cone, respectively. Building upon the theory we have developed in this chapter, we investigate how we can derive closed-form structured nonlinear valid inequalities and closed convex hull descriptions for two-term disjunctions on $\mathbb{K}$ in these particular cases.

Chapter 6

Convex Hulls of Disjunctions on Second-Order Cones

Acknowledgments. This chapter is based on joint work with Fatma Kılınç-Karzan [84]. A preliminary version appeared in [83].

6.1 Introduction

6.1.1 Motivation

Let $\mathbb{L}^k = \left\{ x \in \mathbb{R}^k : \sqrt{x_1^2 + \ldots + x_{k-1}^2} \le x_k \right\}$ denote the k-dimensional second-order cone. In this chapter we consider general two-term disjunctions on direct products of second-order cones and nonnegative rays. To be precise, let $\mathbb{K} \subset \mathbb{R}^n$ be defined as $\mathbb{K} = \prod_{j=1}^{p_1+p_2} \mathbb{K}^j$ where $\mathbb{K}^j = \mathbb{L}^{n^j}$ for $j \in \{1, \ldots, p_1\}$ and $\mathbb{K}^j = \mathbb{R}_+$ for $j \in \{p_1+1, \ldots, p_1+p_2\}$. Associated with a disjunction $\langle c_1, x \rangle \ge c_{1,0} \vee \langle c_2, x \rangle \ge c_{2,0}$ on $\mathbb{K}$, we define the sets

$$\mathbb{C}_i = \{x \in \mathbb{K} : \quad \langle c_i, x \rangle \ge c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}. \tag{6.1}$$

The purpose of this chapter is to provide an explicit outer description of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ with closed-form convex inequalities in the space of the original variables. To this end, we specialize the results of Chapter 5 to our setting. In the greater part of this chapter, we focus on the case where $\mathbb{K}$ is a single second-order cone. We note that, although we consider two-term disjunctions on $\mathbb{K}$ in this chapter, our results also extend to two-term disjunctions on sets of the form $\{x \in \mathbb{R}^n : Ax - b \in \mathbb{K}\}$ where the matrix A has full row rank through the affine transformation discussed in [8].

The reader is referred to Section 5.1.2 for a detailed discussion of disjunctive inequalities in mixed-integer conic programming. Prior to our study, similar results which characterize the convex hulls of two-term disjunctions on a single second-order cone appeared in [8, 89]. Nevertheless, our work is set apart from [8, 89] by the fact that we study two-term disjunctions on the second-order cone in full generality and do not restrict our attention to split disjunctions, which are defined by *parallel* hyperplanes. Our analysis shows that the resulting convex hulls can turn out to be significantly more complex in the absence of this assumption. Furthermore, our proof techniques originate from the conic programming duality perspective of Chapter 5 and are completely different from what was employed in the aforementioned papers.

Chapter 8 extends the results of this chapter in two directions: First, we show that a convex inequality of the form developed in this chapter can describe the convex hull of homogeneous two-term disjunctions on the second-order cone. Second, we show that such an inequality can characterize the closed convex hull of two-term disjunctions on affine *cross-sections* of the second-order cone under certain conditions. Similar and complementary results on describing the convex hull of intersections of the second-order cone or its affine cross-sections with a single homogeneous quadratic have recently been obtained in [38, 88].

6.1.2 Notation and Terminology

We assume that $\mathbb{R}^n$ is equipped with the standard inner product $\langle \alpha, x \rangle = \alpha^\top x$. The standard (Euclidean) norm $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ is defined as $\|x\|_2 = \sqrt{\langle x, x \rangle}$.

For any positive integer k, let $[k] = \{1, \ldots, k\}$. In this chapter, we consider a regular cone $\mathbb{K} \subset \mathbb{R}^n$ such that $\mathbb{K} = \prod_{j=1}^{p_1+p_2} \mathbb{K}^j$ where $\mathbb{K}^j = \mathbb{L}^{n^j}$ for $j \in [p_1]$ and $\mathbb{K}^{p_1+j} = \mathbb{R}_+$ for $j \in [p_2]$. The dual cone of $\mathbb{V} \subset \mathbb{R}^n$ is $\mathbb{V}^* = \{\alpha \in \mathbb{R}^n : \langle x, \alpha \rangle \geq 0 \ \forall x \in \mathbb{V}\}$. We remind the reader that $\mathbb{K}$ is self-dual, that is, the dual cone of $\mathbb{K}$ is again $\mathbb{K}$ itself. We let $\operatorname{conv} \mathbb{V}$, $\overline{\operatorname{conv}}\, \mathbb{V}$, and $\operatorname{span} \mathbb{V}$ represent the convex hull, closed convex hull, and linear span of a set $\mathbb{V} \subset \mathbb{R}^n$, respectively. We let $\operatorname{int} \mathbb{V}$, and $\operatorname{bd} \mathbb{V}$ represent the topological interior and boundary of $\mathbb{V} \subset \mathbb{R}^n$, respectively. We use $\operatorname{rec} \mathbb{V}$ to refer to the recession cone of a closed convex set $\mathbb{V}$. For $i \in [n]$, we let e^i be the i-th unit vector in $\mathbb{R}^n$, and for a vector $x \in \mathbb{R}^n$, we use $\tilde{x}$ to denote the subvector $\tilde{x} = (x_1; \ldots; x_{n-1})$.

Throughout the chapter, we consider sets $\mathbb{C}_1$ and $\mathbb{C}_2$ of the form (6.1). If $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$ and the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Conditions 5.1 and 5.2, we say that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. If $\{x \in \mathbb{K} : \ \langle c_1, x \rangle > c_{1,0}, \ \langle c_2, x \rangle > c_{2,0}\} = \emptyset$, we say that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the disjointness condition.

6.1.3 Outline of the Chapter

The set $\mathbb{C}_1 \cup \mathbb{C}_2$, the object of our analysis in this chapter, is a special case of that studied in Chapter 5. In this chapter, we build upon the results of Chapter 5 and characterize the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ with closed-form nonlinear inequalities.

Define $\mathbb{B} = \{(\beta_1, \beta_2) \in \mathbb{R}^2 : \ \beta_1, \beta_2 > 0, \ \beta_2 c_2 - \beta_1 c_1 \notin \pm \operatorname{int} \mathbb{K}^*\}$ as in (5.11). Let $(\beta_1, \beta_2) \in \mathbb{B}$ and $\mu_0(\beta_1, \beta_2) = \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$. Proposition 5.19 shows that any point $x \in \mathbb{C}_1 \cup \mathbb{C}_2$ satisfies the convex inequality

$$f_{\mathbb{K},\beta_2 c_2 - \beta_1 c_1}(x) \geq 2\mu_0(\beta_1, \beta_2) - \langle \beta_1 c_1 + \beta_2 c_2, x \rangle \tag{6.2}$$

Furthermore, when $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup, Remark 5.17 and Proposition 5.19 guarantee that

$$\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \{x \in \mathbb{K} : \ f_{\mathbb{K},\beta_2 c_2 - \beta_1 c_1}(x) \geq 2\mu_0(\beta_1, \beta_2) - \langle \beta_1 c_1 + \beta_2 c_2, x \rangle \ \ \forall (\beta_1, \beta_2) \in \mathbb{B}\}.$$

In Section 6.2.1, we study the fundamental case where $\mathbb{K}$ is a single second-order cone. We develop an equivalent closed-form expression for (6.2) and show that it admits a second-order cone representation in a lifted space with one additional variable. Under a certain disjointness condition, the additional variable in the second-order cone representation of (6.2) can be eliminated, leading to a valid second-order cone inequality in the space of the original variables. In Section 6.2.2, we extend these results to the case where $\mathbb{K}$ is a direct product of multiple second-order cones and nonnegative rays. Throughout Section 6.2, we also investigate the relationship of two-term disjunctions on $\mathbb{K}$ with non-convex sets defined by rank-two quadratics.

In Section 6.3, we seek to provide an explicit closed convex hull description of $\mathbb{C}_1 \cup \mathbb{C}_2$ in the case where $\mathbb{K}$ is a single second-order cone. As a simple consequence of Theorem 5.9, Proposition 5.19, and our analysis in Section 6.2, we show that the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ can be described with a single closed-form convex inequality for certain choices of disjunction on $\mathbb{K}$. For general two-term disjunctions, we outline a procedure to reach closed convex hull descriptions in Section 6.3.2. We illustrate our results with three examples.

6.2 Disjunctions on Direct Products on Second-Order Cones

In this section, we consider a fixed $(\beta_1, \beta_2) \in \mathbb{B}$. Let $d_i = \beta_i c_i$ for $i \in \{1, 2\}$ and $\mu_0 = \min\{\beta_1 c_{1,0}, \beta_2 c_{2,0}\}$. As in Section 5.4, we consider the relaxed disjunction $\langle d_1, x \rangle \geq \mu_0 \vee \langle d_2, x \rangle \geq \mu_0$ on $\mathbb{K}$. We concentrate our analysis on the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ where

$$\mathbb{D}_i = \{x \in \mathbb{K} : \quad \langle d_i, x \rangle \geq \mu_0\} \quad \text{for} \quad i \in \{1, 2\}. \tag{6.3}$$

If the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition, then $\mathbb{D}_1 \cup \mathbb{D}_2 = \mathbb{F} \cap \mathbb{K}$ where $\mathbb{F} \subset \mathbb{R}^n$ is a non-convex set defined by a rank-two quadratic of the form

$$\mathbb{F} = \{x \in \mathbb{R}^n : \quad (\mu_0 - \langle d_1, x\rangle)(\mu_0 - \langle d_2, x\rangle) \leq 0\}. \tag{6.4}$$

Throughout this section, we denote $r = d_2 - d_1$. We are mainly interested in sets $\mathbb{D}_1$ and $\mathbb{D}_2$ which satisfy Condition 5.1. This implies $r \notin \pm\mathbb{K}$ in our analysis.

6.2.1 Disjunctions on a Single Second-Order Cone

Let $\mathbb{K} = \mathbb{L}^n$. In this section we develop closed-form convex inequalities for the set $\mathbb{D}_1 \cup \mathbb{D}_2$, where $\mathbb{D}_1$ and $\mathbb{D}_2$ are defined as in (6.3). We specialize Propositions 5.19 and 5.31 to this setting in Theorem 6.3. This result is based on the following lemma.

Lemma 6.1. *For any $r \notin \pm\mathbb{L}^n$ and $x \in \mathbb{L}^n$, we have*

$$f_{\mathbb{L}^n,r}(x) = \sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)}. \tag{6.5}$$

Proof. Let $r \notin \pm\mathbb{L}^n$ and $x \in \mathbb{L}^n$. Recall from Remark 5.22(ii) that

$$f_{\mathbb{L}^n,r}(x) = f'_{\mathbb{L}^n,r}(x) = \inf_{\alpha_1,\alpha_2}\{\langle \alpha_1 + \alpha_2, x\rangle : \quad \alpha_1 - \alpha_2 = r, \quad \alpha_1, \alpha_2 \in \operatorname{bd}\mathbb{L}^n\}.$$

Because $r \notin \pm\mathbb{L}^n$, Moreau's decomposition theorem implies that there exist orthogonal nonzero vectors $\alpha_1^*, \alpha_2^* \in \operatorname{bd}\mathbb{L}^n$ such that $r = \alpha_1^* - \alpha_2^*$. Thus, the minimization problem above is feasible. Defining a new variable $\pi = \alpha_1 + \alpha_2$, we can rewrite $f_{\mathbb{L}^n,r}(x)$ as

$$f_{\mathbb{L}^n,r}(x) = \inf_{\pi}\{\langle \pi, x\rangle : \quad \|\tilde{\pi} + \tilde{r}\|_2 = \pi_n + r_n, \quad \|\tilde{\pi} - \tilde{r}\|_2 = \pi_n - r_n\}$$

Let $\mathbb{P} = \{\pi \in \mathbb{R}^n : \ \|\tilde{\pi} + \tilde{r}\|_2 = \pi_n + r_n, \ \|\tilde{\pi} - \tilde{r}\|_2 = \pi_n - r_n\}$. Note that

$$\mathbb{P} = \{\pi \in \mathbb{R}^n : \quad \|\tilde{\pi} + \tilde{r}\|_2 = \|\tilde{\pi} - \tilde{r}\|_2 + 2r_n, \quad \|\tilde{\pi} - \tilde{r}\|_2 = \pi_n - r_n\}.$$

After taking the square of both sides of the first equation above, noting $r \notin \pm\mathbb{L}^n$, and replacing the term $\|\tilde{\pi} - \tilde{r}\|_2$ with $\pi_n - r_n$, we arrive at

$$\mathbb{P} = \left\{\pi \in \mathbb{R}^n : \quad \left\langle \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix}, \pi \right\rangle = 0, \quad \|\tilde{\pi} - \tilde{r}\|_2 = \pi_n - r_n \right\}.$$

Then we have

$$f_{\mathbb{L}^n,r}(x) = \inf_{\pi}\left\{\langle \pi, x\rangle : \quad \left\langle \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix}, \pi \right\rangle = 0, \quad \|\tilde{\pi} - \tilde{r}\|_2 = \pi_n - r_n \right\}.$$

Unfortunately, the optimization problem stated above is non-convex due to the second equality constraint. We show below that the natural convex relaxation for this problem is tight. Indeed, consider the relaxation

$$\inf_{\pi} \left\{ \langle \pi, x \rangle : \quad \left\langle \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix}, \pi \right\rangle = 0, \quad \|\tilde{\pi} - \tilde{r}\|_2 \leq \pi_n - r_n \right\}$$

The feasible region of this relaxation is the intersection of a hyperplane with a second-order cone shifted by the vector r. Any solution which is feasible to the relaxation but not the original problem can be expressed as a convex combination of solutions feasible to the original problem. Because we are optimizing a linear function, this shows that the relaxation is equivalent to the original problem. Thus, we have

$$f_{\mathbb{L}^n,r}(x) = \inf_{\pi} \left\{ \langle \pi, x \rangle : \quad \left\langle \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix}, \pi \right\rangle = 0, \quad \pi - r \in \mathbb{L}^n \right\}.$$

Consider α_1^* and α_2^* defined at the beginning of the proof. Note that $\tilde{r}^\top(\tilde{\alpha}_1^* + \tilde{\alpha}_2^*) - r_n(\tilde{\alpha}_1^* + \tilde{\alpha}_2^*) = 0$. The minimization problem in the last line above is feasible since $\pi^* = 2\alpha_2^* + r$ is a feasible solution. Indeed, it is strictly feasible since $\alpha_1^* + \alpha_2^*$ is a recession direction of the feasible region and belongs to the interior of $\mathbb{L}^n$. Hence, the optimal value of the minimization problem is equal to that of its dual problem. Furthermore, the dual problem is solvable whenever it is feasible. Then

$$\begin{aligned} f_{\mathbb{L}^n,r}(x) &= \max_{\rho,\tau} \left\{ \langle r, \rho \rangle : \quad \rho + \tau \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix} = x, \quad \rho \in \mathbb{L}^n \right\} \\ &= \max_{\tau} \left\{ \langle r, x \rangle - \tau \left(\|\tilde{r}\|_2^2 - r_n^2 \right) : \quad x - \tau \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix} \in \mathbb{L}^n, \quad \rho \in \mathbb{L}^n \right\}. \end{aligned}$$

There will be an optimal solution to the problem above on the boundary of the feasible region. Because $\|\tilde{r}\|_2^2 - r_n^2 > 0$, an optimal solution to this problem is

$$\tau_- = \frac{\langle r, x \rangle - \sqrt{\langle r, x \rangle^2 + (\|\tilde{r}\|_2^2 - r_n^2)(x_n^2 - \|\tilde{x}\|_2^2)}}{\|\tilde{r}\|_2^2 - r_n^2}.$$

The conclusion (6.5) follows. □

The following remark shows that the conclusion of Lemma 6.1 still holds when $r \in \pm \operatorname{bd} \mathbb{L}^n$.

Remark 6.2. *Suppose $r \in \pm \operatorname{bd} \mathbb{L}^n$. Remark 5.22(i) shows that, for any $x \in \mathbb{L}^n$, we have $f_{\mathbb{L}^n,r}(x) = |\langle r, x \rangle| = \sqrt{\langle r, x \rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)}$.*

Lemma 6.1 and Remark 6.2 yield the following result.

Theorem 6.3. *Let $\mathbb{K} = \mathbb{L}^n$. Suppose $r \notin \pm \operatorname{int} \mathbb{L}^n$. Then a point $x \in \mathbb{L}^n$ satisfies* (5.20) *if and only if it satisfies*

$$\sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)} \geq 2\mu_0 - \langle d_1 + d_2, x\rangle. \tag{6.6}$$

Similarly, a point $x \in \mathbb{L}^n$ satisfies (5.31) *if and only if it satisfies*

$$\sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)} \geq |2\mu_0 - \langle d_1 + d_2, x\rangle|. \tag{6.7}$$

As a result of Theorem 6.3 and Proposition 5.19, the inequality (6.6) provides a convex relaxation for $\mathbb{D}_1 \cup \mathbb{D}_2$ in the space of the original variables. In addition, if $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Theorem 5.9, the inequality (6.6) and the cone constraint $x \in \mathbb{L}^n$ together characterize the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$. Recall from Corollary 5.34 that, if $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition, a point $x \in \mathbb{L}^n$ satisfies (5.20) if and only if it satisfies (5.31). Thus, in the case of disjoint disjunctions, the inequalities (6.6) and (6.7) are equivalent. On the other hand, by Theorem 6.3 and Proposition 5.31(i), any point $x \in \mathbb{F} \cap \mathbb{L}^n$ satisfies (6.7), where $\mathbb{F} \subset \mathbb{R}^n$ is defined as in (6.4). Moreover, if $\mathbb{F}$ satisfies the conditions of Proposition 5.31(ii), then (6.7) produces the closed convex hull of $\mathbb{F} \cap \mathbb{L}^n$.

Remark 6.4. *Let $\mathbb{K} = \mathbb{L}^n$. Consider $\mathbb{D}_1$ and $\mathbb{D}_2$ defined as in* (6.3). *The inequality* (6.6) *has a simple geometrical meaning when the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition. Consider a point $x \in \mathbb{R}^n$ which is on the hyperplane defined by $\langle d_1, x\rangle = \mu_0$. Then the disjointness condition implies $\langle d_2, x\rangle \leq \mu_0$. Replacing $\langle d_1, x\rangle$ with μ_0 on both sides of* (6.6)*, we can see that when $r = d_2 - d_1 \notin \pm\mathbb{L}^n$, such a point x satisfies* (6.6) *if and only if $x \in \pm\mathbb{L}^n$. Similarly, a point x which is on the hyperplane defined by $\langle d_2, x\rangle = \mu_0$ satisfies* (6.6) *if and only if $x \in \pm\mathbb{L}^n$. Thus, the region defined by* (6.6) *has the same cross-section as $\pm\mathbb{L}^n$ at the hyperplanes defined by the equations $\langle d_1, x\rangle = \mu_0$ and $\langle d_2, x\rangle = \mu_0$.*

In the next two results, we show that (6.6) and (6.7) have simple second-order cone representations.

Lemma 6.5. *Suppose $r \notin \pm\mathbb{L}^n$. Then a point $x \in \mathbb{L}^n$ satisfies* (6.6) *if and only if there exists $z \geq \mu_0$ such that*

$$\left(\|\tilde{r}\|_2^2 - r_n^2\right)\left(x_n^2 - \|\tilde{x}\|_2^2\right) \geq 4(z - \langle d_1, x\rangle)(z - \langle d_2, x\rangle). \tag{6.8}$$

Similarly, a point $x \in \mathbb{L}^n$ satisfies (6.7) *if and only if it satisfies* (6.8) *together with $z = \mu_0$.*

Proof. Lemma 6.1 shows

$$[f_{\mathbb{L}^n, r}(x)]^2 - \langle r, x\rangle^2 = \left(\|\tilde{r}\|_2^2 - r_n^2\right)\left(x_n^2 - \|\tilde{x}\|_2^2\right).$$

Then the two claims follow from Propositions 5.27(ii) and 5.32(ii), respectively. □

Proposition 6.6. *Suppose $r \notin \pm\mathbb{L}^n$. For any $z \in \mathbb{R}$, a point $x \in \mathbb{L}^n$ satisfies* (6.8) *if and only if it satisfies*

$$\left(\|\tilde{r}\|_2^2 - r_n^2\right) x - 2(z - \langle d_1, x\rangle)\begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix} \in \mathbb{L}^n. \tag{6.9}$$

Proof. Fix $z \in \mathbb{R}$. Because $r \notin \pm\mathbb{L}^n$, any point $x \in \mathbb{L}^n$ satisfies (6.8) if and only if it satisfies

$$\left(\|\tilde{r}\|_2^2 - r_n^2\right)^2 \left(x_n^2 - \|\tilde{x}\|_2^2\right) - 4\left(\|\tilde{r}\|_2^2 - r_n^2\right)(z - \langle d_1, x\rangle)(z - \langle d_2, x\rangle) \geq 0.$$

The left-hand side of this inequality is identical to the following quadratic form which has a single positive eigenvalue:

$$\left(\left(\|\tilde{r}\|_2^2 - r_n^2\right) x_n + 2(z - \langle d_1, x\rangle) r_n\right)^2 - \left\|\left(\|\tilde{r}\|_2^2 - r_n^2\right)\tilde{x} - 2(z - \langle d_1, x\rangle)\tilde{r}\right\|_2^2.$$

For ease of exposition, let us define the functions $\mathcal{A}, \mathcal{B} : \mathbb{R}^n \to \mathbb{R}$ as

$$\mathcal{A}(x) = \left\|\left(\|\tilde{r}\|_2^2 - r_n^2\right)\tilde{x} - 2(z - \langle d_1, x\rangle)\tilde{r}\right\|_2 \quad \text{and} \quad \mathcal{B}(x) = \left(\|\tilde{r}\|_2^2 - r_n^2\right) x_n + 2(z - \langle d_1, x\rangle) r_n.$$

We have just shown that a point $x \in \mathbb{L}^n$ satisfies (6.8) if and only if it satisfies $\mathcal{A}(x)^2 \leq \mathcal{B}(x)^2$. To finish the proof, we show that $x \in \mathbb{L}^n$ satisfies either $\mathcal{A}(x) + \mathcal{B}(x) > 0$ or $\mathcal{A}(x) = \mathcal{B}(x) = 0$. Suppose $\mathcal{A}(x) + \mathcal{B}(x) \leq 0$ for some $x \in \mathbb{L}^n$. Using the triangle inequality, we can write

$$\begin{aligned}
0 &\geq \mathcal{A}(x) + \mathcal{B}(x) \\
&= \left\|\left(\|\tilde{r}\|_2^2 - r_n^2\right)\tilde{x} - 2(z - \langle d_1, x\rangle)\tilde{r}\right\|_2 + \left(\|\tilde{r}\|_2^2 - r_n^2\right) x_n + 2(z - \langle d_1, x\rangle) r_n \\
&\geq -\left(\|\tilde{r}\|_2^2 - r_n^2\right)\|\tilde{x}\|_2 + 2|z - \langle d_1, x\rangle|\,\|\tilde{r}\|_2 + \left(\|\tilde{r}\|_2^2 - r_n^2\right) x_n - 2|z - \langle d_1, x\rangle||r_n| \\
&= \left(\|\tilde{r}\|_2^2 - r_n^2\right)(x_n - \|\tilde{x}\|_2) + 2|z - \langle d_1, x\rangle|\left(\|\tilde{r}\|_2 - |r_n|\right).
\end{aligned}$$

Because $x \in \mathbb{L}^n$ and $r \notin \pm\mathbb{L}^n$, the last expression above must be equal to zero. Hence, $\|\tilde{x}\|_2 = x_n$ and $\langle d_1, x\rangle = z$. This implies $\mathcal{A}(x) + \mathcal{B}(x) = (\|\tilde{r}\|_2^2 - r_n^2)(\|\tilde{x}\|_2 + x_n)$ which is strictly positive unless $x = 0$, but then $\mathcal{A}(x) = \mathcal{B}(x) = 0$. □

Remark 6.7. *Suppose the hypotheses of Proposition 6.6 are satisfied. Changing the roles of d_1 and d_2, the proof of Proposition 6.6 can be repeated to show that a point $x \in \mathbb{L}^n$ satisfies* (6.8) *if and only if it satisfies*

$$\left(\|\tilde{r}\|_2^2 - r_n^2\right) x + 2(z - \langle d_2, x\rangle)\begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix} \in \mathbb{L}^n.$$

The following is a consequence of Proposition 6.6 and Corollary 5.34.

Corollary 6.8. *Let $\mathbb{K} = \mathbb{L}^n$. Consider $\mathbb{D}_1$ and $\mathbb{D}_2$ defined as in* (6.3)*. Suppose $r \notin \pm\mathbb{L}^n$.*

i. Let $x \in \mathbb{L}^n$ be such that $\langle d_1, x\rangle \le \mu_0 \vee \langle d_2, x\rangle \le \mu_0$. Then x satisfies (6.6) *if and only if it satisfies*

$$\left(\|\tilde{r}\|_2^2 - r_n^2\right) x - 2(\mu_0 - \langle d_1, x\rangle)\begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix} \in \mathbb{L}^n. \tag{6.10}$$

ii. Suppose $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition. Then a point $x \in \mathbb{L}^n$ satisfies (6.6) *if and only if it satisfies* (6.10).

6.2.2 Extension to Direct Products of Second-Order Cones

Corollary 6.9 extends Theorem 6.3 to the case where $\mathbb{K} \subset \mathbb{R}^n$ is a direct product of multiple second-order cones and nonnegative rays.

Corollary 6.9. *Let $\mathbb{K} \subset \mathbb{R}^n$ such that $\mathbb{K} = \prod_{j=1}^{p_1+p_2} \mathbb{K}^j$ where $\mathbb{K}^j = \mathbb{L}^{n^j}$ for $j \in [p_1]$ and $\mathbb{K}^{p_1+j} = \mathbb{R}_+$ for $j \in [p_2]$. Let*

$$\mathbb{P}_1^+ = \left\{ j \in [p_1] \colon -r^j \in \mathbb{L}^{n^j} \right\},\ \mathbb{P}_1^- = \left\{ j \in [p_1] \colon r^j \in \mathbb{L}^{n^j} \right\},\ \mathbb{P}_1^\circ = \left\{ j \in [p_1] \colon r^j \notin \pm\mathbb{L}^{n^j} \right\}.$$

i. A point $x \in \mathbb{K}$ satisfies (6.6) *if and only if it satisfies*

$$\begin{aligned}\sum_{j\in\mathbb{P}_1^\circ} f_{\mathbb{L}^{n^j}, r^j}(x^j) &+ \sum_{j\in\mathbb{P}_1^\circ} \langle d_1^j + d_2^j, x^j\rangle_j \\ &+ 2\sum_{j\in\mathbb{P}_1^+} \langle d_1^j, x^j\rangle_j + 2\sum_{j\in\mathbb{P}_1^-} \langle d_2^j, x^j\rangle_j + 2\sum_{j=p_1+1}^{p_1+p_2} \max\left\{d_1^j, d_2^j\right\} x^j \ge 2\mu_0 \end{aligned} \tag{6.11}$$

where $f_{\mathbb{L}^{n^j}, r^j}(x^j) = \sqrt{\langle r^j, x^j\rangle_j^2 + \left(\|\tilde{r}^j\|^2 - (r_{n^j}^j)^2\right)\left((x_{n^j}^j)^2 - \|\tilde{x}^j\|^2\right)}$ for $j \in \mathbb{P}_1^\circ$.

ii. A point $x \in \mathbb{K}$ satisfies (6.11) *if and only if there exist $z^j \in \mathbb{R}$, $j \in \mathbb{P}_1^\circ$, such that*

$$\left(\|\tilde{r}^j\|^2 - (r_{n^j}^j)^2\right) x^j - 2\left(z^j - \langle d_1^j, x^j\rangle_j\right)\begin{pmatrix} \tilde{r}^j \\ -r_{n^j}^j \end{pmatrix} \in \mathbb{L}^{n^j} \qquad \forall j \in \mathbb{P}_1^\circ, \tag{6.12a}$$

$$\sum_{j\in\mathbb{P}_1^\circ} z^j + \sum_{j\in\mathbb{P}_1^+} \langle d_1^j, x^j\rangle_j + \sum_{j\in\mathbb{P}_1^-} \langle d_2^j, x^j\rangle_j + \sum_{j=p_1+1}^{p_1+p_2} \max\left\{d_1^j, d_2^j\right\} x^j \ge \mu_0. \tag{6.12b}$$

Proof. Fix $x \in \mathbb{K}$. Proposition 6.1, together with Proposition 5.27 and Remarks 5.29 and 5.22(i), shows that the inequality (5.20) reduces to (6.11). To show statement (*ii*), consider Proposition 5.27(ii). Remark 5.28 shows that the auxiliary variables z^j can be eliminated from (5.27) for $j \in \mathbb{P}_1^+ \cup \mathbb{P}_1^-$. Furthermore, as discussed in Lemma 6.5 and Proposition 6.6, the inequalities $\left[f_{\mathbb{L}^{n^j},r^j}(x^j)\right]^2 - \langle r^j, x^j\rangle_j^2 \geq 4\left(z^j - \langle d_1^j, x^j\rangle_j\right)\left(z^j - \langle d_2^j, x^j\rangle_j\right)$ can be represented in second-order conic form as (6.12a) for $j \in \mathbb{P}_1^\circ$. Hence, (5.27) reduces to (6.12). □

6.3 Describing the Closed Convex Hull

In this section, we consider the set $\mathbb{C}_1 \cup \mathbb{C}_2$, where $\mathbb{C}_1$ and $\mathbb{C}_2$ are defined as in (6.1) and $\mathbb{K} = \mathbb{L}^n$. We assume that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. The main purpose of this section is to use the results of Section 6.2.1 to provide a complete closed convex hull description of $\mathbb{C}_1 \cup \mathbb{C}_2$. We first state the following corollary of Proposition 5.19, Remark 5.20, and Theorem 6.3.

Corollary 6.10. *Let* $\mathbb{K} = \mathbb{L}^n$*. Consider* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *defined as in* (6.1) *and* $\mathbb{B}$ *defined as in* (5.11)*. Let* $(\beta_1, \beta_2) \in \mathbb{B}$ *and* $\mu_0(\beta_1, \beta_2) = \min\{c_{1,0}, c_{2,0}\}$*. Any point* $x \in \mathbb{C}_1 \cup \mathbb{C}_2$ *satisfies*

$$f_{\mathbb{L}^n,\beta_2 c_2 - \beta_1 c_1}(x) \geq 2\mu_0(\beta_1, \beta_2) - \langle \beta_1 c_1 + \beta_2 c_2, x\rangle \tag{6.13}$$

where $f_{\mathbb{L}^n,r}(x) = \sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)}$*. Furthermore, this inequality defines a convex region inside the second-order cone.*

6.3.1 When does a Single Convex Inequality Suffice?

The following result is a consequence of Theorems 5.9 and 6.3.

Corollary 6.11. *Let* $\mathbb{K} = \mathbb{L}^n$*. Consider* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *defined as in* (6.1)*. Suppose* $\mathbb{C}_1$ *and* $\mathbb{C}_2$ *satisfy the basic disjunctive setup. Let* $\mu_0 = \min\{c_{1,0}, c_{2,0}\}$*. Suppose one of the conditions below holds:*

i. The points $c_1, c_2 \in \mathbb{R}^n$ *satisfy* $c_1 \in \mathbb{L}^n$ *or* $c_2 \in \mathbb{L}^n$*.*

ii. The convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ *is closed and* $c_{1,0} = c_{2,0} \in \{\pm 1\}$*.*

Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ *is*

$$\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{x \in \mathbb{L}^n : \quad f_{\mathbb{L}^n,c_2-c_1}(x) \geq 2\mu_0 - \langle c_1 + c_2, x\rangle\right\}.$$

where $f_{\mathbb{L}^n,r}(x) = \sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)}$*.*

Corollary 6.11 shows that, under certain conditions, the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is completely described with a single inequality of the form (6.6), in addition to the cone constraint $x \in \mathbb{L}^n$. Nevertheless, it is easy to construct instances where these hypotheses are not satisfied. We explore these cases further in Section 6.3.2.

Let us consider the case $\mu_0 = c_{1,0} = c_{2,0} \in \{0, \pm 1\}$. In this case Lemma 5.3 implies $r = c_2 - c_1 \notin \pm\mathbb{L}^n$. Suppose also that the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy i) the conditions of Corollary 6.11, and ii) the disjointness condition. Statement (i) holds, for instance, when the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ are defined by a split disjunction which excludes the origin; in this case $\mu_0 = 1$ and the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is closed according to Corollary 5.14. The disjointness condition also holds for split disjunctions. Then Corollary 6.11 indicates that the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is completely characterized by the inequality

$$f_{\mathbb{L}^n,r}(x) \geq 2\mu_0 - \langle c_1 + c_2, x\rangle, \tag{6.14}$$

together with the cone constraint $x \in \mathbb{L}^n$. Furthermore, Corollary 6.8 shows that any $x \in \mathbb{L}^n$ satisfies (6.14) if and only if it satisfies

$$\left(\|\tilde{r}\|_2^2 - r_n^2\right) x - 2(c_{1,0} - \langle c_1, x\rangle) \begin{pmatrix} \tilde{r} \\ -r_n \end{pmatrix} \in \mathbb{L}^n.$$

We formulate this conclusion into Corollary 6.12 below. This recovers the related results of [8, 89] on split disjunctions on the second-order cone. Note that, in the case of split disjunctions, $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup if they satisfy Condition 5.1 and $c_{1,0} = c_{2,0} = 1$.

Corollary 6.12. *Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined by a split disjunction $\langle t_1\ell, x\rangle \geq c_{1,0} \vee \langle t_2\ell, x\rangle \geq c_{2,0}$ on $\mathbb{L}^n$ such that $t_1 > 0 > t_2$ and $\mathbb{C}_1 \cup \mathbb{C}_2 \subsetneq \mathbb{L}^n$. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Condition 5.1 and $c_{1,0}, c_{2,0} \in \{0, \pm 1\}$. If $c_{1,0} = c_{2,0} = 1$, then*

$$\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{L}^n : (t_1 - t_2)\left(\|\tilde{\ell}\|_2^2 - \ell_n^2\right) x + 2(1 - \langle t_1\ell, x\rangle) \begin{pmatrix} \tilde{\ell} \\ -\ell_n \end{pmatrix} \in \mathbb{L}^n \right\}.$$

Otherwise, $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) = \mathbb{L}^n$.

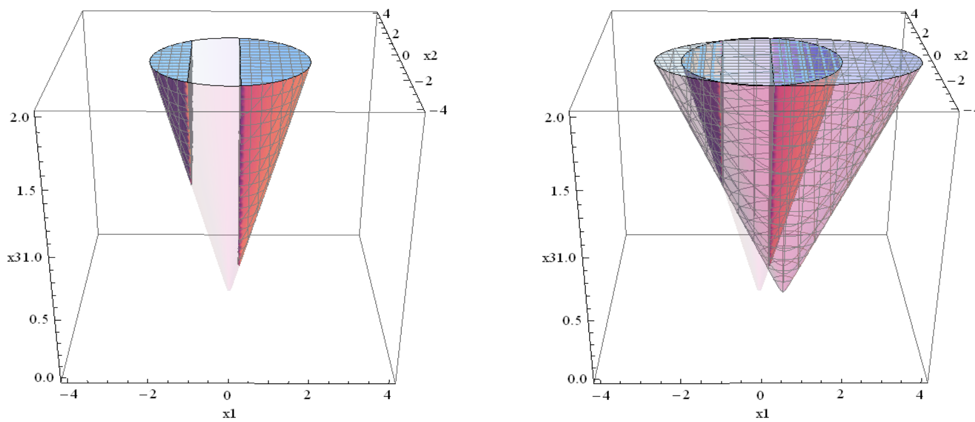
Corollaries 6.8 and 6.11 recover the results of [8, 89] for split disjunctions on the cone the second-order cone and extend them significantly to more general two-term disjunctions. Theorem 8.6 in Chapter 8 complements Corollary 6.11 and shows that a single inequality of the form (6.6) always defines the convex hull of a *homogeneous* two-term disjunction on the second-order cone as long as $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup.

Examples where a Single Inequality Suffices

Example 6.1. As an application of Corollary 6.12, consider the split disjunction $4x_1 \geq 1 \vee -x_1 \geq 1$ on the second-order cone $\mathbb{L}^3$. Corollary 6.12 states that in this case the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is the set of points $x \in \mathbb{L}^3$ that satisfy the second-order cone inequality

$$5x + 2(1 - 4x_1)e^1 \in \mathbb{L}^3.$$

Figures 6.1(a) and (b) show the disjunctive set $\mathbb{C}_1 \cup \mathbb{C}_2$ and the second-order cone inequality which is introduced to convexify $\mathbb{C}_1 \cup \mathbb{C}_2$, respectively.



(a) $\mathbb{C}_1 \cup \mathbb{C}_2$

(b) Conic quadratic inequality describing the convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$

Figure 6.1: Sets associated with the split disjunction $4x_1 \geq 1 \vee -x_1 \geq 1$ on $\mathbb{L}^3$.

Example 6.2. Consider the cone $\mathbb{L}^3$ and the disjunction $x_3 \geq 1 \vee x_1 + x_3 \geq 1$ ($c_1 = e^3$, $c_2 = e^1 + e^3$, $c_{1,0} = c_{2,0} = 1$). Note that $c_1, c_2 \in \mathbb{L}^3$ in this example. Hence, we can use Corollary 6.11 to characterize the closed convex hull:

$$\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{L}^3 : \sqrt{x_3^2 - x_2^2} \geq 2 - (x_1 + 2x_3) \right\}.$$

Figures 6.2(a) and (b) depict the disjunctive set $\mathbb{C}_1 \cup \mathbb{C}_2$ and the associated closed convex hull, respectively. In order to give a better sense of the convexification operation, we plot the points added to $\mathbb{C}_1 \cup \mathbb{C}_2$ to generate the closed convex hull in Figure 6.2(c). We note that in this example the disjointness condition that was required in Corollary 6.8(ii) is violated. Nevertheless, the inequality that we provide is still intrinsically related to the second-order cone inequality (6.10) of Corollary 6.8: The sets described by the two inequalities coincide in the region outside $\mathbb{C}_1 \cap \mathbb{C}_2$ as a consequence of Corollary 6.8(i). We display the corresponding cone for this example in Figure 6.2(d). Note that the resulting second-order cone inequality is in fact not valid for some points in $\mathbb{C}_1 \cap \mathbb{C}_2$.

6.3.2 When are Multiple Convex Inequalities Needed?

As Proposition 5.15 hints, there are cases where a single inequality of the form (6.5) is not sufficient to define the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. In this section, we study these cases when $\mathbb{K} = \mathbb{L}^n$ and outline a procedure to find closed-form expressions describing the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. We first state the following consequence of Corollary 5.8 and Theorem 6.3. The sets $\mathbb{B}_1$ and $\mathbb{B}_2$ are defined as in (5.14).

Corollary 6.13. *Let $\mathbb{K} = \mathbb{L}^n$. Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (6.1)*. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is*

$$\overline{\mathrm{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{L}^n : \begin{array}{ll} f_{\mathbb{L}^n, c_2 - \beta_1 c_1}(x) \geq 2c_{2,0} - \langle \beta_1 c_1 + c_2, x \rangle & \forall\, \beta_1 \in \mathbb{B}_1, \\ f_{\mathbb{L}^n, \beta_2 c_2 - c_1}(x) \geq 2c_{1,0} - \langle c_1 + \beta_2 c_2, x \rangle & \forall\, \beta_2 \in \mathbb{B}_2 \end{array} \right\}.$$

where $f_{\mathbb{L}^n,r}(x) = \sqrt{\langle r, x\rangle^2 + (\|\tilde{r}\|^2 - r_n^2)(x_n^2 - \|\tilde{x}\|^2)}$.

Consider $\beta_1 \in \mathbb{B}_1$ and $\beta_2 \in \mathbb{B}_2$. Let $x \in \mathbb{L}^n$. For ease of notation, let us define the functions $\mathcal{R}, \mathcal{P}, \mathcal{Q} : \mathbb{L}^n \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{R}(x) &= \langle c_1, x\rangle^2 + (\|\tilde{c}_1\|_2^2 - c_{1,n}^2)(x_n^2 - \|\tilde{x}\|_2^2), \\ \mathcal{P}(x) &= \langle c_1, x\rangle\langle c_2, x\rangle + (\tilde{c}_1^\top \tilde{c}_2 - c_{1,n}c_{2,n})(x_n^2 - \|\tilde{x}\|_2^2), \\ \mathcal{Q}(x) &= \langle c_2, x\rangle^2 + (\|\tilde{c}_2\|_2^2 - c_{2,n}^2)(x_n^2 - \|\tilde{x}\|_2^2). \end{aligned}$$

With these definitions, we have

$$\begin{aligned} \mathcal{R}(x)\beta_1^2 - 2\mathcal{P}(x)\beta_1 + \mathcal{Q}(x) &= \langle c_2 - \beta_1 c_1, x\rangle^2 + \left(\|\tilde{c}_2 - \beta_1\tilde{c}_1\|_2^2 - (c_{2,n} - \beta_1 c_{1,n})^2\right)\left(x_n^2 - \|\tilde{x}\|_2^2\right), \\ \mathcal{Q}(x)\beta_2^2 - 2\mathcal{P}(x)\beta_2 + \mathcal{R}(x) &= \langle \beta_2 c_2 - c_1, x\rangle^2 + \left(\|\beta_2\tilde{c}_2 - \tilde{c}_1\|_2^2 - (\beta_2 c_{2,n} - c_{1,n})^2\right)\left(x_n^2 - \|\tilde{x}\|_2^2\right). \end{aligned}$$

We further define the functions $t_1^x : \mathbb{B}_1 \to \mathbb{R}$ and $t_2^x : \mathbb{B}_2 \to \mathbb{R}$ as

$$\begin{aligned} t_1^x(\beta_1) &= \beta_1\langle c_1, x\rangle + f_{\mathbb{L}^n, c_2 - \beta_1 c_1}(x) = \beta_1\langle c_1, x\rangle + \sqrt{\mathcal{R}(x)\beta_1^2 - 2\mathcal{P}(x)\beta_1 + \mathcal{Q}(x)}, \\ t_2^x(\beta_2) &= \beta_2\langle c_2, x\rangle + f_{\mathbb{L}^n, \beta_2 c_2 - c_1}(x) = \beta_2\langle c_2, x\rangle + \sqrt{\mathcal{Q}(x)\beta_2^2 - 2\mathcal{P}(x)\beta_2 + \mathcal{R}(x)}. \end{aligned}$$

Through these definitions and Corollary 6.13, we reach

$$\begin{aligned} \overline{\mathrm{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) &= \left\{ x \in \mathbb{L}^n : \begin{array}{c} t_1^x(\beta_1) \geq 2c_{2,0} - \langle c_2, x\rangle \ \ \forall \beta_1 \in \mathbb{B}_1, \\ t_2^x(\beta_2) \geq 2c_{1,0} - \langle c_1, x\rangle \ \ \forall \beta_2 \in \mathbb{B}_2 \end{array} \right\} \\ &= \left\{ x \in \mathbb{L}^n : \begin{array}{c} \inf_{\beta_1 \in \mathbb{B}_1} t_1^x(\beta_1) \geq 2c_{2,0} - \langle c_2, x\rangle, \\ \inf_{\beta_2 \in \mathbb{B}_2} t_2^x(\beta_2) \geq 2c_{1,0} - \langle c_1, x\rangle \end{array} \right\}. \end{aligned} \tag{6.15}$$

It follows that, for any given $x \in \mathbb{L}^n$, we can check whether $x \in \overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$ by calculating the optimal value of the problems on the left-hand side of the inequalities in (6.15). Furthermore, whenever the minimizer $\beta_1^* = \beta_1^*(x)$ of $\inf_{\beta_1 \in \mathbb{B}_1} t_1^x(\beta_1)$ exists and can be expressed parametrically in terms of c_1, c_2, and x, one can replace the inequality $\inf_{\beta_1 \in \mathbb{B}_1} t_1^x(\beta_1) \geq 2c_{2,0} - \langle c_2, x\rangle$ in (6.15) with $t_1^x(\beta_1^*) \geq 2c_{2,0} - \langle c_2, x\rangle$. Similarly, one can define the minimizer $\beta_2^* = \beta_2^*(x)$ and replace $\inf_{\beta_2 \in \mathbb{B}_2} t_2^x(\beta_2) \geq 2c_{1,0} - \langle c_1, x\rangle$ with $t_2^x(\beta_2^*) \geq 2c_{1,0} - \langle c_1, x\rangle$. We illustrate this procedure on an example in the next section.

Example where Multiple Inequalities are Needed

Example 6.3. Consider the cone $\mathbb{L}^3$ and the disjunction $-x_2 \geq 0 \vee -x_3 \geq -1$ ($c_1 = -e^2$, $c_{1,0} = 0$, $c_2 = -e^3$, $c_{2,0} = -1$). Since $c_{1,0} > c_{2,0}$, Proposition 5.15 implies that any undominated valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$ will be tight on $\mathbb{C}_2$ but not on $\mathbb{C}_1$. Therefore, we follow the approach outlined at the beginning of this section. Noting that $c_2 - \beta_1 c_1 \in -\operatorname{int} \mathbb{L}^3$ for $0 \leq \beta_1 < 1$ and $c_2 - \beta_1 c_1 \notin \pm \operatorname{int} \mathbb{L}^3$ for $\beta_1 \geq 1$, we obtain $\mathbb{B}_1 = [1, \infty)$. For $\beta_1 = 1$, $c_2 - \beta_1 c_1 \in -\operatorname{bd} \mathbb{L}^3$; Remark 5.22 indicates that $x_2 \leq 1$ is a valid linear inequality for $\mathbb{C}_1 \cup \mathbb{C}_2$. It is also clear in this example that $\mathbb{B}_2 = \emptyset$.

Since we are interested in cutting off only points $x \in \mathbb{L}^3$ such that $x_2 \leq 1$ and $x \notin \overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$, consider $x \in \mathbb{L}^3$ such that $0 < x_2 \leq 1$ and $x_3 > 1$. The hypotheses $x \in \mathbb{L}^3$ and $x_2 > 0$ imply $x_3 - |x_1| > 0$. In this setup we have

$$\begin{aligned}
\mathcal{R}(x) &= x_3^2 - x_1^2, \\
\mathcal{P}(x) &= x_2 x_3, \\
\mathcal{Q}(x) &= x_1^2 + x_2^2.
\end{aligned}$$

The resulting t_1^x is a convex function of β_1 and has a critical point at

$$\begin{aligned}
\hat{\beta}_1 = \hat{\beta}_1(x) &= \frac{\mathcal{P}(x)}{\mathcal{R}(x)} - \frac{\langle c_1, x\rangle}{\mathcal{R}(x)} \sqrt{\frac{\mathcal{P}(x)^2 - \mathcal{Q}(x)\mathcal{R}(x)}{\langle c_1, x\rangle^2 - \mathcal{R}(x)}} \\
&= \frac{x_2 x_3}{x_3^2 - x_1^2} + \frac{x_2}{x_3^2 - x_1^2} \sqrt{\frac{x_2^2 x_3^2 - (x_1^2 + x_2^2)(x_3^2 - x_1^2)}{(-1)(x_3^2 - x_1^2 - x_2^2)}} \\
&= \frac{x_2 x_3 + |x_1| x_2}{x_3^2 - x_1^2} = \frac{x_2}{x_3 - |x_1|},
\end{aligned}$$

where the last equation uses the fact that $x \in \mathbb{L}^3$ and thus $x_3 > 1$.

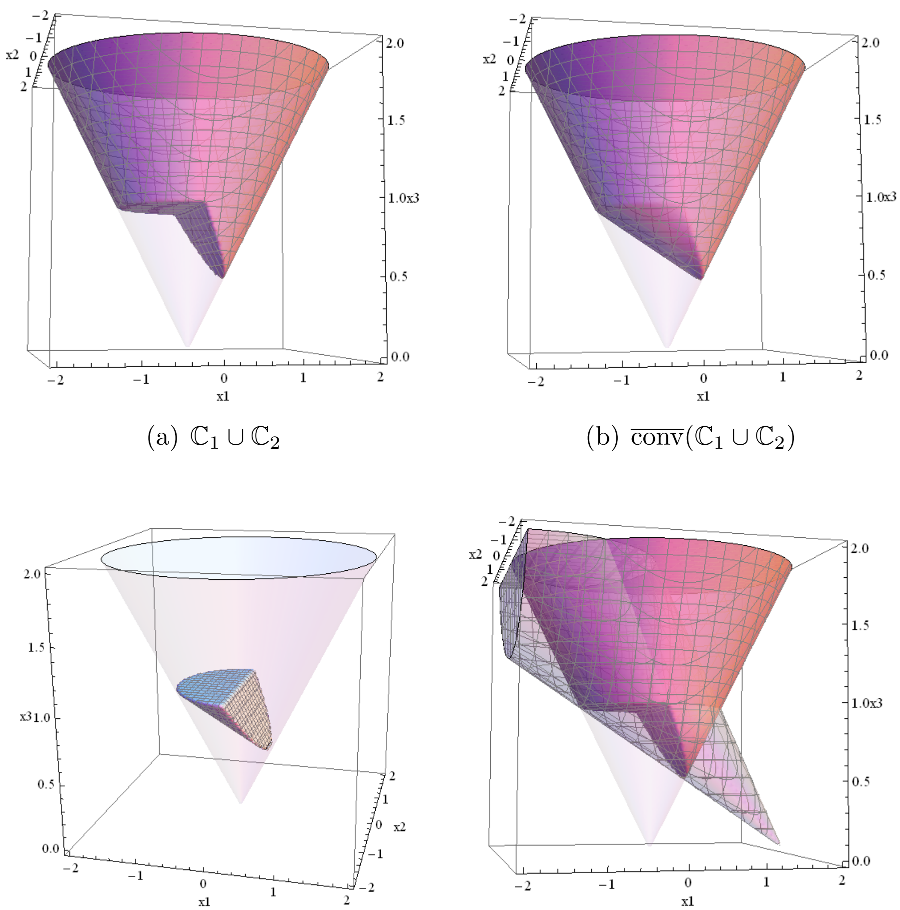
For any $x \in \mathbb{L}^3$ such that $x_2 \leq x_3 - |x_1|$, we have $\hat{\beta}_1 \leq 1$. By the convexity of t_1^x, the minimum of t_1^x occurs at $\beta_1^* = \max\left\{\hat{\beta}_1, 1\right\} = 1$. As discussed above, the inequality $t_1^x(1) \geq 2c_{2,0} - \langle c_2, x\rangle$ reduces to the linear inequality $x_2 \leq 1$. Moreover, for any $x \in \mathbb{L}^3$

such that $x_2 \geq x_3 - |x_1|$, we have $\hat{\beta}_1 \geq 1$. For such points, $\beta_1^* = \hat{\beta}_1$ and $t_1^x(\beta_1^*) = |x_1| - \frac{x_2^2(x_3+|x_1|)}{x_3^2-x_1^2} = |x_1| - \frac{x_2^2}{x_3-|x_1|}$. Therefore, for all $x \in \mathbb{L}^3$ such that $0 < x_2 \leq 1$, $x_3 > 1$, and $x_2 \geq x_3 - |x_1|$, we can impose the inequality $t_1^x(\hat{\beta}_1) \geq 2c_{2,0} - \langle c_2, x\rangle$ which translates into $|x_1| - \frac{x_2^2}{x_3-|x_1|} \geq -2 + x_3$ in this example. Using $0 < x_2 \leq 1$ and $x_3 - |x_1| > 0$, we can rewrite this inequality as $\sqrt{1 - \max\{0, x_2\}^2} \geq 1 + |x_1| - x_3$. Putting this together with $x_2 \leq 1$, we arrive at

$$\begin{aligned}\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) &= \left\{x \in \mathbb{L}^3 : t_1^x(\beta_1) \geq -2 + x_3 \ \ \forall \beta_1 \in [1, \infty)\right\} \\ &= \left\{x \in \mathbb{L}^3 :\ x_2 \leq 1,\ \sqrt{1 - \max\{0, x_2\}^2} \geq 1 + |x_1| - x_3\right\},\end{aligned}$$

where both inequalities are convex on $\mathbb{R}^3$. In fact, both inequalities are second-order cone representable in a lifted space as expected.

In Figures 6.3(a) and (b), we plot the disjunctive set $\mathbb{C}_1 \cup \mathbb{C}_2$ and its closed convex hull, respectively. The closed convex hull is obtained by imposing various convex inequalities of the form (6.6), each corresponding to $d_1 = \beta_1 c_1$, $d_2 = c_2$, and a different value of $\beta_1 \in \mathbb{B}_1$, on $\mathbb{L}^3$. In Figure 6.3(c) we show the second-order cone counterparts (6.10) of these inequalities. Note that these inequalities are not necessarily valid for all points in $\mathbb{C}_1 \cup \mathbb{C}_2$ because the disjointness condition is not satisfied; however, they describe how the boundary of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is formed outside $\mathbb{C}_1 \cup \mathbb{C}_2$. In Figure 6.3(d) we show the cross-section of $\mathbb{C}_1 \cup \mathbb{C}_2$ and the regions defined by the second-order cone inequalities (6.10) at $x_3 = 4$.



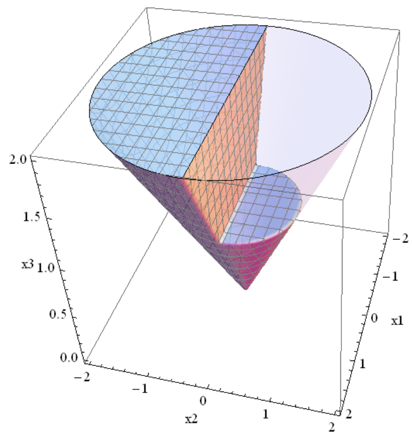
(a) $\mathbb{C}_1 \cup \mathbb{C}_2$

(b) $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$

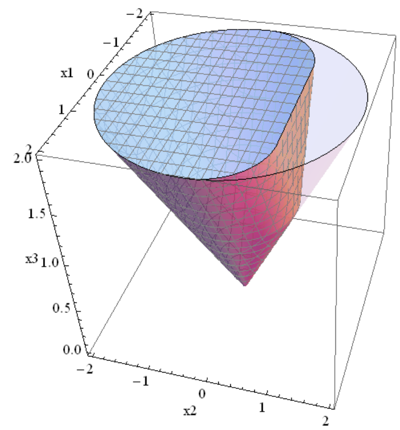
(c) $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) \setminus \mathbb{C}_1 \cup \mathbb{C}_2$

(d) Underlying cone generating the convex inequality

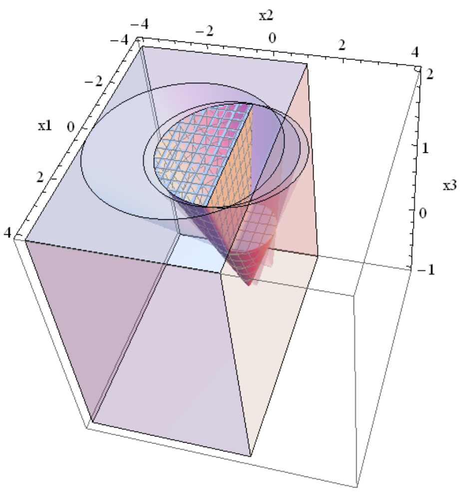
Figure 6.2: Sets associated with the disjunction $x_3 \geq 1 \vee x_1 + x_3 \geq 1$ on $\mathbb{L}^3$.



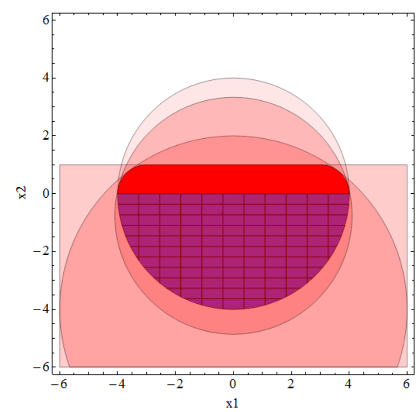
(a) $\mathbb{C}_1 \cup \mathbb{C}_2$



(b) The closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$



(c) Underlying cones generating the convex inequalities



(d) Cross-section at $x_3 = 4$

Figure 6.3: Sets associated with the disjunction $-x_2 \geq 0 \vee -x_3 \geq -1$ on $\mathbb{L}^3$.

Chapter 7

Low-Complexity Relaxations and Convex Hulls of Disjunctions on the Positive Semidefinite Cone

Acknowledgments. This chapter is based on joint work with Fatma Kılınç-Karzan [108].

7.1 Introduction

7.1.1 Motivation

Let $\mathbb{S}^n$ denote the space of symmetric $n \times n$ matrices with real entries. In this chapter, we study two-term disjunctions on the positive semidefinite cone $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : a^\top X\, a \geq 0\ \forall a \in \mathbb{R}^n\}$. Consider the disjunction $\langle D_1, X\rangle \geq \mu_0 \vee \langle D_2, X\rangle \geq \mu_0$. With respect to this disjunction, we define the sets

$$\mathbb{D}_i = \Big\{X \in \mathbb{S}^n_+ : \quad \langle D_i, X\rangle \geq \mu_0\Big\} \quad \text{for} \quad i \in \{1,2\}. \tag{7.1}$$

In addition, we consider the intersection $\mathbb{F} \cap \mathbb{S}^n_+$ where $\mathbb{F} \subset \mathbb{S}^n$ is a non-convex set defined by a rank-two quadratic constraint of the form

$$\mathbb{F} = \{X \in \mathbb{S}^n : \quad (\mu_0 - \langle D_1, X\rangle)(\mu_0 - \langle D_2, X\rangle) \leq 0\}\,. \tag{7.2}$$

As in Chapter 6, the purpose of this chapter is to provide convex relaxations for $\mathbb{D}_1 \cup \mathbb{D}_2$ and $\mathbb{F} \cap \mathbb{S}^n_+$ with structured nonlinear valid inequalities in the space of the original variables. In particular, we are interested in inequalities which explicitly characterize the closed convex hulls of these sets whenever possible. Whenever we consider the sets $\mathbb{D}_1$ and $\mathbb{D}_2$, we are primarily interested in the cases where $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy Condition 5.1. Hence, we assume $R = D_2 - D_1 \notin \pm\mathbb{S}^n_+$ when necessary.

While the class of disjunctions we consider in this chapter is more limited than those we analyzed in Chapters 5 and 6, we note that such disjunctions appear as relaxations of more general two-term disjunctions. Furthermore, convex inequalities that are obtained from sets of the form $\mathbb{D}_1 \cup \mathbb{D}_2$ can be used to derive closed convex hull characterizations of general two-term disjunctions; see Chapter 6.

The reader is referred to Section 5.1.2 for a detailed discussion of disjunctive inequalities in mixed-integer conic programming. To the best of our knowledge, none of the papers from the existing literature provide closed-form inequalities which describe closed convex hulls of two-term disjunctions on the positive semidefinite cone in the space of the original variables.

7.1.2 Notation and Terminology

In this chapter, we distinguish between the elements of $\mathbb{R}^n$ and $\mathbb{S}^n$: We denote the elements of $\mathbb{R}^n$ with lowercase letters and the elements of $\mathbb{S}^n$ with uppercase letters. With this notation, we have $\mathbb{S}^n = \left\{X \in \mathbb{R}^{n\times n} : \quad X^\top = X\right\}$. We assume that $\mathbb{S}^n$ is equipped with the Frobenius inner product $\langle A, X\rangle = \mathrm{Tr}(AX)$. The Frobenius norm $\|\cdot\|_F : \mathbb{S}^n \to \mathbb{R}$ on $\mathbb{S}^n$ is defined as $\|X\|_F = \sqrt{\langle X, X\rangle}$. The ℓ-1 norm $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ is defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$.

For any positive integer k, let $[k] = \{1, \ldots, k\}$. Given a matrix $A \in \mathbb{R}^{n\times n}$, we let $\lambda(A)$ denote the vector of the eigenvalues of A arranged in nonincreasing order and $\lambda_i(A)$ denote its i-th eigenvalue. If $A \in \mathbb{S}^n$, then the eigenvalues of A are real. Furthermore, $A \in \mathbb{S}^n$ is positive semidefinite (resp. positive definite) if and only if $\lambda_i(A) \geq 0$ (resp. $\lambda_i(A) > 0$) for all $i \in [n]$. The dual cone of $\mathbb{V} \subset \mathbb{S}^n$ is $\mathbb{V}^* = \{A \in \mathbb{S}^n : \langle X, A\rangle \geq 0 \ \forall X \in \mathbb{V}\}$. We remind the reader that the positive semidefinite cone is self-dual, that is, its dual is equal to itself. Given a matrix $A \in \mathbb{R}^{n\times n}$ and $\mathbb{J} \subset [n]$, we let $A[\mathbb{J}]$ denote the principal submatrix of A whose rows and columns are indexed by the elements of $\mathbb{J}$. We let $I_n \in \mathbb{S}^n$ represent the $n \times n$ identity matrix. For $i \in [n]$, we let e^i be the i-th unit vector in $\mathbb{R}^n$,

Throughout this chapter, we consider sets $\mathbb{D}_1$ and $\mathbb{D}_2$ defined as in (7.1). If $\mu_0 \in \{0, \pm 1\}$ and the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy Conditions 5.1 and 5.2, we say that $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the basic disjunctive setup. If $\{X \in \mathbb{S}^n_+ : \ \langle D_1, X\rangle > \mu_0, \ \langle D_2, X\rangle > \mu_0\} = \emptyset$, we say that $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition.

7.1.3 Outline of the Chapter

In Section 7.2 we specialize the results of Section 5.4 to the case where $\mathbb{K}$ is the positive semidefinite cone. In Section 7.2.1, we introduce a linear transformation which simplifies our analysis of the sets $\mathbb{D}_1 \cup \mathbb{D}_2$ and $\mathbb{F} \cap \mathbb{S}^n_+$. In Section 7.2.2, we consider general two-term disjunctions on the positive semidefinite cone and investigate the structure of the convex inequalities developed in Section 5.4 in this setting. In Section 7.2.3, we identify a class of elementary disjunctions where these inequalities can be expressed in a simple second-order conic form. For more general disjunctions, we present several techniques to generate low-complexity convex inequalities that are valid for $\mathbb{C}_1 \cup \mathbb{C}_2$. Although, we do not explicitly focus on affine cross-sections of regular cones, our approach immediately leads to valid convex (or conic) inequalities for two-term disjunctions applied to those sets. We comment on such extensions in Section 7.3.

7.2 Disjunctions on the Positive Semidefinite Cone

7.2.1 A Transformation to Simplify Disjunctions

Let $R = D_2 - D_1$. In this section, we establish a linear correspondence which reduces the closed convex hull description of any two-term disjunction on $\mathbb{S}^n_+$ to the closed convex hull description of an associated disjunction for which the matrix $R = D_2 - D_1$ is diagonal. We first prove the following more general result.

Proposition 7.1. *Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear map. Consider $\mathbb{C}_1, \mathbb{C}_2 \subset \mathbb{S}^n$ defined as $\mathbb{C}_i = \{X \in \mathbb{S}^n_+ : \ \mathcal{A}X = b, \ \langle C_i, X\rangle \geq c_{i,0}\}$. Let $Q \in \operatorname{int} \mathbb{S}^n_+$ and $U \in \mathbb{R}^{n\times n}$ be a diagonal matrix and an orthogonal matrix, respectively. Define the linear map $\mathcal{A}' : \mathbb{S}^n \to \mathbb{R}^m$ as $\mathcal{A}'X = \mathcal{A}UQXQU^\top$. Define the matrices $C_i' = QU^\top C_i UQ$ and the sets $\mathbb{C}_i' = \{X \in \mathbb{S}^n_+ : \mathcal{A}'X = b, \ \langle C_i', X\rangle \geq c_{i,0}\}$ for $i \in \{1, 2\}$. Then*

i. $\mathbb{C}_i = UQ\mathbb{C}_i'QU^\top$ *for* $i \in \{1, 2\}$,

ii. $\operatorname{conv}(\mathbb{C}_1 \cup \mathbb{C}_2) = UQ \operatorname{conv}(\mathbb{C}_1' \cup \mathbb{C}_2')QU^\top$.

iii. $\overline{\operatorname{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = UQ\, \overline{\operatorname{conv}}(\mathbb{C}_1' \cup \mathbb{C}_2')QU^\top$.

Proof. First we prove (i). Note that $C_i = UQ^{-1}C_i'Q^{-1}U^\top$ for $i \in \{1, 2\}$. We can write

$$\begin{aligned}
\mathbb{C}_i &= \Big\{X \in \mathbb{S}^n_+ : \quad \mathcal{A}X = b, \quad \langle C_i, X\rangle \geq c_{i,0}\Big\} \\
&= \Big\{UQYQU^\top \in \mathbb{S}^n_+ : \ \mathcal{A}UQYQU^\top = b, \ \Big\langle UQ^{-1}C_i'Q^{-1}U^\top, UQYQU^\top \Big\rangle \geq c_{i,0}\Big\} \\
&= \Big\{UQYQU^\top : \quad \mathcal{A}'Y = b, \quad \langle C_i', Y\rangle \geq c_{i,0}, \quad Y \in \mathbb{S}^n_+\Big\} \\
&= UQ\mathbb{C}_i'QU^\top.
\end{aligned}$$

The third equality above uses the observation that $UQYQU^\top \in \mathbb{S}^n_+$ if and only $Y \in \mathbb{S}^n_+$, which is true because $QU^\top$ is a nonsingular matrix.

Statement (ii) follows from (i) and the observation that convex combinations are invariant under the linear transformations $X \mapsto UQXQU^\top : \mathbb{S}^n \to \mathbb{S}^n$ and $X \mapsto Q^{-1}U^\top XUQ^{-1} : \mathbb{S}^n \to \mathbb{S}^n$. Statement (iii) follows from (ii) and the observation that the linear transformations $X \mapsto UQXQU^\top : \mathbb{S}^n \to \mathbb{S}^n$ and $X \mapsto Q^{-1}U^\top XUQ^{-1} : \mathbb{S}^n \to \mathbb{S}^n$ are continuous. □

Corollary 7.2. *Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear map. Consider $\mathbb{C}, \mathbb{X} \subset \mathbb{S}^n$ defined as $\mathbb{C} = \{X \in \mathbb{S}^n_+ : \mathcal{A}X = b\}$ and $\mathbb{X} = \{X \in \mathbb{S}^n : (c_{1,0} - \langle C_1, X\rangle)(c_{2,0} - \langle C_2, X\rangle) \le 0\}$. Let $Q \in \operatorname{int} \mathbb{S}^n_+$ and $U \in \mathbb{R}^{n\times n}$ be a diagonal matrix and an orthogonal matrix, respectively. Define the linear map $\mathcal{A}' : \mathbb{S}^n \to \mathbb{R}^m$ as $\mathcal{A}'X = \mathcal{A}UQXQU^\top$, the matrices $C'_i = QU^\top C_i UQ$, and the sets $\mathbb{C}' = \{X \in \mathbb{S}^n_+ : \mathcal{A}'X = b\}$ and $\mathbb{X}' = \{X \in \mathbb{E} : (c_{1,0} - \langle C'_1, X\rangle)(c_{2,0} - \langle C'_2, X\rangle) \le 0\}$. Then*

i. $\operatorname{conv}(\mathbb{C} \cap \mathbb{X}) = UQ \operatorname{conv}(\mathbb{C}' \cap \mathbb{X}')QU^\top$.

ii. $\overline{\operatorname{conv}}(\mathbb{C} \cap \mathbb{X}) = UQ\, \overline{\operatorname{conv}}(\mathbb{C}' \cap \mathbb{X}')QU^\top$.

Proof. For $i \in \{1, 2\}$, let $\mathbb{C}^+_i = \{X \in \mathbb{C} : \langle C_i, X\rangle \ge c_{i,0}\}$ and $\mathbb{C}^-_i = \{X \in \mathbb{C} : \langle C_i, X\rangle \le c_{i,0}\}$. Similarly, define $(\mathbb{C}^+_i)' = \{X \in \mathbb{C}' : \langle C'_i, X\rangle \ge c_{i,0}\}$ and $(\mathbb{C}^-_i)' = \{X \in \mathbb{C}' : \langle C'_i, X\rangle \le c_{i,0}\}$. Then $\mathbb{C} \cap \mathbb{X} = (\mathbb{C}^+_1 \cup \mathbb{C}^+_2) \cap (\mathbb{C}^-_1 \cup \mathbb{C}^-_2)$ and $\mathbb{C}' \cap \mathbb{X}' = ((\mathbb{C}^+_1)' \cup (\mathbb{C}^+_2)') \cap ((\mathbb{C}^-_1)' \cup (\mathbb{C}^-_2)')$. To prove statement (i), note that

$$\begin{aligned}
\operatorname{conv}(\mathbb{C} \cap \mathbb{X}) &= \operatorname{conv}(\mathbb{C}^+_1 \cup \mathbb{C}^+_2) \cap \operatorname{conv}(\mathbb{C}^-_1 \cup \mathbb{C}^-_2) \\
&= UQ\Big[\operatorname{conv}((\mathbb{C}^+_1)' \cup (\mathbb{C}^+_2)') \cap \operatorname{conv}((\mathbb{C}^-_1)' \cup (\mathbb{C}^-_2)')\Big]QU^\top \\
&= UQ \operatorname{conv}(\mathbb{C}' \cap \mathbb{X}')QU^\top.
\end{aligned}$$

The first and third equalities above hold as a result of Proposition 5.2; and the second equality follows from Proposition 7.1(ii). Statement (ii) follows similarly from the same results. □

Remark 7.3. *Based on Proposition 7.1, we can assume without any loss of generality that the matrices $D_1, D_2 \in \mathbb{S}^n$ which define the sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are such that the matrix $R = D_2 - D_1$ is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. To see this, consider the eigenvalue decomposition of $R = U\Lambda U^\top$ where $U \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and $\Lambda \in \mathbb{S}^n$ is a diagonal matrix whose entries are the eigenvalues of R sorted in nonincreasing order. Let $Q \in \operatorname{int} \mathbb{S}^n_+$ be the diagonal matrix with diagonal entries $Q_{ii} = \frac{1}{\sqrt{|\Lambda_{ii}|}}$ if Λ_{ii} is nonzero and $Q_{ii} = 1$ otherwise. By Proposition 7.1(iii), we have $\overline{\operatorname{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = UQ\, \overline{\operatorname{conv}}(\mathbb{D}'_1 \cup \mathbb{D}'_2)QU^\top$ where $\mathbb{D}'_i = \left\{X \in \mathbb{S}^n_+ : \langle D'_i, X\rangle \ge \mu_0\right\}$ and $D'_i = QU^\top D_i UQ$ for $i \in \{1, 2\}$. Furthermore, $R' = D'_2 - D'_1 = QU^\top RUQ = Q\Lambda Q$ is a* *diagonal matrix with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. When $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy Condition 5.1, Lemma 5.3 implies $R \notin \pm\mathbb{S}^n_+$, in which case R' has at least one diagonal entry equal to 1 and one diagonal entry equal to -1. Analogously, based on Corollary 7.2, we can assume that the matrices $D_1, D_2 \in \mathbb{S}^n$ which define $\mathbb{F}$ are such that the matrix $R = D_2 - D_1$ is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order.*

In order to simplify the presentation of certain results in the rest of the chapter, we sometimes make the assumption that R is a diagonal matrix whose diagonal elements are from $\{0, \pm 1\}$ and sorted in nonincreasing order. Proposition 7.1, Corollary 7.2, and Remark 7.3 show that this assumption is without any loss of generality.

7.2.2 General Two-Term Disjunctions on the Positive Semidefinite Cone

We specialize Propositions 5.19 and 5.31 to disjunctions on the positive semidefinite cone in Theorem 7.5. This result is based on the following lemma.

Lemma 7.4. *For any $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$, we have $f_{\mathbb{S}^n_+,R}(X) = \left\| \lambda\Big(X^{1/2} R X^{1/2}\Big) \right\|_1$.*

Proof. The dual cone of $\mathbb{S}^n_+$ is again $\mathbb{S}^n_+$. Hence, by Proposition 5.19, we have

$$f_{\mathbb{S}^n_+,R}(X) = \max_P \left\{ \langle R, P \rangle : \; X - P \in \mathbb{S}^n_+, \; X + P \in \mathbb{S}^n_+ \right\}.$$

First consider the case $X \in \operatorname{int} \mathbb{S}^n_+$. Then there exists a matrix $X^{1/2} \in \operatorname{int} \mathbb{S}^n_+$ such that $X = X^{1/2} X^{1/2}$. A matrix $P \in \mathbb{S}^n$ satisfies $X - P \in \mathbb{S}^n_+$ and $X + P \in \mathbb{S}^n_+$ if and only if it satisfies $I_n - X^{-1/2} P X^{-1/2} \in \mathbb{S}^n_+$ and $I_n + X^{-1/2} P X^{-1/2} \in \mathbb{S}^n_+$. Therefore, after introducing a new variable $Q = X^{-1/2} P X^{-1/2}$, we can write

$$\begin{aligned} f_{\mathbb{S}^n_+,R}(X) &= \max_Q \left\{ \langle R, X^{1/2} Q X^{1/2} \rangle : \; I_n - Q \in \mathbb{S}^n_+, \; I_n + Q \in \mathbb{S}^n_+ \right\} \\ &= \max_Q \left\{ \Big\langle X^{1/2} R X^{1/2}, Q \Big\rangle : \; I_n - Q \in \mathbb{S}^n_+, \; I_n + Q \in \mathbb{S}^n_+ \right\} \\ &= \max_Q \left\{ \Big\langle X^{1/2} R X^{1/2}, Q \Big\rangle : \; \|\lambda(Q)\|_\infty \le 1 \right\} = \left\| \lambda\Big(X^{1/2} R X^{1/2}\Big) \right\|_1. \end{aligned}$$

Now consider the more general case $X \in \mathbb{S}^n_+$. For $\epsilon > 0$, let $X^\epsilon = X + \epsilon I_n$. Then $X^\epsilon \in \operatorname{int} \mathbb{S}^n_+$ and $\lambda_i\Big((X^\epsilon)^{1/2}\Big) = \sqrt{\lambda_i(X) + \epsilon}$ for all $i \in [n]$. Furthermore, $\lim_{\epsilon \downarrow 0} \left\| (X^\epsilon)^{1/2} R (X^\epsilon)^{1/2} - X^{1/2} R X^{1/2} \right\|_F = 0$. The function $A \mapsto \|\lambda(A)\|_1 : \mathbb{S}^n \to \mathbb{R}$ is convex and finite everywhere; therefore, it is continuous. It follows that $\lim_{\epsilon \downarrow 0} \left\| \lambda\Big((X^\epsilon)^{1/2} R (X^\epsilon)^{1/2}\Big) \right\|_1 = \left\| |\lambda\Big(X^{1/2} R X^{1/2}\Big) \right\|_1$. On the other hand, according to

Remark 5.20, the function $-f_{\mathbb{S}^n_+,R}(X)$ is a closed convex function of X; therefore, $\lim_{\epsilon\downarrow 0} f_{\mathbb{S}^n_+,R}(X^\epsilon) = f_{\mathbb{S}^n_+,R}(X)$ (see, for instance, [69, Proposition B.1.2.5]). Putting these together, we get

$$f_{\mathbb{S}^n_+,R}(X) = \lim_{\epsilon\downarrow 0} f_{\mathbb{S}^n_+,R}(X^\epsilon) = \lim_{\epsilon\downarrow 0} \left\|\lambda\Big((X^\epsilon)^{1/2}R(X^\epsilon)^{1/2}\Big)\right\|_1 = \left\|\lambda\Big(X^{1/2}RX^{1/2}\Big)\right\|_1.$$

□

We note that, for any $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$, the eigenvalues of $X^{1/2}RX^{1/2}$ are real because it is real symmetric. Lemma 7.4 implies the following result.

Theorem 7.5. *Let $\mathbb{K} = \mathbb{S}^n_+$. Then a point $X \in \mathbb{S}^n_+$ satisfies* (5.18) *if and only if it satisfies*

$$\left\|\lambda\Big(X^{1/2}RX^{1/2}\Big)\right\|_1 \geq 2\mu_0 - \langle D_1 + D_2, X\rangle. \tag{7.3}$$

Similarly, a point $X \in \mathbb{S}^n_+$ satisfies (5.31) *if and only if it satisfies*

$$\left\|\lambda\Big(X^{1/2}RX^{1/2}\Big)\right\|_1 \geq |2\mu_0 - \langle D_1 + D_2, X\rangle|. \tag{7.4}$$

Theorem 7.5 and Proposition 5.19 indicate that (7.3) is a convex inequality that is valid for $\mathbb{D}_1 \cup \mathbb{D}_2$, where $\mathbb{D}_1, \mathbb{D}_2 \subset \mathbb{S}^n_+$ are defined as in (7.1). Furthermore, if $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the conditions of Theorem 5.9, the inequality (7.3) describes the closed convex hull of $\mathbb{D}_1 \cup \mathbb{D}_2$, together with the cone constraint $X \in \mathbb{S}^n_+$. If $\mathbb{D}_1$ and $\mathbb{D}_2$ satisfy the disjointness condition, then Corollary 5.34 shows that a point $X \in \mathbb{S}^n_+$ satisfies (7.3) if and only if it satisfies (7.4). On the other hand, Theorem 7.5 and Proposition 5.31(i) indicate that (7.4) provides a convex relaxation for $\mathbb{F} \cap \mathbb{S}^n_+$, where $\mathbb{F} \subset \mathbb{S}^n$ is defined as in (7.2). Furthermore, if $\mathbb{F}$ satisfies the conditions of Proposition 5.31(ii), then (7.4) describes the closed convex hull of $\mathbb{F} \cap \mathbb{S}^n_+$.

The lemma below can be used to simplify the term $\left\|\lambda\Big(X^{1/2}RX^{1/2}\Big)\right\|_1$ on the left-hand side of (7.3); we refer to [71, Theorem 1.3.22] for a proof of this result.

Lemma 7.6. *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $m \leq n$. Then the n eigenvalues of BA are the m eigenvalues of AB together with $n - m$ zeroes.*

Corollary 7.7. *For any $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$, we have $\lambda\Big(X^{1/2}RX^{1/2}\Big) = \lambda(RX)$. In particular:*

i. The eigenvalues of RX are real.

ii. $f_{\mathbb{S}^n_+,R}(X) = \|\lambda(RX)\|_1$.

Corollary 7.8. *Let $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$. Suppose R is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $\operatorname{supp}(R) \subset [n]$ be the set of indices of the nonzero elements of the diagonal of R. Then* *i. The eigenvalues of* $R[\mathrm{supp}(R)]X[\mathrm{supp}(R)]$ *are real,*

ii.

$$\begin{aligned} f_{\mathbb{S}^n_+,R}(X) &= \left\| \lambda(X[\mathrm{supp}(R)]^{1/2}R[\mathrm{supp}(R)]X[\mathrm{supp}(R)]^{1/2}) \right\|_1 \\ &= \left\| \lambda(R[\mathrm{supp}(R)]X[\mathrm{supp}(R)]) \right\|_1. \end{aligned}$$

Proof. Let t^+, t^-, and t^0 be the number of diagonal elements of R which are equal to $+1$, -1, and 0, respectively. Then $t^+ + t^- = |\mathrm{supp}(R)|$. Let $P \in \mathbb{R}^{n\times(t^+ + t^-)}$ be the matrix whose i-th row is e^i if $i \in [t^+]$, e^{i-t^0} if $i \in [n] \setminus [t^+ + t^0]$, and the zero vector otherwise. Then $R = PR[\mathrm{supp}(R)]P^\top$ and

$$X^{1/2}RX^{1/2} = X^{1/2}PR[\mathrm{supp}(R)]P^\top X^{1/2}.$$

Note that the eigenvalues of $X^{1/2}PR[\mathrm{supp}(R)]P^\top X^{1/2}$ are real because it is real symmetric. By Lemma 7.6, the n eigenvalues of $X^{1/2}PR[\mathrm{supp}(R)]P^\top X^{1/2}$ are the $t^+ + t^-$ eigenvalues of $R[\mathrm{supp}(R)]P^\top XP = R[\mathrm{supp}(R)]X[\mathrm{supp}(R)]$ together with t^0 zeroes. Noting $X[\mathrm{supp}(R)] \in \mathbb{S}^{t^+ + t^-}_+$ and applying Lemma 7.6 again, we see that the eigenvalues of $R[\mathrm{supp}(R)]X[\mathrm{supp}(R)]$ are the same as the eigenvalues of $X[\mathrm{supp}(R)]^{1/2}R[\mathrm{supp}(R)]X[\mathrm{supp}(R)]^{1/2}$. □

We use the next result in the proof of Lemma 7.10, which provides an alternate representation of $\left\| \lambda\left(X^{1/2}RX^{1/2}\right) \right\|_1$.

Lemma 7.9. *Let* $R \in \mathbb{S}^n$ *and* $X \in \mathbb{S}^n_+$. *The number of positive (resp. negative) eigenvalues of* $X^{1/2}RX^{1/2}$ *is less than or equal to the number of positive (resp. negative) eigenvalues of* R.

Proof. Consider the eigenvalue decomposition of $X = U_x D_x U_x^\top$ with an orthogonal matrix U_x and a diagonal matrix D_x. Note $\lambda(X^{1/2}RX^{1/2}) = \lambda(D_x^{1/2}U_xRU_x^\top D_x^{1/2})$. Let I_x be a diagonal matrix which has $(I_x)_{ii} = (D_x)_{ii}$ if $(D_x)_{ii} > 0$ and $(I_x)_{ii} = 1$ if $(D_x)_{ii} = 0$. Let P_x be a diagonal matrix which has $(P_x)_{ii} = 1$ if $(D_x)_{ii} > 0$ and $(I_x)_{ii} = 0$ if $(D_x)_{ii} = 0$. Then $D_x^{1/2}U_xRU_x^\top D_x^{1/2} = P_x(I_x^{1/2}U_xRU_x^\top I_x^{1/2})P_x$. The matrix $I_x^{1/2}U_xRU_x^\top I_x^{1/2}$ has the same inertia as R because $I_x^{1/2}U_x$ is nonsingular. Because $P_x(I_x^{1/2}U_xRU_x^\top I_x^{1/2})P_x$ is a principal submatrix of $I_x^{1/2}U_xRU_x^\top I_x^{1/2}$, we deduce the result from Cauchy's interlacing eigenvalue theorem. □

Lemma 7.10. *Let* $R \in \mathbb{S}^n$ *and* $X \in \mathbb{S}^n_+$. *Suppose* $R \notin \pm\mathbb{S}^n_+$ *and it is diagonal with diagonal elements from* $\{0, \pm 1\}$ *sorted in nonincreasing order. Let* $n^+ = \max\{k : R_{kk} = 1\}$, $n^- = \min\{k : R_{kk} = -1\}$, *and* $\mathbb{J} = \{(i,j) : 1 \le i \le n^+,\ n^- \le j \le n\}$. *Then*

$$\left\| \lambda\left(X^{1/2}RX^{1/2}\right) \right\|_1 = \sqrt{\langle R, X\rangle^2 - 4 \sum_{(i,j)\in\mathbb{J}} \lambda_i\left(X^{1/2}RX^{1/2}\right)\lambda_j\left(X^{1/2}RX^{1/2}\right)}.$$

Proof. Note that $\langle R, X\rangle = \mathrm{Tr}(RX) = \sum_{i=1}^n \lambda_i(RX) = \sum_{i=1}^n \lambda_i\left(X^{1/2}RX^{1/2}\right)$ where the last equality follows from Corollary 7.7. Furthermore, $X^{1/2}RX^{1/2}$ has at most n^+ positive and at most $n - n^- + 1$ negative eigenvalues because of Lemma 7.9. Hence, we can write

$$\begin{aligned}
\left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1^2 - \langle R, X\rangle^2 &= \left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1^2 - \left(\sum_{i=1}^n \lambda_i\left(X^{1/2}RX^{1/2}\right)\right)^2 \\
&= \left[\sum_{i=1}^{n^+} \lambda_i\left(X^{1/2}RX^{1/2}\right) - \sum_{i=n^-}^{n} \lambda_i\left(X^{1/2}RX^{1/2}\right)\right]^2 \\
&\qquad - \left[\sum_{i=1}^{n^+} \lambda_i\left(X^{1/2}RX^{1/2}\right) + \sum_{i=n^-}^{n} \lambda_i\left(X^{1/2}RX^{1/2}\right)\right]^2 \\
&= -4\left[\sum_{i=1}^{n^+} \lambda_i\left(X^{1/2}RX^{1/2}\right)\right]\left[\sum_{i=n^-}^{n} \lambda_i\left(X^{1/2}RX^{1/2}\right)\right] \\
&= -4\sum_{(i,j)\in\mathbb{J}} \lambda_i\left(X^{1/2}RX^{1/2}\right)\lambda_j\left(X^{1/2}RX^{1/2}\right).
\end{aligned}$$

The result follows from the nonnegativity of $\left\|\lambda\left(X^{1/2}RX^{1/2}\right)\right\|_1$. □

Lemmas 7.4 and 7.10, along with Propositions 5.27(ii) and 5.32(ii), have the following consequence.

Corollary 7.11. *Suppose $R \notin \pm\mathbb{S}_+^n$ and it is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ = \max\{k : R_{kk} = 1\}$, $n^- = \min\{k : R_{kk} = -1\}$, and $\mathbb{J} = \{(i,j) : 1 \le i \le n^+,\ n^- \le j \le n\}$. Then a point $X \in \mathbb{S}_+^n$ satisfies (7.3) if and only if there exists $z \ge \mu_0$ such that*

$$-\sum_{(i,j)\in\mathbb{J}} \lambda_i\left(X^{1/2}RX^{1/2}\right)\lambda_j\left(X^{1/2}RX^{1/2}\right) \ge (z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle). \tag{7.5}$$

Similarly, a point $X \in \mathbb{S}_+^n$ satisfies (7.4) if and only if it satisfies (7.5) together with $z = \mu_0$.

Proof. Lemmas 7.4 and 7.10 show

$$\begin{aligned}
[f_{\mathbb{S}_+^n,R}(X)]^2 - \langle R, X\rangle^2 &= \|\lambda(X^{1/2}RX^{1/2})\|_1^2 - \langle R, X\rangle^2 \\
&= -\sum_{(i,j)\in\mathbb{J}} \lambda_i\left(X^{1/2}RX^{1/2}\right)\lambda_j\left(X^{1/2}RX^{1/2}\right).
\end{aligned}$$

Then the two claims follow from Propositions 5.27(ii) and 5.32(ii), respectively. □

7.2.3 Elementary Disjunctions on the Positive Semidefinite Cone

Although it provides a closed-form equivalent for (5.18) in the case of disjunctions on the positive semidefinite cone, (7.3) can pose challenges from a computational perspective. In this section, we identify a class of two-term disjunctions for which (7.3) can be exactly represented in a tractable form.

We say that the disjunction $\langle D_1, X\rangle \geq \mu_0 \vee \langle D_2, X\rangle \geq \mu_0$ is *elementary* when the matrix $R = D_2 - D_1 \in \mathbb{S}^n$ has exactly one positive and one negative eigenvalue. In this section we consider sets $\mathbb{D}_1, \mathbb{D}_2 \subset \mathbb{S}^n_+$ which are defined by an elementary disjunction $\langle D_1, X\rangle \geq \mu_0 \vee \langle D_2, X\rangle \geq \mu_0$. By Remark 7.3, we assume without any loss of generality that R is diagonal and has exactly one positive entry $R_{11} = 1$ and one negative entry $R_{nn} = -1$. In this case, using Lemma 7.9, the matrix $X^{1/2}RX^{1/2}$ has at most one positive and at most one negative eigenvalue for any $X \in \mathbb{S}^n_+$. The largest and smallest eigenvalues of $X^{1/2}RX^{1/2}$ are

$$\lambda_1\Big(X^{1/2}RX^{1/2}\Big) = \frac{1}{2}\left(X_{11} - X_{nn} + \sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)}\right), \tag{7.6a}$$

$$\lambda_n\Big(X^{1/2}RX^{1/2}\Big) = \frac{1}{2}\left(X_{11} - X_{nn} - \sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)}\right). \tag{7.6b}$$

Hence, Lemma 7.4 and Theorem 7.5 reduce to the statement below for elementary disjunctions on the positive semidefinite cone.

Corollary 7.12. *Suppose $R = D_2 - D_1$ is a diagonal matrix with exactly one positive entry $R_{11} = 1$ and one negative entry $R_{nn} = -1$. Then $f_{\mathbb{S}^n_+,R}(X) = \sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)}$ for any $X \in \mathbb{S}^n_+$. Furthermore, a point $X \in \mathbb{S}^n_+$ satisfies* (7.3) *if and only if it satisfies*

$$\sqrt{(X_{11} - X_{nn})^2 + 4(X_{11}X_{nn} - X_{1n}^2)} \geq 2\mu_0 - \langle D_1 + D_2, X\rangle. \tag{7.7}$$

Proof. The proof follows from noting that $\|\lambda(X^{1/2}RX^{1/2})\|_1 = \lambda_1(X^{1/2}RX^{1/2}) - \lambda_n(X^{1/2}RX^{1/2})$ where $\lambda_1(X^{1/2}RX^{1/2})$ and $\lambda_n(X^{1/2}RX^{1/2})$ are as in (7.6). □

Corollary 7.11 leads to equivalent second-order cone representations for (7.7) in the case of both disjoint and non-disjoint disjunctions.

Theorem 7.13. *Suppose $R = D_2 - D_1$ is a diagonal matrix with exactly one positive entry $R_{11} = 1$ and one negative entry $R_{nn} = -1$. Then a point $X \in \mathbb{S}^n_+$ satisfies* (7.3) *if and only if there exists $z \geq \mu_0$ such that*

$$X[\{1, n\}] - (z - \langle D_1, X\rangle)R[\{1, n\}] \in \mathbb{S}^2_+. \tag{7.8}$$

Similarly, a point $X \in \mathbb{S}^n_+$ *satisfies* (7.4) *if and only if it satisfies* (7.8) *together with* $z = \mu_0$. *Furthermore, the inequality* (7.8) *can be represented as a second-order cone constraint.*

Proof. Fix $X \in \mathbb{S}^n_+$. The first part of Corollary 7.11 shows that X satisfies (7.3) if and only if there exists $z \geq \mu_0$ such that

$$(X_{11}X_{nn} - X_{1n}^2) \geq (z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle).$$

This inequality can be rewritten as

$$\begin{aligned}
&[X_{11}X_{nn} - X_{1n}^2] \geq (z - \langle D_1, X\rangle)(z - \langle D_1, X\rangle - \langle R, X\rangle) \\
&\Leftrightarrow [X_{11}X_{nn} - X_{1n}^2] \geq (z - \langle D_1, X\rangle)^2 - (z - \langle D_1, X\rangle)[X_{11} - X_{nn}] \\
&\Leftrightarrow X_{11}X_{nn} + (z - \langle D_1, X\rangle)[X_{11} - X_{nn}] - (z - \langle D_1, X\rangle)^2 - X_{1n}^2 \geq 0 \\
&\Leftrightarrow [X_{11} - (z - \langle D_1, X\rangle)]\,[X_{nn} + (z - \langle D_1, X\rangle)] - X_{1n}^2 \geq 0. \qquad (7.9)
\end{aligned}$$

The left-hand side of (7.9) is equal to the determinant of the matrix

$$\begin{pmatrix} X_{11} - (z - \langle D_1, X\rangle) & X_{1n} \\ X_{1n} & X_{nn} + (z - \langle D_1, X\rangle) \end{pmatrix}.$$

This matrix equals $X[\{1, n\}] - (z - \langle D_1, X\rangle)R[\{1, n\}]$ which also appears in (7.8).

To finish the proof, we show that the diagonal elements of the matrix on the left-hand side of (7.8) are nonnegative for any $X \in \mathbb{S}^n_+$ and $z \in \mathbb{R}$ which satisfy (7.9). That is, we show $X_{11} - (z - \langle D_1, X\rangle) \geq 0$ and $X_{nn} + (z - \langle D_1, X\rangle) \geq 0$. When X and z satisfy $\langle D_1, X\rangle = z$, the hypothesis that $X \in \mathbb{S}^n_+$ implies this immediately. Therefore, we can assume $\langle D_1, X\rangle \neq z$. Note that (7.9) implies

$$[X_{11} - (z - \langle D_1, X\rangle)]\,[X_{nn} + (z - \langle D_1, X\rangle)] \geq 0.$$

Because $\langle D_1, X\rangle \neq z$ and $X_{11}, X_{nn} \geq 0$ for $X \in \mathbb{S}^n_+$, at least one of the terms in the product above is positive; this also implies the nonnegativity of the other term. Hence, (7.9) is equivalent to (7.8) for any $X \in \mathbb{S}^n_+$ and $z \in \mathbb{R}$.

The second part of Corollary 7.11 shows that X satisfies (7.4) if and only if it satisfies (7.8) together with $z = \mu_0$. □

Remark 7.14. *Suppose the hypotheses of Theorem 7.13 are satisfied. Reversing the roles of* D_1 *and* D_2 *in the proof of Theorem 7.13, the inequality* (7.8) *can be equivalently represented as*

$$X[\{1, n\}] + (z - \langle D_2, X\rangle)R[\{1, n\}] \in \mathbb{S}^2_+.$$

7.2.4 Low-Complexity Inequalities for General Two-Term Disjunctions

In this section, in a spirit similar to Remark 5.24, we study structured conic inequalities valid for two-term disjunctions on $\mathbb{S}^n_+$. Section 7.2.3 showed that (7.3) admits an exact second-order cone representation when we consider elementary disjunctions on the positive semidefinite cone. However, the structure of (7.3) can be more complicated in the case of general two-term disjunctions. In this section, we introduce and discuss simpler conic inequalities which provide good relaxations to (7.3) at a significantly lower cost of computational complexity.

Relaxing the Inequality

We are going to use a classical result from matrix analysis to arrive at the results of this section. We state this result as Lemma 7.15 below; see [71, Theorem 1.2.16] for a proof.

Lemma 7.15. *Let $A \in \mathbb{R}^{n\times n}$. Then*

$$\sum_{1\le i<j\le n} \det(A[\{i,j\}]) = \sum_{1\le i<j\le n} \lambda_i(A)\lambda_j(A).$$

Using Lemma 7.15, we prove the following result.

Lemma 7.16. *Let $R \in \mathbb{S}^n$ and $X \in \mathbb{S}^n_+$. Suppose $R \notin \pm\mathbb{S}^n_+$ and R is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ = \max\{k : \ R_{kk} = 1\}$, $n^- = \min\{k : \ R_{kk} = -1\}$, and $\mathbb{J} = \{(i,j) : \ 1 \le i \le n^+, \ n^- \le j \le n\}$. Then*

$$\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) \ge -\sum_{(i,j)\in\mathbb{J}} \lambda_i\Big(X^{1/2}RX^{1/2}\Big)\lambda_j\Big(X^{1/2}RX^{1/2}\Big). \tag{7.10}$$

Proof. Let $Y = RX$. From Corollary 7.7, $\lambda(Y) = \lambda(X^{1/2}RX^{1/2})$; therefore, the right-hand side of (7.10) is exactly equal to $-\sum_{(i,j)\in\mathbb{J}} \lambda_i(Y)\lambda_j(Y)$. Define the sets $\mathbb{J}^+ = \{(i,j) : \ 1 \le i < j \le n^+\}$ and $\mathbb{J}^- = \{(i,j) : \ n^- \le i < j \le n\}$. Note that $\det(Y[\{i,j\}]) = \det(X[\{i,j\}])$ if $(i,j) \in \mathbb{J}^+ \cup \mathbb{J}^-$, $\det(Y[\{i,j\}]) = -\det(X[\{i,j\}])$ if $(i,j) \in \mathbb{J}$, and $\det(Y[\{i,j\}]) = 0$ otherwise. Furthermore, Y has at most n^+ positive and at most $n - n^- + 1$ negative eigenvalues. Then

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) &= -\sum_{(i,j)\in\mathbb{J}} \det(Y[\{i,j\}]) \\
&= -\sum_{1\le i<j\le n} \det(Y[\{i,j\}]) + \sum_{(i,j)\in\mathbb{J}^+} \det(Y[\{i,j\}]) + \sum_{(i,j)\in\mathbb{J}^-} \det(Y[\{i,j\}]) \\
&= -\sum_{1\le i<j\le n} \lambda_i(Y)\lambda_j(Y) + \sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) + \sum_{(i,j)\in\mathbb{J}^-} \det(X[\{i,j\}]) \\
&= -\sum_{(i,j)\in\mathbb{J}} \lambda_i(Y)\lambda_j(Y) + \left[\sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) - \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(Y)\lambda_j(Y)\right] \\
&\qquad + \left[\sum_{(i,j)\in\mathbb{J}^-} \det(X[\{i,j\}]) - \sum_{(i,j)\in\mathbb{J}^-} \lambda_i(Y)\lambda_j(Y)\right].
\end{aligned}$$

In order to reach (7.10), we show

$$\sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) \ge \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(Y)\lambda_j(Y), \tag{7.11a}$$

$$\sum_{(i,j)\in\mathbb{J}^-} \det(X[\{i,j\}]) \ge \sum_{(i,j)\in\mathbb{J}^-} \lambda_i(Y)\lambda_j(Y). \tag{7.11b}$$

Let $P^+ \in \mathbb{S}^n_+$ be the diagonal matrix with diagonal entries $P^+_{ii} = 1$ if $i \in [n^+]$ and zero otherwise. Let $P^- \in \mathbb{S}^n_+$ be the matrix $P^- = P^+ - R$. Define $X^+ = P^+XP^+$ and $X^- = P^-XP^-$. Then $X^+, X^- \in \mathbb{S}^n_+$. Furthermore, X^+ (resp. X^-) has at most n^+ (resp. $n - n^- + 1$) nonzero (positive) eigenvalues. We first prove (7.11a). Note that

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{J}^+} \det(X[\{i,j\}]) &= \sum_{1\le i<j\le n} \det(X^+[\{i,j\}]) = \sum_{1\le i<j\le n} \lambda_i(X^+)\lambda_j(X^+) \\
&= \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(X^+)\lambda_j(X^+),
\end{aligned}$$

where the second equation follows from Lemma 7.15 and the last one from the fact that X^+ has at most n^+ positive eigenvalues. From $(P^+)^2 = P^+$ and Lemma 7.6, we have $\lambda(X^+) = \lambda(P^+XP^+) = \lambda(P^+X) = \lambda(X^{1/2}P^+X^{1/2})$. From Corollary 7.7, we have $\lambda(Y) = \lambda(X^{1/2}RX^{1/2})$. Note $X^{1/2}P^+X^{1/2} - X^{1/2}RX^{1/2} = X^{1/2}P^-X^{1/2} \in \mathbb{S}^n_+$; hence, $\lambda(X^{1/2}P^+X^{1/2}) \ge \lambda(X^{1/2}RX^{1/2})$. Note from Lemma 7.9 that $X^{1/2}RX^{1/2}$ has at most $n - n^- + 1$ negative eigenvalues; hence, the largest n^+ eigenvalues of $X^{1/2}RX^{1/2}$ are all nonnegative. Then we have $\sum_{(i,j)\in\mathbb{J}^+} \lambda_i(X^+)\lambda_j(X^+) \ge \sum_{(i,j)\in\mathbb{J}^+} \lambda_i(Y)\lambda_j(Y)$ because the first n^+ coordinates of both $\lambda(X^+)$ and $\lambda(Y)$ are nonnegative and $\lambda(X^+) \ge \lambda(Y)$. This proves (7.11a). The proof of (7.11b) follows in a similar manner. □

Remark 7.17. *Suppose the hypotheses of Lemma 7.16 are satisfied. Then Remark 5.20(ii) and Lemmas 7.4, 7.10, and 7.16 imply that, for any $X \in \mathbb{S}^n_+$, we have*

$$\sqrt{\langle R, X\rangle^2 + 4 \sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}])} \geq \|\lambda(X^{1/2}RX^{1/2})\|_1 \geq |\langle R, X\rangle|.$$

If the rank of $X \in \mathbb{S}^n_+$ is one, then $\det(X[\{i,j\})=0$ for all $(i,j) \in \mathbb{J}$; therefore, both inequalities above hold at equality.

An appealing feature of (7.3) is that any rank-one matrix $X \in \mathbb{S}^n_+$ satisfies (7.3) if and only if $X \in \mathbb{D}_1 \cup \mathbb{D}_2$. Recall Remark 5.20 and the ensuing discussion. Next we use Remark 7.17 to construct a relaxation of (7.3) which shares the same feature.

Proposition 7.18. *Suppose $R \notin \pm\mathbb{S}^n_+$ and it is diagonal with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $n^+ = \max\{k : \ R_{kk} = 1\}$, $n^- = \min\{k : \ R_{kk} = -1\}$, and $\mathbb{J} = \{(i,j) : \ 1 \leq i \leq n^+, \ n^- \leq j \leq n\}$. Let $g_{\mathbb{S}^n_+,R} : \mathbb{S}^n \to \mathbb{R} \cup \{-\infty\}$ be defined as*

$$g_{\mathbb{S}^n_+,R}(X) = \begin{cases} \sqrt{\langle R, X\rangle^2 + 4\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}])} & \text{if } X \in \mathbb{S}^n_+, \\ -\infty & \text{otherwise.} \end{cases}$$

i. Any point $X \in \mathbb{S}^n_+$ which satisfies (7.3) also satisfies

$$g_{\mathbb{S}^n_+,R}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle. \tag{7.12}$$

Similarly, any point $X \in \mathbb{S}^n_+$ which satisfies (7.4) also satisfies

$$g_{\mathbb{S}^n_+,R}(X) \geq |2\mu_0 - \langle D_1 + D_2, X\rangle|. \tag{7.13}$$

ii. Any point $X \in \mathbb{S}^n_+$ satisfies (7.12) if and only if there exists $z \geq \mu_0$ such that

$$\left[\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle)\right]\left[\sum_{j=n^-}^{n} X_{jj} + (z - \langle D_1, X\rangle)\right] \geq \sum_{(i,j)\in\mathbb{J}} X_{ij}^2, \tag{7.14a}$$

$$\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle) \geq 0, \qquad \sum_{j=n^-}^{n} X_{jj} + (z - \langle D_1, X\rangle) \geq 0. \tag{7.14b}$$

Similarly, any point $X \in \mathbb{S}^n_+$ satisfies (7.13) if and only if it satisfies (7.14) together with $z = \mu_0$. Furthermore, (7.14) can be represented as a single second-order cone constraint.

Proof. By Remark 7.17, $g_{\mathbb{S}^n_+,R}(X) \geq f_{\mathbb{S}^n_+,R}(X)$ for all $X \in \mathbb{S}^n_+$. Then statement (i) follows from Theorem 7.5. As in Proposition 5.27(ii), we can show that a point $X \in \mathbb{S}^n_+$ satisfies (7.12) if and only if there exists $z \geq \mu_0$ such that

$$\left[g_{\mathbb{S}^n_+,R}(X)\right]^2 - \langle R, X\rangle^2 \geq 4(z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle). \tag{7.15}$$

Similarly, as in Proposition 5.32(ii), we can show that a point $X \in \mathbb{S}^n_+$ satisfies (7.13) if and only if it satisfies (7.15) together with $z = \mu_0$. We show that (7.15) can be represented as (7.14). The inequality (7.15) is identical to $\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) \geq (z - \langle D_1, X\rangle)(z - \langle D_2, X\rangle)$. Following steps similar to those in the proof of Theorem 7.13, we rewrite it as

$$\begin{aligned}
&\sum_{(i,j)\in\mathbb{J}} \det(X[\{i,j\}]) \geq (z - \langle D_1, X\rangle)(z - \langle D_1, X\rangle - \langle R, X\rangle) \\
\Leftrightarrow &\sum_{(i,j)\in\mathbb{J}} [X_{ii}X_{jj} - X_{ij}^2] \geq (z - \langle D_1, X\rangle)^2 - (z - \langle D_1, X\rangle)\left[\sum_{i=1}^{n^+} X_{ii} - \sum_{j=n^-}^{n} X_{jj}\right] \\
\Leftrightarrow &\left[\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle)\right]\left[\sum_{j=1}^{n^-} X_{jj} + (z - \langle D_1, X\rangle)\right] - \sum_{(i,j)\in\mathbb{J}} X_{ij}^2 \geq 0.
\end{aligned}$$

The final form is the same as (7.14a). Furthermore, as in the proof of Theorem 7.13, we can show $\sum_{i=1}^{n^+} X_{ii} - (z - \langle D_1, X\rangle) \geq 0$ and $\sum_{j=n^-}^{n} X_{jj} + (z - \langle D_1, X\rangle) \geq 0$ for any $X \in \mathbb{S}^n_+$ and $z \in \mathbb{R}$ satisfying (7.14a). Observing that the inequalities (7.14) can be written as a rotated second-order cone constraint completes the proof. □

Remark 7.19. *We note that, under the hypotheses of Proposition 7.18, the inequality* (7.12) *defines a convex region in* $\mathbb{S}^n_+$. *To see this, note that the set of points satisfying* (7.12) *and* $X \in \mathbb{S}^n_+$ *is precisely the projection of the set of points satisfying* (7.14) *and* $X \in \mathbb{S}^n_+$ *onto the space of* X *variables. Because projection of a convex set is convex, this immediately proves the convexity of the region defined by* (7.12) *inside* $\mathbb{S}^n_+$.

Remark 7.20. *We note that the results of Section 7.2.3 immediately follow from Proposition 7.18 because in the particular case of elementary disjunctions,* (7.10) *holds at equality. This can be seen by noting that* $\mathbb{J}^+ = \mathbb{J}^- = \emptyset$ *in the proof of Lemma 7.16. Therefore, in the case of elementary disjunctions,* (7.12) *does not only define a relaxation of* (7.3)*; it is also equivalent to* (7.3). *Despite this connection, we have opted to keep Section 7.2.3 due to its more transparent derivation.*

Example 7.1. Consider the split disjunction $-\frac{1}{2}(X_{11} + X_{22} - X_{33}) \geq 1 \vee \frac{1}{2}(X_{11} + X_{22} - X_{33}) \geq 1$ on $\mathbb{S}^3_+$. The sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are defined as in (7.1) with $D_1 =$

$-\frac{1}{2}\left(\left(e^1\right)\left(e^1\right)^\top + \left(e^2\right)\left(e^2\right)^\top - \left(e^3\right)\left(e^3\right)^\top\right)$, $D_2 = -D_1$, and $\mu_0 = 1$. Proposition 7.18(ii) shows that the inequalities

$$\left[\frac{1}{2}(X_{11}+X_{22}+X_{33})-1\right]\left[\frac{1}{2}(X_{11}+X_{22}+X_{33})+1\right] \geq X_{13}^2+X_{23}^2,$$
$$\frac{1}{2}(X_{11}+X_{22}+X_{33})-1 \geq 0, \qquad \frac{1}{2}(X_{11}+X_{22}+X_{33})+1 \geq 0$$

are valid for $\mathbb{D}_1 \cup \mathbb{D}_2$. Furthermore, these inequalities can be represented as the second-order cone constraint

$$\begin{pmatrix} 2X_{13} \\ 2X_{23} \\ 2 \\ X_{11}+X_{22}+X_{33} \end{pmatrix} \in \mathbb{L}^4. \tag{7.16}$$

Let $\mathbb{G}$ denote the region defined by (7.16). Figure 7.1 shows the intersection of various two-dimensional linear spaces with $\mathbb{D}_1 \cup \mathbb{D}_2$, $\mathbb{S}^3_+$, and $\mathbb{G}$. Each two-dimensional linear space has the form $\mathbb{W} = \{x\pi\pi^\top + y\psi\psi^\top : (x,y) \in \mathbb{R}^2\}$ where $\pi, \psi \in \mathbb{R}^3$ are chosen such that $\pi_1 = \frac{\sqrt{5}}{2}$, $\psi_3 = \sqrt{2}$, and the remaining components of π and ψ are random numbers from the interval $[-1,1]$. The intersection of $\mathbb{W}$ with $\mathbb{S}^3_+$ corresponds to the nonnegative orthant in the (x,y) space. Each image depicts the intersection of $\mathbb{W}$ with $\mathbb{D}_1 \cup \mathbb{D}_2$ (blue meshed area) and $\mathbb{G}$ (red unmeshed area) in the (x,y) space.

We remind the reader that (7.16) is valid for all of $\mathbb{D}_1 \cup \mathbb{D}_2$ and not just $\mathbb{D}_1 \cup \mathbb{D}_2 \cap \mathbb{W}$. Hence, even in the cases where $\overline{\operatorname{conv}}(\mathbb{D}_1 \cup \mathbb{D}_2) = \mathbb{S}^3_+ \cap \mathbb{G}$, we cannot in general expect to have $\overline{\operatorname{conv}}((\mathbb{D}_1 \cup \mathbb{D}_2) \cap \mathbb{W}) = \mathbb{S}^3_+ \cap \mathbb{G} \cap \mathbb{W}$.

In the next remark, we discuss how we can utilize our results for elementary disjunctions in the light of Remark 5.24 to build structured relaxations of (7.3).

Remark 7.21. *Suppose $R \notin \pm\mathbb{S}^n_+$ is a diagonal matrix with diagonal elements from $\{0, \pm 1\}$ sorted in nonincreasing order. Let $R_+, R_- \in \mathbb{S}^n_+$ and $R_1, \ldots, R_\ell \notin \pm\mathbb{S}^n_+$ be such that $R = R_+ - R_- + \sum_{k=1}^{\ell} R_k$ and $\operatorname{rank}(R_k) = 2$. Remark 5.23 indicates that any $X \in \mathbb{D}_1 \cup \mathbb{D}_2$ satisfies the convex inequality*

$$f_{\mathbb{S}^n_+, R_+}(X) + f_{\mathbb{S}^n_+, -R_-}(X) + \sum_{k=1}^{\ell} f_{\mathbb{S}^n_+, R_k}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle.$$

Note that, for any $X \in \mathbb{S}^n_+$, $f_{\mathbb{S}^n_+, R_+}(X) = \langle R_+, X\rangle$ and $f_{\mathbb{S}^n_+, -R_-}(X) = \langle R_-, X\rangle$. Now, for each $k \in [\ell]$, consider the eigenvalue decomposition of $R_k = U_k D_k U_k^\top$, and define $Q_k \in \operatorname{int}\mathbb{S}^n_+$ as in Remark 7.3. Then $J = Q_k U_k^\top R_k U_k Q_k$ is a diagonal matrix with exactly one

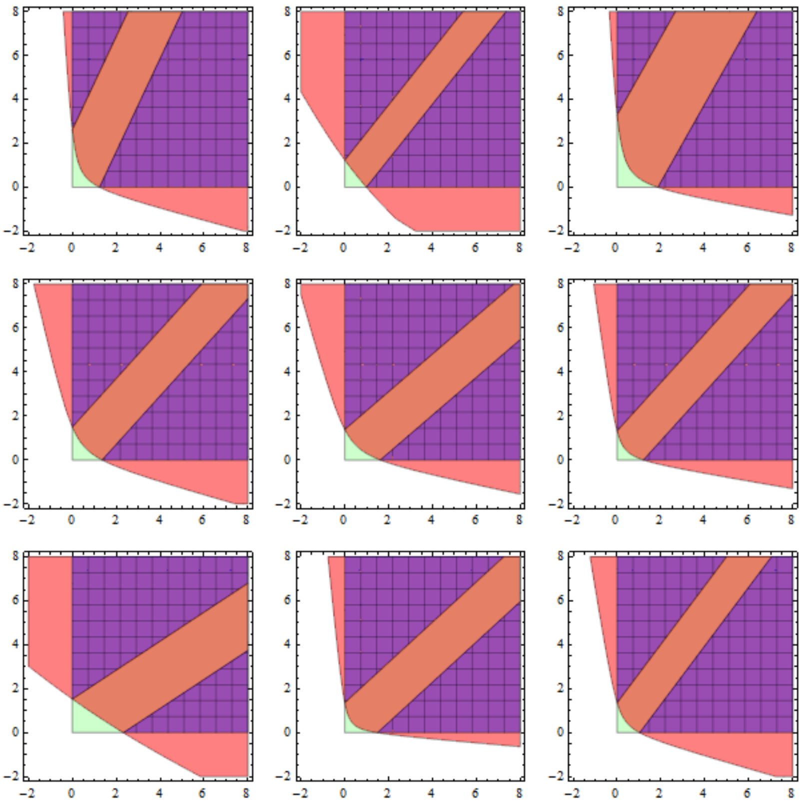


Figure 7.1: Sets associated with the disjunction $-\frac{1}{2}(X_{11}+X_{22}-X_{33}) \geq 1 \vee \frac{1}{2}(X_{11}+X_{22}-X_{33}) \geq 1$ on $\mathbb{S}^3_+$.

positive entry $J_{11}=1$ and exactly one negative entry $J_{nn}=-1$. Furthermore, Lemmas 7.4 and 7.6 show

$$\begin{aligned} f_{\mathbb{S}^n_+,R_k}(X) &= \left\|\lambda\Big(R_k X\Big)\right\|_1 = \left\|\lambda\Big(J(Q_k^{-1}U_k^\top X U_k Q_k^{-1})\Big)\right\|_1 \\ &= f_{\mathbb{S}^n_+,J}\Big(Q_k^{-1}U_k^\top X U_k Q_k^{-1}\Big). \end{aligned}$$

The function $f_{\mathbb{S}^n_+,J}(\cdot)$ has the form given in Corollary 7.12. It follows that any inequality constructed through this approach admits a second-order conic representation in a lifted space. We note that there is a lot of flexibility in the choice of the matrices R_+, R_-, and R_k and each selection will lead to a different valid inequality.

Relaxing the Disjunction

Another approach to using our results on elementary disjunctions for arbitrary two-term disjunctions might be through relaxing the underlying disjunction. To illustrate this point, consider a disjunction $\langle D_1, X\rangle \geq \mu_0 \vee \langle D_2, X\rangle \geq \mu_0$. Let $R_+, R_- \in \mathbb{S}^n_+$ be such that $R' = R - R_+ + R_- \notin \pm\mathbb{S}^n_+$ and has rank two. Define $D'_1 = D_1 + R_-$ and $D'_2 = D_2 + R_+$. The matrices D'_1 and D'_2 define a relaxation $\langle D'_1, X\rangle \geq \mu_0 \vee \langle D'_2, X\rangle \geq \mu_0$ of the original disjunction because any $X \in \mathbb{S}^n_+$ satisfying $\langle D_i, X\rangle \geq \mu_0$ also satisfies $\langle D'_i, X\rangle \geq \mu_0$ for $i \in \{1, 2\}$. Therefore, any inequality valid for the relaxed disjunction is also valid for the original. Because $R' \notin \pm\mathbb{S}^n_+$ and has rank two, it has exactly one positive and one negative eigenvalue. The relaxed disjunction is elementary, and the results of Section 7.2.3 can be used to derive structured nonlinear valid inequalities for $\mathbb{D}_1 \cup \mathbb{D}_2$. In particular, this approach leads to the inequality

$$\begin{aligned}
& f_{\mathbb{S}^n_+,R'}(X) \geq 2\mu_0 - \langle D'_1 + D'_2, X\rangle = 2\mu_0 - \langle D_1 + D_2, X\rangle - \langle R_+ + R_-, X\rangle \\
\iff & \langle R_+ + R_-, X\rangle + f_{\mathbb{S}^n_+,R'}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle \\
\iff & f_{\mathbb{S}^n_+,R_+}(X) + f_{\mathbb{S}^n_+,-R_-}(X) + f_{\mathbb{S}^n_+,R'}(X) \geq 2\mu_0 - \langle D_1 + D_2, X\rangle.
\end{aligned}$$

We note, however, that the inequality above can also be obtained through the approach outlined in Remark 7.21. Therefore, the approach of Remark 7.21 is a more powerful method to build structured relaxations of (7.3).

7.3 Conclusion

In this chapter we have considered two-term disjunctions on the positive semidefinite cone and intersections of the positive semidefinite cone with rank-two non-convex quadratics. We have developed structured nonlinear valid inequalities for such sets by building upon the results of Section 5.4.

In Chapter 8 we extend the results of Chapter 6 to affine cross-sections of the second-order cone. Nonetheless, studying the closed convex hulls of disjunctions on cross-sections of general regular cones remains a topic of future research. Our results in Chapters 5 and 7 immediately extend to cases where the base convex set is the intersection of a regular cone $\mathbb{K}$ with homogeneous half-spaces through [38, Lemma 5] (or its generalization given in [70, Lemma 3.6]) and to cases where it corresponds to certain cross-sections of $\mathbb{K}$ through [38, Lemma 7]. Particular cross-sections of the positive semidefinite cone deserve specific interest from the point of view of combinatorial optimization. For instance, in the case of the maximum cut problem, it is well-known that the elliptope $\{X \in \mathbb{S}^n_+ : X_{ii} = 1 \; \forall i \in [n]\}$ provides a good outer approximation to the cut polytope, which is the convex hull of

(± 1) characteristic vectors of all cuts in a complete graph on n vertices. Goemans and Williamson [60] used this observation to develop the approximation algorithm with the best known approximation guarantee for the maximum cut problem. Furthermore, the elliptope provides a valid integer programming formulation for the maximum cut problem in the sense that any $X \in \{\pm 1\}^{n \times n}$ in the elliptope corresponds to the characteristic vector of a cut. On this cross-section of the positive semidefinite cone, we can easily transform any two-term disjunction into an elementary disjunction. Thus, the results of Section 7.2.3 can be relevant. We hope that these results will be instrumental to the development of more practical algorithms for maximum cut and other hard combinatorial problems.

Chapter 8

Convex Hulls of Disjunctions on Cross-Sections of the Second-Order Cone

Acknowledgments. This chapter is based on joint work with Gérard Cornuéjols [107].

8.1 Introduction

8.1.1 Motivation

In Chapter 6 we derived a family of convex inequalities which collectively describe the closed convex hull of a general two-term disjunction on the second-order cone and identified conditions which ensure that a single inequality from this family is sufficient for the convex hull description. The purpose of this chapter is similar: In this chapter, we consider general two-term disjunctions on an affine *cross-section* of the second-order cone:

$$\mathbb{C} = \{x \in \mathbb{L}^n : \ Ax = b\} \tag{8.1}$$

As before, associated with a disjunction $c_1^\top x \geq c_{1,0} \vee c_2^\top x \geq c_{2,0}$, we define the sets

$$\mathbb{C}_i = \{x \in \mathbb{C} : \ \langle c_i, x\rangle \geq c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}.$$

In order to derive the tightest convex inequalities that can be obtained from the disjunction $\mathbb{C}_1 \cup \mathbb{C}_2$, we study the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. In particular, we are interested in convex inequalities that may be added to the description of $\mathbb{C}$ to obtain a characterization of the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. Our starting point is the results of Chapter 6 about two-term disjunctions on the second-order cone. We extend the main result of Chapter 6 to cross-sections of the second-order cone. Such cross-sections include ellipsoids, paraboloids, and hyperboloids as special cases.

The reader is referred to Section 5.1.2 for a detailed discussion of disjunctive inequalities in mixed-integer conic programming. Prior to our study, similar results about two-term disjunctions on cross-sections of the second-order cone appeared in [27, 52, 89]. Our results generalize the work of [52, 89], which considered only split disjunctions on cross-sections of the second-order cone, and the work of [27], which analyzed the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ under the assumptions that $\mathbb{C}_1 \cap \mathbb{C}_2 = \emptyset$ and the sets $\{x \in \mathbb{C} : \langle c_1, x\rangle = c_{1,0}\}$ and $\{x \in \mathbb{C} : \langle c_2, x\rangle = c_{2,0}\}$ are bounded. Our results show that the associated convex hulls can be significantly more complicated in the absence of these assumptions. Similar and complementary results on describing the convex hull of intersections of the second-order cone or its affine cross-sections with a single homogeneous quadratic have recently been obtained in [38, 88].

8.1.2 Notation and Terminology

We assume that $\mathbb{R}^n$ is equipped with the standard inner product $\langle \alpha, x\rangle = \alpha^\top x$. The standard (Euclidean) norm $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ is defined as $\|x\|_2 = \sqrt{\langle x, x\rangle}$. The dual cone of $\mathbb{V} \subset \mathbb{R}^n$ is $\mathbb{V}^* = \{\alpha \in \mathbb{R}^n : \langle x, \alpha\rangle \ge 0 \ \forall x \in \mathbb{V}\}$. We remind the reader that the second-order cone is self-dual, that is, its dual is equal to itself. Throughout the chapter, we let $\operatorname{conv}\mathbb{V}$, $\overline{\operatorname{conv}}\,\mathbb{V}$, $\operatorname{cone}\mathbb{V}$, and $\operatorname{span}\mathbb{V}$ represent the convex hull, closed convex hull, conical hull, and linear span of a set $\mathbb{V} \subset \mathbb{R}^n$, respectively. We let $\operatorname{int}\mathbb{V}$, $\operatorname{bd}\mathbb{V}$, and $\dim\mathbb{V}$ represent the topological interior, boundary, and dimension of $\mathbb{V}$, respectively. We use $\operatorname{rec}\mathbb{V}$ to refer to the recession cone of a closed convex set $\mathbb{V}$. Given a vector $u \in \mathbb{R}^n$, we let $\tilde{u} = (u_1; \ldots; u_{n-1})$ denote the subvector obtained by dropping its last entry.

8.1.3 Outline of the Chapter

In Section 8.2 we show that the set $\mathbb{C}$ can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. In Section 8.3 we provide a complete convex hull description of homogeneous two-term disjunctions on the (whole) second-order cone. In Section 8.4 we prove the main result of this chapter, Theorem 8.8, which characterizes the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ with a single convex inequality under certain conditions. We end the chapter with two examples which illustrate the applicability of Theorem 8.8.

8.2 Intersection of the Second-Order Cone with an Affine Subspace

In this section, we show that the set $\mathbb{C}$ can be assumed to be the intersection of a lower-dimensional second-order cone with a single hyperplane. Let $\mathbb{W} = \{x \in \mathbb{R}^n : Ax = b\}$ so that $\mathbb{C} = \mathbb{L}^n \cap \mathbb{W}$. We are going to use the following lemma to simplify our analysis.

Lemma 8.1. *Let $\mathbb{V}$ be a p-dimensional linear subspace of $\mathbb{R}^n$. The intersection $\mathbb{L}^n \cap \mathbb{V}$ is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^p$.*

See Section 2.1 of [25] for a similar result. We do not give a formal proof of Lemma 8.1 but just note that it can be obtained by observing that the second-order cone is the conic hull of a (one dimension smaller) sphere, and that the intersection of a sphere with an affine space is either empty, a single point (when the affine space intersects the sphere but not its interior), or a lower dimensional sphere of the same dimension as the affine space (when the affine space intersects the interior of the sphere).

Lemma 8.1 implies that, when $b = 0$, $\mathbb{C}$ is either the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m}$. The closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ can be described easily when $\mathbb{C}$ is a single point or a half-line. Furthermore, the problem of characterizing the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ when $\mathbb{C}$ is a bijective linear transformation of $\mathbb{L}^{n-m}$ can be reduced to that of convexifying an associated two-term disjunction on $\mathbb{L}^{n-m}$. Chapter 6 contains a detailed study of closed convex hulls of two-term disjunctions on the second-order cone.

In the remainder, we focus on the case $b \neq 0$. Note that, whenever this is the case, we can permute and normalize the rows of (A, b) so that its last row reads $(a_m^\top, 1)$, and subtracting a multiple of $(a_m^\top, 1)$ from the other rows if necessary, we can write the remaining rows of (A, b) as $(\tilde{A}, 0)$. Therefore, we can assume without any loss of generality that all components of b are zero except the last one. Isolating the last row of (A, b) from the others, we can then write

$$\mathbb{W} = \left\{x \in \mathbb{R}^n : \ \tilde{A}x = 0, \ a_m^\top x = 1\right\}.$$

Let $\mathbb{V} = \{x \in \mathbb{R}^n : \tilde{A}x = 0\}$. By Lemma 8.1, $\mathbb{L}^n \cap \mathbb{V}$ is the origin, a half-line, or a bijective linear transformation of $\mathbb{L}^{n-m+1}$. Again, the first two cases are easy and not of interest in our analysis. In the last case, we can find a matrix D whose columns form an orthonormal basis for $\mathbb{V}$ and define a nonsingular matrix H such that $\{y \in \mathbb{R}^{n-m+1} : Dy \in \mathbb{L}^n\} =$

$H\mathbb{L}^{n-m+1}$. Then we can represent $\mathbb{C}$ equivalently as

$$\begin{aligned}\mathbb{C} &= \left\{x \in \mathbb{L}^n :\ x = Dy,\ a_m^\top x = 1\right\} \\ &= D\left\{y \in \mathbb{R}^{n-m+1} :\ Dy \in \mathbb{L}^n,\ a_m^\top Dy = 1\right\} \\ &= D\left\{y \in \mathbb{R}^{n-m+1} :\ y \in H\mathbb{L}^{n-m+1},\ a_m^\top Dy = 1\right\} \\ &= DH\left\{z \in \mathbb{L}^{n-m+1} :\ a_m^\top DHz = 1\right\}.\end{aligned}$$

The set $\mathbb{C} = \mathbb{L}^n \cap \mathbb{W}$ is a bijective linear transformation of $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$. Furthermore, the same linear transformation maps any two-term disjunction on $\{z \in \mathbb{L}^{n-m+1} : a_m^\top DHz = 1\}$ to a two-term disjunction on $\mathbb{C}$ and vice versa. Thus, without any loss of generality, we can assume $m = 1$. Under this assumption, we can rewrite (8.1) as

$$\mathbb{C} = \left\{x \in \mathbb{L}^n :\ a^\top x = 1\right\}. \tag{8.2}$$

In the remainder, we study the problem of describing the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ where

$$\mathbb{C}_i = \{x \in \mathbb{C} :\ \langle c_i, x\rangle \geq c_{i,0}\} \quad \text{for} \quad i \in \{1, 2\}. \tag{8.3}$$

In Section 8.4 we show that, under certain conditions, the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ can be described with a single second-order cone inequality, together with the constraint $x \in \mathbb{C}$.

8.3 Homogeneous Disjunctions on the Second-Order Cone

In this section, we analyze the convex hull of a homogeneous two-term disjunction $\langle c_1, x\rangle \geq 0 \vee \langle c_2, x\rangle \geq 0$ on the second-order cone. Associated with this disjunction, we define the sets

$$\mathbb{K}_i = \{x \in \mathbb{L}^n :\ \langle c_i, x\rangle \geq 0\} \quad \text{for} \quad i \in \{1, 2\}. \tag{8.4}$$

Note that each set $\mathbb{K}_i$ is a relaxation of the set $\mathbb{C}_i$, considered in the previous section. The main result of this section characterizes the convex hull of $\mathbb{K}_1 \cup \mathbb{K}_2$. Note that $\mathbb{K}_1$ and $\mathbb{K}_2$ are closed, convex, pointed cones; therefore, the convex hull of $\mathbb{K}_1 \cup \mathbb{K}_2$ is always closed (see, e.g., [96, Corollary 9.1.3]).

Consider sets $\mathbb{K}_1$ and $\mathbb{K}_2$, which are defined as in (8.4) and which satisfy the basic disjunctive setup. By Condition 5.1, we have $\mathbb{K}_1, \mathbb{K}_2 \subsetneq \mathbb{L}^n$, and by Condition 5.2, we have that $\mathbb{K}_1$ and $\mathbb{K}_2$ are full-dimensional. This implies $c_i \notin \pm\mathbb{L}^n$, or equivalently $\|\tilde{c}_i\|^2 > c_{i,n}^2$, for $i \in \{1, 2\}$. After scaling c_1 and c_2 with appropriate positive scalars if necessary, we may assume without any loss of generality that

$$\|\tilde{c}_1\|^2 - c_{1,n}^2 = \|\tilde{c}_2\|^2 - c_{2,n}^2 = 1. \tag{8.5}$$

In the remainder, we let $r = c_2 - c_1$ and $\mathcal{N} = \|\tilde{r}\|^2 - r_n^2$.

Remark 8.2. *Consider $\mathbb{K}_1$ and $\mathbb{K}_2$ defined as in (8.4). Suppose $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy Condition 5.1. Then we have $r = c_2 - c_1 \notin \pm\mathbb{L}^n$. Indeed, $r \in \mathbb{L}^n$ implies that $\langle r, x\rangle \geq 0$ for all $x \in \mathbb{L}^n$, and this implies $\mathbb{K}_2 \subset \mathbb{K}_1$; similarly, $-r \in \mathbb{L}^n$ implies $\mathbb{K}_1 \subset \mathbb{K}_2$. Hence, $\mathcal{N} = \|\tilde{r}\|^2 - r_n^2 > 0$.*

We recall the following results from Chapter 6 which will be useful in proving the results of this chapter. The first result is a restatement of Corollary 6.10 for the disjunction $\mathbb{K}_1 \cup \mathbb{K}_2$ under consideration.

Corollary 8.3. *Consider $\mathbb{K}_1$ and $\mathbb{K}_2$ defined as in (8.4). Suppose $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy Condition 5.1. Any point $x \in \mathbb{K}_1 \cup \mathbb{K}_2$ satisfies*

$$\sqrt{\langle r, x\rangle^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|^2\right)} \geq \langle -c_1 - c_2, x\rangle. \tag{8.6}$$

Furthermore, this inequality defines a convex region inside the second-order cone.

Proof. The fact that $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy Condition 5.1 implies $r = c_2 - c_1 \notin \pm\mathbb{L}^n$. The hypotheses of Corollary 6.10 are satisfied after setting $\beta_1 = \beta_2 = 1$. The result follows. □

The next proposition shows that (8.6) can be written in second-order cone form inside the second-order cone except in the region where both clauses of the disjunction are strictly satisfied. It is a restatement of Remark 6.7 and Corollary 6.8.

Proposition 8.4. *Consider $\mathbb{K}_1$ and $\mathbb{K}_2$ defined as in (8.4). Suppose $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy Condition 5.1. Let $x' \in \mathbb{L}^n$ be such that $\langle c_1, x'\rangle \leq 0 \vee \langle c_2, x'\rangle \leq 0$. Then the following statements are equivalent:*

i. x' satisfies (8.6).

ii. x' satisfies the second-order cone inequality

$$\mathcal{N}x - 2\langle c_1, x\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n. \tag{8.7}$$

iii. x' satisfies the second-order cone inequality

$$\mathcal{N}x + 2\langle c_2, x\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n. \tag{8.8}$$

Remark 8.5. *When c_1 and c_2 satisfy* (8.5)*, the inequalities* (8.7) *and* (8.8) *describe cylindrical second-order cones whose lineality spaces contain* span$\left\{\binom{-\tilde{r}}{r_n}\right\}$*. To see this, note that*

$$\mathcal{N} = 2 - 2(\tilde{c}_1^\top \tilde{c}_2 - c_{1,n}c_{2,n}) = 2\left\langle c_1, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle = -2\left\langle c_2, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle.$$

Recall that c_1 and c_2 can always be scaled so that they satisfy (8.5) when $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy Condition 5.2. The next theorem is the main result of this section. It shows that (8.6), together with the constraint $x \in \mathbb{L}^n$, characterizes the convex hull of $\mathbb{K}_1 \cup \mathbb{K}_2$ when c_1 and c_2 satisfy (8.5). Because this assumption is without any loss of generality, Theorem 8.6 complements the results of Section 6.3.1, settling the case for two-term disjunctions on the second-order cone when $c_{1,0} = c_{2,0} = 0$ in (6.1).

Theorem 8.6. *Consider $\mathbb{K}_1$ and $\mathbb{K}_2$ defined as in* (8.4)*. Suppose $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy the basic disjunctive setup. Suppose also that c_1 and c_2 are scaled so that they satisfy* (8.5)*. Then*

$$\operatorname{conv}(\mathbb{K}_1 \cup \mathbb{K}_2) = \{x \in \mathbb{L}^n : x \text{ satisfies } (8.6)\}. \tag{8.9}$$

Proof. Let $\mathbb{D}$ denote the set on the right-hand side of (8.9). We already know from Corollary 8.3 that (8.6) is valid for the convex hull of $\mathbb{K}_1 \cup \mathbb{K}_2$. Hence, $\operatorname{conv}(\mathbb{K}_1 \cup \mathbb{K}_2) \subset \mathbb{D}$. Let $x' \in \mathbb{D}$. If $x' \in \mathbb{K}_1 \cup \mathbb{K}_2$, then clearly $x' \in \operatorname{conv}(\mathbb{K}_1 \cup \mathbb{K}_2)$. Therefore, suppose $x' \in \mathbb{L}^n \setminus (\mathbb{K}_1 \cup \mathbb{K}_2)$ is a point that satisfies (8.6). By Proposition 8.4, x' satisfies

$$\mathcal{N}x' - 2\langle c_1, x'\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n \quad \text{and} \quad \mathcal{N}x' + 2\langle c_2, x'\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n.$$

We will show that x' belongs to the convex hull of $\mathbb{K}_1 \cup \mathbb{K}_2$.

By Remarks 8.2 and 8.5, $0 < \mathcal{N} = 2\left\langle c_1, \binom{-\tilde{r}}{r_n}\right\rangle = -2\left\langle c_2, \binom{-\tilde{r}}{r_n}\right\rangle$. Let

$$\begin{aligned} \alpha_1 &= \frac{\langle c_1, -x'\rangle}{\left\langle c_1, \binom{-\tilde{r}}{r_n}\right\rangle}, \qquad & \alpha_2 &= \frac{\langle c_2, -x'\rangle}{\left\langle c_2, \binom{-\tilde{r}}{r_n}\right\rangle}, \\ x_1 &= x' + \alpha_1 \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix}, & x_2 &= x' + \alpha_2 \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix}. \end{aligned} \tag{8.10}$$

It is not difficult to see that $\langle c_1, x_1\rangle = \langle c_2, x_2\rangle = 0$. Furthermore, $x' \in \operatorname{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. Therefore, the only thing we need to show is $x_1, x_2 \in \mathbb{L}^n$. By Remark 8.5, we have

$$\mathcal{N}\begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} - 2\left\langle c_1, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix}\right\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = \mathcal{N}\begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} + 2\left\langle c_2, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix}\right\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = 0.$$

Hence, we reach

$$\begin{aligned}\mathcal{N}x_1 - 2\langle c_1, x_1\rangle \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix} &= \mathcal{N}x' - 2\langle c_1, x'\rangle \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix} \in \mathbb{L}^n \quad \text{and}\\ \mathcal{N}x_2 + 2\langle c_2, x_2\rangle \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix} &= \mathcal{N}x' + 2\langle c_2, x'\rangle \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix} \in \mathbb{L}^n.\end{aligned}$$

Now observing that $\langle c_1, x_1\rangle = \langle c_2, x_2\rangle = 0$ and $\mathcal{N} > 0$ shows $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in \mathbb{K}_1$ and $x_2 \in \mathbb{K}_2$. □

In the next section, we will show that the inequality (8.6) can also be used to characterize the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ when $\mathbb{C}_1$ and $\mathbb{C}_2$ are as in (8.3).

8.4 Disjunctions on Cross-Sections of the Second-Order Cone

8.4.1 The Main Result

Consider the set $\mathbb{C}$ defined as in (8.2) and the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in (8.3). The set $\mathbb{C}$ is an ellipsoid when $a \in \operatorname{int}\mathbb{L}^n$, a paraboloid when $a \in \operatorname{bd}\mathbb{L}^n$, a hyperboloid when $a \notin \pm\mathbb{L}^n$, and empty when $a \in -\mathbb{L}^n$. In this section, we prove the main result of this chapter, Theorem 8.8, which characterizes the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ under some mild conditions.

In the rest of this chapter, we assume $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. This assumption is useful later when we use Corollary 8.3 whose proof relies on conic duality. By Condition 5.1, we have $\mathbb{C}_1, \mathbb{C}_2 \subsetneq \mathbb{C}$, and by Condition 5.2, we have $\dim \mathbb{C}_1 = \dim \mathbb{C}_2 = n-1$. We also assume, without any loss of generality, that $c_{1,0} = c_{2,0} = 0$; note that this can always be ensured by subtracting a multiple of $\langle a, x\rangle = 1$ from $\langle c_i, x\rangle \geq c_{i,0}$ if necessary. With this assumption, the hypothesis that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup implies $c_i \notin \pm\mathbb{L}^n$, or equivalently $\|\tilde{c}_i\|^2 > c_{i,n}^2$, for $i \in \{1, 2\}$. As in the previous section, we assume that c_1 and c_2 have been scaled by positive scalars so that they satisfy (8.5).

Consider the relaxations $\mathbb{K}_1$ and $\mathbb{K}_2$ of $\mathbb{C}_1$ and $\mathbb{C}_2$ to the whole second-order cone:

$$\mathbb{K}_i = \{x \in \mathbb{L}^n : \langle c_i, x\rangle \geq 0\} \quad \text{for} \quad i \in \{1, 2\}.$$

Clearly, $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy the basic disjunctive setup because $\mathbb{C}_1$ and $\mathbb{C}_2$ do. Define $\mathcal{N}$ and r as in Section 8.3 using c_1 and c_2. Noting that $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy the basic disjunctive setup and c_1 and c_2 satisfy (8.5), all results of Section 8.3 hold for $\mathbb{K}_1$ and $\mathbb{K}_2$. In particular, Corollary 8.3 implies that the inequality (8.6) is valid for the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$. In Theorem 8.8, we are going to show that (8.6) is also sufficient to describe the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ when the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy certain conditions. The proof of Theorem 8.8 requires the following technical lemma.

Lemma 8.7. *Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (8.3) *for $c_{1,0} = c_{2,0} = 0$. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup, Suppose also that c_1 and c_2 are scaled so that they satisfy* (8.5)*. Assume $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle \neq 0$, and let $x^* = \frac{\binom{-\tilde{r}}{r_n}}{\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle}$. Let $x' \in \mathbb{C} \setminus (\mathbb{C}_1 \cup \mathbb{C}_2)$ satisfy* (8.6)*.*

a. If $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle > 0$, then $\langle c_1, x' - x^* \rangle < 0$. If in addition

$$\begin{gathered}(a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \quad (-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \\ (-a + \text{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset,\end{gathered} \tag{8.11}$$

then $\langle c_2, x' - x^* \rangle \geq 0$.

b. If $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle < 0$, then $\langle c_2, x' - x^* \rangle < 0$. If in addition

$$\begin{gathered}(a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \quad (-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset, \quad \textit{or} \\ (-a + \text{cone}\{c_1\}) \cap -\mathbb{L}^n \neq \emptyset,\end{gathered} \tag{8.12}$$

then $\langle c_1, x' - x^* \rangle \geq 0$.

Proof. By Remarks 8.2 and 8.5, we have $\mathcal{N} = 2\left\langle c_1, \binom{-\tilde{r}}{r_n} \right\rangle = -2\left\langle c_2, \binom{-\tilde{r}}{r_n} \right\rangle > 0$. From this, we get

$$\mathcal{N}x^* - 2\langle c_1, x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = \frac{1}{\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle} \left(\mathcal{N} - 2\left\langle c_1, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \right) \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = 0, \tag{8.13}$$

$$\mathcal{N}x^* + 2\langle c_2, x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = \frac{1}{\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle} \left(\mathcal{N} + 2\left\langle c_2, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \right) \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = 0. \tag{8.14}$$

Furthermore, $a^\top x' = a^\top x^* = 1$.

a. Having $x' \notin \mathbb{C}_1$ implies $\langle c_1, x' \rangle < 0$. Furthermore, it follows from $\left\langle c_1, \binom{-\tilde{r}}{r_n} \right\rangle = \frac{\mathcal{N}}{2} > 0$ that

$$\langle c_1, x^* \rangle = \frac{\left\langle c_1, \binom{-\tilde{r}}{r_n} \right\rangle}{\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle} > 0.$$

Thus, we get $\langle c_1, x' - x^* \rangle < 0$.

Now suppose $(a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset$. Then there exist $\lambda \geq 0$ and $0 \leq \theta \leq 1$ such that $a + \lambda(\theta c_1 + (1-\theta)c_2) \in \mathbb{L}^n$. The point x' does not belong to either $\mathbb{C}_1$ or $\mathbb{C}_2$ and satisfies (8.6). By Proposition 8.4, it satisfies (8.8) as well. Using (8.14), we can write

$$\mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n. \tag{8.15}$$

Because $\mathbb{L}^n$ is self-dual, we get

$$\begin{aligned}
0 &\leq \left\langle a + \lambda(\theta c_1 + (1-\theta)c_2), \mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \\
&= 2\langle c_2, x' - x^* \rangle \left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle + \lambda \left\langle \theta c_1 + (1-\theta)c_2, \mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \\
&= 2\langle c_2, x' - x^* \rangle \left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle - \lambda\theta \left\langle r, \mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \\
&\qquad + \lambda \langle c_2, x' - x^* \rangle \left(\mathcal{N} + 2 \left\langle c_2, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \right) \\
&= 2\langle c_2, x' - x^* \rangle \left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle - \lambda\theta \left\langle r, \mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \\
&= 2\langle c_2, x' - x^* \rangle \left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle - \lambda\theta\mathcal{N}\langle r, x' - x^* \rangle - 2\lambda\theta\langle c_2, x' - x^* \rangle \left\langle r, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle \\
&= 2\langle c_2, x' - x^* \rangle \left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle + \lambda\theta\mathcal{N}\langle c_1 + c_2, x' - x^* \rangle \\
&= \left(2 \left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle + \lambda\theta\mathcal{N} \right) \langle c_2, x' - x^* \rangle + \lambda\theta\mathcal{N}\langle c_1, x' - x^* \rangle
\end{aligned}$$

using $\langle a, x' - x^* \rangle = 0$ to obtain the first equality, $\mathcal{N} + 2\left\langle c_2, \left(\begin{smallmatrix} -\tilde{r} \\ r_n \end{smallmatrix}\right)\right\rangle = 0$ to obtain the third equality, and $\left\langle -r, \left(\begin{smallmatrix} -\tilde{r} \\ r_n \end{smallmatrix}\right)\right\rangle = \mathcal{N}$ to obtain the fifth equality. Now it follows from $2\left\langle a, \left(\begin{smallmatrix} -\tilde{r} \\ r_n \end{smallmatrix}\right)\right\rangle + \lambda\theta\mathcal{N} > 0$, $\langle c_1, x' - x^* \rangle < 0$, and $\lambda\theta\mathcal{N} \geq 0$ that $\langle c_2, x' - x^* \rangle \geq 0$.

Now suppose $(-a + \text{cone}\{c_1, c_2\}) \cap \mathbb{L}^n \neq \emptyset$. Let $\lambda \geq 0$ and $0 \leq \theta \leq 1$ be such that $-a + \lambda(\theta c_1 + (1-\theta)c_2) \in \mathbb{L}^n$. By Proposition 8.4, x' satisfies (8.7), and using (8.13), we can write

$$\mathcal{N}(x' - x^*) - 2\langle c_1, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n.$$

As before, because $\mathbb{L}^n$ is self-dual, we get

$$0 \leq \left\langle -a + \lambda(\theta c_1 + (1-\theta)c_2), \mathcal{N}(x' - x^*) - 2\langle c_1, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle.$$

The right-hand side of this inequality is identical to

$$\left(2\left\langle a, \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix}\right\rangle + \lambda(1-\theta)\mathcal{N}\right)\langle c_1, x' - x^*\rangle + \lambda(1-\theta)\mathcal{N}\langle c_2, x' - x^*\rangle.$$

It follows from $2\left\langle a, \binom{-\tilde{r}}{r_n}\right\rangle + \lambda(1-\theta)\mathcal{N} > 0$, $\langle c_1, x' - x^*\rangle < 0$, and $\lambda(1-\theta)\mathcal{N} \geq 0$ that $\langle c_2, x' - x^*\rangle \geq 0$.
Finally suppose $(-a + \text{cone}\{c_2\}) \cap -\mathbb{L}^n \neq \emptyset$. Let $\theta \geq 0$ be such that $-a + \theta c_2 \in -\mathbb{L}^n$. Then using (8.15), we obtain

$$\begin{aligned}
0 &\geq \left\langle -a + \theta c_2, \mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^*\rangle \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix}\right\rangle \\
&= -2\langle c_2, x' - x^*\rangle \left\langle a, \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix}\right\rangle + \theta\langle c_2, x' - x^*\rangle \left(\mathcal{N} + 2\left\langle c_2, \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix}\right\rangle\right) \\
&= -2\langle c_2, x' - x^*\rangle \left\langle a, \begin{pmatrix}-\tilde{r}\\ r_n\end{pmatrix}\right\rangle.
\end{aligned}$$

It follows from $\left\langle a, \binom{-\tilde{r}}{r_n}\right\rangle > 0$ that $\langle c_2, x' - x^*\rangle \geq 0$.

b. If $\left\langle a, \binom{-\tilde{r}}{r_n}\right\rangle < 0$, then $\left\langle a, -\binom{-\tilde{r}}{r_n}\right\rangle > 0$. Since $-\binom{-\tilde{r}}{r_n} = \begin{pmatrix}\tilde{c}_2 - \tilde{c}_1\\ -c_{2,n} + c_{1,n}\end{pmatrix}$, part (b) follows from part (a) by interchanging the roles of $\mathbb{C}_1$ and $\mathbb{C}_2$.

□

In the next result we show that the inequality (8.6) is sufficient to describe the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ when conditions (8.11) and (8.12) hold.

Theorem 8.8. *Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in (8.3) for $c_{1,0} = c_{2,0} = 0$. Suppose the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup and the vectors c_1 and c_2 satisfy (8.5). Suppose also that one of the following conditions is satisfied:*
a. $\left\langle a, \binom{-\tilde{r}}{r_n}\right\rangle = 0$.
b. $\left\langle a, \binom{-\tilde{r}}{r_n}\right\rangle > 0$ *and (8.11) holds.*
c. $\left\langle a, \binom{-\tilde{r}}{r_n}\right\rangle < 0$ *and (8.12) holds.*
Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is

$$\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{x \in \mathbb{C} : \sqrt{\langle r, x\rangle^2 + \mathcal{N}\left(x_n^2 - \|\tilde{x}\|^2\right)} \geq \langle -c_1 - c_2, x\rangle\right\}. \tag{8.16}$$

Proof. Let $\mathbb{D}$ denote the set on the right-hand side of (8.16). The inequality (8.6) is valid for the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ by Corollary 8.3. Hence, $\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) \subset \mathbb{D}$. Let $x' \in \mathbb{D}$. If $x' \in \mathbb{C}_1 \cup \mathbb{C}_2$, then clearly $x' \in \overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2)$. Therefore, suppose $x' \in \mathbb{C} \setminus (\mathbb{C}_1 \cup \mathbb{C}_2)$ is a point that satisfies (8.6). By Proposition 8.4, it satisfies (8.7) and (8.8) as well. We are going to show that in each case x' belongs to the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$.

a. Suppose $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle = 0$. By Remarks 8.2 and 8.5, $\mathcal{N} = 2\left\langle c_1, \binom{-\tilde{r}}{r_n} \right\rangle = -2\left\langle c_2, \binom{-\tilde{r}}{r_n} \right\rangle > 0$. Define α_1, α_2, x_1, and x_2 as in (8.10). It is not difficult to see that $\langle a, x_1 \rangle = \langle a, x_2 \rangle = 1$ and $\langle c_1, x_1 \rangle = \langle c_2, x_2 \rangle = 0$. Furthermore, $x' \in \text{conv}\{x_1, x_2\}$ because $\alpha_2 < 0 < \alpha_1$. One can show that $x_1, x_2 \in \mathbb{L}^n$ using the same arguments as in the proof of Theorem 8.6. This proves $x_1 \in \mathbb{C}_1$ and $x_2 \in \mathbb{C}_2$.

b. Suppose $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle > 0$ and (8.11) holds. Let $x^* = \frac{\binom{-\tilde{r}}{r_n}}{\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle}$. Then by Lemma 8.7, we have $\langle c_1, x' - x^* \rangle < 0$ and $\langle c_2, x' - x^* \rangle \geq 0$.
First, suppose $\langle c_2, x' - x^* \rangle > 0$, and let

$$\begin{aligned} \alpha_1 &= \frac{\langle c_1, -x' \rangle}{\langle c_1, x' - x^* \rangle}, & \alpha_2 &= \frac{\langle c_2, -x' \rangle}{\langle c_2, x' - x^* \rangle}, \\ x_1 &= x' + \alpha_1 (x' - x^*), & x_2 &= x' + \alpha_2 (x' - x^*). \end{aligned} \tag{8.17}$$

As in part (a), $\langle a, x_1 \rangle = \langle a, x_2 \rangle = 1$, $\langle c_1, x_1 \rangle = \langle c_2, x_2 \rangle = 0$, and $x' \in \text{conv}\{x_1, x_2\}$ because $\alpha_1 < 0 < \alpha_2$. To show $x_1, x_2 \in \mathbb{L}^n$, first note $\mathcal{N}x^* - 2\langle c_1, x^* \rangle \binom{-\tilde{r}}{r_n} = \mathcal{N}x^* + 2\langle c_2, x^* \rangle \binom{-\tilde{r}}{r_n} = 0$ as in (8.13) and (8.14). Using this and $\langle c_1, x_1 \rangle = \langle c_2, x_2 \rangle = 0$, we get

$$\begin{aligned} \mathcal{N}x_1 &= \mathcal{N}x_1 - 2\langle c_1, x_1 \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = (1 + \alpha_1) \left(\mathcal{N}x' - 2\langle c_1, x' \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right), \\ \mathcal{N}x_2 &= \mathcal{N}x_2 + 2\langle c_2, x_2 \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} = (1 + \alpha_2) \left(\mathcal{N}x' + 2\langle c_2, x' \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right). \end{aligned}$$

Clearly, $1 + \alpha_2 > 0$; hence, $\mathcal{N}x_2 \in \mathbb{L}^n$. Furthermore,

$$1 + \alpha_1 = \frac{\langle c_1, -x^* \rangle}{\langle c_1, x' - x^* \rangle} = \frac{-\left\langle c_1, \binom{-\tilde{r}}{r_n} \right\rangle}{\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle \langle c_1, x' - x^* \rangle} = \frac{-\mathcal{N}}{2\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle \langle c_1, x' - x^* \rangle} > 0,$$

where we have used the relationships $\mathcal{N} > 0$, $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle > 0$, and $\langle c_1, x' - x^* \rangle < 0$ to reach the inequality. It follows that $\mathcal{N}x_1 \in \mathbb{L}^n$ as well. Because $\mathcal{N} > 0$, we get $x_1, x_2 \in \mathbb{L}^n$. This proves $x_1 \in \mathbb{C}_1$ and $x_2 \in \mathbb{C}_2$.

Now suppose $\langle c_2, x' - x^* \rangle = 0$. Define α_1 and x_1 as in (8.17). All of the arguments that we have just used to show $\alpha_1 < 0$ and $x_1 \in \mathbb{C}_1$ continue to hold. Using $\mathcal{N}x^* + 2\langle c_2, x^* \rangle \binom{-\tilde{r}}{r_n} = 0$, we can write

$$\mathcal{N}(x' - x^*) = \mathcal{N}(x' - x^*) + 2\langle c_2, x' - x^* \rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n.$$

Because $\mathcal{N} \succ 0$, we get $x' - x^* \in \mathbb{L}^n$. Together with $\langle c_2, x' - x^* \rangle = 0$ and $\langle a, x' - x^* \rangle = 0$, this implies $x' - x^* \in \operatorname{rec} \mathbb{C}_2$. Then $x' = x_1 - \alpha_1(x' - x^*) \in \mathbb{C}_1 + \operatorname{rec} \mathbb{C}_2$ because $\alpha_1 < 0$. The claim now follows from the fact that the last set is contained in the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ (see, e.g., [96, Theorem 9.8]).

c. Suppose $\left\langle a, \binom{-\tilde{r}}{r_n} \right\rangle < 0$ and (8.12) holds. Since $-\binom{-\tilde{r}}{r_n} = \begin{pmatrix} \tilde{c}_2 - \tilde{c}_1 \\ -c_{2,n} + c_{1,n} \end{pmatrix}$, part (c) follows from part (b) by interchanging the roles of $\mathbb{C}_1$ and $\mathbb{C}_2$.

□

The following result shows that when $\mathbb{C}$ is an ellipsoid or a paraboloid, the closed convex hull of any two-term disjunction can be obtained by adding an inequality of the form (8.6) to the description of $\mathbb{C}$.

Corollary 8.9. *Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (8.3) *for $c_{1,0} = c_{2,0} = 0$. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Suppose also that c_1 and c_2 satisfy* (8.5). *If $a \in \mathbb{L}^n$, then* (8.16) *holds.*

Proof. The result follows from Theorem 8.8 after observing that conditions (8.11) and (8.12) are trivially satisfied for any c_1 and c_2 when $a \in \mathbb{L}^n$. □

The case of a split disjunction is particularly relevant in the solution of mixed-integer second-order cone programs, and it has been studied by several groups recently, in particular Dadush et al. [52], Andersen and Jensen [8], Belotti et al. [27], and Modaresi et al. [89]. Theorem 8.8 has the following consequence for split disjunctions on $\mathbb{C}$.

Corollary 8.10. *Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined by a split disjunction $\langle t_1 \ell, x \rangle \geq \ell_{1,0} \vee \langle t_2 \ell, x \rangle \geq \ell_{2,0}$ on $\mathbb{C}$ such that $t_1 > 0 > t_2$ and $\mathbb{C}_1 \cup \mathbb{C}_2 \subsetneq \mathbb{C}$. Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup. Then* (8.16) *holds for*

$$c_i = \frac{t_i \ell - \ell_{i,0} a}{\sqrt{\|t_i \tilde{\ell} - \ell_{i,0} \tilde{a}\|_2^2 - (t_i \ell_n - \ell_{i,0} a_n)^2}} \quad \textit{for} \quad i \in \{1, 2\}.$$

Proof. First note $\mathbb{C}_i = \{x \in \mathbb{C} : \langle t_i \ell, x\rangle \geq \ell_{i,0}\} = \{x \in \mathbb{C} : \langle c_i, x\rangle \geq 0\}$ for $i \in \{1,2\}$. For the given split disjunction, we have $\mathbb{C}_1 \cup \mathbb{C}_2 \subsetneq \mathbb{C}$ only if $\frac{\ell_{1,0}}{t_1} > \frac{\ell_{2,0}}{t_2}$. Let $\lambda_i = \left(\sqrt{\|t_i\tilde{\ell} - \ell_{i,0}\tilde{a}\|_2^2 - (t_i\ell_n - \ell_{i,0}a_n)^2}\right)^{-1}$ for $i \in \{1,2\}$. Also, let $\theta_1 = \frac{-t_2}{\lambda_1(t_1\ell_{2,0} - t_2\ell_{1,0})}$ and $\theta_2 = \frac{t_1}{\lambda_2(t_1\ell_{2,0} - t_2\ell_{1,0})}$. Then

$$a + \theta_1 c_1 + \theta_2 c_2 = a + \frac{-t_2(t_1\ell - \ell_{1,0}a)}{t_1\ell_{2,0} - t_2\ell_{1,0}} + \frac{t_1(t_2\ell - \ell_{2,0}a)}{t_1\ell_{2,0} - t_2\ell_{1,0}} = 0 \in \mathbb{L}^n.$$

The result now follows from Theorem 8.8 after observing that $\theta_1, \theta_2 > 0$ implies that conditions (8.11) and (8.12) are satisfied. □

We say that the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the *disjointness condition* when $\{x \in \mathbb{C} : \langle c_1, x\rangle > 0, \langle c_2, x\rangle > 0\} = \emptyset$. Under this condition, Proposition 8.4 says that (8.6) can be expressed in second-order cone form and directly implies the following result.

Corollary 8.11. *Consider $\mathbb{C}_1$ and $\mathbb{C}_2$ defined as in* (8.3) *for $c_{1,0} = c_{2,0} = 0$. Suppose the sets $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the basic disjunctive setup.*

i. Let $x \in \mathbb{C}$ be such that $\langle c_1, x\rangle \leq 0 \vee \langle c_2, x\rangle \leq 0$. Then x satisfies (8.6) *if and only if it satisfies* (8.7) *(or, equivalently* (8.8)*).*

ii. Suppose that $\mathbb{C}_1$ and $\mathbb{C}_2$ satisfy the disjointness condition, the vectors c_1 and c_2 satisfy (8.5)*, and the conditions of Theorem 8.8 hold. Then the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ is*

$$\begin{aligned}\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) &= \left\{x \in \mathbb{C} : \mathcal{N}x - 2\langle c_1, x\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n\right\} \\ &= \left\{x \in \mathbb{C} : \mathcal{N}x + 2\langle c_2, x\rangle \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \in \mathbb{L}^n\right\}.\end{aligned}$$

Remark 8.12. *Conditions* (8.11) *and* (8.12) *are directly related to the sufficient conditions which guarantee the closedness of the convex hull of a two-term disjunction on a regular cone, explored in Chapter 5. In particular, one can show using Corollary 5.14 that the convex hull of a disjunction $\ell_1^\top x \geq \ell_{1,0} \vee \ell_2^\top x \geq \ell_{2,0}$ on the second-order cone is closed if*

i. $\ell_{1,0} = \ell_{2,0} \in \{\pm 1\}$ and there exists $0 < \mu < 1$ such that $\mu\ell_1 + (1-\mu)\ell_2 \in \mathbb{L}^n$, or

ii. $\ell_{1,0} = \ell_{2,0} = -1$ and $\ell_1, \ell_2 \in -\operatorname{int}\mathbb{L}^n$.

In our present context, exploiting conditions (i) and (ii) after letting $\ell_i = a + \theta_i c_i$ and $\ell_{i,0} = 1$ (or, $\ell_i = -a + \theta_i c_i$ and $\ell_{i,0} = -1$) for some $\theta_i > 0$ leads to (8.11) *and* (8.12)*.*

8.4.2 Two Examples

In this section we illustrate Theorem 8.8 with two examples.

A Two-Term Disjunction on a Paraboloid

Example 8.1. Consider the disjunction $-2x_1 - x_2 - 2x_4 \geq 0 \vee x_1 \geq 0$ on the paraboloid $\mathbb{C} = \{x \in \mathbb{L}^4 : x_1 + x_4 = 1\}$. Let $\mathbb{C}_1 = \{x \in \mathbb{C} : -2x_1 - x_2 - 2x_4 \geq 0\}$ and $\mathbb{C}_2 = \{x \in \mathbb{C} : x_1 \geq 0\}$. Noting that $\mathbb{C}$ is a paraboloid and $\mathbb{C}_1$ and $\mathbb{C}_2$ are disjoint, we can use Corollary 8.11 to characterize the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ with a second-order cone inequality:

$$\overline{\text{conv}}(\mathbb{C}_1 \cup \mathbb{C}_2) = \left\{ x \in \mathbb{C} : \quad 3x + x_1 \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \end{pmatrix} \in \mathbb{L}^4 \right\}$$

Figure 8.1 depicts the paraboloid $\mathbb{C}$ in mesh and the disjunction $\mathbb{C}_1 \cup \mathbb{C}_2$ in blue. The second-order cone disjunctive inequality added to convexify this set is shown in red.

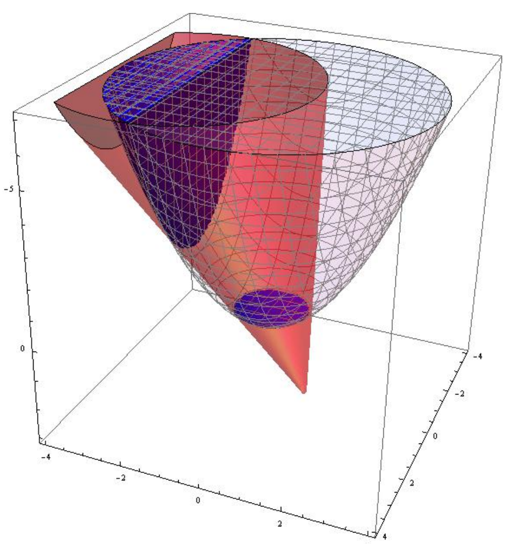


Figure 8.1: Sets associated with the disjunction $-2x_1 - x_2 - 2x_4 \geq 0 \vee x_1 \geq 0$ on the paraboloid $\mathbb{C} = \{x \in \mathbb{L}^4 : x_1 + x_4 = 1\}$.

A Two-Term Disjunction on a Hyperboloid

Example 8.2. Consider the disjunction $-2x_1 - x_2 \geq 0 \vee \sqrt{2}x_1 - x_3 \geq 0$ on the hyperboloid $\mathbb{C} = \{x \in \mathbb{L}^3 : x_1 = 2\}$. Let $\mathbb{C}_1 = \{x \in \mathbb{C} : -2x_1 - x_2 \geq 0\}$ and $\mathbb{C}_2 = \{x \in \mathbb{C} : \sqrt{2}x_1 - x_3 \geq$

$0\}$. Note that in this setting

$$\left\langle a, \begin{pmatrix} -\tilde{r} \\ r_n \end{pmatrix} \right\rangle = \frac{1}{10} \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2\sqrt{5}+5\sqrt{2} \\ -\sqrt{5} \\ -5 \end{pmatrix} \right\rangle > 0,$$

but none of the conditions (8.11) are satisfied. The second-order cone inequality

$$(5+2\sqrt{10})x + (\sqrt{2}x_1 - x_3) \begin{pmatrix} -2\sqrt{5}+5\sqrt{2} \\ -\sqrt{5} \\ -5 \end{pmatrix} \in \mathbb{L}^3 \tag{8.18}$$

of Theorem 8.8 is valid for $\mathbb{C}_1 \cup \mathbb{C}_2$ but not sufficient to characterize its closed convex hull. Indeed, the inequality $x_2 \leq 2$ is valid for the closed convex hull of $\mathbb{C}_1 \cup \mathbb{C}_2$ but is not implied by (8.18). Figure 8.2 depicts the hyperboloid $\mathbb{C}$ in mesh and the disjunction $\mathbb{C}_1 \cup \mathbb{C}_2$ in blue. The second-order cone disjunctive inequality (8.18) is shown in red.

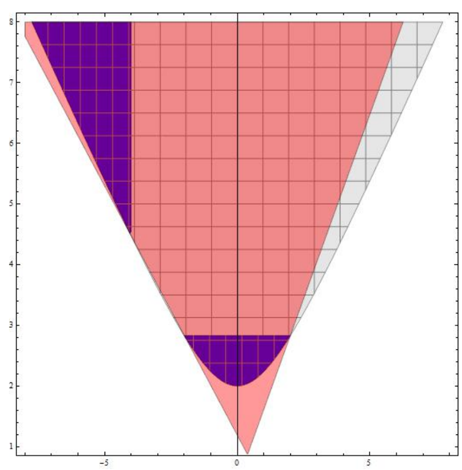


Figure 8.2: Sets associated with the disjunction $-2x_1 - x_2 \geq 0 \vee \sqrt{2}x_1 - x_3 \geq 0$ on the hyperboloid $\mathbb{C} = \{x \in \mathbb{L}^3 : x_1 = 2\}$.

Bibliography

[1] IBM ILOG CPLEX Optimization Studio 12.6.3. http://www-01.ibm.com/support/knowledgecenter/SSSA5P/welcome.

[2] Gurobi Optimizer 6.5. http://www.gurobi.com/documentation.

[3] MOSEK 7. https://mosek.com/resources/doc.

[4] FICO Xpress Optimization Suite 7.7. http://www.fico.com/en/products/fico-xpress-optimization-suite.

[5] J. Aczél. *Lectures on functional equations and their applications*. Academic Press, New York, NY, 1966. ISBN 978-0-4864-4523-6.

[6] M. S. Aktürk, A. Atamtürk, and S. Gürel. A strong conic quadratic reformulation for machine-job assignment with controllable processing times. *Operations Research Letters*, 37(3):187–191, 2009.

[7] F. Alizadeh and D. Goldfarb. Second-order cone programming. *Mathematical Programming*, 95(1):3–51, 2003.

[8] K. Andersen and A. N. Jensen. Intersection cuts for mixed integer conic quadratic sets. In M. Goemans and J. Correa, editors, *Integer Programming and Combinatorial Optimization*, volume 7801 of *Lecture Notes in Computer Science*, pages 37–48. Springer, 2013.

[9] K. Andersen, Q. Louveaux, R. Weismantel, and L. A. Wolsey. Cutting planes from two rows of a simplex tableau. In G. Cornuéjols, R. E. Burkard, and G. J. Woeginger, editors, *Integer Programming and Combinatorial Optimization*, volume 4513 of *Lecture Notes in Computer Science*, pages 1–15. Springer, 2007.

[10] A. Atamtürk and V. Narayanan. Conic mixed-integer rounding cuts. *Mathematical Programming Ser. A*, 122(1):1–20, 2010.

[11] A. Atamtürk and V. Narayanan. Lifting for conic mixed-integer programming. *Mathematical Programming Ser. A*, 126(2):351–363, 2011.

[12] A. Atamtürk, G. Berenguer, and Z. J. M. Shen. A conic integer programming ap-

proach to stochastic joint location-inventory problems. *Operations Research*, 60: 366–381, 2012.

[13] A. Bachem, E. L. Johnson, and R. Schrader. A characterization of minimal valid inequalities for mixed integer programs. *Operations Research Letters*, 1(2):63–66, 1982.

[14] E. Balas. Intersection cuts - a new type of cutting planes for integer programming. *Operations Research*, 19:19–39, 1971.

[15] E. Balas. Disjunctive programming: Properties of the convex hull of feasible points. *Discrete Applied Mathematics*, 89:1–44, 1998. GSIA Management Science Research Report MSRR 348, Carnegie Mellon University, 1974.

[16] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295–324, 1993.

[17] A. Basu and J. Paat. Operations that preserve the covering property of the lifting region. *SIAM Journal on Optimization*, 25(4):2313–2333, 2015.

[18] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Minimal inequalities for an infinite relaxation of integer programs. *SIAM Journal on Discrete Mathematics*, 24 (1):158–168, 2010.

[19] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Maximal lattice-free convex sets in linear subspaces. *Mathematics of Operations Research*, 35(3):704–720, 2010.

[20] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. A counterexample to a conjecture of Gomory and Johnson. *Mathematical Programming Ser. A*, 133:25–38, 2012.

[21] A. Basu, M. Campêlo, M. Conforti, G. Cornuéjols, and G. Zambelli. Unique lifting of integer variables. *Mathematical Programming Ser. A*, 141(1-2):561–576, 2013.

[22] A. Basu, R. Hildebrand, and M. Köppe. Light on the infinite group problem. http://arxiv.org/pdf/1410.8584v2.pdf, October 2014.

[23] A. Basu, M. Conforti, and M. Di Summa. A geometric approach to cut-generating functions. *Mathematical Programming Ser. B*, 151(1):153–189, 2015.

[24] P. Belotti. Disjunctive cuts for nonconvex MINLP. In Jon Lee and Sven Leyffer, editors, *Mixed Integer Nonlinear Programming*, volume 154 of *The IMA Volumes in Mathematics and its Applications*, pages 117–144. Springer, New York, NY, 2012.

[25] P. Belotti, J. C. Góez, I. Pólik, T. K. Ralphs, and T. Terlaky. On families of quadratic surfaces having fixed intersections with two hyperplanes. *Discrete Applied Mathematics*, 161(16):2778–2793, 2013.

[26] P. Belotti, C. Kirches, S. Leyffer, J. Linderoth, J. Luedtke, and A. Mahajan. Mixed-integer nonlinear optimization. *Acta Numerica*, 12:1–131, 2013.

[27] P. Belotti, J. C. Goez, I. Polik, T. K. Ralphs, and T. Terlaky. A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization. In M. Al-Baali, L. Grandinetti, and A. Purnama, editors, *Numerical Analysis and Optimization*, volume 134 of *Springer Proceedings in Mathematics and Statistics*, pages 1–35. Springer, 2014.

[28] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization*. MPS-SIAM Series on Optimization. SIAM, Philadehia, PA, 2001. ISBN 978-0-898714-91-3.

[29] H. Y. Benson and Ü. Sağlam. Mixed-integer second-order cone programming: A survey. Tutorials in Operations Research, pages 13–36. INFORMS, Hanover, MD, 2013.

[30] D. Bienstock and A. Michalka. Cutting-planes for optimization of convex functions over nonconvex sets. *SIAM Journal on Optimization*, 24(2):643–677, 2014.

[31] R. E. Bixby, M. Fenelon, Z. Gu, E. Rothberg, and R. Wunderling. Mixed integer programming: A progress report. In Martin Grotschel, editor, *The Sharpest Cut: The Impact of Manfred Padberg and His Work*, MPS-SIAM Series on Optimization, pages 309–326. SIAM, Philadelphia, PA, 2004.

[32] C. E. Blair. Minimal inequalities for mixed integer programs. *Discrete Mathematics*, 24(2):147–151, 1978.

[33] P. Bonami. Lift-and-project cuts for mixed integer convex programs. In O. Günlük and G. J. Woeginger, editors, *Integer Programming and Combinatorial Optimization*, volume 6655 of *Lecture Notes in Computer Science*, pages 52–64. Springer, 2011.

[34] P. Bonami, M. Conforti, G. Cornuéjols, M. Molinaro, and G. Zambelli. Cutting planes from two-term disjunctions. *Operations Research Letters*, 41:442–444, 2013.

[35] V. Borozan and G. Cornuéjols. Minimal valid inequalities for integer constraints. *Mathematics of Operations Research*, 34(3):538–546, 2009.

[36] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, Cambridge, UK, 2004. ISBN 978-0-521-83378-3.

[37] R. Brandenberg and L. Roth. New algorithms for k-center and extensions. *Journal of Combinatorial Optimization*, 18:376–392, 2009.

[38] S. Burer and F. Kılınç-Karzan. How to convexify the intersection of a second-order cone with a non-convex quadratic. http://www.optimization-online.org/DB_FILE/2014/06/4383.pdf, June 2014.

[39] S. Burer and A. Saxena. The MILP road to MIQCP. In *Mixed Integer Nonlinear Programming*, pages 373–405. Springer, 2012.

[40] M. Çezik and G. Iyengar. Cuts for mixed 0-1 conic programming. *Mathematical Programming Ser. A*, 104(1):179–202, 2005.

[41] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4(4):305–337, 1973.

[42] K. Ciesielski. *Set Theory for the Working Mathematician*. Number 39 in London Mathematical Society Student Texts. Cambridge University Press, Cambridge, UK, 1997. ISBN 978-0-5215-9441-7.

[43] M. Conforti, G. Cornuéjols, and G. Zambelli. Equivalence between intersection cuts and the corner polyhedron. *Operations Research Letters*, 38:153–155, 2010.

[44] M. Conforti, G. Cornuéjols, and G. Zambelli. A geometric perspective on lifting. *Operations Research*, 59:569–577, 2011.

[45] M. Conforti, G. Cornuéjols, and G. Zambelli. Corner polyhedron and intersection cuts. *Surveys in Operations Research and Management Science*, 6(2):105–120, 2011.

[46] M. Conforti, G. Cornuéjols, and G. Zambelli. *Integer Programming*. Graduate Texts in Mathematics. Springer, Heidelberg, Switzerland, 2014. ISBN 978-3-319-11008-0.

[47] M. Conforti, G. Cornuéjols, A. Daniilidis, C. Lemaréchal, and J. Malick. Cut-generating functions and S-free sets. *Mathematics of Operations Research*, 40:276–301, 2015.

[48] W. Cook, R. Kannan, and A. Schrijver. Chvátal closures for mixed integer programming problems. *Mathematical Programming*, 47:155–174, 1990.

[49] G. Cornuéjols and F. Margot. On the facets of mixed integer programs with two integer variables and two constraints. *Mathematical Programming Ser. A*, 120:429–456, 2009.

[50] G. Cornuéjols, T. Kis, and M. Molinaro. Lifting Gomory cuts with bounded variables. *Operations Research Letters*, 41:142–146, 2013.

[51] G. Cornuéjols, L. Wolsey, and S. Yıldız. Sufficiency of cut-generating functions. *Mathematical Programming Ser. A*, 152:643–651, 2015.

[52] D. Dadush, S. S. Dey, and J. P. Vielma. The split closure of a strictly convex body. *Operations Research Letters*, 39:121–126, 2011.

[53] S. S. Dey and L. A. Wolsey. Two row mixed-integer cuts via lifting. *Mathematical Programming Ser. B*, 124:143–174, 2010.

[54] S. S. Dey and L. A. Wolsey. Constrained infinite group relaxation of MIPs. *SIAM*

Journal on Optimization, 20:2890–2912, 2010.

[55] S. S. Dey, J.-P. P. Richard, Y. Li, and L. A. Miller. On the extreme inequalities of infinite group problems. *Mathematical Programming Ser. A*, 121:145–170, 2010.

[56] S. Drewes. *Mixed Integer Second Order Cone Programming*. PhD thesis, Technische Universität Darmstadt, 2009.

[57] S. Drewes and S. Pokutta. Cutting-planes for weakly-coupled 0/1 second order cone programs. *Electronic Notes in Discrete Mathematics*, 36:735–742, 2010.

[58] S. Drewes and S. Pokutta. Symmetry-exploiting cuts for a class of mixed-0/1 second order cone programs. *Discrete Optimization*, 13:23–35, 2014.

[59] S. Elhedhli. Service system design with immobile servers, stochastic demand, and congestion. *Manufacturing and Service Operations Management*, 8(1):92–97, 2006.

[60] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM*, 42(6):1115–1145, 1995.

[61] R. E. Gomory. Outline of an algorithm for integer solutions to linear programs. *Bulletin of the American Mathematical Society*, 64(5):275–278, 1958.

[62] R. E. Gomory. An algorithm for the mixed integer problem. Technical Report RM-2597-PR, RAND Corporation, 1960.

[63] R. E. Gomory and E. L. Johnson. Some continuous functions related to corner polyhedra. *Mathematical Programming*, 3:23–85, 1972.

[64] R. E. Gomory and E. L. Johnson. Some continuous functions related to corner polyhedra, ii. *Mathematical Programming*, 3:359–389, 1972.

[65] R. E. Gomory and E. L. Johnson. T-space and cutting planes. *Mathematical Programming Ser. B*, 96:341–375, 2003.

[66] R.G. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and Applications*, 2:451–558, 1969.

[67] A. Hassibi and S. Boyd. Integer parameter estimation in linear models with applications to GPS. *IEEE Transactions on Signal Processing*, 46(11):2938–2952, 1998.

[68] H. Hijazi, P. Bonami, and A. Ouorou. Robust delay-constrained routing in telecommunications. *Annals of Operations Research*, 206:163–181, July 2013.

[69] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Grundlehren Text Editions. Springer, Berlin, Germany, 2004. ISBN 978-3-540-42205-1.

[70] N. Ho-Nguyen and F. Kılınç-Karzan. A second-order cone based approach for solving

the trust region subproblem and its variants. http://arxiv.org/pdf/1603.03366v1.pdf, March 2016.

[71] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, UK, 2012. ISBN 978-0-5215-4823-6.

[72] R. A. Jabr, R. Singh, and B. C. Pal. Minimum loss network reconfiguration using mixed-integer convex programming. *IEEE Transactions on Power Systems*, 27(2): 1106–1115, 2012.

[73] R. G. Jeroslow. Minimal inequalities. *Mathematical Programming*, 17(1):1–15, 1979.

[74] N. J. Jobst, M. D. Horniman, C. A. Lucas, and G. Mitra. Computational aspects of alternative portfolio selection models in the presence of discrete asset choice constraints. *Quantitative Finance*, 1:489–501, 2001.

[75] E. L. Johnson. On the group problem for mixed integer programming. *Mathematical Programming Study*, 2:137–179, 1974.

[76] E. L. Johnson. Characterization of facets for multiple right-hand side choice linear programs. *Mathematical Programming Study*, 14:112–142, 1981.

[77] J. J. Júdice, H. D. Sherali, I. M. Ribeiro, and A. M. Faustino. A complementarity-based partitioning and disjunctive cut algorithm for mathematical programming problems with equilibrium constraints. *Journal of Global Optimization*, 136:89–114, 2006.

[78] P. Kannappan. *Functional Equations and Inequalities with Applications.* Springer Monographs in Mathematics. Springer, New York, 2009.

[79] M. R. Kılınç, J. Linderoth, and J. Luedtke. Effective separation of disjunctive cuts for convex mixed integer nonlinear programs. http://www.optimization-online.org/DB_FILE/2010/11/2808.pdf, February 2010.

[80] F. Kılınç-Karzan. On minimal valid inequalities for mixed integer conic programs. *Mathematics of Operations Research*, 2015. http://dx.doi.org/10.1287/moor.2015.0737.

[81] F. Kılınç-Karzan and D. E. Steffy. On sublinear inequalities for mixed integer conic programs. http://www.optimization-online.org/DB_FILE/2015/07/5002.pdf, December 2014.

[82] F. Kılınç-Karzan and B. Yang. Sufficient conditions and necessary conditions for the sufficiency of cut-generating functions. http://www.andrew.cmu.edu/user/fkilinc/files/draft-sufficiency-web.pdf, December 2015.

[83] F. Kılınç-Karzan and S. Yıldız. Two-term disjunctions on the second-order cone. In J. Lee and J. Vygen, editors, *Integer Programming and Combinatorial Optimization*, volume 8494 of *Lecture Notes in Computer Science*, pages 345–356. Springer, 2014.

[84] F. Kılınç-Karzan and S. Yıldız. Two-term disjunctions on the second-order cone. *Mathematical Programming Ser B.*, 154(1):463–491, 2015.

[85] M. Laurent and F. Rendl. Semidefinite programming and integer programming. *Handbooks in Operations Research and Management Science*, 12:393–514, 2005.

[86] M. S. Lobo, M. Fazel, and S. Boyd. Portfolio optimization with linear and fixed transaction costs. *Annals of Operations Research*, 152(1):341–365, 2007.

[87] R. R. Meyer. On the existence of optimal solutions to integer and mixed integer programming problems. *Mathematical Programming*, 7:223–235, 1974.

[88] S. Modaresi and J. P. Vielma. Convex hull of two quadratic or a conic quadratic and a quadratic inequality. http://www.optimization-online.org/DB_FILE/2014/11/4641.pdf, November 2014.

[89] S. Modaresi, M. R. Kılınç, and J. P. Vielma. Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets. *Mathematical Programming Ser. A*, 2015. http://dx.doi.org/10.1007/s10107-015-0866-5.

[90] S. Modaresi, M. R. Kılınç, and J. P. Vielma. Split cuts and extended formulations for mixed integer conic quadratic programming. *Operations Research Letters*, 43:10–15, 2015.

[91] D. A. Morán, S. S. Dey, and J. P. Vielma. A strong dual for conic mixed integer programs. *SIAM Journal on Optimization*, 22(3):1136–1150, 2012.

[92] G. L. Nemhauser and L. A. Wolsey. A recursive procedure to generate all cuts for 0-1 mixed integer programs. *Mathematical Programming*, 46:379–390, 1990.

[93] G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*. Wiley - Interscience Series in Discrete Mathematics and Optimization. Wiley, New York, NY, 1999. ISBN 978-0-471-35943-2.

[94] M. Pilanci, M. J. Wainwright, and L. El Ghaoui. Sparse learning via boolean relaxations. *Mathematical Programming Ser. B*, 151(1):63–87, 2015.

[95] J.-P. P. Richard and S. S. Dey. The group-theoretic approach in mixed integer programming. In M. Jünger, M. T. Liebling, D. Naddef, G. L. Nemhauser, R. W. Pulleyblank, G. Reinelt, G. Rinaldi, and L. A. Wolsey, editors, *50 Years of Integer Programming 1958-2008*, pages 727–801. Springer, Berlin, Heidelberg, 2010.

[96] R. T. Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics. Princeton

University Press, New Jersey, NJ, 1970. ISBN 978-0-6910-1586-6.

[97] A. Schrijver. *Theory of Linear and Integer Programming.* Wiley - Interscience Series in Discrete Mathematics and Optimization. Wiley, New York, NY, 1998. ISBN 978-0-471-98232-6.

[98] R. Shioda and L. Tunçel. Clustering via minimum volume ellipsoids. *Computational Optimization and Applications*, 37:247–295, July 2007.

[99] R. A. Stubbs and S. Mehrotra. A branch-and-cut method for 0-1 mixed convex programming. *Mathematical Programming*, 86(3):515–532, 1999.

[100] R. A. Stubbs and S. Mehrotra. Generating convex polynomial inequalities for mixed 0-1 programs. *Journal of Global Optimization*, 24(3):311–332, November 2002.

[101] M. Tawarmalani, J.-P. P. Richard, and K. Chung. Strong valid inequalities for orthogonal disjunctions and bilinear covering sets. *Mathematical Programming Ser. B*, 124(1-2):481–512, 2010.

[102] J. A. Taylor and F. S. Hover. Conic AC transmission system planning. *IEEE Transactions on Power Systems*, 28(2):952–959, 2013.

[103] A. Vinel and P. Krokhmal. On valid inequalities for mixed integer p-order cone programming. *Journal of Optimization Theory and Applications*, 160:439–456, 2014.

[104] X. Wang. The master equality polyhedron: Two-slope facets and separation algorithm. Master's thesis, University of Waterloo, 2015.

[105] H. M. Weingartner. Capital budgeting of interrelated projects: Survey and synthesis. *Management Science*, 12(7):485–516, 1966.

[106] S. Yıldız and G. Cornuéjols. Cut-generating functions for integer variables. http://integer.tepper.cmu.edu/webpub/draftfinal.pdf, December 2014.

[107] S. Yıldız and G. Cornuéjols. Disjunctive cuts for cross-sections of the second-order cone. *Operations Research Letters*, 43:432–437, 2015.

[108] S. Yıldız and F. Kılınç-Karzan. Low-complexity relaxations and convex hulls of disjunctions on the positive semidefinite cone and general regular cones. http://www.optimization-online.org/DB_FILE/2016/04/5398.pdf, April 2016.

[109] K. Yonekura and Y. Kanno. Global optimization of robust truss topology via mixed integer semidefinite programming. *Optimization and Engineering*, 11:355–379, September 2010.

[110] G. Zambelli. On degenerate multi-row Gomory cuts. *Operations Research Letters*, 37:21–22, 2009.