Discrete Resource Allocation Problems: Market Design and Axiomatic Mechanism Design

by

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Dissertation

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To my parents

Mevlüt & Ayşe Ekici

and my brother

İbrahim Deniz Ekici

Dissertation Abstract

Chapter 1: Reclaim-proof Allocation of Indivisible Objects

This paper studies axioms defining a "desirable allocation" in indivisible object allocation problems. The existing axioms in the literature are conditions of *ex-ante* robustness (individual-rationality and group-rationality) and *ex-post* robustness (Pareto-efficiency) to blocking coalitions. We introduce an all-encompassing stringent axiom. An allocation is *reclaim-proof* if it is *interim* robust to blocking coalitions. Interim robustness to blocking coalitions has practical appeal in allocation problems in which the assignments are to be made in multiple rounds. Our main results unify and extend several disparate results in the literature. We show that an allocation is reclaim-proof if and only if it is induced by a YRMH-IGYT mechanism (introduced by Abdulkadiroğlu and Sönmez, *Journal of Economic Theory' 1999*) and if and only if it is a Walrasian allocation.

Chapter 2: Fair and Efficient Discrete Resource Allocation: A Market Approach

In a variety of cases, a set of indivisible objects must be allocated to a set of agents where each agent is entitled to receive exactly one object. Examples include the allocation of tasks to workers, spots at public schools to pupils, and kidneys to patients with renal failure. We consider the mixed ownership case of this problem (some objects are initially owned by some agents while the other objects are unowned) and introduce a market-based mechanism that is procedurally reminiscent of the Walrasian Mechanism from equal-division. Our mechanism is strategy-proof and procedurally fair, and it leads to Pareto-efficient allocations. We obtain that it is equivalent to a well-known priority-order based mechanism. The equivalence result in the classical paper by Abdulkadiroğlu and Sönmez (*Econometrica' 1998*) follows as a corollary.

Chapter 3: House Swapping

An increasingly more popular practice that allows vacationers to save from accommodation costs is house swapping. A vacationer is endowed with preferences over (house, guest) pairs where "house" stands for the house she is to receive for vacation, and "guest" stands for the person who is to receive her house. We show under additively-separable preferences that in a house-swapping market a pairwise-stable allocation is not guaranteed to exist, and possibly no Pareto-efficient allocation may be attainable via only executing two-way swaps. If preferences are "guest-diseparable," then there exists a core allocation. More restrictively, if preferences are "guest-dichotomous," there exists a unique core allocation and the mechanism that selects it is strategy-proof.

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All errors are of my own.

Contents

Chapter 1

Reclaim-proof Allocation of Indivisible Objects

1.1 Introduction

We study the classical problem of allocating n indivisible objects to n agents. Each agent is entitled to receive exactly one object, her preferences over objects are *strict*, and monetary transfers are not allowed. An *allocation* is a one-to-one matching of objects to agents. There are many real-life applications of this problem, such as the allocation of offices to faculty members, spots at public schools to pupils, dormitory rooms to college students, and organs for transplant to patients. We follow the convention in the literature and refer to our objects as "houses."

There are three variants of this problem in the literature, varying in the initial ownership structure. In a *housing market* each house is initially owned by a distinct agent (Shapley and Scarf [34]); in a *house allocation problem* every house is initially unowned (Hylland and Zeckhauser [17]); and in the general case, a *house allocation problem with existing tenants*, initially there may be both owned and unowned houses (Abdulkadiroğlu and Sönmez [2]).

A *mechanism* (or an *allocation rule*) is a systematic rule that selects an allocation at any admissible preference profile that may potentially be reported by agents. Two questions are important in judging the desirability of a mechanism: Does it select a "desirable allocation" under any admissible preference profile of agents? And for the previous question to bear significance, does it induce agents to report their preferences truthfully? The content of this paper relates to the first question. We study axioms defining a desirable allocation.

The proposed axioms in the literature are various conditions of robustness to "blocking coalitions." In the general sense, an allocation μ is *blocked* by a coalition of agents if there exists a way in which the coalition members trade their owned houses such that under the induced coalitional allocation each coalition member is weakly and at least one coalition member is strictly better off in comparison to under μ. The proposed axioms in the literature are as follows:

- In the context of a housing market, the prominent axiom is "group-rationality," which is an allocation's *ex-ante* robustness to blocking coalitions. Formally, an allocation μ is *group rational* (or, *in the core*) if no coalition of agents blocks it by trading their initially owned houses. Roth and Postlewaite [29] showed that in a housing market there exists a unique group-rational allocation, which is also the unique Walrasian allocation (i.e., competitive allocation), and it is induced by the *top trading cycle* mechanism (in short, TTC), credited to Gale in [34].[1]

- In the context of a house allocation problem, the prominent axiom is "Pareto-efficiency," which is an allocation's *ex-post* robustness to blocking coalitions. Formally, an allocation μ is *Pareto efficient* if no coalition of agents blocks it by trading the houses that they are assigned at μ (i.e., by trading the houses that they own after the assignments have been made as specified by μ). Svensson [37] showed that in a house allocation problem the set of Pareto-efficient allocations coincides with the set of allocations induced by the class of *serial dictatorship* mechanisms.[2]

[1] TTC is described in Subsection [1.2.2]. Roth [28] proved that TTC is strategy-proof, and Ma [20] showed that it is the only mechanism that is Pareto-efficient, individually-rational, and strategy-proof.

[2] Serial dictatorships, described in Subsection [1.2.2], are also known as *priority rules* or *queue allocation rules*. Svensson [37] showed that serial dictatorships are strategy-proof. Svensson [38] and Ergin [15] provided two characterizations for this class of mechanisms. See also Satterthwaite and Sonnenschein [33].

- In the context of a house allocation problem with existing tenants, as properties of a desirable allocation Abdulkadiroğlu and Sönmez [2] considered two axioms—Pareto-efficiency and "individual-rationality." Individual-rationality is an allocation's *ex-ante* robustness to *blocking individuals* and is implied by group-rationality. Formally, an allocation μ is *individually rational* if no agent blocks it by keeping her initially owned house (i.e., no agent strictly prefers her initially owned house to the house that she is assigned at μ). They introduced the class of *You request my house - I get your turn* (YRMH-IGYT) mechanisms, which induce Pareto-efficient and individually-rational allocations. The class of YRMH-IGYT mechanisms generalizes TTC and serial dictatorships: This class reduces to TTC in a housing market and to the class of serial dictatorships in a house allocation problem.[3]

This paper introduces a new axiom that subsumes and strengthens Pareto-efficiency and group-rationality. The following simple example will be useful in our discussion.

Example 1.1 *Consider the following problem: Houses h_1, h_2, and h_3 are to be allocated to agents a_1, a_2, and a_3. Agents' strict preference orders are as given in the table below. Initially a_3 owns h_3 (shown in boldface in the table) and h_1 and h_2 are unowned. The allocation π is the one at which a_1, a_2, a_3 are assigned h_3, h_1, h_2, respectively (displayed in boxes in the table).*

a_1	a_2	a_3
$\boxed{h_3}$	h_3	h_1
h_2	$\boxed{h_1}$	$\boxed{h_2}$
h_1	h_2	$\boldsymbol{h}_3$

$\Diamond$

In Example 1.1 the allocation π does not admit blocking coalitions ex-ante and ex-post (i.e., it is group rational[4] and Pareto efficient). Nevertheless, if π is not implemented in one shot and rather if its assignments are made over a time horizon, an "interim blocking coalition" may arise: Suppose a_2 is assigned first as specified in π (i.e., she receives h_1). Then the "interim ownership

[3]Abdulkadiroğlu and Sönmez 2 showed that the YRMH-IGYT mechanisms are strategy-proof. See Sönmez and Ünver [36] for a characterization of this class of mechanisms.

[4]Note that in Example 1.1 group-rationality reduces to individual-rationality because initially only a_3 owns a house.

structure" arises where h_2 is unowned and a_2 and a_3 respectively own h_1 and h_3, one another's most favorite houses. Then the coalition of a_2 and a_3 blocks μ as each receives her most favorite house when they trade, which makes both strictly better off in comparison to under μ.

When an allocation μ is to be implemented, there is an initial ownership structure, call ω, and a final ownership structure μ^{-1} characterized by μ, but if the assignments prescribed by μ are made in multiple rounds rather than simultaneously, "interim ownership structures" arise as well from ω to μ^{-1}. A house may be unowned in an interim ownership structure, but if it is owned, it must be owned by its initial owner or final owner. Motivated by the possibility that interim ownership structures may arise in allocation problems with a dynamic nature, we introduce a very stringent axiom. An allocation μ is *reclaim-proof* if no coalition of agents blocks it by trading the houses that they initially own (ex-ante robustness to blocking coalitions) or the houses that they own under μ (ex-post robustness to blocking coalitions) or a combination of the previous two (interim robustness to blocking coalitions). In Example 1.1 π is not reclaim-proof because a_2 and a_3 block it by trading h_1 (the house a_2 owns under π) and h_3 (the house a_3 initially owns). Clearly, an allocation is robust to blocking coalitions at every admissible interim ownership structure if and only if it is reclaim-proof.

In real-life applications an allocation's group-rationality and Pareto-efficiency is a satisfactory combination for its successful implementation if its prescribed assignments are to be made simultaneously. In some real-life applications, however, assignments are non-simultaneously made for various reasons. For instance, in the allocation of office space to faculty members, vacant (unowned) offices may be made available at different dates by the outgoing faculty members, so not all incoming faculty members may move into their offices at the same time. Similarly, in the allocation of dormitory rooms to college students, students may arrive at school and hence move into their rooms at different dates. In kidney exchange practices with Good Samaritan Donors, logistical constraints do not allow undertaking too many simultaneous transplantation operations (see Roth, Sönmez and Ünver [30]). In such real-life applications an allocation's interim robustness to blocking coalitions is a desirable property for its successful implementation.

The main results of our paper identify new merits of the class of YRMH-IGYT mechanisms. First, we show that in a house allocation problem with existing tenants the set of allocations induced by the class of YRMH-IGYT mechanisms is precisely the set of reclaim-proof allocations (Theorem 1.1). Second, we show that an allocation is reclaim-proof if and only if it is a Walrasian allocation (Theorem 1.2). This second result, we believe, provides further justification for reclaim-proofness as an axiom. The results by Roth and Postlewaite [29] (that in a housing market TTC induces the unique group-rational allocation, which is also the unique Walrasian allocation), Svensson [37] (that in a house allocation problem the set of allocations induced by serial dictatorships is the set of Pareto-efficient allocations) and Abdulkadiroğlu and Sönmez [2] (that in the general setting the YRMH-IGYT mechanisms induce Pareto-efficient and individually-rational allocations) are corollaries of our more general results. Thus our results unify and extend these disparate results in the literature on indivisible object allocation problems. A possible direction for future research is to explore how reclaim-proofness can play a role in characterizing the class of YRMH-IGYT mechanisms.

The organization of the rest of the paper is as follows: Section 1.2 presents the model and the related results in the literature; Section 1.3 introduces reclaim-proofness and presents our results.

1.2 The Model

1.2.1 Preliminaries

A *house allocation problem with existing tenants* is a fourtuple $< a_0 \cup A, H, P, \omega >$ where:

- a_0 is the social planner and $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set of agents;
- $H = \{h_1, h_2, \ldots, h_n\}$ is a finite set of houses;
- $P : (P_a)_{a \in A}$ is the profile of agents' *strict* preferences over H;

- ω is an *ownership structure*, that is, a mapping from H to $a_0 \cup A$ such that $|\omega^{-1}(a)| \leq 1$ for every $a \in A$ (i.e., only the social planner may own more than one house).

In what follows we consider a representative problem $\Pi :< a_0 \cup A, H, P, \omega >$. Π is a *house allocation problem* if $\omega^{-1}(a_0) = H$ (every house is owned by the social planner) and a *housing market* if $\omega^{-1}(a_0) = \emptyset$ (each house is owned by a distinct agent).

For ease of reference we refer to an agent who initially owns a house as the *existing tenant* of that house. We denote the set of existing tenants by $A_E \subseteq A$. Also, we sometimes refer to houses owned by the social planner as *unowned houses*, meaning that they are not owned by agents.

We assume that every agent prefers being assigned a house to not being assigned a house. For $a \in A$ and $h, h' \in H$ we write $h\, P_a\, h'$ if a prefers h to h'. We use R_a to represent the "at least as good as" relation for $a \in A$ (i.e., $h\, R_a\, h'$ means $h\, P_a\, h'$ or $h = h'$).

An *allocation* $\mu : A \rightarrow H$ is a one-to-one mapping from the set of of agents to the set of houses. Note that the inverse mapping μ^{-1} is an ownership structure. We denote the domain of admissible allocations by $\mathcal{M}$.

An allocation $\mu \in \mathcal{M}$ is

- *individually rational* if $\mu(a)\, R_a\, \omega^{-1}(a)$ for every $a \in A_E$;
- *group rational* if there exist no subset of agents $C \subseteq A_E$ and a one-to-one mapping $bl : C \rightarrow \omega^{-1}(C)$ such that $bl(a)\, R_a\, \mu(a)$ for every $a \in C$ and $bl(a)\, P_a\, \mu(a)$ for an agent $a \in C$;
- *Pareto efficient* if there is no allocation $\mu' \in \mathcal{M}$ such that $\mu'(a)\, R_a\, \mu(a)$ for every $a \in A$ and $\mu'(a)\, P_a\, \mu(a)$ for an agent $a \in A$.

The above axioms are various conditions of robustness to "blocking coalitions." In the general sense, an allocation μ is "blocked" by a coalition of agents $C \subseteq A$ if there exists a way in which agents in C trade their owned houses such that the coalitional allocation induced Pareto dominates the coalitional allocation of these agents under μ. In the implementation of an allocation μ, if μ is group rational it is not blocked ex-ante (i.e., when the ownership structure is ω); if μ is Pareto efficient it is not blocked ex-post (i.e., when the ownership structure is μ^{-1}). Individual-rationality is the condition of ex-ante robustness to *blocking individuals* and is implied by group-rationality.

An allocation $\mu \in \mathcal{M}$ is a *Walrasian allocation* if there exist a non-negative price function $pr : H \to \mathbb{R}^+$ and a non-negative transfer function $tr : A \to \mathbb{R}^+$ such that

1. $pr(\mu(a)) \leq tr(a)$ for every $a \in A \setminus A_E$;

2. $pr(\mu(a)) \leq tr(a) + pr(\omega^{-1}(a))$ for every $a \in A_E$;

3. $\sum_{a \in A} tr(a) \leq \sum_{h \in \omega^{-1}(a_0)} pr(h)$;

4. if $h \, P_a \, \mu(a)$ for any $a \in A \setminus A_E$ and $h \in H$ then $tr(a) < pr(h)$;

5. if $h \, P_a \, \mu(a)$ for any $a \in A_E$ and $h \in H$ then $tr(a) + pr(\omega^{-1}(a)) < pr(h)$.

The first three conditions above state that at a Walrasian allocation everyone affords her expenditure: For an agent who initially does not own a house, her expenditure is the value of the house she is assigned at μ and her income is the transfer payment she receives from the social planner. For an agent who initially owns a house, her expenditure is the value of the house she is assigned at μ and her income is the sum of the transfer payment that she receives from the social planner and the value of the house that she initially owns. For the social planner, her expenditure is the sum of transfer payments that she makes and her income is the total value of the houses that she initially owns. If we add up the inequalities in (1), (2) and (3) across all agents and the social planner, the left-hand side and the right-hand side are the same, so the inequalities in (1), (2) and (3) are binding. The last two conditions above state that at a Walrasian allocation agents buy their most preferred affordable houses.

1.2.2 Related Results in the Literature

In the context of a housing market the reputable allocation rule is the following *top trading cycle* (TTC) mechanism, credited to Gale in [34]:

> *Imagine a diagram consisting of agents and houses. Let each agent "point" to her most preferred house in the diagram. Let each house "point" to its owner. There exists at least one "cycle."*[5] *Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram and in the reduced diagram, proceed similarly.*

TTC is a strategy-proof[6] as shown in [28] and it is inherently linked to ex-ante robustness to blocking coalitions:

Theorem: (Roth and Postlewaite [29]): *TTC induces the unique group-rational allocation in a housing market, which is also the unique Walrasian allocation.*

A "priority-order" $f : A \to \{1, 2, ..., n\}$ is a bijection that orders agents. We denote the set of admissible priority-orders by $\mathcal{F}$. In the context of a house allocation problem the reputable class of mechanisms is *serial dictatorships.* Each priority-order $f \in \mathcal{F}$ defines a distinct serial dictatorship mechanism, which proceeds as described below:

> *Assign the agent who is ordered first in f to her most preferred house; assign the agent who is ordered next in f to her most preferred house among remaining ones; and so on.*

Serial dictatorships are strategy-proof as shown in [37] and they are inherently linked to ex-post robustness to blocking coalitions:

Theorem: (Svensson [37]): *The set of allocations induced by the class of serial dictatorships in a house allocation problem is precisely the set of Pareto-efficient allocations.*

In the general setting, a house allocation problem with existing tenants, Abdulkadiroğlu and Sönmez [2] proposed the class of *You request my house - I get your turn* (YRMH-IGYT)

[5] A cycle is characterized by an ordered list $i_1, i_2, \cdots, i_k$ of agents where i_s points to i_{s+1}'s house for $s = 1, \cdots, k-1$ and i_k points to i_1's house.

[6] i.e., truthful preference revelation is a dominant strategy.

mechanisms. Each priority-order $f \in \mathcal{F}$ defines a distinct YRMH-IGYT mechanism, which proceeds as described below:

> *Assign the agent ordered first in f to her most preferred house; assign the agent ordered second in f to her most preferred house among remaining ones; and so on, until an agent requests a house owned by an existing tenant. If at that point the existing tenant whose house is requested has already been assigned a house, do not disturb the procedure. Otherwise, update the remainder of the priority-order by inserting that existing tenant to the top and proceed. If at any point a loop forms, it is formed exclusively by a subset of existing tenants who consecutively request the houses owned by one another. In such cases remove all agents in the loop by assigning them the houses they request and proceed.*

A YRMH-IGYT mechanism is strategy-proof; it reduces to TTC in a housing market and to the serial dictatorship defined by the same priority-order in a house allocation problem [2]. Its induced allocation has been shown to be ex-ante robust to blocking individuals and ex-post robust to blocking coalitions:

Theorem: (Abdulkadiroğlu and Sönmez [2]): *The allocation induced by a YRMH-IGYT mechanism is Pareto efficient and individually rational.*

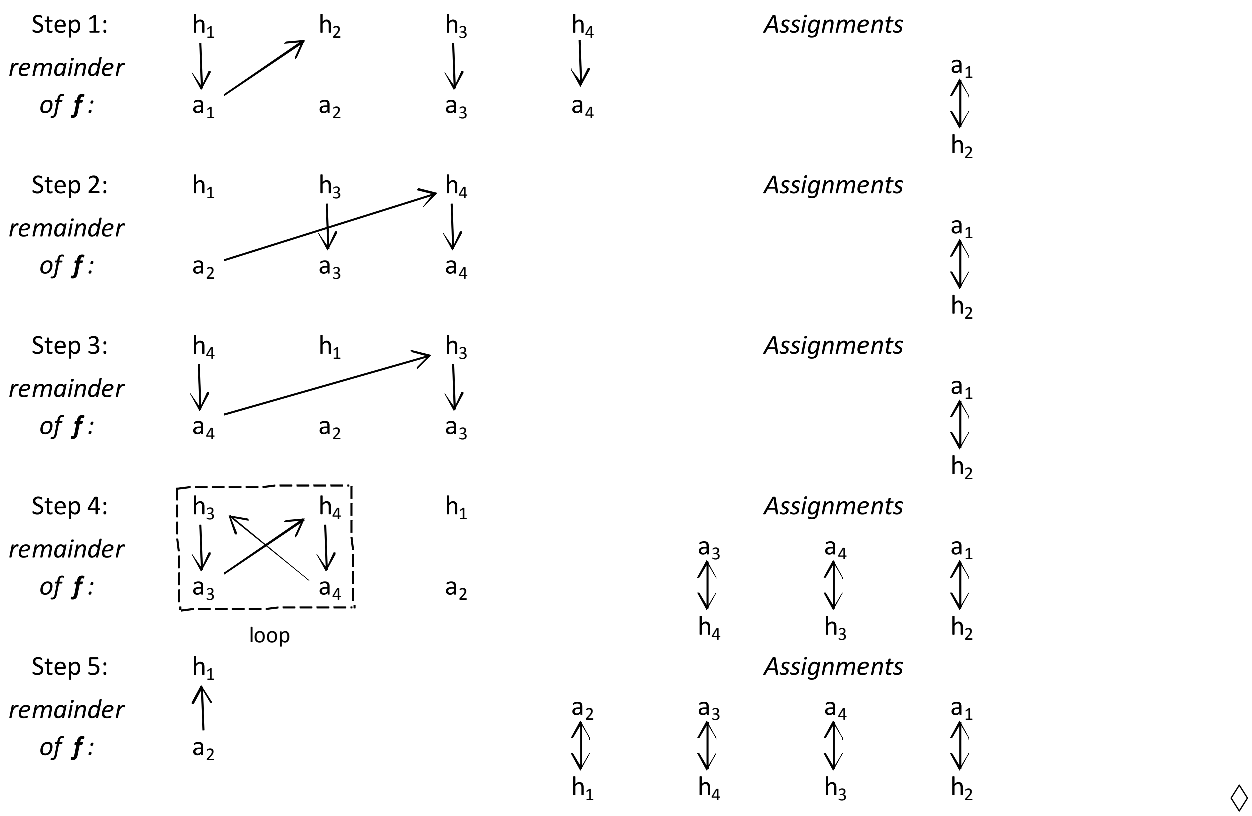
1.2.3 More on YRMH-IGYT mechanisms

We illustrate in a simple example below how a YRMH-IGYT mechanism proceeds.

Example 1.2 *Consider the following problem: Houses h_1, h_2, h_3, h_4 are to be allocated to agents a_1, a_2, a_3, a_4. Agents' strict preference orders are as given in the table below. Initially a_1, a_3, a_4 respectively own h_1, h_3, h_4 (shown in boldface in the table) and h_2 is unowned.*

a_1	a_2	a_3	a_4
h_2	h_4	h_4	h_2
h_4	h_2	h_2	h_3
h_3	h_3	$\boldsymbol{h}_3$	$\boldsymbol{h}_4$
$\boldsymbol{h}_1$	h_1	h_1	h_1

Consider the YRMH-IGYT mechanism defined by the priority-order f such that $f(a_i) = i$ for $i = 1, 2, 3, 4$. We illustrate in a series of figures below how it proceeds. In figures unidirectional arrows point from owned houses to their owners and from agents to their most preferred houses among remaining ones; bidirectional arrows indicate the assignments made by the mechanism.



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As in Example 1.2, when the YRMH-IGYT mechanism defined by a priority-order f is executed, the order in which agents are assigned houses is not necessarily the same as f. By this observation, we define the *assignment order* $f^{ao} : A \cup H \to \{1, 2, ..., n\}$ implied by a priority-order $f \in \mathcal{F}$, which is constructed as described below:

> *Order first in f^{ao} the agent and the house that are assigned first by the YRMH-IGYT mechanism defined by f; order second in f^{ao} the agent and the house that are assigned next; and so on. Whenever a loop is encountered, specify in f^{ao} the orders of agents in the loop and the houses they are assigned the same and proceed.*

The assignment order implied by a priority-order becomes a very useful tool in showing our results in Section 1.3. As an illustration, the table below presents the priority-order f in Example

1.2 and the assignment order f^{ao} that it implies.

	f	f^{ao}
1	a_1	a_1, h_2
2	a_2	a_3, h_4, a_4, h_3
3	a_3	a_2, h_1
4	a_4	

1.3 Reclaim-proof Allocations and Results

As argued in Section 1.1, in some real-life applications, in the implementation of an allocation its prescribed assignments may not be made simultaneously for various reasons, and therefore, between the initial and final (ex-ante and ex-post) ownership structures, various other "interim" ownership structures may arise. This is formalized by Definition 1.1.

Definition 1.1 *In our representative problem* Π*, given an allocation* $\mu \in \mathcal{M}$ *to be implemented, an ownership structure* $\omega' : H \to a_0 \cup A$ *is an* ***interim ownership structure*** *if* $\omega'(h) \in a_0 \cup \omega(h) \cup \mu^{-1}(h)$ *for every* $h \in H$*.*

In words, in an interim ownership structure ω' a house h is either unowned ($\omega'(h) = a_0$) or it is owned by its initial or final owner ($\omega(h)$ or $\mu^{-1}(h)$). We do not argue that every admissible interim ownership structure has to arise if assignments are not made simultaneously, but any interim ownership structure may arise. The initial and final ownership structures ω and μ^{-1} are admissible interim ownership structures. However, robustness to blocking coalitions in ω (group-rationality) and μ^{-1} (Pareto-efficiency) does not entail robustness to blocking coalitions in every admissible interim ownership structure, which is entailed by the axiom that we introduce next.

Definition 1.2 *In our representative problem* Π*, an allocation* $\mu \in \mathcal{M}$ *is* ***reclaim-proof*** *if there exists no fourtuple* $\langle C, H^C, rc, rbl \rangle$ *(a "reclaim blocking fourtuple") where:*

1. $C \subseteq A$ *is a subset of agents (a "reclaim blocking coalition");*

2. $H^C \subseteq H$ *is a subset of houses such that* $\left|H^C\right| = |C|$*;*

3. $rc : C \to H^C$ *is a one-to-one mapping (a "reclaim function") such that* $rc(a) \in \omega^{-1}(a) \cup \mu(a)$ *for every* $a \in C$*;*

4. $rbl : C \to H^C$ *is a one-to-one mapping (a "reclaim blocking allocation") such that* $rbl(a)\, R_a\, \mu(a)$ *for every* $a \in C$ *and* $rbl(a)\, P_a\, \mu(a)$ *for an agent* $a \in C$*.*

In words, an agent joins a reclaim blocking coalition by trading either the house that she initially owns (if there is any) or the house that she is assigned at μ. Since these are the two houses that she may own in an interim ownership structure, an allocation is robust to blocking coalitions in every conceivable interim ownership structure if and only if it is reclaim-proof. Since in μ's implementation ω and μ^{-1} are admissible interim ownership structures, reclaim-proofness implies group-rationality and Pareto-efficiency. The converse is not true, however. As an illustration, in Example 1.1 π is group rational and Pareto efficient but not reclaim-proof. The two reclaim-proof allocations in Example 1.1 are

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ h_2 & h_3 & h_1 \end{pmatrix} \text{ and } \begin{pmatrix} a_1 & a_2 & a_3 \\ h_3 & h_2 & h_1 \end{pmatrix}.$$

Theorem 1.1 exposes the inherent link between the class of YRMH-IGYT mechanisms and the set of reclaim-proof allocations.

Theorem 1.1 *In our representative problem* Π*, an allocation* $\mu \in \mathcal{M}$ *is reclaim-proof if and only if it is induced by a YRMH-IGYT mechanism.*

Proof. We prove the theorem in two parts:

(I) Suppose μ is not reclaim-proof but it is induced by the YRMH-IGYT mechanism defined by a priority-order f. Let $\langle C, H^C, rc, rbl\rangle$ be a reclaim blocking fourtuple. Consider the assignment order f^{ao} implied by f. For an agent $a \in C$ if $rc(a) = \mu(a)$ then clearly $f^{ao}(a) = f(rc(a))$. For an agent $a \in C$ if $rc(a) = \omega^{-1}(a)$, then, by construction of the YRMH-IGYT mechanism,

$f^{ao}(a) = f(rc(a))$ if a joins a loop in its execution[7] and $f^{ao}(a) < f(rc(a))$ if she does not. Then

$$\sum_{a \in C} f^{ao}(a) \leq \sum_{a \in C} f(rc(a)) = \sum_{h \in H^C} f^{ao}(h).$$

For an agent $a \in C$ if $rbl(a) = \mu(a)$ then clearly $f^{ao}(a) = f^{ao}(rbl(a))$. For an agent $a \in C$ if $rbl(a) \neq \mu(a)$ (and there exists such an agent) then $rbl(a)\, P_a\, \mu(a)$, which implies by construction of the YRMH-IGYT mechanism that $f^{ao}(a) > f^{ao}(rbl(a))$. Then

$$\sum_{a \in C} f^{ao}(a) > \sum_{a \in C} f^{ao}(rbl(a)) = \sum_{h \in H^C} f^{ao}(h),$$

which contradicts the previous inequality.

(II) Suppose μ is reclaim-proof. We construct a priority-order $f \in \mathcal{F}$ such that the YRMH-IGYT mechanism defined by f induces μ. Consider a diagram consisting of every agent and every house:

(*) Find an agent a such that $\mu(a)$ is a's most preferred house and it is initially unowned. Order a first in f and remove a and $\mu(a)$ from the diagram. Among remaining agents, find an agent a such that $\mu(a)$ is a's most preferred house among remaining ones and initially it is either unowned or owned by a removed agent. Order a second in f and remove a and $\mu(a)$ from the diagram. Proceed similarly whenever it is possible to do so, and whenever it is not, proceed as in (**).

(**) Let the sets of remaining agents and houses be $A' \subseteq A$ and $H' \subseteq H$, and let $H^O \subseteq H'$ be the set of remaining houses that are initially owned by agents in A'. Let every remaining agent $a \in A'$ point to her most preferred house in H'; every house $h \in H^O$ point to its initial owner $\omega^{-1}(h)$; and every house $h \in H \setminus H^O$ point to $\mu^{-1}(h)$. There exists a cycle. Suppose in this cycle there exists an agent who is not assigned at μ her most preferred house in H'. But then this cycle defines a reclaim blocking fourtuple $\langle C, H^C, rc, rbl \rangle$: C is the set of agents in the cycle; for $a \in C$, $rc(a) \in H^C$ is the house that points to a in the cycle; and $rbl(a)$ is the house

[7] Note that if a receives $\omega^{-1}(a)$ it means she forms a loop by herself and the equality $f^{ao}(a) = f(rc(a))$ still holds.

to which a points. Since μ is reclaim-proof this cannot be true and so every agent in the cycle is assigned at μ the house to which she points. But then every house in this cycle should be in H^O because if there were a house $h \in H \setminus H^O$ in the cycle then the agent who points to h would be an agent as required in (*): She would be assigned at μ her most preferred house among remaining ones which is initially either unowned or owned by a removed agent. So the agents in the cycle are existing tenants who are assigned the initially owned houses of one another, which are their most preferred houses among remaining ones. Remove the agents and houses in the cycle from the diagram by ordering these agents next in f (in any order).

Evidently the YRMH-IGYT mechanism defined by f induces μ. Note that when this mechanism is executed, the agents ordered in f as described in (**) form the loops. ■

Theorem 1.2 exposes the inherent link between reclaim-proof allocations and Walrasian allocations.

Theorem 1.2 *In our representative problem Π, an allocation $\mu \in \mathcal{M}$ is reclaim-proof if and only if it is a Walrasian allocation.*

Proof. We prove the theorem in two parts:

(I) Let $\mu \in \mathcal{M}$ be a reclaim-proof allocation. Then, by Theorem 1.1, there exists a priority-order f such that the YRMH-IGYT mechanism defined by f induces μ. Consider the assignment order f^{ao} implied by f. Let us define a non-negative price function $pr : H \to \mathbb{R}^+$ where $pr(h) = n - f^{ao}(h)$ for every $h \in H$, and a non-negative transfer function $tr : A \to \mathbb{R}^+$ where $tr(a) = pr(\mu(a))$ for every $a \in A \setminus A_E$ and $tr(a) = pr(\mu(a)) - pr(\omega^{-1}(a))$ for every $a \in A_E$. The allocation μ is a Walrasian allocation for pr and tr.

(II) Let $\mu \in \mathcal{M}$ be a Walrasian allocation for a non-negative price function $pr : H \to \mathbb{R}^+$ and a non-negative transfer function $tr : A \to \mathbb{R}^+$. Consider a priority-order f such that $f(a) < f(a')$ for any $a, a' \in A$ if $pr(\mu(a)) > pr(\mu(a'))$. The YRMH-IGYT mechanism defined by f induces μ.

■

Chapter 2

Fair and Efficient Discrete Resource Allocation: A Market Approach

2.1 Introduction

We consider the problem of allocating n indivisible objects to n agents where each agent is entitled to receive exactly one object and agents' preferences over objects are strict. Monetary transfers are not allowed.[1] There are numerous real-life applications of this problem, such as the allocation of tasks to workers, spots at public schools to pupils, kidneys to patients with renal failure, dormitory rooms to college students, and legislators to committees [1, 2, 3, 5, 17, 30]. The purpose of this paper is to design a mechanism (a systematic allocation rule) which has a market-based approach, is fair in a certain sense, and leads to efficient (Pareto-optimal) allocations (matchings of agents and houses). A core issue in designing a mechanism is that preferences are elicited from agents who may respond strategically rather than truthfully.

As is the convention in this literature, we employ the paradigm of allocating "houses" to agents. There are three cases of this problem in the literature, varying in the initial ownership structure. In the pure exchange case, called a *housing market*, each agent initially owns a house

[1] We do not take a normative standpoint against the use of money. In many real-life applications, however, the use of money is not permissible.

(see Shapley and Scarf [34]). In the pure distributional case, called a *house allocation problem*, houses are initially unowned (see Hylland and Zeckhauser [17]). In the mixed ownership case, called a *house allocation problem with existing tenants*, the previous two cases are generalized: There are k "newcomers" who initially do not own any houses; k "vacant houses," which are initially unowned; and $n - k$ "existing tenants" who own the $n - k$ "occupied houses" (see Abdulkadiroğlu and Sönmez [2]).

This paper considers the mixed ownership case. Besides being more general, it arouses interest for a number of interesting real-life applications of it. A prominent example is *kidney exchange with Good Samaritan Donors* [30, 32]. In the most serious forms of renal disease, the preferred treatment is kidney transplantation. As of March 2009, there were about 79,000 patients waiting for a kidney transplant in the United States. While some patients ("existing tenants") have friends or relatives willing to donate them their kidneys ("occupied houses"), there are also patients ("newcomers") who do not have donors. There are also kidneys obtained from Good Samaritan Donors and cadavers ("vacant houses") which are donated to patients collectively. In many cases a patient cannot be transplanted the kidney of her donor due to medical incompatibilities. A common practice is then kidney exchange in which patients are transplanted the kidneys of one another's donors. A kidney exchange may also involve patients without donors and kidneys obtained from Good Samaritan Donors and cadavers. Another real-life application of the mixed ownership case is *on-campus housing* practices at colleges [2, 9]. Each returning student ("existing tenant") occupies a room from the previous year. There are also incoming freshmen ("newcomers"), who initially do not occupy any rooms, and vacant rooms, vacated by the graduating class.

The mechanism that we introduce is inspired from the "Walrasian Mechanism from equal-division," which is arguably the most widely advocated mechanism to allocate a bundle of infinitely divisible goods to a set of agents fairly and efficiently. This mechanism proceeds in a simple manner. First, the bundle is equally divided among agents ("equal-division"), which results in an exchange market. (If there are also individual endowments, agents' equal-division shares of the bundle are added to their individual endowments.) In the induced exchange market the Walrasian Mechanism is operated resulting in a Walrasian allocation, which is Pareto efficient under standard assumptions on preferences.[2] This mechanism has been shown to be compatible with many equity criteria that has been proposed in the literature (see Thomson [39]).

Clearly, "indivisible" objects cannot be equally "divided." A "probabilistic equal-division" idea can be employed, however, using random distribution. In an indivisible object allocation problem, the first mechanism that involves random distribution has been introduced by Abdulkadiroğlu and Sönmez [1]. In the context of a house allocation problem, they proposed the *core from random endowments* mechanism (in short, CFRE), which proceeds as follows:

> Distribute n houses to n agents uniformly at random (each agent receives exactly one house). In the induced exchange market, reallocate houses to agents by executing the *top trading cycles* mechanism[3] (in short, TTC), as described below:
>
> Step 1: Let each agent "point" to her most preferred house and let each house "point" to its owner. Since the number of agents and houses is finite, there exists at least one "cycle." A cycle is characterized by a list $a^1, a^2, \cdots, a^j$ of agents where agent a^1 points to the house owned by a^2, which points to a^2; a^2 points to the house owned by a^3, which points to a^3;$\cdots$; a^{j-1} points to the house owned by a^j, which points to a^j; and a^j points to the house owned by a^1, which points to a^1. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.
>
> Step t > 1: Let each remaining agent point to her most preferred house among remaining ones and let each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.

Roth [28] showed that TTC is strategy-proof (truthful preference revelation is a dominant strategy), from which it immediately follows that CFRE is also strategy-proof.

[2] A Walrasian allocation is an allocation that can be attained in a Walrasian equilibrium. The Walrasian mechanism is the rule that maps an exchange economy to a Walrasian allocation.

[3] This mechanism is credited to David Gale in Shapley and Scarf [34].

We should highlight the parallels between CFRE and the Walrasian Mechanism from equal-division. In the Walrasian Mechanism from equal-division, the exchange market is induced by (physical) equal-division of the bundle. In CFRE, the exchange market is induced by probabilistic equal-division of the (unowned) houses; each house is given to each agent with exactly the same probability, $1/n$. In the Walrasian Mechanism from equal-division, in the induced exchange market the Walrasian Mechanism is executed resulting in a Walrasian allocation, which is Pareto efficient. In CFRE, in the induced exchange market TTC is executed, which is the exact counterpart of the Walrasian Mechanism in this context, as it produces the unique Walrasian allocation, which is also Pareto efficient (see Roth and Postlewaite [29]). In the Walrasian Mechanism from equal-division, under certain assumptions that guarantee core convergence (see Aumann [4]) and the uniqueness of a Walrasian allocation (see Mas-Colell [21]), in the induced exchange market the allocation produced by the Walrasian Mechanism is the unique core allocation. In CFRE, in the induced exchange market the allocation produced by TTC is also the unique core allocation (see Roth and Postlewaite [29]).

We want to apply a similar approach in the mixed ownership case. The random distribution in this context is a more delicate issue, however. To highlight the key challenges, consider the following two mechanisms, the first of which is due to Sönmez and Ünver [35]:

(1) Distribute k vacant houses to k newcomers uniformly at random (each newcomer receives exactly one vacant house). Then reallocate houses to agents by executing TTC.

(2) Distribute k vacant houses to n agents uniformly at random (out of n agents, only k of them receive a vacant house). Then reallocate houses to agents by executing TTC.

In (1), random distribution results in a housing market (each agent owns exactly one house), and TTC produces its unique core allocation. There is no probabilistic equal-division, however, as a vacant house is given to a newcomer with $1/k$ probability but to an existing tenant with zero probability. To emphasize the fairness shortcoming of this mechanism, consider an existing tenant whose occupied house is the least desired house of every agent. Then in (1) she is assigned her least desired house. In a sense, she is punished for owning a house, which is not what we desire. Arguably, this feature of the mechanism may also cause an incentive shortcoming. The agent who owns the least desired house may respond strategically by first giving up her house and then participating in the mechanism as a newcomer.

In (2), there is probabilistic equal-division, as each vacant house is given to each agent with exactly $1/n$ probability. There is an efficiency shortcoming of this mechanism, however. After vacant houses have been distributed, if an existing tenant e receives a vacant house and a newcomer a does not, then, in the induced exchange market, e owns two houses (a vacant house besides her occupied house) while a owns none. When TTC is executed, a cannot join a cycle and remains unassigned, and when e is removed after joining a cycle, one of her two houses remains, which becomes "wasted."

We introduce a mechanism that resolves the fairness and efficiency tension in (1) and (2) in an intuitive way. The *core from random distribution* mechanism (in short, CFRD) proceeds as follows:

> Besides the k vacant houses, introduce $n-k$ "inheritance rights" associated with $n-k$ existing tenants. Distribute k vacant houses and $n-k$ inheritance rights to n agents uniformly at random (each agent receives exactly one vacant house or one inheritance right). If an agent receives an inheritance right, she becomes the "inheritor" of the associated existing tenant; if she receives a vacant house, she owns that vacant house. Therefore, in the induced exchange market, a newcomer owns a house or an inheritance right, and an existing tenant owns two houses (a vacant house besides her occupied house) or an inheritance right and a house. We call this an *inheritors augmented housing market.* In this market reallocate houses to agents by executing the following *inheritors augmented top trading cycles* mechanism (in short, IATTC):
>
> Step 1: Let each agent point to her most preferred house and let each house point to its owner. Since the number of agents and houses is finite, there exists at least one cycle.

Assign the agents in cycles the houses they point to and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones and let each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.

The innovation in CFRD is the role played by inheritance rights in the execution of IATTC. When an existing tenant who owns two houses is removed after joining a cycle, her remaining house is not wasted; it is given to her inheritor (or to the inheritor of her inheritor and so on). Note that this has no negative welfare implications for her. She can trade any of her two houses to join a cycle and be assigned the best house that she can. Only after she leaves the market her remaining house is given to another agent.

It can be shown that IATTC is strategy-proof.[4] Then it immediately follows that CFRD is also strategy-proof. Note that there is a probabilistic equal-division in CFRD; each vacant house is given to each agent with exactly $1/n$ probability. Further, IATTC is the counterpart of the Walrasian Mechanism in this context, as it produces a Walrasian allocation.[5] In Proposition 2.1 we show that, as TTC produces the unique core allocation in a housing market, IATTC produces the unique core allocation in an inheritors augmented housing market. (The core allocation notion in this context is more subtle, however. The definition should incorporate

[4] We do not include a proof for this, as it follows as a corollary of Theorem 2.1.

[5] This is due to Theorem 2.1 and the fact that a YRMH-IGYT mechanism produces a Walrasian allocation (Ekici [14]).

the rights of inheritors; see Definition 2.2.) Indeed, in an inheritors augmented housing market if every agent owns exactly one house, IATTC proceeds just like TTC, and in this sense, an inheritors augmented housing market and IATTC can be seen as generalizations of a housing market and TTC.

Our main theoretical contribution is a surprising equivalence result, similar to the one in Abdulkadiroğlu and Sönmez [1]. In a house allocation problem, they showed that CFRE is equivalent to *random-priority* (also known as "random serial dictatorship"). That is, given any preference profile, any given allocation is produced by the two mechanisms with exactly the same probability. Random-priority proceeds as follows:

> Choose a priority-order (an ordering of agents) uniformly at random. Then execute the associated *priority-rule* as follows: Assign the first agent her most preferred house, the second agent her most preferred house among remaining ones, and so on.

In Theorem 2.1 we show that CFRD is equivalent to *randomized You request my house - I get your turn mechanism* (in short, RYRMH-IGYT), a mechanism due to Abdulkadiroğlu and Sönmez [2] and which proceeds as follows:

> Choose a priority-order uniformly at random. Then execute the associated *You request my house - I get your turn mechanism* (in short, YRMH-IGYT) as follows: Assign the first agent her most preferred house, the second agent her most preferred house among remaining ones, and so on, until an agent requests the occupied house of an existing tenant. If at that point that existing tenant has already been assigned a house, do not disturb the procedure. Otherwise, update the remainder of the order by inserting that existing tenant to the top and proceed. If at any point a loop forms, it is formed exclusively by a subset of existing tenants who request the occupied houses of one another. In such cases remove the existing tenants in the loop by assigning them the houses they request and proceed.

In a house allocation problem, RYRMH-IGYT reduces to random-priority and CFRD reduces to CFRE. Therefore, the equivalence result in Abdulkadiroğlu and Sönmez [1] is a corollary of our more general equivalence result.

Our equivalence result contributes to our understanding of the links between the allocation problems of infinitely divisible goods and indivisible objects. In indivisible object allocation problems priority-order based mechanisms—RYRMH-IGYT and random-priority, are popular, perhaps owing to their simplicity. Procedurally, however, they cannot be given interpretations from a market point of view. On the other hand, CFRE and CFRD have clear procedural market interpretations. First, they induce exchange markets via probabilistic equal-division, and then they produce Walrasian allocations. They proceed analogously to the Walrasian Mechanism from equal-division. The equivalence results in our paper and in [1] expose that, although this is not explicit in their formulations, RYRMH-IGYT and random-priority share the same analogy to the Walrasian Mechanism from equal-division.

The rest of the paper is organized as follows. The next subsection briefly mentions the related literature. Section 2.2 introduces the model, describes RYRMH-IGYT and CFRD, and presents our equivalence result. The proof is bijective and fairly involved, which we cover exclusively in Section 2.3 (where we present an alternative specification of IATTC and explore its properties). Section 2.4 concludes the paper. We present the proof of Proposition 2.1 in the Appendix.

Related Literature

In addition to [1], there are several other papers with equivalence results in the literature:

Sönmez and Ünver [35] showed that the TTC based mechanism in (1) in Section 2.1 is equivalent to the following priority-order based mechanism: Choose a priority-order by ordering k newcomers at the top uniformly at random and placing $n-k$ existing tenants at the bottom (in any order); then execute the YRMH-IGYT mechanism defined by that priority-order.

In two recent papers random-priority has been shown to be equivalent to certain mechanisms that execute TTC based upon "inheritance tables." An inheritance table is a collection of orderings of agents. Each ordering relates to a house. While TTC is executed, an agent points to her most preferred house in the market (as usual) and a house points to the agent in the market who is ordered highest in its ordering. Pathak and Sethuraman [25] showed that, if TTC is executed based upon a randomly generated inheritance table where every agent is included in every ordering, the resulting mechanism is equivalent to random-priority. They also extended this equivalence to the houses-with-quotas case (e.g., a public school with a quota of q can be assigned to q students). Also, they show that, in the houses-with-quotas case, if TTC is executed based upon a randomly generated inheritance table where the ordering for a house with quota q includes only q agents and each agent is included in the ordering of only one house, the resulting mechanism is again equivalent to random-priority. Carroll [7] later showed a general equivalence result that implies and extends the preceding ones.

There is a conceptual difference between CFRD and the above-mentioned TTC based mechanisms. The execution of TTC in these mechanisms is based upon a randomly generated inheritance table, which is unlike IATTC, whose execution is based upon randomly generated "inheritor relationships between agents." This innovation in CFRD promises a new line of research. Future research papers may study how to execute IATTC in the houses-with-quotas case, or, when an existing tenant may initially own multiple houses, which may potentially lead to the design of other IATTC based mechanisms that are equivalent to RYRMH-IGYT. The tools that we introduce in Section 2.3 may become useful in these efforts.

2.2 House Allocation with Existing Tenants

2.2.1 Preliminaries

A *house allocation problem with existing tenants* is a five-tuple $< A_N, H_V, A_E, H_O, P >$ where

- $A_N : \{a_1, a_2, ..., a_k\}$ is a finite set of "newcomers";
- $H_V : \{h_1, h_2, \ldots, h_k\}$ is a finite set of "vacant houses";
- $A_E : \{e_{k+1}, e_{k+2}, ..., e_n\}$ is a finite set of "existing tenants";
- $H_O : \{o_{k+1}, o_{k+2}, ..., o_n\}$ is a finite set of "occupied houses" such that existing tenant e_s owns (or, equivalently, occupies) o_s for $s = k+1, \cdots, n$;

– $P : (P_a)_{a \in A_N \cup A_E}$ is the profile of agents' *strict* preference relations over the set of houses.

A house allocation problem with existing tenants is a *housing market* if every agent is an existing tenant (i.e., $k = 0$), and it is a *house allocation problem* if every agent is a newcomer (i.e., $k = n$).

We fix A_N, H_V, A_E, H_O throughout the paper, and we denote $A_N \cup A_E$ and $H_V \cup H_O$ respectively by A and H.

We assume that every agent prefers being assigned *any* house to not being assigned a house. For $a \in A$ and $h, h' \in H$ we write $h \, P_a \, h'$ if a prefers h to h'. We denote the domain of admissible preference relations by $\mathcal{P}$ (so, $P \in \mathcal{P}^n$). We use R_a to represent the "at least as good as" relation for $a \in A$ derived from P_a (i.e., $h \, R_a \, h'$ means $h \, P_a \, h'$ or $h = h'$).

An *allocation* $\mu : A \rightarrow H$ is a bijection from the set of agents to the set of houses. We denote the set of admissible allocations by $\mathcal{M}$.

Given $P \in \mathcal{P}^n$, an allocation $\mu \in \mathcal{M}$ is:

- *Pareto efficient* if there exists no $\mu' \in \mathcal{M}$ such that $\mu'(a) \, R_a \, \mu(a)$ for every $a \in A$ and $\mu'(a) \, P_a \, \mu(a)$ for an agent $a \in A$.
- *individually rational* if $\mu(e_s) \, R_{e_s} \, o_s$ for $s = k+1, \cdots, n$.
- *group rational* if there exists no triplet $\langle C, H^C, bl \rangle$ where $C \subseteq A_E$; $H^C \subseteq H$ is the set of houses owned by agents in C; and $bl : C \rightarrow H^C$ is a bijection such that $bl(a) \, R_a \, \mu(a)$ for every $a \in C$ and $bl(a) \, P_a \, \mu(a)$ for an agent $a \in C$. If there exists such a triplet then C is called a "blocking coalition" and we say that μ is "blocked" by C.

In the context of a housing market a group-rational allocation is also called a *core allocation*.

An allocation $\mu \in \mathcal{M}$ is a *Walrasian allocation* (with transfers) if there exists a non-negative price function $pr : H \rightarrow \mathbb{R}^+$ and a non-negative transfer function $tr : A \rightarrow \mathbb{R}^+$ such that

1. the budget function $w : A \to \mathbb{R}^+$ is given by $w(a) = tr(a)$ for $a \in A_N$ and $w(e_s) = tr(e_s) + pr(o_s)$ for $s = k+1, \cdots, n$;

2. $pr(\mu(a)) \leq w(a)$ for every $a \in A$;

3. $\sum_{a \in A} tr(a) \leq \sum_{h \in H_V} pr(h)$;

4. if $h \, P_a \, \mu(a)$ for any $a \in A$ and $h \in H$, then $w(a) < pr(h)$.

In words, at a Walrasian allocation, vacant houses are sold in the market and the raised revenue is distributed to agents as transfers, existing tenants raise additional revenue by selling their occupied houses, and then agents buy in the market their most preferred affordable houses. Ekici [14] showed that the inequalities in (2) and (3) are binding.

A *random assignment* $\lambda : \mathcal{M} \to \mathbb{R}$ is a probability distribution over allocations. We denote the domain of admissible random assignments by Λ. Note that for every $\lambda \in \Lambda$,

$$\lambda(\mu) \geq 0 \text{ for every } \mu \in \mathcal{M}, \text{ and } \sum_{\mu \in \mathcal{M}} \lambda(\mu) = 1.$$

A *mechanism* (or, *an allocation rule*) is a systematic way to choose an allocation at any given preference profile. Formally, a "deterministic" mechanism $\varphi^D : \mathcal{P}^n \to \mathcal{M}$ is a function that maps the domain of admissible preference profiles to the codomain of allocations (so at $P \in \mathcal{P}^n$ its allocation choice is certain), and a "lottery" mechanism $\varphi^L : \mathcal{P}^n \to \Lambda$ is a function that maps the domain of admissible preference profiles to the codomain of random assignments (so, at $P \in \mathcal{P}^n$ it chooses an allocation randomly based upon $\varphi^L(P)$). We have given these definitions in the context of a house allocation problem with existing tenants, but we will also talk about mechanisms in more restricted domains, such as in a housing market or in a house allocation problem. Therefore, in what follows, a "mechanism" should be understood as a systematic way to choose an allocation in the context of a well-specified class of problems.

A lottery mechanism is ex-post (Pareto) efficient if it maps every preference profile to a random assignment at which positive probability weights are given to only Pareto-efficient allocations. For a lottery mechanism, the properties of ex-post individual-rationality and ex-post group-rationality are defined accordingly.

In what follows we consider a representative problem $\Pi :< A_N, H_V, A_E, H_O, P >$, which stands for the class of house allocation problems with existing tenants.

2.2.2 Randomized You request my house - I get your turn

This subsection presents a lottery mechanism in the context of a house allocation problem with existing tenants. It is derived from the class of "You request my house - I get your turn" (YRMH-IGYT) mechanisms, introduced by Abdulkadiroğlu and Sönmez [2].

A *priority-order* is a bijection from the set of agents A to the set of numbers $\{1, 2, ..., n\}$. We denote a generic priority-order by f, and the domain of admissible priority-orders by $\mathcal{F}$. For instance, if $f(a) = 4$ for $a \in A$ and $f \in \mathcal{F}$, it means that agent a is ordered fourth in the priority-order f.

Each priority-order $f \in \mathcal{F}$ defines a separate YRMH-IGYT mechanism, which allocates houses to agents at a given preference profile as described below.

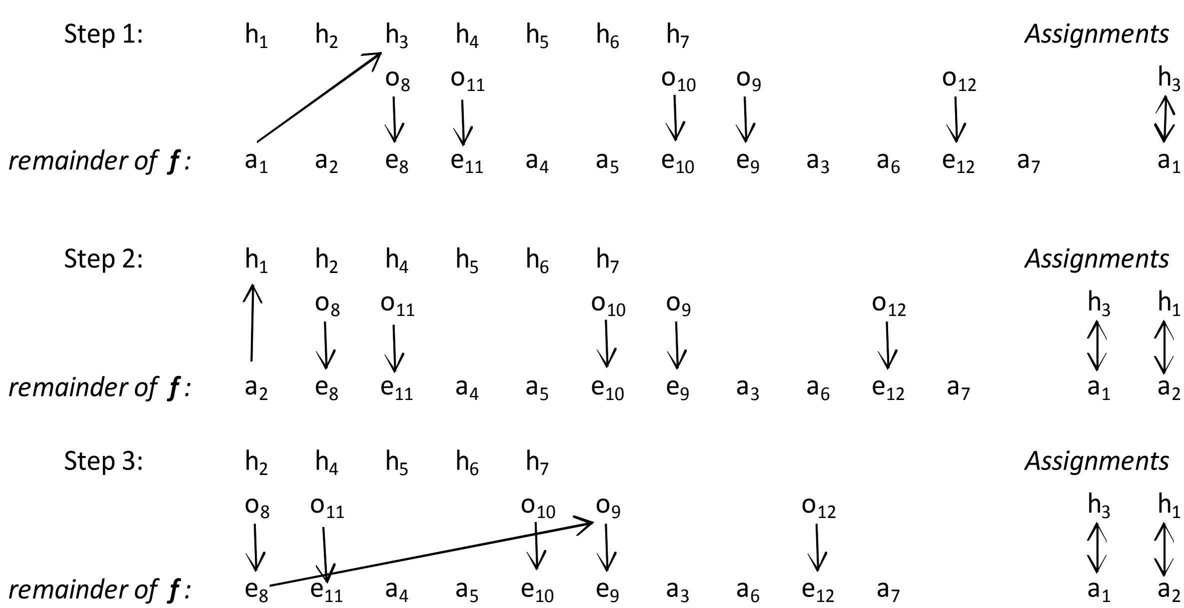
The YRMH-IGYT mechanism defined by $f \in \mathcal{F}$: *Assign the agent ordered first in f to her most preferred house; assign the agent ordered second in f to her most preferred house among remaining ones; and so on, until an agent requests the occupied house of an existing tenant. If at that point the existing tenant whose occupied house is requested has already been assigned a house, do not disturb the procedure. Otherwise, update the remainder of the priority-order by inserting that existing tenant to the top and proceed. If at any point a loop forms, it is formed exclusively by a subset of existing tenants who request the occupied houses of one another. In such cases remove all agents in the loop by assigning them the houses they request and proceed.*

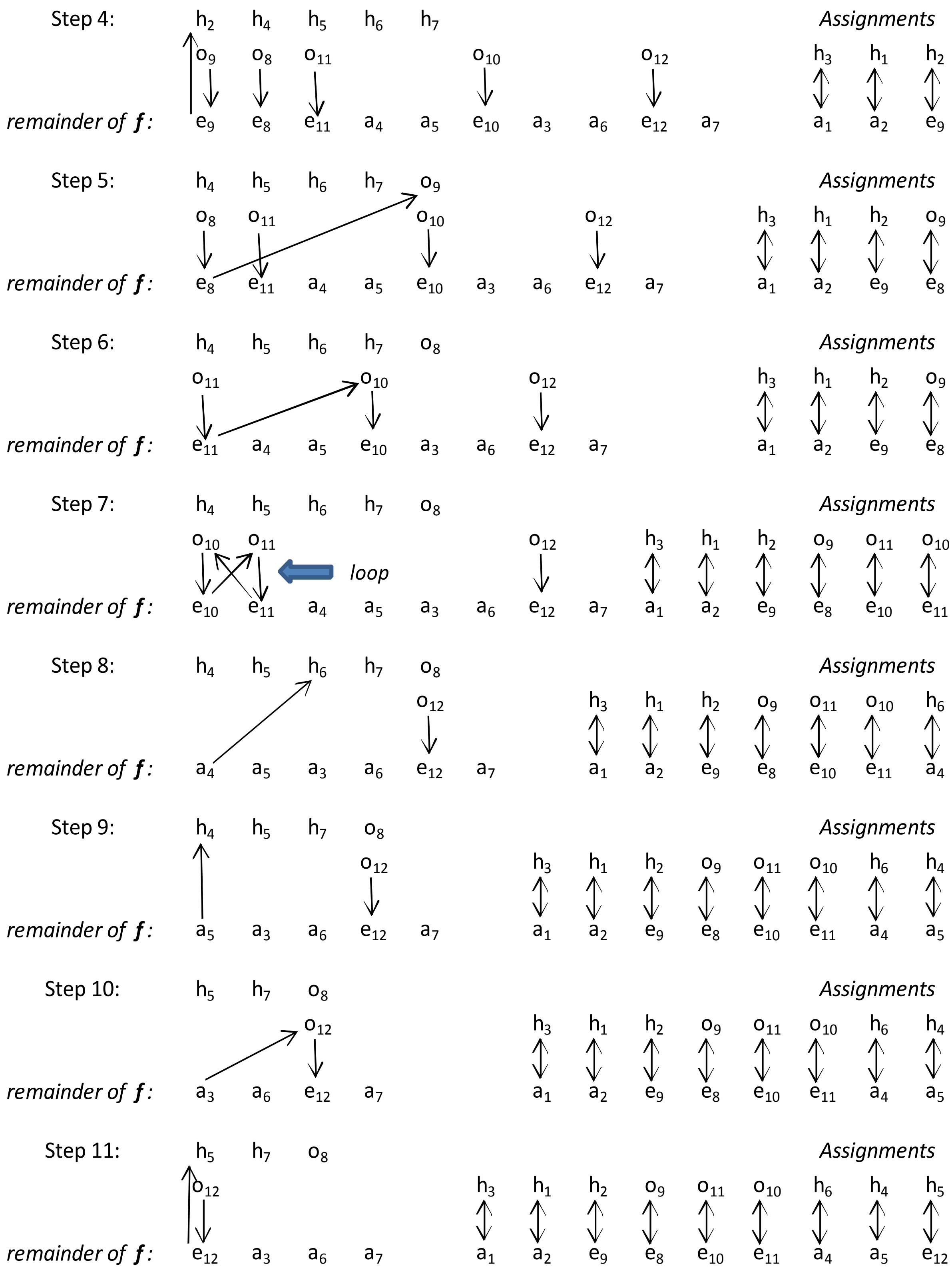
There are appealing properties of the class of YRMH-IGYT mechanisms. They are strategy-proof [2] (truthful preference revelation is a dominant strategy); for any given preference profile, they lead to Pareto-efficient and individually-rational allocations [2], and the set of allocation induced by them coincides with the set of Walrasian allocations [14]. The following example demonstrates the workings of a YRMH-IGYT mechanism.

Example 2.1 *Consider a house allocation problem with existing tenants in which the preference profile of agents is as in the following table:*

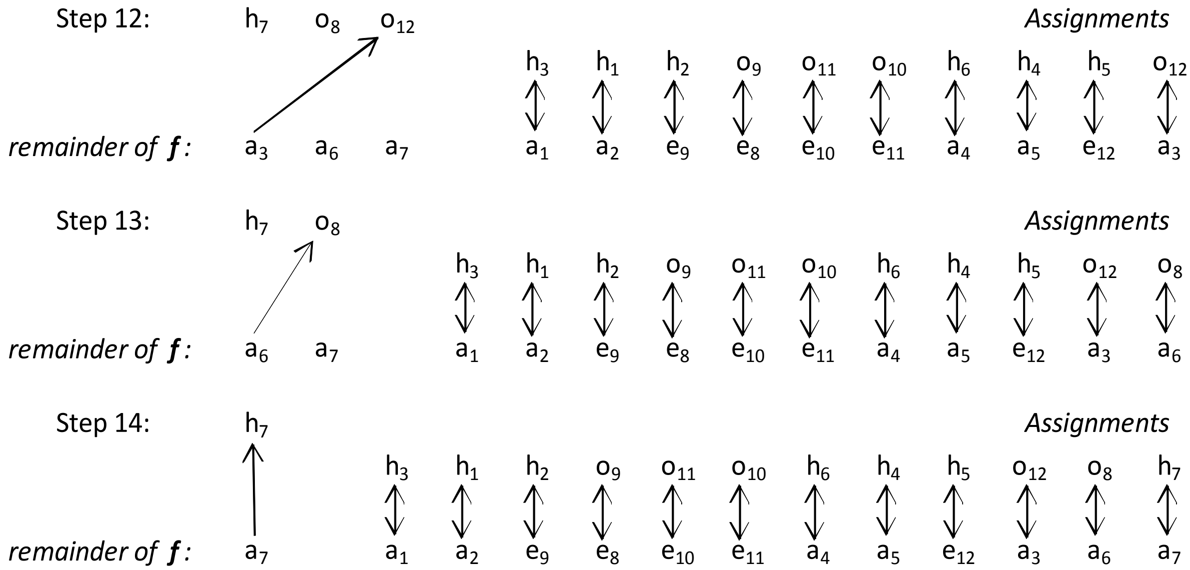
a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_3	h_1	h_3	h_6	h_4	o_{11}	o_8	o_9	h_2	h_1	o_{10}	h_5
$\vdots$	$\vdots$	o_{10}	$\vdots$	$\vdots$	o_8	h_1	$\vdots$	$\vdots$	o_{11}	$\vdots$	$\vdots$
		o_{12}			$\vdots$	o_{10}			$\vdots$		
		$\vdots$				h_7					
						$\vdots$					

Let in the priority-order f agents be ordered as $a_1, a_2, e_8, e_{11}, a_4, a_5, e_{10}, e_9, a_3, a_6,\ e_{12}, a_7$ (so $f(a_1) = 1, \cdots, f(a_7) = 12$). We illustrate in a series of figures below how the YRMH-IGYT mechanism defined by f proceeds. In figures, unidirectional arrows point from occupied houses to their existing tenants and from agents to their most preferred houses among remaining ones; bidirectional arrows indicate the assignments made by the mechanism.





Step 4: h_2 h_4 h_5 h_6 h_7
o_9 o_8 o_{11} o_{10} o_{12}
remainder of $\boldsymbol{f}$: e_9 e_8 e_{11} a_4 a_5 e_{10} a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2
a_1 a_2 e_9
Step 5: h_4 h_5 h_6 h_7 o_9
o_8 o_{11} o_{10} o_{12}
remainder of $\boldsymbol{f}$: e_8 e_{11} a_4 a_5 e_{10} a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2 o_9
a_1 a_2 e_9 e_8
Step 6: h_4 h_5 h_6 h_7 o_8
o_{11} o_{10} o_{12}
remainder of $\boldsymbol{f}$: e_{11} a_4 a_5 e_{10} a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2 o_9
a_1 a_2 e_9 e_8
Step 7: h_4 h_5 h_6 h_7 o_8
o_{10} o_{11} o_{12}
loop
remainder of $\boldsymbol{f}$: e_{10} e_{11} a_4 a_5 a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2 o_9 o_{11} o_{10}
a_1 a_2 e_9 e_8 e_{10} e_{11}
Step 8: h_4 h_5 h_6 h_7 o_8
o_{12}
remainder of $\boldsymbol{f}$: a_4 a_5 a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2 o_9 o_{11} o_{10} h_6
a_1 a_2 e_9 e_8 e_{10} e_{11} a_4
Step 9: h_4 h_5 h_7 o_8
o_{12}
remainder of $\boldsymbol{f}$: a_5 a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2 o_9 o_{11} o_{10} h_6 h_4
a_1 a_2 e_9 e_8 e_{10} e_{11} a_4 a_5
Step 10: h_5 h_7 o_8
o_{12}
remainder of $\boldsymbol{f}$: a_3 a_6 e_{12} a_7
Assignments
h_3 h_1 h_2 o_9 o_{11} o_{10} h_6 h_4
a_1 a_2 e_9 e_8 e_{10} e_{11} a_4 a_5
Step 11: h_5 h_7 o_8
o_{12}
remainder of $\boldsymbol{f}$: e_{12} a_3 a_6 a_7
Assignments
h_3 h_1 h_2 o_9 o_{11} o_{10} h_6 h_4 h_5
a_1 a_2 e_9 e_8 e_{10} e_{11} a_4 a_5 e_{12}



$\Diamond$

In our representative problem Π, let $\mu^{Y-I,f}$ denote the allocation chosen by the YRMH-IGYT mechanism defined by $f \in \mathcal{F}$, and let $\mathcal{F}^{\mu} \subseteq \mathcal{F}$ be the subset of priority-orders for which the resulting YRMH-IGYT mechanisms choose $\mu \in \mathcal{M}$ (i.e., $\mathcal{F}^{\mu} = \{f \in \mathcal{F} | \mu^{Y-I,f} = \mu\}$).

Despite its other appealing properties, a YRMH-IGYT mechanism suffers on grounds of fairness. If in a priority-order $f \in \mathcal{F}$ we have $f(a) < f(a')$ for two agents $a, a' \in A$, then the YRMH-IGYT mechanism defined by f clearly favors a over a'. A natural way to introduce fairness is randomization. That is, one may first choose a priority-order uniformly at random and then execute the YRMH-IGYT mechanism defined by the chosen priority-order. This is what we call the *randomized You request my house - I get your turn* mechanism (in short, RYRMH-IGYT).

RYRMH-IGYT: *Choose a priority-order of agents $f \in \mathcal{F}$ uniformly at random, and then allocate houses to agents by executing the YRMH-IGYT mechanism defined by f.*

Having been derived from the class of YRMH-IGYT mechanisms, RYRMH-IGYT is strategy-proof, ex-post efficient, and ex-post group rational.

In our representative problem Π, let λ^{rY-I} denote the random assignment induced by RYRMH-

IGYT. Then

$$\lambda^{rY-I}(\mu) = \frac{|\mathcal{F}^{\mu}|}{n!}.$$

2.2.3 Core from Random Distribution

This subsection introduces an alternative lottery mechanism in the context of a house allocation problem with existing tenants. For this purpose we first recall Gale's reputable *top trading cycles* mechanism (in short, TTC), an allocation rule defined in the context of a housing market that proceeds as follows

TTC: *Step 1: Let each agent "point" to her most preferred house, and each house "point" to its owner. Since the number of agents and houses is finite, there exists at least one "cycle." A cycle is characterized by a list $a^1, a^2, \cdots, a^j$ of agents where agent a^1 points to the house owned by a^2, which points to a^2; a^2 points to the house owned by a^3, which points to a^3;$\cdots$; a^{j-1} points to the house owned by a^j, which points to a^j; and a^j points to the house owned by a^1, which points to a^1. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.*

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones and let each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.

TTC is strategy-proof [28] and in a housing market it chooses the unique core allocation, which is also the unique Walrasian allocation [29].

We also need to introduce what we call an "inheritors augmented housing market," which is generated from our representative problem Π.

Definition 2.1 *From* $\Pi :< A_N, H_V,\ A_E, H_O, P >$ *an* ***inheritors augmented housing market*** $\Pi^v :< A_N, H_V,\ A_E, H_O, P, v >$ *is generated by specifying a bijection* $v : A_N \cup A_E \to H_V \cup I$ *such that:*

– *$I : \{i_{k+1}, \cdots, i_n\}$ is the set of "inheritance rights," where i_s is the inheritance right associated with existing tenant e_s;*

– *agent $a \in A_N \cup A_E$ owns $v(a)$ (which is a vacant house or an inheritance right).*

In words, an inheritors augmented housing market is generated from a house allocation problem with existing tenants by distributing to agents vacant houses and inheritance rights associated with existing tenants. Note that in an inheritors augmented housing market it is possible that an existing tenant owns two houses (a vacant house besides her occupied house) and a newcomer owns none (she then owns only an inheritance right).

For $a \in A$ if $v(a) = i_s$, then we call a the "inheritor" of e_s and e_s the "bequeather" of a. If we talk of "bequeathers" of a, the agents we mean by it are a's bequeather, a's bequeather's bequeather, and so on. We denote by $\mathcal{V}$ the domain of admissible bijections from $A_N \cup A_E$ to $H_V \cup I$. Note that $|\mathcal{V}| = n!$, and by separately augmenting $n!$ bijections to our representative problem Π we can generate $n!$ distinct inheritors augmented housing markets. If we talk about an "allocation" in an inheritors augmented housing market, we mean by it, as usual, a bijection from A to H.

The essential component of our alternative lottery mechanism is the "inheritors augmented top trading cycles" mechanism (in short, IATTC), which is an allocation rule in the context of an inheritors augmented housing market. IATTC, described below, resembles TTC.

IATTC: *Step 1: Let each agent point to her most preferred house and each house point to its owner. Since the number of agents and houses is finite, there exists at least one cycle. Assign the agents in cycles the houses they point to, and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.*

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones and each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to, and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.

The verify that IATTC is well-defined, the key observation is that, when IATTC is executed in Π^v ($v \in \mathcal{V}$), a newcomer who does not own a house in Π^v will always inherit a house from one of her bequeathers: Suppose a^1 is a newcomer who does not own a house in Π^v, and let a^2 be the existing tenant who is her bequeather. If in Π^v a^2 owns two houses, say, h and h', then a^2 does not have a bequeather, and the only bequeather of a^1 is a^2. In the execution of IATTC, a^2 joins a cycle by trading h or h', and when she leaves, a^1 inherits the house that remains from her, and she later joins a cycle by trading this house. If in Π^v a^2 owns a house, say h, and an inheritance right, let the bequeather of a^2 be a^3. If in Π^v a^3 owns two houses, say, h' and h'', then a^3 does not have a bequeather, the only bequeather of a^2 is a^3, and the only bequeathers of a^1 are a^2 and a^3. In the execution of IATTC, a^3 joins a cycle by trading one of her two houses, say, h''. When a^3 leaves, there are two possibilities. The first possibility is that a^2 is still in the market and inherits h', and she later joins a cycle by trading h or h'. The house that remains from a^2 is then inherited by a^1, who later joins a cycle by trading this house. The second possibility is that a^2 has left earlier by joining a cycle in which she traded h, in which case, as the inheritor of inheritor of a^3, a^1 inherits h', and she later joins a cycle by trading h'. By similarly iterating this argument, we conclude that every newcomer who does not own a house in Π^v eventually inherits a house and later joins a cycle by trading this house.

The following example demonstrates the workings of IATTC.

Example 2.2 *Consider an inheritors augmented housing market generated from the house allocation problem with existing tenants in Example 2.1 by distributing to agents vacant houses and* *inheritance rights as in the following table:*

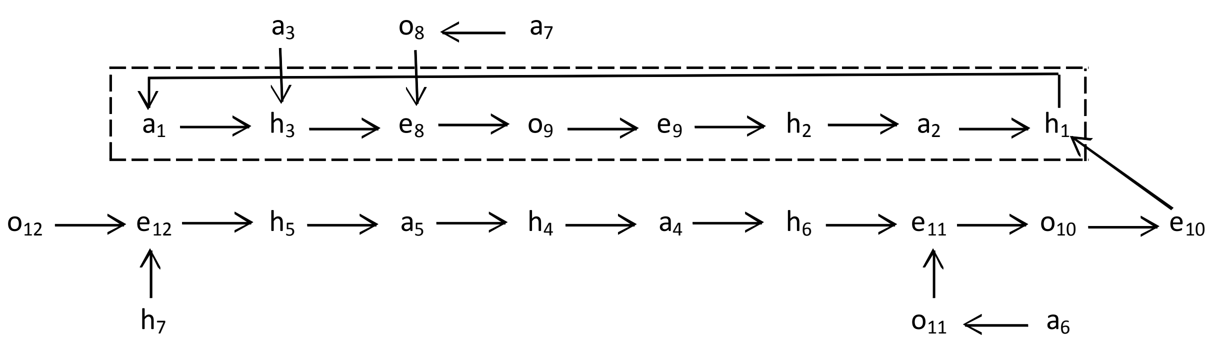
a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3	i_9	i_8	h_6	h_7

Thus, the distribution of houses and inheritance rights to agents in this inheritors augmented housing market is as in the following table:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3, o_8	i_9, o_9	i_8, o_{10}	h_6, o_{11}	h_7, o_{12}

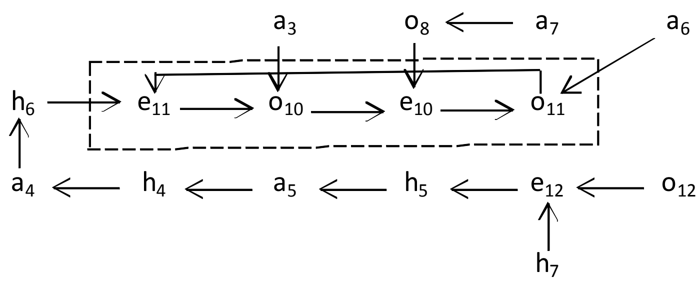
We will illustrate how IATTC proceeds in a series of figures. For visual ease we indicate cycles in the figures in dashed rectangles.

When each house points to its owner and each agent points to her most preferred house, the resulting figure is as follows:



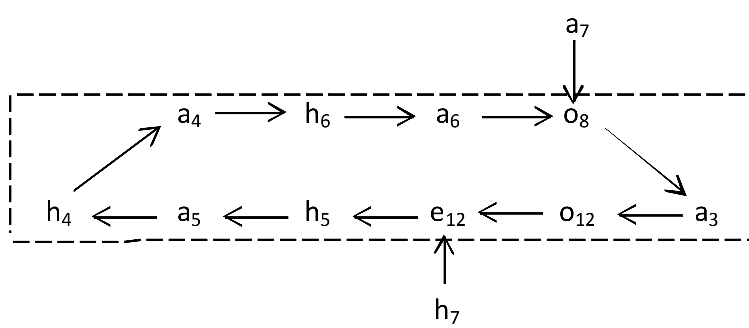
As prescribed by the cycle in the figure, a_1, e_8, e_9, a_2 are assigned h_3, o_9, h_2, h_1, respectively. The house o_8 of e_8 remains, which is received by her inheritor e_{10}.

When each remaining house points to its owner and each remaining agent points to her most *preferred house among remaining ones, the resulting figure is as follows:*



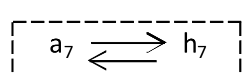
As prescribed by the cycle in the figure, e_{10} *and* e_{11} *are assigned* o_{11} *and* o_{10}*, respectively. The houses* h_6 *of* e_{11} *and* o_8 *of* e_{10} *remain, which are received by their inheritors* a_6 *and* a_3*, respectively.*

When each remaining house points to its owner and each remaining agent points to her most preferred house among remaining ones, the resulting figure is as follows:



As prescribed by the cycle in the figure, $e_{12}, a_5, a_4, a_6, a_3$ *are assigned* $h_5, h_4, h_6, o_8, o_{12}$*, respectively. The house* h_7 *of* e_{12} *remains, which is received by her inheritor* a_7*.*

The last cycle is formed by a_7 *and* h_7*; thus,* a_7 *is assigned* h_7*:*



The allocation chosen by IATTC is the same as the one we obtained in Example 2.1. $\Diamond$

In our representative problem Π, let $\mu^{iattc,v}$ denote the allocation chosen by IATTC in Π^v, and let $\mathcal{V}^\mu \subseteq \mathcal{V}$ be the set of bijections from $A_N \cup A_E$ to $H_V \cup I$ such that, in the inheritors augmented housing markets generated by augmenting them to Π, the allocation chosen by IATTC is μ (i.e., $\mathcal{V}^\mu = \{v \in \mathcal{V} | \mu^{iattc,v} = \mu\}$).

We should point out that, in essence, an inheritors augmented housing market generalizes a housing market, and IATTC generalizes TTC. If Π is a housing market (i.e., $k = 0$) and Π^v is an inheritors augmented housing market generated from Π, in Π^v every agent owns exactly one house; the execution of IATTC in Π^v is exactly the same as the execution of TTC in Π (because no house ever remains from an agent who is removed by joining a cycle); and hence, the distribution of inheritance rights to agents, v, is of no significance. In this generalization IATTC retains the theoretical properties of TTC. IATTC is strategy-proof,[6] and it chooses the unique core allocation of an inheritors augmented housing market. The definition of a "core allocation" in the context of an inheritors augmented housing market is more subtle, however. The definition needs to take into account the rights of inheritors to the houses of their bequeathers. We introduce this subtle core allocation notion in Definition 2.2, and explore it in Proposition 2.1.

Definition 2.2 *An allocation $\mu : A \to H$ is a **core allocation** in $\Pi^v : < A_N, H_V, A_E, H_O, P, v >$ ($v \in \mathcal{V}$) if there exists no four-tuple $< C, H^C, bl, Claim >$ where $C \subseteq A$, $H^C \subseteq H$, and $bl : C \to H^C$ and $Claim : C \to H^C$ are bijections such that:*

(i) $bl(a) \, R_a \, \mu(a)$ for every $a \in C$ and $bl(a) \, P_a \, \mu(a)$ for an agent $a \in C$;

(ii) for any $a \in C$, a owns $Claim(a)$; or a is the inheritor of an agent $a' \in C$ who owns $Claim(a)$; or a is the inheritor of an agent $a' \in C$ who is the inheritor of another agent $a'' \in C$ who owns $Claim(a)$; or so on.

If there exists such $<C, H^C, bl, Claim >$ we say that μ is "blocked" by the four-tuple $< C, H^C, bl, Claim >$ and we call C a "blocking coalition."

[6] This can be proved by arguments parallel to the ones in Roth [28], where he shows that TTC is strategy-proof.

The subtle part in the above definition is the *Claim* function. It states that in a blocking coalition every agent needs to claim (bring into the coalition) a distinct house, which should be a house that she or one of her bequeathers owns. In case an agent, say a, claims a house owned by one of her bequeathers, say a'', inheritors of a'' that are more closely related to her than a should also be in the blocking coalition. That is, if a'' is the bequeather of a' and a' is the bequeather of a, then a' should also be in the blocking coalition. This requirement ensures that the allocation is not blocked to benefit the distant inheritor a at the expense of the more immediate inheritor a'. If every agent owns precisely one house, Definition 2.2 reduces to the familiar core allocation notion in a housing market.

Proposition 2.1 *For $v \in \mathcal{V}$ the allocation $\mu^{iattc,v}$ is the unique core allocation in the inheritors augmented housing market Π^v.*

Proof. See Appendix. ■

We are now ready to introduce our alternative lottery mechanism in the context of a house allocation problem with existing tenants.

Core from Random Distribution: *Distribute n "items"—k vacant houses and $n-k$ inheritance rights associated with $n-k$ existing tenants, to n agents uniformly at random (each agent receives exactly one vacant house or one inheritance right). In the generated inheritors augmented housing market, reallocate houses to agents by executing IATTC.*

We shortly call this mechanism CFRD. From the theoretical properties of its main component, IATTC, it is not difficult to show that CFRD is strategy-proof, ex-post efficient, and ex-post group rational.

In our representative problem Π, let λ^{cfrd} denote the random assignment induced by CFRD. Then

$$\lambda^{cfrd}(\mu) = \frac{|\mathcal{V}^\mu|}{n!}.$$

Theorem 2.1 presents the main result of our paper. Its proof is bijective and fairly involved, which we cover exclusively in Section 2.3.

Theorem 2.1 *RYRMH-IGYT and CFRD are equivalent. That is, for any $\mu \in \mathcal{M}$,*

$$\lambda^{rY-I}(\mu) = \lambda^{cfrd}(\mu).$$

2.3 The Proof of Theorem 2.1

This section provides an alternative specification of IATTC. As we proceed, we introduce some tools, make certain observations about this alternative specification, and present four lemmas, which help us prove Theorem 2.1. The proof involves the construction of a bijection as in Abdulkadiroğlu and Sönmez [1], but our construction is fairly more involved due to the presence of existing tenants.

Recall that, in CFRD, first n items (k vacant houses and $n-k$ inheritance rights) are distributed to n agents uniformly at random, and then, in the generated inheritors augmented housing market, houses are reallocated to agents by executing IATTC. In the execution of IATTC an existing tenant is assigned a house by joining a cycle, in which the house that she trades comes from one of two resources. It is either her occupied house, or a house that she receives due to the item that she received in the random distribution (i.e., a vacant house that she received in the random distribution, or a house that is accrued to her because of an inheritance right that she received in the random distribution). The distinguishing feature of our alternative specification of IATTC is that, it "monitors" the potential benefits to an existing tenant from these two resources by representing her in the exchange market with two copies of her, one of them owning her occupied house, and the other owning the item that she received in the random distribution. This separation allows us to construct a priority-order of agents from the distribution of items to agents. Our construction turns out to be a bijective mapping and leads to the proof of Theorem 2.1.

From a given inheritors augmented housing market $\Pi^v : < A_N, H_V, A_E, H_O, P, v >$, we construct its "ab-representation" $\Pi^{v,ab} : < A_N, H_V, A_E^a, A_E^b, H_O, P, v >$ in the following manner:

– We preserve the set of newcomers A_N: $a \in A_N$ owns $v(a)$.

– We replace the set of existing tenants A_E by two disjoint sets, A_E^a and A_E^b: Each existing tenant $e_s \in A_E$ in Π^v is now "represented" in $\Pi^{v,ab}$ by two distinct agents, $a_s \in A_E^a$ who owns $v(e_s)$, and $b_s \in A_E^b$ who owns o_s. The preferences of a_s and b_s are the same as the preferences of e_s. Although technically a_s and b_s are two separate agents, they are both to serve the interests of e_s, and hence we call them the "sisters" of one another.

In the ab-representation, we refer to the agents in $A_N \cup A_E^a$ as "a-type" agents, and to the agents in A_E^b as "b-type" agents. As an illustration, we present below how inheritance rights and houses are distributed to agents in the inheritors augmented housing market in Example 2.2 and in its ab-representation:

inheritors augmented housing market in Example 2.2											
A_N							A_E				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	o_8, h_3	o_9, i_9	o_{10}, i_8	o_{11}, h_6	o_{12}, h_7

$$\Downarrow$$

ab-representation																
a-type agents												*b-type agents*				
A_N							A_E^a					A_E^b				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	b_8	b_9	b_{10}	b_{11}	b_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3	i_9	i_8	h_6	h_7	o_8	o_9	o_{10}	o_{11}	o_{12}

We are now ready to introduce the "ab-representation specification" of IATTC, or, shortly, IATTC$^{\text{ab}}$.

IATTC$^{\text{ab}}$: *Given an inheritors augmented housing market Π^v, construct its ab-representation $\Pi^{v,ab}$. In $\Pi^{v,ab}$, reallocate houses and inheritance rights to a-type and b-type agents by the following iterative procedure:*

Step 0,1: (b-step) Let every remaining house and inheritance right point to its owner. Among remaining agents, let only b-type agents point. A b-type agent $b_s \in A_E^b$ points to her most preferred house among remaining ones if $a_s \in A_E^a$ has not been assigned a house yet, and she points to i_s if a_s has already been assigned a house. If there exists one or more cycles, remove the agents in cycles by assigning them the houses and inheritance rights they point to.

Step 0,r: (b-step) Same as Step 0,1. (Continue until there exists no cycles)

Step 1,0: (a-step) Let every remaining house and inheritance right point to its owner. Now, let every remaining agent (both a-type and b-type) point. A newcomer $a \in A_N$ points to her most preferred house among remaining ones. Of two sister agents $a_s \in A_E^a$ and $b_s \in A_E^b$, if neither has been assigned a house yet, let them both point to their most preferred house among remaining ones; if one of them has been assigned a house before, let the remaining one point to i_s. There exists at least one cycle. Remove the agents in cycles by assigning them the houses and inheritance rights they point to.

Step 1,1: (b-step) Same as Step 0,1.

Step 1,r: (b-step) Same as Step 0,1. (Continue until there exists no cycles)

Step t,0: (a-step) Same as Step 1,0.

Step t,1: (b-step) Same as Step 0,1.

Step t,r: (b-step) Same as Step 0,1. (Continue until there exists no cycles)

Stop when the procedure assigns every a-type and b-type agent a house or an inheritance right. Then, in Π^v, let the houses assigned to agents be as follows: For a newcomer $a \in A_N$, the house assigned to her is the house the above procedure assigns $a \in A_N$ in $\Pi^{v,ab}$; and for an existing tenant $e_s \in A_E$, the house assigned to her is the house the above procedure assigns $a_s \in A_E^a$ or $b_s \in A_E^b$ (the procedure assigns a house to only one of them, the other one is assigned i_s).

Notice that IATTC$^{\rm ab}$ proceeds just like TTC, by identifying cycles and then carrying out the trades in cycles, but it gives precedence to the trades in cycles that involve only b-type agents. First at b-steps trades are carried out in cycles that involve only b-type agents, and then when no such cycle remains, IATTC$^{\rm ab}$ moves to an a-step at which it carries out the trades in cycles that involve both a-type and b-type agents.

Two observations are useful to better understand the design of IATTC$^{\rm ab}$.

† **Observation 1:** *Suppose for an existing tenant $e_s \in A_E$ in Π^v ($v \in \mathcal{V}$) it happens that $v(e_s) \in H_V$ (so, e_s owns two houses, $v(e_s)$ and o_s). Notice how IATTC and IATTC^{ab} proceed analogously:*

When IATTC is executed in Π^v, at initial steps $v(e_s)$ and o_s point to e_s and e_s points to her most preferred house among remaining ones; when IATTC^{ab} is executed in $\Pi^{v,ab}$, at initial steps $v(e_s)$ and o_s respectively point to a_s and b_s (the agents that represent e_s), and a_s and b_s point to e_s's most preferred house among remaining ones.

In IATTC's execution in Π^v, when e_s joins a cycle in which she exchanges $v(e_s)$ or o_s, in parallel to that, in IATTC^{ab}'s execution in $\Pi^{v,ab}$, a_s or b_s joins the analogous cycle in which she exchanges $v(e_s)$ or o_s.

In IATTC's execution in Π^v, the house that remains from e_s is given to her inheritor; in IATTC^{ab}'s execution in $\Pi^{v,ab}$, the analogous thing happens: The remaining house ($v(e_s)$ or o_s) points to the remaining sister agent (a_s or b_s); the remaining sister agent points to i_s; i_s points to the a-type or b-type agent that represents the inheritor of e_s; and hence, in essence, the remaining house is transferred to the inheritor of e_s.

For $v \in \mathcal{V}$ let $\mu^{ab,v}$ denote the allocation chosen by IATTC$^{\rm ab}$ in Π^v. Given Observation 1 the following lemma is evident.

Lemma 2.1 *IATTC and IATTC^{ab} are equivalent. That is, for any $v \in \mathcal{V}$,*

$$\mu^{iattc,v} = \mu^{ab,v}.$$

† **Observation 2:** *In IATTC^{ab}, at a b-step only b-type agents point (to houses or inheritance rights), and thus:*

(i) a cycle at a b-step consists of only b-type agents and occupied houses;

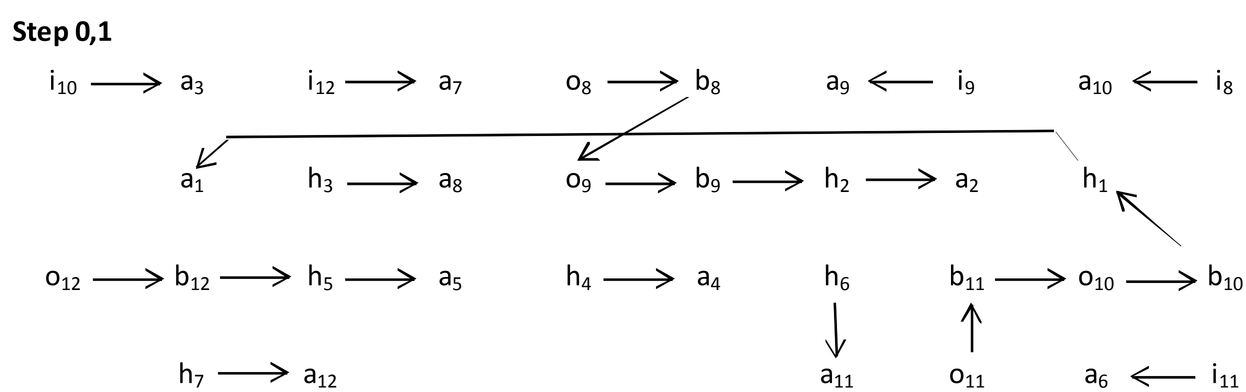
(ii) a-type agents, vacant houses, and inheritance rights are part of the cycles at a-steps, but a cycle at an a-step may also include b-type agents and occupied houses.

The separation of the steps in IATTC$^{\mathrm{ab}}$ as a-steps and b-steps is fundamental to our proof of Theorem 2.1. In the following example we demonstrate the workings of IATTC$^{\mathrm{ab}}$.

Example 2.3 *Consider the ab-representation of the inheritors augmented housing market in Example 2.2. The table below presents the distribution of houses and inheritance rights to a-type and b-type agents:*

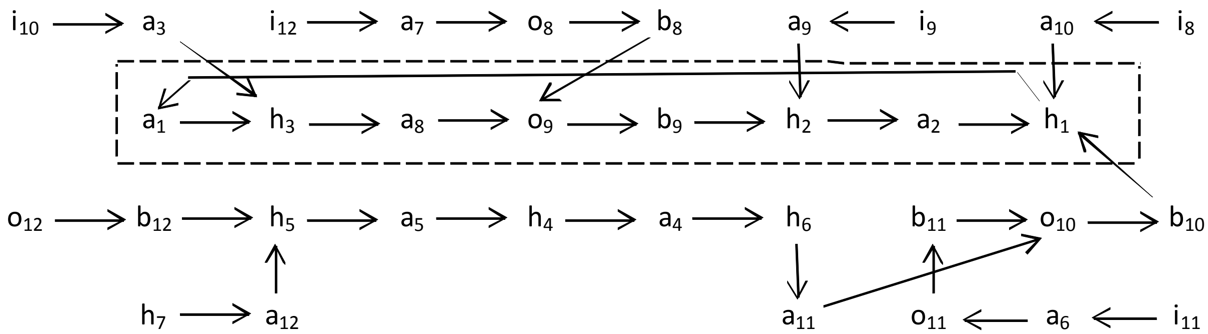
a-type agents												*b-type agents*				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	b_8	b_9	b_{10}	b_{11}	b_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3	i_9	i_8	h_6	h_7	o_8	o_9	o_{10}	o_{11}	o_{12}

We illustrate in a series of figures below how IATTC^{ab} proceeds. While looking into the figures, recall that remaining houses and inheritance rights point (to their owners) at both a-steps and b-steps; remaining b-type agents also point (to houses and inheritance rights) at both a-steps and b-steps; but remaining a-type agents point (to houses and inheritance rights) only at a-steps. For visual ease we indicate cycles in the figures in dashed rectangles.



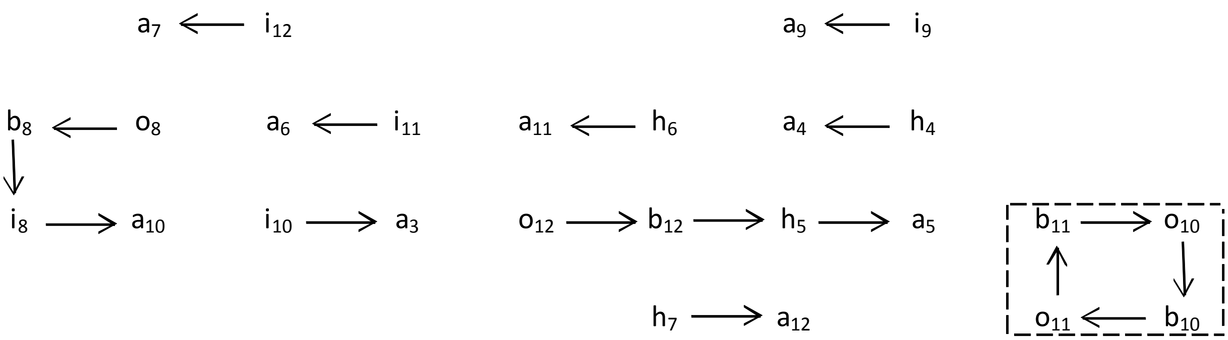
There are no cycles at Step 0,1. IATTC^{ab} proceeds to Step 1,0.

Step 1,0



There is one cycle at Step 1,0. As prescribed by the cycle, a_1, a_8, b_9, a_2 are assigned h_3, o_9, h_2, h_1, respectively. IATTC^{ab} proceeds to Step 1,1.

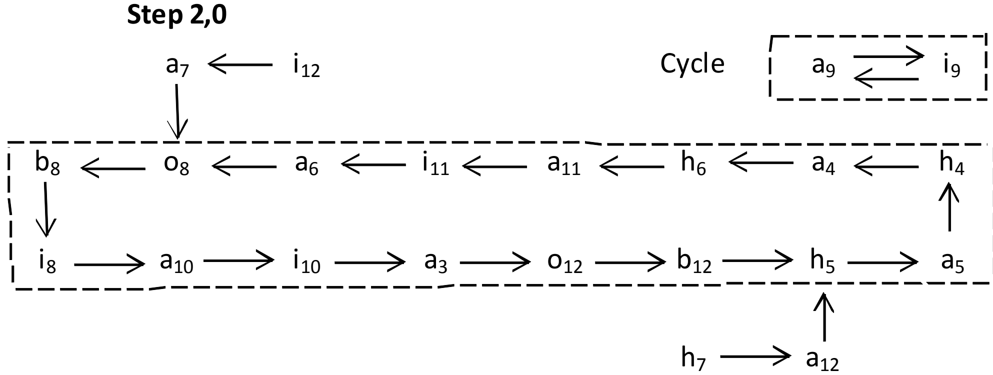
Step 1,1



There is one cycle at Step 1,1. As prescribed by the cycle, b_{10} and b_{11} are assigned o_{11} and o_{10}, respectively. IATTC^{ab} proceeds to Step 1,2.

At Step 1,2 there are two remaining b-type agents, b_8 and b_{12}, who respectively point to i_8 and h_5. The resulting figure would be the same as the preceding figure except that the cycle in *the figure is removed. There are no cycles and thus IATTC^{ab} proceeds to Step 2,0.*

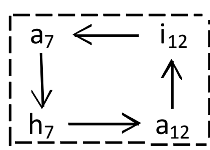
Step 2,0



There are two cycles at Step 2,0. As prescribed by the cycles, $a_3, b_{12}, a_5, a_4, a_{11}, a_6, b_8, a_{10}, a_9$ *are assigned* $o_{12}, h_5, h_4, h_6, i_{11}, o_8, i_8, i_{10}, i_9$, *respectively. IATTC^{ab} proceeds to Step 2,1.*

Since there is no remaining b-type agent, there are no cycles at Step 2,1, and the mechanism proceeds to Step 3,0.

Step 3,0



There is one cycle at Step 3,0. As prescribed by the cycle, a_7 *and* a_{12} *are assigned* h_6 *and* i_{12}, *respectively, and the procedure terminates.*

The houses the procedure assigns to a-type and b-type agents, and the implied assignments to agents made by IATTC^{ab} in the inheritors augmented housing market, are as follows:

Assignments of a-type agents												*Assignments of b-type agents*				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	b_8	b_9	b_{10}	b_{11}	b_{12}
h_3	h_1	o_{12}	h_6	h_4	o_8	h_7	o_9	i_9	i_{10}	i_{11}	i_{12}	i_8	h_2	o_{11}	o_{10}	h_5

$$\Downarrow$$

Assignments of newcomers and existing tenants

a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_3	h_1	o_{12}	h_6	h_4	o_8	h_7	o_9	h_2	o_{11}	o_{10}	h_5

$\diamondsuit$

In what follows we introduce some tools about IATTC$^{\text{ab}}$, and based upon these tools we make certain observations.

‡ **TOOL 1, sets defined by the order of cycle groups:** In the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$ ($v \in \mathcal{V}$), houses and inheritance rights are assigned to a-type and b-type agents in a well-defined order of cycle groups. Based upon this order of cycle groups, we define below certain sets of agents, houses, and inheritance rights:

– $A^v_{t,r}$: the set of a-type and b-type agents that are assigned a house or an inheritance right in a cycle at Step t,r.

– $A^v_0 = A^v_{0,1} \cup A^v_{0,2} \cup \cdots$ and $A^v_t = A^v_{t,0} \cup A^v_{t,1} \cup \cdots$ for $t \geq 1$.

– $H^v_{t,r}$: the set of houses assigned to agents in $A^v_{t,r}$.

– $H^v_0 = H^v_{0,1} \cup H^v_{0,2} \cup \cdots$ and $H^v_t = H^v_{t,0} \cup H^v_{t,1} \cup \cdots$ for $t \geq 1$.

– $I^v_{t,0}$: the set of inheritance rights assigned to agents in $A^v_{t,0}$ for $t \geq 1$. (Recall from Observation 2 (i) that in the cycles at b-steps there are no inheritance rights.)

‡ **TOOL 2, a-blocks at an a-step:** In the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$ ($v \in \mathcal{V}$), we define an "a-block" at an a-step Step t,0 ($t \geq 1$) as an ordered list $bl^v_t(a) : (a, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, y)$ (or $bl^v_t(a) : (a, y)$) where

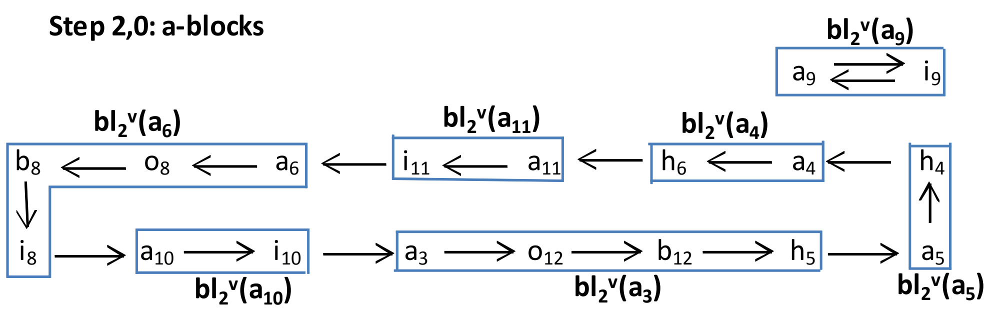
– $a \in A^v_{t,0} \cap (A_N \cup A^a_E)$; $y \in (H^v_{t,0} \cap H_V) \cup I^v_{t,0}$;

– $k+1 \leq \pi_p \leq n$ for $p = 1, \cdots, q$;

– at Step t,0 a points to o_{π_1}, b_{π_1} points to o_{π_2}, $\cdots$, and b_{π_q} points to y (if $bl^v_t(a) : (a, y)$ then simply a points to y).

We call a the "source" and y the "sink" of the a-block $bl^v_t(a)$. With some abuse of notation, we denote the set $\{a, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, y\}$ (or $\{a, y\}$) also by $bl^v_t(a)$.

More simply, an a-block is a segment of a cycle that arises at an a-step in the execution of IATTC^{ab}. It starts with the only a-type agent of that a-block, and ends with a vacant house or an inheritance right owned by another a-type agent. At an a-step sinks of a-blocks (vacant houses and inheritance rights) point to the sources of a-blocks (a-type agents), and hence the cycles form. As an illustration, in the figure below we indicate in enclosed boxes the a-blocks at Step 2,0 in Example 2.3.



The following observation summarizes our preceding discussion on a-blocks.

† **Observation 3:** *In the execution of IATTC^{ab} in $\Pi^{v,ab}$ ($v \in V$), the cycles that arise at an a-step Step t,0 ($t \geq 1$) consist of a-blocks. The sinks of a-blocks point to the sources of a-blocks, and hence the cycles form. So,*

(i) $\bigcup_{a \in A^v_{t,0} \cap (A_N \cup A^a_E)} bl^v_t(a) = A^v_{t,0} \cup H^v_{t,0} \cup I^v_{t,0}$;

(ii) and for $a, a' \in A^v_{t,0} \cap (A_N \cup A^a_E)$ *and* $a \neq a'$, $bl^v_t(a) \cap bl^v_t(a') = \emptyset$.

The following is another simple observation pertaining to a-blocks, which later proves useful.

† **Observation 4:** *Suppose we are given the list of sets* $\left(A^v_{j,0} \cap (A_N \cup A^a_E)\right)^t_{j=1}$ *but we do not know* $v \in V$. *(That is, we are given the sets of a-type agents that are assigned houses at Step 1,0, Step 2,0,* $\cdots$*, Step t,0 when IATTC* ^{ab} *is executed in* $\Pi^{v,ab}$*.) Then, we can determine*

(i) to which house or inheritance right a remaining agent points to up to Step t+1,0;

(ii) the assignments made by IATTC ^{ab} *up to Step t+1,0;*

and we can identify

(iii) the a-block $bl^v_j(a)$ *for any* $a \in A^v_{j,0} \cap (A_N \cup A^a_E)$ *and* $j \in \{1, \cdots, t\}$.

Explanation: *The execution of IATTC* ^{ab} *in* $\Pi^{v,ab}$ *at Step 0,1, Step 0,2, and so on, is independent of* v. *Then, to which house or inheritance right a remaining agent points to, and the assignments made, can be determined up to Step 1,0. For the subsequent steps, we can iteratively apply the following arguments for j=1,*$\cdots$*,t:*

Given the assignments made prior to Step j,0, we know to which house or inheritance right a remaining agent points to at Step j,0. So, for any $a \in A^v_{j,0} \cap (A_N \cup A^a_E)$*, we can identify* $bl^v_j(a)$*. But then we can also determine the assignments made at Step j,0: Each agent in an a-block is assigned the house she points to in that a-block.*

Given the assignments made at Step j,0 and prior to it, the execution of IATTC ^{ab} *in* $\Pi^{v,ab}$ *at Step j,1, Step j,2, and so on, is independent of* v. *Then, to which house or inheritance right a remaining agent points to, and the assignments made, can be determined for Step j,1, Step j,2, and so on.*

‡ **TOOL 3, chains at an a–step:** Consider the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$ ($v \in V$). The elements of $A^v_{t+1,0} \cup H^v_{t+1,0} \cup I^v_{t+1,0}$ ($t \geq 1$), which form the cycle(s) at Step t+1,0, form at Step t,0 what we call "chains." Formally, a chain at Step t,0 is an ordered list $ch^v_t(x_1) : (x_1, y_1, \cdots, x_q, y_q)$ ($t \geq 1$, $q \geq 1$) where

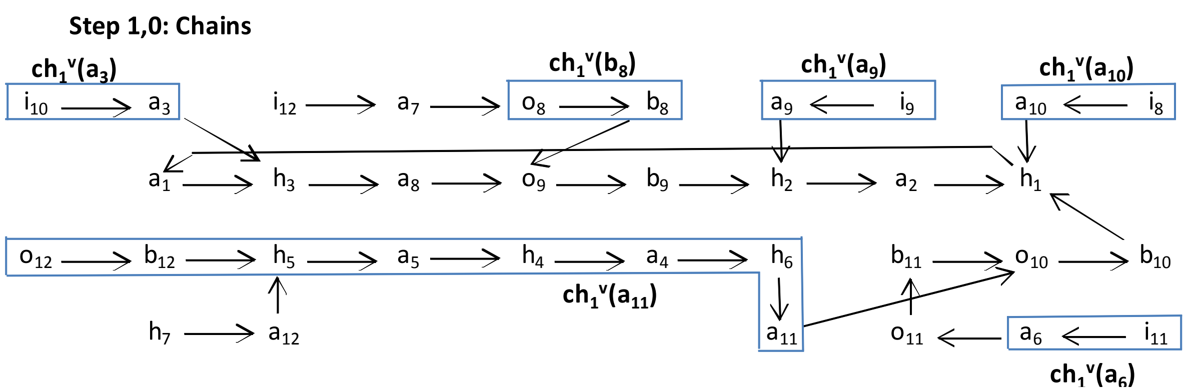
– $x_p \in A^v_{t+1,0}$ and $y_p \in H^v_{t+1,0} \cup I^v_{t+1,0}$ for $p = 1, \cdots, q$;

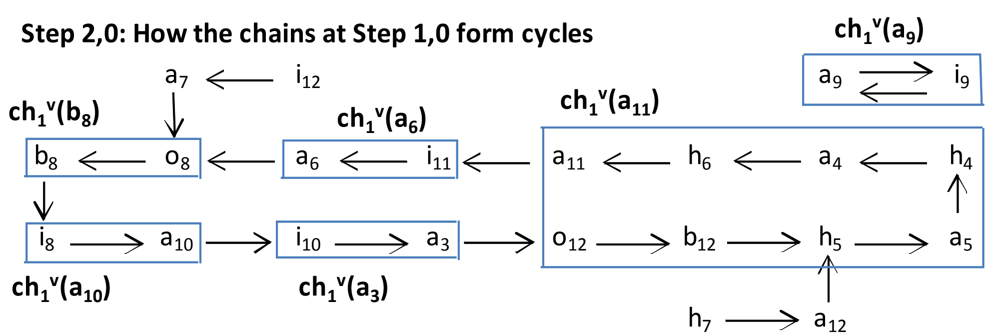
– at Step t,0 x_1 points to a house or an inheritance right in $H_t^v \cup I_{t,0}^v$; y_1 points to x_1 (i.e., x_1 owns y_1); x_2 points to y_1;$\cdots$; y_q points to x_q (i.e., x_q owns y_q);

– there exists no $x_{q+1} \in A_{t+1,0}^v$ who points to y_q at Step t,0.

We call x_1 the "head" of $ch_t^v(x_1)$ and y_q the "tail" of $ch_t^v(x_1)$. With some abuse of notation, we denote the set $\{x_1, y_1, \cdots, x_q, y_q\}$ also by $ch_t^v(x_1)$.

More simply, a chain at Step t,0 is a connected (by pointers) elements of $A_{t+1,0}^v \cup H_{t+1,0}^v \cup I_{t+1,0}^v$. At Step t+1,0, the heads of chains at Step t,0 point to the tails of chains at Step t,0, hence the cycles form. As an illustration, in the figures below we indicate in enclosed boxes the chains at Step 1,0 in Example 2.3, and how they form the cycles at Step 2,0.





Notice that at Step t,0 ($t \geq 1$) the head of a chain, which by definition points to a house or an inheritance right, indeed always points to a house: By construction of IATTC^{ab}, an inheritance right i_s is pointed by only one agent—a_s or b_s, whoever is assigned later. But if the head of a chain at Step t,0 points to an inheritance right, it means she is not assigned that inheritance right, which would be a contradiction. (As an illustration, notice that in the second preceding figure all heads of chains point to houses.)

The following observation summarizes our preceding discussion on chains.

† **Observation 5:** *In the execution of IATTC^{ab} in $\Pi^{v,ab}$ ($v \in V$), let X be the set of heads of chains at Step t,0 ($t \geq 1$). Then,*

(i) a head of a chain $x \in X$ points to a house in $H^v_{t,0}$ at Step t,0, and to a house or an inheritance right in $H^v_{t+1,0} \cup I^v_{t+1,0}$ at Step t+1,0;

(ii) an agent $a \in A^v_{t+1,0} \,/\, X$ points to the same house or inheritance right in $H^v_{t+1,0} \cup I^v_{t+1,0}$ at Step t,0 and Step t+1,0.

For the chains at Step t,0, at Step t+1,0 their heads point to their tails, and hence the cycles at Step t+1,0 form. Then,

(iii) $\bigcup\limits_{x \in X} ch^v_t(x) = A^v_{t+1,0} \cup H^v_{t+1,0} \cup I^v_{t+1,0}$*;*

(iv) and for $x, x' \in X$ *and* $x \neq x'$, $ch^v_t(x) \cap ch^v_t(a') = \emptyset$.

From the head to the tail of a chain, we call the a-type agent ordered first the "a-head" of the chain, and the a-type agent ordered last the "a-tail" of the chain. For instance, looking into the second preceding figure above, the a-head and a-tail of $ch^v_1(a_{11})$ are respectively a_{11} and a_5. Looking into that figure, also note that in a chain (i) the head and a-head can be the same (e.g., $ch^v_1(a_{11})$); (ii) there may be only one a-type agent and so its a-head and a-tail can be the same (e.g., $ch^v_1(a_3)$); (iii) there may be no a-type agents, in which case we call it an "empty chain" (e.g., $ch^v_1(b_8)$).

‡ **TOOL 4, the chain-order o^v_{ch} of a-type agents**

For $v \in \mathcal{V}$, the "chain-order" $o^v_{ch} : A_N \cup A^a_E \to \{1, 2, \cdots, n\}$ of a-type agents is a bijection that orders a-type agents according to the following three rules:

Chain-order Rule 1: *In the chain-order* o^v_{ch}, *order a-type agents in* $A^v_{1,0} \cap (A_N \cup A^a_E)$ *before a-type agents in* $A^v_{2,0} \cap (A_N \cup A^a_E)$*; order a-type agents in* $A^v_{2,0} \cap (A_N \cup A^a_E)$ *before a-type agents in* $A^v_{3,0} \cap (A_N \cup A^a_E)$*; and so on.*

Chain-order Rule 2: *In the chain-order* o^v_{ch}, *order a-type agents in* $A^v_{1,0} \cap (A_N \cup A^a_E)$ *in order of the indices of vacant houses and inheritance rights that they are assigned at* v.

Chain-order Rule 3: *In the chain-order* o^v_{ch}, *order the a-type agents in* $A^v_{t+1,0} \cap (A_N \cup A^a_E))$ $(t \geq 1)$ *in the following manner: Consider the chains at Step t,0. Order the a-type agents in a non-empty chain from its a-head to its a-tail, consecutively. Order non-empty chains in order of the indices of vacant houses and inheritance rights that their a-tails are assigned at* v.

As an illustration, consider the execution of IATTC$^{\text{ab}}$ in the ab-representation of the inheritors augmented housing market in Example 2.3:

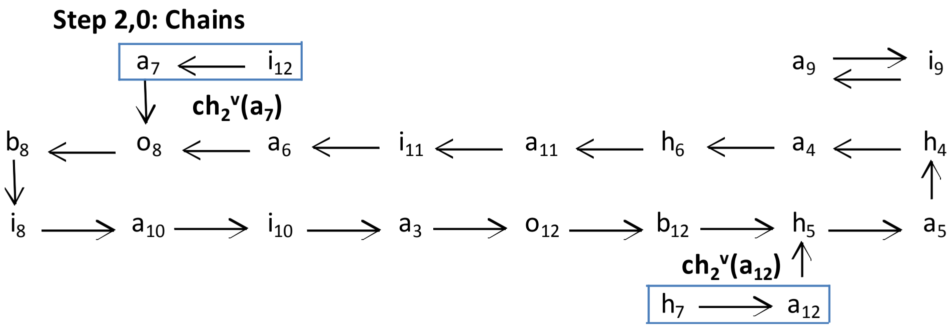
By Chain-order Rule 1, in o^v_{ch} the a-type agents assigned at Step 1,0 (i.e., a_1, a_2, a_8) are ordered before the a-type agents assigned at Step 2,0 (i.e., $a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6$), who are ordered before the a-type agents assigned at Step 3,0 (i.e., a_{12}, a_7).

By Chain-order Rule 2, in o^v_{ch} the three a-type agents assigned at Step 1,0 are ordered as a_1, a_2, a_8. (Note that a_1, a_2, a_8 own respectively h_1, h_2, h_3, whose indices are respectively $1, 2, 3$.)

By Chain-order Rule 3 and looking into the second preceding figure above, the order of non-empty chains at Step 1,0 is $ch^v_1(a_{11}), ch^v_1(a_{10}), ch^v_1(a_9), ch^v_1(a_3), ch^v_1(a_6)$ (a-tails of these chains own respectively $h_5, i_8, i_9, i_{10}, i_{11}$; indices are respectively $5, 8, 9, 10, 11$), and hence the chain-order of a-type agents in $A^v_{2,0}$ is $a_{11}, a_4, a_5, a_{10},\ a_9, a_3, a_6$.

The figure below shows the chains at Step 2,0 in Example 2.3, formed by the elements of

$A^v_{3,0} \cup H^v_{3,0} \cup I^v_{3,0}$.



By Chain-order Rule 3 and looking into the preceding figure, the order of non-empty chains at Step 2,0 is $ch^v_2(a_{12}), ch^v_2(a_7)$ (a-tails of these chains respectively own h_7, i_{12}; indices are respectively 7 and 12), and hence the chain-order of a-type agents in $A^v_{3,0}$ is a_{12}, a_7.

Therefore, the chain-order of a-type agents that we obtain in Example 2.3 is:

$$o^v_{ch}: \quad \underbrace{a_1, a_2, a_8}_{A^v_{1,0} \cap (A_N \cup A^a_E)} \qquad \underbrace{a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6}_{A^v_{2,0} \cap (A_N \cup A^a_E)} \qquad \underbrace{a_{12}, a_7}_{A^v_{3,0} \cap (A_N \cup A^a_E)}$$

‡ **TOOL 5, the chain priority-order f^v_{ch} of newcomers and existing tenants:** For $v \in \mathcal{V}$, from the chain-order o^v_{ch} of a-type agents, we derive the "chain priority-order" $f^v_{ch} \in \mathcal{F}$ of newcomers and existing tenants in a straightforward way. Simply set $f^v_{ch}(a) = o^v_{ch}(a)$ for $a \in A_N$, and $f^v_{ch}(e_s) = o^v_{ch}(a_s)$ for $e_s \in A_E$.

For instance, the chain order given above, and the chain priority-order derived from it, are as follows:

$$o^v_{ch} \;:\; a_1, a_2, a_8, a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6, a_{12}, a_7$$

$$f^v_{ch} \;:\; a_1, a_2, e_8, e_{11}, a_4, a_5, e_{10}, e_9, a_3, a_6, e_{12}, a_7$$

Observe that f^v_{ch} is precisely the same priority-order as the one considered in Example 2.1. Also, recall that the allocations chosen in Example 2.3 by IATTC^{ab}, and in Example 2.1 by the

YRMH-IGYT mechanism defined by f^v_{ch}, are the same. Lemma 2.2 states that this holds in general.

Lemma 2.2 *For any $v \in \mathcal{V}$,*

$$\mu^{ab,v} = \mu^{Y-I,f^v_{ch}}.$$

Proof. It is plain to see that the lemma holds once the following observation is made. Considering the execution of the YRMH-IGYT mechanism defined by f^v_{ch} in Π, and the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$, a loop in the former one corresponds to a cycle at a b-step in the latter one, and an out-of-loop assignment made in the former one corresponds to an a-block at an a-step in the latter one. We elaborate below.

In the execution of the YRMH-IGYT mechanism defined by f^v_{ch} in Π, let it be the turn of agent $a \in A$ to request a house. Also, in the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$, let Step t,0 be when the a-type agent that represents a is assigned a house or an inheritance right. When a requests a house, one of the following five cases occurs:

(1) Agent a requests a house that triggers the formation of one or more loops (if a is an existing tenant, she may also be part of one of these loops). In the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$, these loops correspond to certain cycles that arise at b-steps prior to Step t,0: The agents in the cycles are the b-type agents that represent the existing tenants in the loop, and they are assigned the same houses at $\mu^{ab,v}$ and $\mu^{Y-I,f^v_{ch}}$.

(2) Agent a requests a vacant house $h \in H_V$. In the execution of IATTC$^{\text{ab}}$ in $\Pi^{v,ab}$, this case corresponds to the a-block $bl^v_t(a) : (a, h)$ at Step t,0. Agent a is assigned h at both $\mu^{ab,v}$ and $\mu^{Y-I,f^v_{ch}}$.

(3) Agent a requests the occupied house o_s of $e_s \in A_E$, who has already been assigned a house before. In the execution of IATTC$^{\text{ab}}$, this case corresponds to the a-block $bl^v_t(a) : (a, o_s, b_s, i_s)$ at Step t,0. Agent a is assigned o_s at both $\mu^{ab,v}$ and $\mu^{Y-I,f^v_{ch}}$.

(4) Agent a requests the occupied house o_{π_1} of $e_{\pi_1} \in A_E$; e_{π_1} moves to the top of the remainder of the priority-order and requests the occupied house o_{π_2} of $e_{\pi_2} \in A_E$; $\cdots$; e_{π_q} moves to the top of the remainder of the priority-order and requests a vacant house $h \in H_V$. In the execution of

IATTC$^{\mathrm{ab}}$ in $\Pi^{v,ab}$, this case corresponds to the a-block $bl_t^v(a) : (a, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, h)$ at Step t,0. Agents $a, e_{\pi_1}, \cdots, e_{\pi_q}$ are assigned the houses $o_{\pi_1}, \cdots, o_{\pi_q}, h$, respectively, at both $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v}.

(5) The same thing happens as in (4) except that e_{π_q} requests the occupied house o_s of $e_s \in A_E$, who has already been assigned a house before. In the execution of IATTC$^{\mathrm{ab}}$ in $\Pi^{v,ab}$, this case corresponds to the a-block $bl_t^v(a) : (a, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, o_s, b_s, i_s)$ at Step t,0. Agents $a, e_{\pi_1}, \cdots, e_{\pi_q}$ are assigned the occupied houses $o_{\pi_1}, \cdots, o_{\pi_q}, o_s$, respectively, at both $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v}. ■

The following lemma states that if the executions of IATTC$^{\mathrm{ab}}$ in two inheritors augmented housing markets induce the same chain priority-order, then a-type agents join cycles at the same a-steps for the two inheritors augmented housing markets.

Lemma 2.3 *For $v_1, v_2 \in \mathcal{V}$ if $f_{ch}^{v_1} = f_{ch}^{v_2}$, then $A_{t,0}^{v_1} \cap (A_N \cup A_E^a) = A_{t,0}^{v_2} \cap (A_N \cup A_E^a)$ for every $t \geq 1$.*

Proof. If $f_{ch}^{v_1} = f_{ch}^{v_2}$, then $\mu^{ab,v_1} = \mu^{ab,v_2}$ (by Lemma 2.1 and Lemma 2.2). Also, $o_{ch}^{v_1} = o_{ch}^{v_2}$ (by definition). Let $\delta : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ be the bijection such that $o_{ch}^{v_1}(a_{\delta(s)}) = o_{ch}^{v_2}(a_{\delta(s)}) = s$ for $s = 1, 2, \cdots, n$. The proof is by induction.

<u>Base Case:</u>

Suppose $A_{1,0}^{v_1} \cap (A_N \cup A_E^a) \neq A_{1,0}^{v_2} \cap (A_N \cup A_E^a)$. W.l.o.g. let

$$\begin{aligned} A_{1,0}^{v_1} \cap (A_N \cup A_E^a) &= \{a_{\delta(1)}, a_{\delta(2)}, \cdots, a_{\delta(\alpha)}\}, \\ A_{1,0}^{v_2} \cap (A_N \cup A_E^a) &= \{a_{\delta(1)}, a_{\delta(2)}, \cdots, a_{\delta(\alpha)}, a_{\delta(\alpha+1)}, \cdots, a_{\delta(\beta)}\} \text{ (i.e., } \beta > \alpha \geq 1). \end{aligned}$$

We present our arguments in four steps:

(1) Show that $bl_1^{v_1}(a_{\delta(s)}) = bl_1^{v_2}(a_{\delta(s)})$ for $s = 1, 2, \cdots, \alpha$:

The execution of IATTC$^{\mathrm{ab}}$ prior to Step 1,0 is independent of v_1 and v_2 (i.e., the same assignments are made prior to Step 1,0). Then, for Π^{v_1} and Π^{v_2} the same a-type agents, b-type agents, houses, and inheritance rights remain at Step 1,0, and remaining a-type and b-type agents point to the same houses and inheritance rights. So, $bl_1^{v_1}(a_{\delta(s)}) = bl_1^{v_2}(a_{\delta(s)})$ for $s = 1, 2, \cdots, \alpha$.

Let $bl_1^{v_2}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, y)$ where $k+1 \leq \pi_p \leq n$ for $p = 1, \cdots, q$ and $y \in H_V \cup I$. (The arguments are essentially the same if $bl_1^{v_2}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, y)$).

(2) Show that $y \notin H_1^{v_1} \cup I_{1,0}^{v_1}$:

Since $y \in bl_1^{v_2}(a_{\delta(\alpha+1)})$, we get $y \notin bl_1^{v_2}(a_{\delta(s)})$ for $s = 1, \cdots, \alpha$ (by Observation 3 (ii)), and hence $y \notin bl_1^{v_1}(a_{\delta(s)})$ for $s = 1, \cdots, \alpha$. Then, $y \notin H_{1,0}^{v_1} \cup I_{1,0}^{v_1}$ (see Observation 3 (i)). Since $y \in H_V \cup I$, we get $y \notin H_{1,r}^{v_1}$ also for $r \geq 1$ (by Observation 2 (i)). Then, $y \notin H_1^{v_1} \cup I_{1,0}^{v_1}$.

(3) Show that $bl_2^{v_1}(a_{\delta(\alpha+1)}) = bl_1^{v_2}(a_{\delta(\alpha+1)})$:

From $bl_1^{v_2}(a_{\delta(\alpha+1)})$ we know that for Π^{v_2} at Step 1,0 $a_{\delta(\alpha+1)}$ points to o_{π_1}; o_{π_1} points to b_{π_1}; $\cdots$; and b_{π_q} points to y. Since the execution of IATTC$^{\text{ab}}$ prior to Step 1,0 is independent of v_1 and v_2, also for Π^{v_1} at Step 1,0 $a_{\delta(\alpha+1)}$ points to o_{π_1}; o_{π_1} points to b_{π_1}; $\cdots$; and b_{π_q} points to y. Since $y \notin H_1^{v_1} \cup I_{1,0}^{v_1}$, for Π^{v_1} this sequence remains unaffected until Step 2,0. Then, $bl_2^{v_1}(a_{\delta(\alpha+1)}) = bl_1^{v_2}(a_{\delta(\alpha+1)})$.

(4) Find a contradiction:

Since in $A_{2,0}^{v_1} \cap (A_N \cup A_E^a)$ the a-type agent who comes first in $o_{ch}^{v_1}$ is $a_{\delta(\alpha+1)}$, at Step 1,0 $a_{\delta(\alpha+1)}$ should be the a-head of a non-empty chain. In this chain either $a_{\delta(\alpha+1)}$ is the head, or a b-type agent in $bl_2^{v_1}(a_{\delta(\alpha+1)})$ is the head, say b_{π_j} for some $j \in \{1, 2, \cdots, q\}$.

If $a_{\delta(\alpha+1)}$ is the head of the chain, then at μ^{ab,v_1} she is not assigned her most preferred house in $H \setminus H_0^{v_1}$. But for Π^{v_2} since $a_{\delta(\alpha+1)}$ is assigned at Step 1,0, at μ^{ab,v_2} she is assigned her most preferred house in $H \setminus H_0^{v_2}$ $(= H \setminus H_0^{v_1})$, which contradicts that $\mu^{ab,v_1} = \mu^{ab,v_2}$.

If b_{π_j} is the head of the chain, then from

$$bl_1^{v_2}(a_{\delta(\alpha+1)}) : (a_{\pi(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, y),$$

in the execution of IATTC$^{\text{ab}}$ in Π^{v_2}, b_{π_j} points at Step 1,0 to $o_{\pi_{j+1}}$ (or y). From the fact that in the execution of IATTC$^{\text{ab}}$ in Π^{v_1} at Step 1,0 b_{π_j} is the head of a chain, by Observation 5 (i) she points at Step 1,0 to a house not in $bl_2^{v_1}(a_{\pi(\alpha+1)})$ $(= bl_1^{v_2}(a_{\delta(\alpha+1)}))$, which contradicts that for Π^{v_1} and Π^{v_2} at Step 1,0 remaining agents point to same houses an inheritance rights.

Inductive Step: (The arguments are exactly parallel to the base case. For the sake of completeness, we reproduce them below, where changes have been made as necessary).

Suppose $A^{v_1}_{j,0} \cap (A_N \cup A^a_E) = A^{v_2}_{j,0} \cap (A_N \cup A^a_E)$ for $j = 1, \cdots, t$ but $A^{v_1}_{t+1,0} \cap (A_N \cup A^a_E) \neq A^{v_2}_{t+1,0} \cap (A_N \cup A^a_E)$. W.l.o.g. let

$$\begin{aligned} A^{v_1}_{t+1,0} \cap (A_N \cup A^a_E) &= \{a_{\delta(l)}, a_{\delta(l+1)}, \cdots, a_{\delta(\alpha)}\}, \\ A^{v_2}_{t+1,0} \cap (A_N \cup A^a_E) &= \{a_{\delta(l)}, a_{\delta(l+1)}, \cdots, a_{\delta(\alpha)}, a_{\delta(\alpha+1)}, \cdots, a_{\delta(\beta)}\} \text{ (i.e., } \beta > \alpha \geq l). \end{aligned}$$

We present our arguments in four steps:

(1) Show that $bl^{v_1}_{t+1}(a_{\delta(s)}) = bl^{v_2}_{t+1}(a_{\delta(s)})$ for $s = l, l+1, \cdots, \alpha$:

By Observation 4, for Π^{v_1} and Π^{v_2} the same assignments are made by IATTC$^{\text{ab}}$ prior to Step t+1,0; the same a-type agents, b-type agents, houses, and inheritance rights remain at Step t+1,0; remaining a-type and b-type agents point to the same houses and inheritance rights; and $bl^{v_1}_{t+1}(a_{\delta(s)}) = bl^{v_2}_{t+1}(a_{\delta(s)})$ for $s = l, l+1, \cdots, \alpha$.

Let $bl^{v_2}_{t+1}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, y)$ where $k+1 \leq \pi_p \leq n$ for $p = 1, \cdots, q$ and $y \in H_V \cup I$. (The arguments are essentially the same if $bl^{v_2}_{t+1}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, y)$).

(2) Show that $y \notin H^{v_1}_{t+1} \cup I^{v_1}_{t+1,0}$:

Since $y \in bl^{v_2}_{t+1}(a_{\delta(\alpha+1)})$, we get $y \notin bl^{v_2}_{t+1}(a_{\delta(s)})$ for $s = l, \cdots, \alpha$ (by Observation 3 (ii)), and hence $y \notin bl^{v_1}_{t+1}(a_{\delta(s)})$ for $s = l, \cdots, \alpha$. Then, $y \notin H^{v_1}_{t+1,0} \cup I^{v_1}_{t+1,0}$ (see Observation 3 (i)). Since $y \in H_V \cup I$, we get $y \notin H^{v_1}_{t+1,r}$ also for $r \geq 1$ (by Observation 2 (i)). Then, $y \notin H^{v_1}_{t+1} \cup I^{v_1}_{t+1,0}$.

(3) Show that $bl^{v_1}_{t+2}(a_{\delta(\alpha+1)}) = bl^{v_2}_{t+1}(a_{\delta(\alpha+1)})$:

From $bl^{v_2}_{t+1}(a_{\delta(\alpha+1)})$ we know that for Π^{v_2} at Step t+1,0 $a_{\delta(\alpha+1)}$ points to o_{π_1}; o_{π_1} points to b_{π_1}; $\cdots$; and b_{π_q} points to y. Given that $A^{v_1}_{j,0} \cap (A_N \cup A^a_E) = A^{v_2}_{j,0} \cap (A_N \cup A^a_E)$ for $j = 1, \cdots, t$ the execution of IATTC$^{\text{ab}}$ prior to Step t+1,0 is the same for Π^{v_1} and Π^{v_2} (see Observation 4). Then, also for Π^{v_1} at Step t+1,0 $a_{\delta(\alpha+1)}$ points to o_{π_1}; o_{π_1} points to b_{π_1}; $\cdots$; and b_{π_q} points to y. Since $y \notin H^{v_1}_{t+1} \cup I^{v_1}_{t+1,0}$, for Π^{v_1} this sequence remains unaffected until Step t+2,0. Then, $bl^{v_1}_{t+2}(a_{\delta(\alpha+1)}) = bl^{v_2}_{t+1}(a_{\delta(\alpha+1)})$.

(4) Find a contradiction:

Since in $A^{v_1}_{t+2,0} \cap (A_N \cup A^a_E)$ the a-type agent who comes first in $o^{v_1}_{ch}$ is $a_{\delta(\alpha+1)}$, at Step t+1,0 $a_{\delta(\alpha+1)}$ should be the a-head of a non-empty chain. In this chain either $a_{\delta(\alpha+1)}$ is the head, or a b-type agent in $bl^{v_1}_{t+2}(a_{\delta(\alpha+1)})$ is the head, say b_{π_j} for some $j \in \{1, 2, \cdots, q\}$.

If $a_{\delta(\alpha+1)}$ is the head of the chain, then at μ^{ab,v_1} she is not assigned her most preferred house in $H \setminus (H^{v_1}_0 \cup H^{v_1}_1 \cup \cdots \cup H^{v_1}_t)$. But for Π^{v_2} since $a_{\delta(\alpha+1)}$ is assigned at Step t+1,0, at μ^{ab,v_2} she is assigned her most preferred house in $H \setminus (H^{v_2}_0 \cup H^{v_2}_1 \cup \cdots \cup H^{v_2}_t)$ $(= H \setminus (H^{v_1}_0 \cup H^{v_1}_1 \cup \cdots \cup H^{v_1}_t))$, which contradicts that $\mu^{ab,v_1} = \mu^{ab,v_2}$.

If b_{π_j} is the head of the chain, then from

$$bl^{v_2}_{t+1}(a_{\delta(\alpha+1)}) : (a_{\pi(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \cdots, o_{\pi_q}, b_{\pi_q}, y),$$

in the execution of IATTC$^{\text{ab}}$ in Π^{v_2}, b_{π_j} points at Step t+1,0 to $o_{\pi_{j+1}}$ (or y). From the fact that in the execution of IATTC$^{\text{ab}}$ in Π^{v_1} at Step t+1,0 b_{π_j} is the head of a chain, by Observation 5 (i) she points at Step t+1,0 to a house not in $bl^{v_1}_{t+2}(a_{\pi(\alpha+1)})$ $(= bl^{v_2}_{t+1}(a_{\delta(\alpha+1)}))$, which contradicts that for Π^{v_1} and Π^{v_2} at Step t+1,0 remaining agents point to same houses an inheritance rights.

■

Lemma 2.4 *If $f^{v_1}_{ch} = f^{v_2}_{ch}$ for $v_1, v_2 \in \mathcal{V}$, then $v_1 = v_2$.*

Proof. Let $f^{v_1}_{ch}$ $(= f^{v_2}_{ch})$ be given but not v_1. By Lemma 2.3 $f^{v_1}_{ch}$ uniquely identifies the sets $A^{v_1}_{t,0} \cap (A_N \cup A^a_E)$ $(= A^{v_2}_{t,0} \cap (A_N \cup A^a_E))$ for $t = 1, 2, \cdots$ (i.e., to identify them we do not need v_1 and v_2). Also, from $f^{v_1}_{ch}$ we can derive $o^{v_1}_{ch}$ $(= o^{v_2}_{ch})$ (by definition). We will show that, from $A^{v_1}_{t,0} \cap (A_N \cup A^a_E)$ for $t = 1, 2, \cdots$ and $o^{v_1}_{ch}$, we can uniquely identify v_1 (and hence also v_2), which proves the lemma.

Let $\delta : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ be the bijection such that $o^{v_1}_{ch}(a_{\delta(s)}) = s$ for $s = 1, 2, \cdots, n$. By Observation 4 (ii), from the sets $A^{v_1}_{t,0} \cap (A_N \cup A^a_E)$ for $t = 1, 2, \cdots$ we can determine the a-blocks at every a-step. The proof is in two parts.

(I) Let the a-blocks at Step 1,0 be

$$\begin{aligned}
bl_1^{v_1}(a_{\delta(1)}) &: (a_{\delta(1)}, \cdots, y^1), \\
bl_1^{v_1}(a_{\delta(2)}) &: (a_{\delta(2)}, \cdots, y^2), \\
&\vdots \\
bl_1^{v_1}(a_{\delta(\alpha)}) &: (a_{\delta(\alpha)}, \cdots, y^{\alpha}).
\end{aligned}$$

By Chain-order Rule 2, at v_1 the a-type agents $a_{\delta(1)}, \cdots, a_{\delta(\alpha)}$ are assigned the vacant houses and inheritance rights $y^1, \cdots, y^{\alpha}$ in order of the indices of vacant houses and inheritance rights. Since this is well-defined, we can uniquely identify how $y^1, \cdots, y^{\alpha}$ are assigned to $a_{\delta(1)}, \cdots, a_{\delta(\alpha)}$ at v_1.

(II) For $t \geq 1$ let the a-blocks at Step t+1,0 be

$$\begin{aligned}
bl_{t+1}^{v_1}(a_{\delta(m_0)}) &: (a_{\delta(m_0)}, \cdots, y^{m_0}), \\
bl_{t+1}^{v_1}(a_{\delta(m_0+1)}) &: (a_{\delta(m_0+1)}, \cdots, y^{m_0+1}), \\
&\vdots \\
bl_{t+1}^{v_1}(a_{\delta(m_1-1)}) &: (a_{\delta(m_1-1)}, \cdots, y^{m_1-1}) \\
bl_{t+1}^{v_1}(a_{\delta(m_1)}) &: (a_{\delta(m_1)}, \cdots, y^{m_1}) \\
&\vdots \\
&\vdots \\
bl_{t+1}^{v_1}(a_{\delta(m_q-1)}) &: (a_{\delta(m_q-1)}, \cdots, y^{m_q-1}) \\
bl_{t+1}^{v_1}(a_{\delta(m_q)}) &: (a_{\delta(m_q)}, \cdots, y^{m_q}) \\
&\vdots \\
bl_{t+1}^{v_1}(a_{\delta(\beta)}) &: (a_{\delta(\beta)}), \cdots, y^{\beta}) \\
(i.e.,\ 1 &\leq m_0 < m_1 < \cdots < m_q \leq \beta)
\end{aligned}$$

such that $a_{\delta(m_0)}, \cdots, a_{\delta(m_q)}$ are a-heads of chains at Step t,0. (By Observation 5 (i), we can determine the heads of chains at Step t,0. They are the agents in $A^{v_1}_{t+1,0}$ who point to a house in $H^v_{t,0}$ at Step t,0, and to a house or an inheritance right in $H^v_{t+1,0} \cup I^v_{t+1,0}$ at Step t+1,0. Then we can also determine the a-heads of chains at Step t,0. An a-type agent $a \in A^{v_1}_{t+1,0} \cap (A_N \cup A^a_E)$ is the a-head of a chain at Step t,0 if an agent in $bl^{v_1}_{t+1}(a)$ is the head of a chain at Step t,0.)

Then, the a-type agents $a_{\delta(m_0)}, \cdots, a_{\delta(m_1-1)}$ are in the same chain at Step t,0, and at v_1, by Chain-order Rule 3, y^{m_0+1} is assigned to $a_{\delta(m_0)}$, y^{m_0+2} is assigned to $a_{\delta(m_0+1)})$, $\cdots$, and y^{m_1-1} is assigned to $a_{\delta(m_1-2)}$. Similarly, $a_{\delta(m_1)}, \cdots, a_{\delta(m_2-1)}$ are in the same chain at Step t,0, and at v_1, by Chain-order Rule 3, y^{m_1+1} is assigned to $a_{\delta(m_1)}$, y^{m_1+2} is assigned to $a_{\delta(m_1+1)})$, $\cdots$, and y^{m_2-1} is assigned to $a_{\delta(m_2-2)}$; and so on.

Then, at v_1 the a-tails of chains at Step t,0 (i.e., $a_{\delta(m_1-1)}, a_{\delta(m_2-1)}, \cdots, a_{\delta(m_q-1)}, a_{\delta(\beta)}$) are assigned the remaining vacant houses and inheritance rights (i.e., $y^{m_0}, y^{m_1}, \cdots, y^{m_q}$). By Chain-order Rule 3 these houses and inheritance rights are assigned to $a_{\delta(m_1-1)}, a_{\delta(m_2-1)}, \cdots, a_{\delta(m_q-1)}, a_{\delta(\beta)}$ in order of their indices. Since this is well-defined, we can also uniquely identify how $y^{m_0}, y^{m_1}, \cdots, y^{m_q}$ are assigned to $a_{\delta(m_1-1)}, a_{\delta(m_2-1)}, \cdots, a_{\delta(m_q-1)}, a_{\delta(\beta)}$ at v_1. ■

Given Lemma 2.2 and Lemma 2.4 the proof of Theorem 2.1 is easy.

Proof of Theorem 2.1. Consider the mapping $f_{ch} : \mathcal{V} \to \mathcal{F}$ such that $f_{ch}(v) = f^v_{ch}$ for $v \in \mathcal{V}$. By Lemma 2.4 f_{ch} is an injection. By the facts that f_{ch} is an injection and $|\mathcal{V}| = |\mathcal{F}| = n!$, f_{ch} is a bijection. By Lemma 2.2 and the fact that f_{ch} is a bijection, we get $\lambda^{rY-I} = \lambda^{cfrd}$. ■

2.4 Conclusion

We studied the house allocation problem with existing tenants: n indivisible objects are to be allocated to n agents; k objects are initially unowned; the remaining $n-k$ objects are owned by $n-k$ agents; each agent needs precisely one object; agents' preferences are strict; and monetary transfers are not allowed. There are various real-life applications of this problem, such as the allocation of dormitory rooms to incoming and returning students at a college, and kidney exchange practices that also involve kidneys obtained from Good Samaritan Donors and cadavers.

For this class of problems, Abdulkadiroğlu and Sönmez [2] proposed *randomized You request my house – I get your turn mechanism* (in short, RYRMH-IGYT), a priority-order based lottery mechanism that is strategy-proof, ex-post efficient, and ex-post group rational. This paper proposes a market-based alternative mechanism, *core from random distribution* (in short, CFRD), which can also be shown to be strategy-proof, ex-post efficient, and ex-post group rational. CFRD proceeds in two steps. In the first step, it generates an exchange market by distributing to n agents k vacant houses and $n - k$ inheritance rights (associated with existing tenants) uniformly at random. In the second step, it chooses the unique core allocation of the generated exchange market (see Definition 2.2 and Proposition 2.1) by executing the *inheritors augmented top trading cycles* mechanism (in short, IATTC). In the execution of IATTC an inheritance right helps restore efficiency, by allowing for the unneeded house of the associated existing tenant (in case she owns two houses) to be inherited by another agent.

There are interesting parallels between CFRD in a house allocation problem with existing tenants, and the Walrasian Mechanism from equal-division in a classical exchange economy with infinitely divisible goods. In the latter, there is (physical) equal-division of commonly-owned bundle, followed by the selection of a Walrasian allocation in the induced exchange market. In the former, there is probabilistic equal-division of vacant houses and inheritance rights through random distribution, followed by the selection of the unique core allocation in the induced exchange market, which is also a Walrasian allocation.

Our main result is that RYRMH-IGYT and CFRD are equivalent (Theorem 2.1). Besides being mathematically interesting, this equivalence result increases the appeal of RYRMH-IGYT on normative grounds, by exposing that it shares the same parallels to the Walrasian Mechanism from equal-division as CFRD. In a house allocation problem, RYRMH-IGYT reduces to random-priority, and CFRD reduces to CFRE. Therefore, the seminal equivalence result in the literature by Abdulkadiroğlu and Sönmez [1] is a corollary of our more general equivalence result.

In two recent papers, Pathak and Sethuraman [25] and Carroll [7] show the equivalence of random-priority to certain mechanisms that execute TTC based upon randomly generated "inheritance tables." In CFRD, however, the execution of IATTC is based upon randomly generated "inheritor relationships between agents." This innovation in CFRD promises a new line of research. Future research papers may study how to execute IATTC in the houses-with-quotas case, or when an existing tenant may initially own multiple houses, which may potentially lead to the design of other IATTC based lottery mechanisms that are equivalent to RYRMH-IGYT. The tools that we introduced in Section 2.3 may become useful in these efforts.

Appendix

Proof of Proposition 2.1.

Observe that in the execution of IATTC in Π^v, groups of cycles form in an order: First group of cycles forms—call Group 1, agents in these cycles are assigned to the houses they point and then they are removed from the market, houses that remain from the removed agents are inherited by agents that are still in the market; second group of cycles forms—call Group 2, agents in these cycles are assigned to the houses they point and then they are removed from the market, houses that remain from the removed agents are inherited by the agents that are still in the market; and so on.

Let A and H be partitioned into $\{A_s\}_{s=1}^T$ and $\{H_s\}_{s=1}^T$ according to cycle groups: A_s and H_s are respectively the sets of agents and houses that join a cycle in Group s for $s = 1, \cdots, T$.

Let $\# : A \cup H \to \{1, 2, \cdots, T\}$ be the function such that $\#(x) = s$ if $x \in A_s \cup H_s$. (It specifies to which group of cycles a house or an agent belongs.)

Let $PointedBy : A \to H$ be the function such that for an agent $a \in A$, $PointedBy(a)$ is the house that she trades in the cycle that she joins. Clearly, $\#(a) = \#(PointedBy(a))$ for any $a \in A$.

The proof is in two parts:

<u>(I) For any $\mu \in \mathcal{M}$ if $\mu \neq \mu^{iattc,v}$, then μ is not a core allocation in Π^v:</u>

Suppose $\mu \neq \mu^{iattc,v}$ but μ is a core allocation in Π^v.

If $\mu(a) \neq \mu^{iattc,v}(a)$ for an agent $a \in A_1$, then a finds $\mu(a)$ less preferable than $\mu^{iattc,v}(a)$ (because $\mu^{iattc,v}(a)$ is a's most preferred house in H). Then μ is clearly blocked by the four-tuple $< A_1, H_1, bl, Claim >$ where $bl(a) = \mu^{iattc,v}(a)$ and $Claim(a) = PointedBy(a)$ for every $a \in A_1$. Then we should have $\mu(a) = \mu^{iattc,v}(a)$ for every $a \in A_1$.

Given that $\mu(a) = \mu^{iattc,v}(a)$ for every $a \in A_1$, if $\mu(a) \neq \mu^{iattc,v}(a)$ for an agent $a \in A_2$, then a finds $\mu(a)$ less preferable than $\mu^{iattc,v}(a)$ (because $\mu^{iattc,v}(a)$ is a's most preferred house in $H \setminus H_1$ and $\mu(a) \in H \setminus H_1$). Then μ is clearly blocked by the four-tuple $< A_1 \cup A_2, H_1 \cup H_2, bl, Claim >$ where $bl(a) = \mu^{iattc,v}(a)$ and $Claim(a) = PointedBy(a)$ for every $a \in A_1 \cup A_2$. Then we should have $\mu(a) = \mu^{iattc,v}(a)$ for every $a \in A_1 \cup A_2$.

If we iterate similarly we conclude that $\mu = \mu^{iattc,v}$, which is a contradiction.

<u>(II) $\mu^{iattc,v}$ is a core allocation in Π^v:</u>

Suppose $\mu^{iattc,v}$ is blocked by the four-tuple $\langle C, H^C, bl, Claim \rangle$.

In the execution of IATTC, whenever an agent is assigned a house, that house is her most preferred house among remaining ones. Therefore, for $a \in A$ if $bl(a)\, R_a\, \mu^{iattc,v}(a)$, then $\#(bl(a)) \leq \#(\mu^{iattc,v}(a))$, and if $bl(a)\, P_a\, \mu^{iattc,v}(a)$, then $\#(bl(a)) < \#(\mu^{iattc,v}(a))$. Then,

$$\sum_{h \in H^C} \#(h) < \sum_{a \in C} \#(a). \qquad (\bigstar)$$

By Definition 2.2 (ii), agents and houses in $C \cup H^C$ can be partitioned into groups, where a group consists of a list of agents $a^1, a^2, \cdots, a^m \subseteq C$ and a list of houses $h^1, h^2, \cdots, h^m \subseteq H^C$, and for which one of the following three cases hold.

CASE 1: a^1 is the inheritor of a^2, a^2 is the inheritor of a^3, $\cdots$, a^{m-1} is the inheritor of a^m; a^2 owns h^1, a^3 owns h^2, $\cdots$, a^{m-1} owns h^{m-2}, and a^m owns h^{m-1} and h^m. A graphical representation, in which agents point to their bequeathers and to the houses they own, is as follows:

$$\begin{tikzcd}
a^1 \arrow[r] & a^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & a^{m-1} \arrow[r] \arrow[d] & a^m \arrow[d] \arrow[dr] & \\
& h^1 & & h^{m-2} & h^{m-1} & h^m
\end{tikzcd}$$

Then, in the execution of IATTC in Π^v, for a^m, the house that she trades in the cycle that she joins (i.e., $PointedBy(a^m)$) is h^{m-1} or h^m; for a^{m-1}, it is h^{m-2} or what remains to her from $\{h^{m-1}, h^m\}$; $\cdots$; for a^2, it is h^1 or what remains to her from $\{h^2, \cdots, h^{m+1}\}$; and for a^1, it is either a house that she owns but not in H^C, say h', or the house that remains to her from $\{h^1, \cdots, h^m\}$. If for a^1 the latter holds, we get

$$\bigcup_{s=1}^{m} PointedBy(a^s) \;=\; \{h^1, \cdots, h^m\}$$

$$\text{and so} \sum_{h \in \{h^1, \cdots, h^m\}} \#(h) \;=\; \sum_{a \in \{a^1, \cdots, a^m\}} \#(a).$$

If for a^1 the former holds, then let h'' be the house in $\{h^1, \cdots, h^m\}$ that has not been traded by any agent in $\{a^2, \cdots, a^m\}$. Then,

$$\bigcup_{s=2}^{m} PointedBy(a^s) \;=\; \{h^1, \cdots, h^m\} \,/\, h''$$

$$\text{and so} \sum_{h \in \{h^1, h^2, \cdots, h^m\} \,/\, h''} \#(h) \;=\; \sum_{a \in \{a^2, \cdots, a^m\}} \#(a).$$

Since h'' joins a cycle after every agent in $\{a^1, a^2, \cdots, a^m\}$ joins a cycle, we get $\#(h'') > \#(a^1)$. Then,

$$\sum_{h \in \{h^1, \cdots, h^m\}} \#(h) > \sum_{a \in \{a^1, \cdots, a^m\}} \#(a).$$

In either case, for Case 1, we get

$$\sum_{h \in \{h^1, \cdots, h^{m+1}\}} \#(h) \geq \sum_{a \in \{a^1, \cdots, a^m\}} \#(a).$$

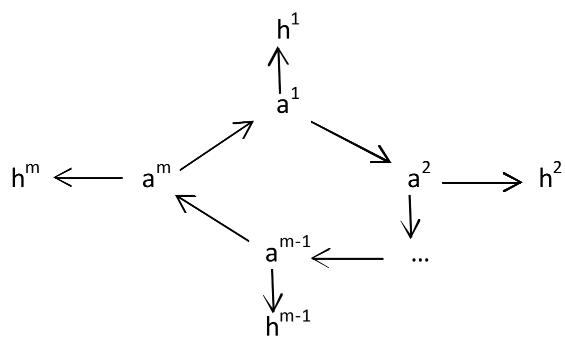
CASE 2: a^1 is the inheritor of a^2, a^2 is the inheritor of a^3, $\cdots$, a^{m-1} is the inheritor of a^m;

a^1 owns h^1, a^2 owns h^2, $\cdots$, a^{m-1} owns h^{m-1}, and a^m owns h^m. A graphical representation, in which agents point to their bequeathers and to the houses they own, is as follows:

$$\begin{tikzcd}
a^1 \arrow[r] \arrow[d] & a^2 \arrow[r] \arrow[d] & \cdots \arrow[r] & a^{m-1} \arrow[r] \arrow[d] & a^m \arrow[d] \\
h^1 & h^2 & & h^{m-1} & h^m
\end{tikzcd}$$

In the execution of IATTC, an agent joins a cycle before or at the same time as a house that she owns. Then, $\#(h^s) \geq \#(a^s)$ for $s = 1, 2, \cdots, m$.

CASE 3: a^1 is the inheritor of a^2, a^2 is the inheritor of a^3, $\cdots$, a^{m-1} is the inheritor of a^m, a^m is the inheritor of a^1; a^1 owns h^1, a^2 owns h^2, $\cdots$, a^m owns h^m. A graphical representation, in which agents point to their bequeathers and to the houses they own, is as follows:



By the same argument as in Case 2, we get $\#(h^s) \geq \#(a^s)$ for $s = 1, 2, \cdots, k$.

From the arguments in Case 1, Case 2, and Case 3, we get,

$$\sum_{h \in H^C} \#(h) \geq \sum_{h \in C} \#(a),$$

which contradicts (★). ■

Chapter 3

House Swapping

3.1 Introduction

A practice that is becoming increasingly more widespread among vacationers is *house swapping*. Thousands of individuals (or families) from across the world swap their houses for vacation purposes. A two-way swap of primary houses is the most common swap form: Two swappers—call i_1 and i_2, swap their primary houses for concurrent time periods (usually in summer), and so during the swap period, free of charge, i_1 stays in i_2's house and i_2 stays in i_1's house. A swap may also involve more than two individuals. In a three-way swap of primary houses, for instance, three swappers—call in order i_1, i_2, and i_3, swap their primary houses for concurrent time periods, and so during the swap period, free of charge, i_1 stays in i_2's house, i_2 stays in i_3's house, and i_3 stays in i_1's house. Sometimes people also swap their second houses rather than their primary houses, in which case houses are initially vacant and can be offered for use without the need for owners to leave, and therefore the stays need not be concurrent. In another swap form, sometimes called a "hospitality exchange," the stays of "swappers" in houses take place in the presence of the owners of houses.

The predominant motivation behind the house-swapping practice is saving from accommodation costs, but frequent swappers express some other motives as well, such as the lure of "mingling with the natives for a richer travel experience," or, "making friends around the globe"

[18]. In a rather interesting case, after the war in 2003, many Shia and Sunni families in Iraq were swapping their houses in order to escape from sectarian violence [8]. The origins of the house-swapping practice are believed to date back to 1950's, born in academia, but its rise in popularity went hand in hand with the advent and spread of internet, where house-swap agency websites serve as markets by bringing together potential swappers. At the time this paper had been written, *homelink.org* reported over 13 000 listings in 78 countries, and *homeexchange.com* reported over 37 000 listings in over 130 countries. According to an article in the New York Times on June 29, 2006, the American representative of Intervac reported for the agency a rise in membership in the preceding two years of 40 percent, and the founder of Digsville reported a past annual growth rate in membership of approximately 40 percent [27].

This paper makes a first attempt to understand the theoretical properties of a house-swapping market. A house-swapping market resembles the celebrated "housing market" of Shapley and Scarf [34]. In modelling both markets, there are n agents each of whom owns a house; agents seek to exchange their houses where each agent is to receive exactly one house; and monetary transfers between agents are not allowed. The two markets vary in the nature of exchanges, however. While in a housing market houses are *traded*—i.e. exchanged on a permanent basis, in a house-swapping market they are *swapped*—i.e. exchanged on a temporary basis. If houses are traded, ownerships of houses change and the outcome of exchanges is final; if houses are swapped, ownerships remain unchanged and agents return to their original houses when the swap periods terminate. This difference in exchange types gives rise to different types of preferences for agents in the two models. In the standard housing market model, an agent's well-being depends solely upon which house she receives; in a model designed to address real-life house-swapping markets, however, additional factors may figure in the well-being of an agent i:

- *guest*—i.e. the agent who is to receive i's house.[1] It is conceivable that i may be unwilling to entrust her house to the use of another agent whom she deems unreliable.
- *swap periods*—i.e. during which time periods i is to stay in another agent's house and her

[1] In the way use this word, the *guest* of an agent i is the agent who is to stay in i's house, whether or not i is to be present along her stay.

guest is to stay in her house. It is conceivable that i may prefer to be on vacation in a specific period of the year. Complementarities in i's preferences concerning swap periods are also natural, such as between two swap periods (e.g., if i is to swap her primary house, most probably she will demand the swaps to take place concurrently) or a swap period and a house (e.g., i may wish to stay in a house in Munich, Germany, particularly during Oktoberfest).

- *swap size*—i.e. whether i engages in a two-way swap or a three-way swap or so on. A prearranged swap will break down even if a single participating agent reneges later on.[2] Arguably, therefore, a swap is the more susceptible to failure the more agents it involves.[3]

Indeed, the argument can be made that, for reasons similar to as above, timing and size regarding *trades* may similarly figure in agents' well-beings in an extended housing market model. The core difference between "swapping" and "trading" houses is therefore the dependence of agents' preferences to their guests in the former. In an attempt to understand the possible theoretical implications of this dependence, we consider a simple house-swapping model where timing and swap size considerations are absent and agents' preferences are simply over (house, guest) pairs. Our setting is appropriate to study, for instance, a market where people swap their primary houses for a specific period of the year and concerns about prearranged swaps later on breaking down are absent or have been satisfactorily addressed by some measures (e.g., at *homelink.org*, a vacationer is insured against breakdown possibilities; if the owner of her assigned vacation house later on reneges, she is compensated so as to be able to go ahead with her vacation at the same destination).

Absent timing and swap size considerations, in our setting an *allocation* is a one-to-one mapping from the set of agents to the set of houses, and an agent's well-being at an allocation depends upon her assigned (house, guest) pair. In order to successfully implement an allocation,

[2] i.e., if i refuses to deliver her house to agent j, j will refuse to deliver her house to agent k, and so on.

[3] Besides its small size, a two-way swap may be appealing perhaps for an additional reason, that two agents may see staying in exactly one another's houses as a confidence-building measure ensuring the safe use of their houses.

the primary constraint is that agents should participate voluntarily, or, to put it differently, the allocation must not admit any blocking coalitions. A coalition of agents *blocks* an allocation μ if there exists a way in which the coalition members can exchange their houses such that at the outcome every coalition member is weakly better off and at least one is strictly better off in comparison to under μ. An allocation is *individually rational* if it is not blocked by any single agent. More strictly, an allocation is *pairwise stable* if it is not blocked by any single agent or any pair of agents. Even stricter, an allocation is a *core allocation* (or, *in the core*) if it is not blocked by any coalition of agents. Clearly a core allocation is also Pareto efficient, as an allocation μ which is Pareto dominated by another allocation μ' is blocked by the coalition of all agents who can exchange their houses so as to attain μ'.

In light of these definitions we study the theoretical properties of a house-swapping market. As we show, the positive theoretical results in the literature obtained in the housing market model do not extend to our setting. In a housing market, under the assumption of strict preferences over houses, there exists a unique allocation in the core [29], obtained by the following *top trading cycles* procedure (in short, TTC), credited to Gale in [34]:

> *Imagine a diagram consisting of agents and houses. Let each agent "point" to her most preferred house in the diagram. Let each house "point" to its owner. There exists at least one "cycle."*[4] *Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram and in the reduced diagram, proceed similarly.*

In a house-swapping market, however, under additively-separable preferences we obtain that the allocation induced by TTC may even fail to be individually rational (Example 3.1). This is not very surprising, because by its design, TTC operates as if agents seek to exchange their houses so as to receive the best houses that they can, which fully disregards their guest preferences.

In a setting as in ours where preferences are simply over (house, guest) pairs, agents should be willing to execute swaps of any size so long as their assigned (house, guest) pairs improve.

[4] A cycle is characterized by an ordered list $i_1, i_2, \cdots, i_k$ of agents where i_s points to i_{s+1}'s house for $s = 1, \cdots, k-1$ and i_k points to i_1's house.

Nevertheless, in decentralized settings large-size swaps may rarely be executed for another reason. In a decentralized setting a swap is organized by the participating agents themselves, and the coordination difficulty involved in this arrangement is proportional to the swap size. A decentralized house-swapping market may therefore be dominated by two-way swaps, as they are associated with the lowest coordination difficulty. Virtually every house-swapping agency website operates in a decentralized fashion.[5] A relevant issue is, therefore, the potential loss in efficiency associated with such coordination difficulties. In an attempt to show why such loss in efficiency may be unavoidable, we confine our attention to allocations that result from only two-way swaps and under additively-separable preferences we show that it is possible for every such allocation to be Pareto inefficient (Example 3.2).

An important question that concerns implementation is whether or not allocations exist that are robust to blocking coalitions. Existence of a pairwise-stable allocation should be of particular concern, as the formation of a coalition is trivial if it consists of a single agent and relatively easy to coordinate if it consists of a pair of agents. Alas, under additively-separable preferences we obtain that in a house-swapping market a pairwise-stable allocation is not guaranteed to exist, and as a corollary, the core can be empty unlike in a housing market (Example 3.3).

Having shown these negative theoretical results under additive-separability, we proceed to add more structure on preferences. We say that an agent's preferences are *guest-diseparable* if she classifies her potential guests as *acceptable* or *unacceptable* such that constraints **C0**, **C1** and **C2** below are satisfied. If in addition constraint **C3** below is also satisfied, we say that her preferences are *guest-dichotomous*.

C0: She is herself an acceptable guest.

C1: She strictly prefers pairs with acceptable guests to pairs with unacceptable guests.

C2: She has a strict ranking of houses such that between two pairs with acceptable guests she prefers the one with the higher-ranked house.

[5] To facilitate the coordination of three-way swaps, *houseexchange.org.uk* provides vacationers with a three-way search option.

C3: She is indifferent between two pairs where houses are the same and guests are both acceptable.

From a practical viewpoint, a guest-diseparable preference relation can be justified by informational arguments. Imagine that a centralized system is to be used to decide how houses are to be swapped based upon preference information elicited from vacationers. On house-swapping agency websites ample information is available pertaining to houses—e.g., size, location and amenities—to evaluate their desirabilities for vacation purposes. Yet information needed to evaluate the realiability of vacationers as guests is naturally limited. On some websites, for instance, there is information on the number of swaps practiced earlier by their users. In some cases, there are also evaluations by past swap partners judging vacationers as guests. Arguably, then, based upon ample information pertaining to houses, a vacationer can fully rank them, but based upon limited information pertaining to people, it is more viable for a vacationer to simply categorize her potential guests as acceptable or unacceptable rather than fully ranking them.

As it turns out, under guest-diseparable preferences, the core is non-empty; the following *acceptable top trading cycles* procedure (in short, ATTC), adapted from TTC, always induces a core allocation (Theorem 3.1):

Imagine a diagram consisting of agents and houses. Let each agent "point" to her most preferred house among those houses in the diagram owned by agents who deem her as an acceptable guest. Let each house "point" to its owner. There exists at least one cycle. Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram and in the reduced diagram, proceed similarly.

Nevertheless, unlike in a housing market under strict preferences, the core allocation is not necessarily unique in a house-swapping market under guest-diseparable preferences (Example 3.4). Further, while in a housing market TTC is strategy-proof (truthtelling is a dominant strategy) under strict preferences [28], in a house-swapping market ATTC can potentially be manipulated under guest-diseparability (Example 3.4). However, if preferences are guest-dichotomous, we show that there exists a unique core allocation, induced by ATTC, and ATTC is strategy-proof (Theorem 3.2).

The rest of the paper is organized as follows: Section 3.2 introduces the model. Section 3.3 shows by examples the aforementioned negative results. Section 3.4 studies the problem under the assumptions of guest-diseparability and guest-dichotomy. The proofs of two theorems are given in the Appendix.

3.2 The Model

A *house-swapping market* is a triplet $\langle I, H, \succsim \rangle$ where

– $I : \{1, 2, ..., n\}$ is a finite set of agents;

– $H : \{h_1, h_2, \ldots, h_n\}$ is a finite set of houses such that $h_i \in H$ is owned by agent i;

– $\succsim$: $(\succsim_i)_{i \in I}$ is a preference profile where $\succsim_i$ represents agent i's preference relation, which is defined over the set of *feasible* (ordered) pairs

$$X^i = \{(h, j) | h \in H, j \in I,\ j = i \text{ if } h = h_i\},$$

and $(h, j) \succsim_i (h', j')$ indicates that i (weakly) prefers $(h, j) \in X^i$ to $(h', j') \in X^i$.

An agent i being assigned a pair $(h,\ j)$ is understood as i receiving the house h and her house h_i being received by agent j. The agent who receives h_i is said to be the "guest" of agent i. As specified above, a pair $(h_i,\ j)$ is feasible (i.e., $(h_i,\ j) \in X^i$) only if $j = i$, because when i receives h_i, by definition her guest is herself.

Let $\succsim_C$: $(\succsim_i)_{i \in C}$ denote the profile of preference relations of a subset of agents $C \subseteq I$ at $\succsim$. Let $\succsim_{-i}$ shortly denote $\succsim_{I \setminus \{i\}}$. Then $\succsim$ can equivalently be written as $(\succsim_C, \succsim_{I \setminus C})$ or $(\succsim_i, \succsim_{-i})$ for $C \subseteq I$ and $i \in I$.

Let $\succ_i$ and $\sim_i$ respectively represent the "strictly more preferred" and "equally preferred" relations of agent i derived from $\succsim_i$. Thus $(h, j) \succ_i (h', j')$ means $(h, j) \succsim_i (h', j')$ holds but

$(h', j') \succsim_i (h, j)$ does not hold, and $(h, j) \sim_i (h', j')$ means both $(h, j) \succsim_i (h', j')$ and $(h', j') \succsim_i (h, j)$ hold.

For an agent i, let P_i denote the domain of admissible preference relations defined over the set X^i. Let $P : (P_1 \times P_2 \times \cdots \times P_n)$ denote the domain of admissible preference profiles for agents.

An *allocation* $\mu : I \to H$ is a bijective mapping from the set of agents to the set of houses. Let $\mathcal{M}$ be the domain of admissible allocations for some fixed I and H. The inverse mapping of an allocation μ is denoted by μ^{-1}. Thus an agent i is assigned at μ the pair $(\mu(i), \mu^{-1}(h_i))$ and $\mu(i) = (h, j)$ is understood as $\mu(i) = h$ and $\mu^{-1}(h_i) = j$.

A "k-way swap" involves an ordered list $i_1, i_2, \cdots i_k$ of k agents such that i_s receives the house $h_{i_{s+1}}$ for $s = 1, \cdots, k-1$ and i_k receives h_{i_1}. An allocation μ results from two-way swaps if for every agent i either $\mu(i) = h_i$ (i receives her own house) or $\mu(i) = h_j$ and $\mu(j) = h_i$ for some $j \in I$ (i.e., i and j execute a two-way swap).

An allocation μ is *blocked* by a coalition of agents $C \subseteq I$ if there exists a one-to-one mapping $bl : C \to H^C$ such that

– $H^C \subseteq H$ is the set of houses owned by agents in C,

– $(bl(i), bl^{-1}(h_i)) \succsim_i (\mu(i), \mu^{-1}(h_i))\ \forall i \in C$,

– $(bl(i), bl^{-1}(h_i)) \succ_i (\mu(i), \mu^{-1}(h_i))\ \exists i \in C$.

We call $\left\langle C, H^C, bl \right\rangle$ above a *blocking triplet* at μ.

An allocation μ is

– *individually rational* if it is not blocked by any agent;

– *pairwise stable* if it is not blocked by any agent or any pair of agents;

– a *core allocation* (or *in the core*) if it is not blocked by any coalition of agents $C \subseteq I$.

An allocation μ is *Pareto-efficient* if it is not Pareto dominated by another allocation. Equivalently, an allocation μ is *Pareto efficient* if it is not blocked by the coalition I. Clearly a core allocation is pairwise stable and Pareto efficient and a pairwise-stable allocation is individually rational.

A *mechanism* $\varphi : P \to \mathcal{M}$ is a one-to-one mapping from the domain of admissible preference profiles to the codomain of admissible allocations. Thus at a preference profile $\succsim$ the allocation selected by a mechanism φ is $\varphi(\succsim)$, and at $\varphi(\succsim)$ the house received by an agent i is $\varphi(\succsim)(i)$ and her guest is $\varphi(\succsim)^{-1}(h_i)$.

A mechanism φ is *strategy-proof* if there does not exist any $\succsim \in P$, $i \in I$, $\succsim_i' \in P_i$ such that

$$(\varphi(\succsim_i', \succsim_{-i})(i), \varphi(\succsim_i', \succsim_{-i})^{-1}(h_i)) \succ_i (\varphi(\succsim)(i), \varphi(\succsim)^{-1}(h_i)).$$

In simple words, a mechanism is strategy-proof if no agent can ever strictly benefit from submitting to it a non-truthful preference relation (i.e., truthful preference revelation is a dominant strategy).

In Section 3.3 and Section 3.4 we study house-swapping markets under additively-separable utility functions. If agent i has an additively-separable utility function $u_i : H \cup I \to \mathbb{R}$, then the utility she derives from a pair $(h, j) \in X^i$ is $u_i(h) + u_i(j)$. Whenever a house-swapping market is denoted by a triplet $\langle I, H, u\rangle$, it is understood that $u : (u_i)_{i\in I}$ is the profile of agents' additively-separable utility functions, and the preference relation $\succsim_i$ for $i \in I$ can be derived from her utility function u_i as follows:

$(h, j) \succsim_i (h', j')$ holds for $(h, j), (h', j') \in X^i$ if and only if $u_i(h) + u_i(j) \geq u_i(h') + u_i(j')$.

The preference relation $\succsim_i$ of an agent i is additively separable if it is consistent with an additively-separable utility function u_i (i.e., the preference relation derived from u_i is $\succsim_i$). A profile of preference relations $\succsim$ is additively separable if $\succsim_i$ is additively separable for every $i \in I$.

3.3 Theoretical Possibilities

A closely related market to a house-swapping market is a *housing market* (introduced by Shapley and Scarf [34]). In a housing market, as usual, there are a set $I : \{1, 2, ..., n\}$ of agents and a set $H : \{h_1, h_2, \ldots, h_n\}$ of houses where $h_i \in H$ is owned by agent i, and houses are to be allocated to agents such that each agent is to receive exactly one house. The two markets differ in agents' preferences, however. The housing market model is intended for markets where houses are "traded" (exchanged permanently), not swapped, and so an agent's preferences are simply over houses, not (house, guest) pairs. A housing market can be thought of as a special house-swapping market $\langle I, H, u\rangle$ where $u_i(j) = 0$ for any $i, j \in I$ (i.e., the "guest" component is inconsequential).

Under strict preferences over houses, there exists a unique core allocation in a housing market [29], which is induced by the following *top trading cycles* mechanism (in short, TTC), credited to Gale in [34]: Imagine a diagram consisting of agents and houses.

> *Step 1: Let each agent "point" to her most preferred house. Let each house "point" to its owner. There exists at least one "cycle," characterized by an ordered list $i_1, i_2, \cdots, i_k$ of agents where i_s points to i_{s+1}'s house for $s = 1, \cdots, k-1$ and i_k points to i_1's house. Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram.*
>
> *Step $t > 1$: Let each remaining agent "point" to her most preferred house among remaining ones. Let each remaining house "point" to its owner. There exists at least one cycle. Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram.*

The algorithm terminates when every agent is assigned a house.

As stated in Proposition 3.1, unlike in a housing market, in a house-swapping market and under additively-separable preferences, the application of TTC may not lead to a desirable allocation.

Proposition 3.1 *In a house-swapping market $\langle I, H, u \rangle$ where agents have strict preferences over houses (i.e., $u_i(h) \neq u_i(h')$ for any $i \in I$, $h, h' \in H$, $h \neq h'$), the allocation induced by TTC may fail to be individually rational.*

Proof. See Example 3.1. ■

Example 3.1 *Let $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3\}$, and the additively-separable utility profile u of agents be as given in the following table:*

u_1		u_2		u_3	
$u_1(h_2) = 3$	$u_1(1) = 4$	$u_2(h_3) = 3$	$u_2(2) = 4$	$u_3(h_1) = 3$	$u_3(3) = 4$
$u_1(h_3) = 2$	$u_1(2) = 3$	$u_2(h_1) = 2$	$u_2(3) = 3$	$u_3(h_2) = 2$	$u_3(1) = 3$
$u_1(h_1) = 0$	$u_1(3) = 0$	$u_2(h_2) = 0$	$u_2(1) = 0$	$u_3(h_3) = 0$	$u_3(2) = 0$

The derived utilities of agents from feasible (house,guest) pairs are as given in parentheses in the following table:

u_1	u_2	u_3
$(h_2, 2)$ (6)	$(h_3, 3)$ (6)	$(h_1, 1)$ (6)
$(h_3, 2)$ (5)	$(h_1, 3)$ (5)	$(h_2, 1)$ (5)
$(h_1, 1)$ (4)	$(h_2, 2)$ (4)	$(h_3, 3)$ (4)
$(h_2, 3)$ (3)	$(h_3, 1)$ (3)	$(h_1, 2)$ (3)
$(h_3, 3)$ (2)	$(h_1, 1)$ (2)	$(h_2, 2)$ (2)

When TTC is executed, at Step 1 agent 1 *points to h_2 which points to agent* 2*, agent* 2 *points to h_3 which points to agent* 3*, and agent* 3 *points to h_1 which points to agent* 1*. A cycle forms and the associated assignments are made. The allocation induced is*

$$\mu : \mu(1) = (h_2, 3), \mu(2) = (h_3, 1), \mu(3) = (h_1, 2).$$

At μ each agent derives a utility of 3*; μ is not individually rational as each agent can derive a utility of* 4 *by being assigned her own house. Indeed there exists a unique core allocation in this example, not selected by TTC, where agents* $1, 2$ *and* 3 *respectively receive h_3, h_1 and h_2 and each derives a utility of* 5*.* ♢

As stated in Proposition 3.2, loss in efficiency may not be avoidable if attention is confined to only two-way swaps.

Proposition 3.2 *In a house-swapping market* $\langle I, H, u\rangle$ *every allocation that results from two-way swaps may be Pareto inefficient.*

Proof. See Example 3.2. ■

Example 3.2 *Let* $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3\}$, *and the additively-separable utility profile* u *of agents be as given in the following table:*

u_1		u_2		u_3	
$u_1(h_2) = 4$	$u_1(1) = 3$	$u_2(h_3) = 4$	$u_2(2) = 3$	$u_3(h_1) = 4$	$u_3(3) = 3$
$u_1(h_1) = 2$	$u_1(3) = 2$	$u_2(h_2) = 2$	$u_2(1) = 2$	$u_3(h_3) = 2$	$u_3(2) = 2$
$u_1(h_3) = 0$	$u_1(2) = 0$	$u_2(h_1) = 0$	$u_2(3) = 0$	$u_3(h_2) = 0$	$u_3(1) = 0$

The derived utilities of agents from feasible (house,guest) pairs are as given in parentheses in the following table:

u_1	u_2	u_3
$(h_2, 3)$ (6)	$(h_3, 1)$ (6)	$(h_1, 2)$ (6)
$(h_1, 1)$ (5)	$(h_2, 2)$ (5)	$(h_3, 3)$ (5)
$(h_2, 2)$ (4)	$(h_3, 3)$ (4)	$(h_1, 1)$ (4)
$(h_3, 3)$ (2)	$(h_1, 1)$ (2)	$(h_2, 2)$ (2)
$(h_3, 2)$ (0)	$(h_1, 3)$ (0)	$(h_2, 1)$ (0)

Clearly, in this example the unique Pareto-efficient allocation, which is also the unique core allocation, is the following where each agent derives a utility of 6*:*

$$\mu : \mu(1) = (h_2, 3), \mu(2) = (h_3, 1), \text{ and } \mu(3) = (h_1, 2)$$

Since μ *results from a three-way swap, there exists no Pareto-efficient allocation that results from two-way swaps.* ◊

As stated in Proposition 3.3, even a blocking pair of agents may not be avoidable in a house-swapping market.

Proposition 3.3 *In a house-swapping market* $\langle I, H, u\rangle$ *there may not exist a pairwise-stable allocation.*

Proof. See Example 3.3. ■

Corollary: *In a house-swapping market* $\langle I, H, u\rangle$ *the core may be empty.*

Example 3.3 *Let* $I = \{1, 2, 3, 4, 5\}$*,* $H = \{h_1, h_2, h_3, h_4, h_5\}$*, and the additively-separable utility profile* u *of agents be as given in the following table:*

u_1		u_2		u_3	
$u_1(h_2)$=10	$u_1(1) = 11$	$u_2(h_1) = 17$	$u_2(2) = 10$	$u_3(h_1) = 10$	$u_3(3) = 11$
$u_1(h_4) = 8$	$u_1(3) = 10$	$u_2(h_3) = 15$	$u_2(5) = 9$	$u_3(h_2) = 8$	$u_3(1) = 10$
$u_1(h_3) = 4$	$u_1(2) = 9$	$u_2(h_5) = 9$	$u_2(3) = 5$	$u_3(h_3) = 5$	$u_3(2) = 9$
$u_1(h_1) = 2.1$	$u_1(4) = 5.5$	$u_2(h_2) = 6$	$u_2(1) = 2$	$u_3(h_4) = 1$	$u_3(4) = 1.5$
$u_1(h_5) = 0.4$	$u_1(5) = 0$	$u_2(h_4) = 0$	$u_2(4) = 0.1$	$u_3(h_5) = 0.2$	$u_3(5) = 0.5$

u_4		u_5	
$u_4(h_1)$=10	$u_4(4)$=10	$u_5(h_2)$=10	$u_5(a_5)$=10
$u_4(h_4)$=8	$u_4(1)$=9	$u_5(h_5)$=8	$u_5(a_2)$=9
$u_4(h_2)$=0.6	$u_4(2)$=0.3	$u_5(h_1)$=0.6	$u_5(a_1)$=0.3
$u_4(h_3)$=0.5	$u_4(3)$=0.2	$u_5(h_3)$=0.5	$u_5(a_3)$=0.2
$u_4(h_5)$=0.4	$u_4(5)$=0.1	$u_5(h_4)$=0.4	$u_5(a_4)$=0.1

The derived utilities of agents from feasible (house, guest) pairs are as given in parentheses in the following table. For the sake of shortness, we list only the most preferred feasible pairs of agents below, up to what is required for our arguments, but one can verify that under u *no agent is indifferent between two distinct feasible pairs.*

u_1	u_2	u_3	u_4	u_5
$(h_2, 3)$ (20)	$(h_1, 5)$ (26)	$(h_1, 1)$ (20)	$(h_1, 1)$ (19)	$(h_2, 2)$ (19)
$(h_2, 2)$ (19)	$(h_3, 5)$ (24)	$(h_1, 2)$ (19)	$(h_4, 4)$ (18)	$(h_5, 5)$ (18)
$(h_4, 3)$ (18)	$(h_1, 3)$ (22)	$(h_2, 1)$ (18)	$\vdots$	$\vdots$
$(h_4, 2)$ (17)	$(h_3, 3)$ (20)	$(h_2, 2)$ (17)		
$(h_2, 4)$ (15.5)	$(h_1, 1)$ (19)	$(h_3, 3)$ (16)		
$(h_3, 3)$ (14)	$(h_5, 5)$ (18)	$\vdots$		
$(h_4, 4)$ (13.5)	$(h_1, 4)$ (17.1)			
$(h_1, 1)$ (13.1)	$(h_3, 1)$ (17)			
$(h_3, 2)$ (13)	$(h_2, 2)$ (16)			
$\vdots$	$\vdots$			

We show that there does not exist a pairwise-stable allocation in four steps:

Step 1: Any allocation at which agent 4 *is not assigned* $(h_1, 1)$ *or* $(h_4, 4)$ *is not individually rational; agent* 4 *blocks it. An allocation where agent* 4 *is assigned* $(h_1, 1)$ *is blocked by agent* 1 *and agent* 3*: Agent* 3 *prefers* $(h_1, 1)$ *to any other pair, and* $(h_3, 3) \succ_1 (h_4, 4)$*. (Note that when agent* 4 *is assigned* $(h_1, 1)$*, agent* 1 *must be assigned* $(h_4, 4)$*.) Therefore, at any pairwise-stable allocation agent* 4 *must be assigned* $(h_4, 4)$*.*

Step 2: Any allocation at which agent 5 *is not assigned* $(h_2, 2)$ *or* $(h_5, 5)$ *is not individually rational; agent* 5 *blocks it. Suppose agent* 5 *is assigned* $(h_2, 2)$*. Then agent* 1 *is not assigned* $(h_2, 3)$ *or* $(h_2, 2)$ *because* h_2 *is received by agent* 5*. But such an allocation is blocked by agent* 1 *and agent* 2 *because* $(h_2, 2)$ *is more preferable to agent* 1 *than her assigned pair (whatever it is), and* $(h_1, 1) \succ_2 (h_5, 5)$*. (Note that when agent* 5 *is assigned* $(h_2, 2)$*, agent* 2 *must be assigned* $(h_5, 5)$*.) Therefore, at any pairwise-stable allocation agent* 5 *must be assigned* $(h_5, 5)$*.*

Step 3: By Steps 1 and 2, at any pairwise-stable allocation agents 1, 2, *and* 3 *must be assigned* $h_1, h_2,$ *and* h_3 *in some order. Consider the allocations that result from two-way swaps. An allocation where agent* 1 *and agent* 2 *swap their houses and agent* 3 *is assigned her own house* *is blocked by agent* 2 *and agent* 3 *because* $(h_3, 3) \succ_2 (h_1, 1)$ *and* $(h_2, 2) \succ_3 (h_3, 3)$. *An allocation where agent* 1 *and agent* 3 *swap their houses and agent* 2 *is assigned her own house is blocked by agent* 1 *and agent* 2, *because* $(h_2, 2) \succ_1 (h_3, 3)$ *and* $(h_1, 1) \succ_2 (h_2, 2)$. *An allocation where agent* 2 *and agent* 3 *swap their houses and agent* 1 *is assigned her own house is blocked by agent* 1 *and agent* 3 *because* $(h_3, 3) \succ_1 (h_1, 1)$ *and* $(h_1, 1) \succ_3 (h_2, 2)$. *If each of agent* 1, *agent* 2, *and agent* 3 *is assigned her own house, then the allocation is blocked by agent* 1 *and agent* 2 *because* $(h_2, 2) \succ_1 (h_1, 1)$ *and* $(h_1, 1) \succ_2 (h_2, 2)$.

Step 4: By Steps 1, 2, and 3, only two allocations are left for consideration:

$$\begin{aligned}
\mu &: \mu(1) = (h_2, 3), \mu(2) = (h_3, 1), \mu(3) = (h_1, 2), \mu(4) = (h_4, 4), \mu(5) = (h_5, 5), \\
\pi &: \pi(1) = (h_3, 2), \pi(2) = (h_1, 3), \pi(3) = (h_2, 1), \pi(4) = (h_4, 4), \pi(5) = (h_5, 5).
\end{aligned}$$

The allocation μ *is blocked by agent* 2 *and agent* 5 *because* $(h_5, 5) \succ_2 (h_3, 1)$ *and* $(h_2, 2) \succ_5 (h_5, 5)$. *Similarly,* π *is blocked by agent* 1 *and agent* 4 *because* $(h_4, 4) \succ_1 (h_3, 2)$ *and* $(h_1, 1) \succ_4 (h_4, 4)$. *Therefore, there does not exist a pairwise-stable allocation.* $\Diamond$

3.4 Guest-diseparable Preferences

As shown by examples in Section 3.3, unlike in a housing market under strict preferences, positive theoretical results cannot be assured in a house-swapping market under additively-separable utility functions. In an attempt to obtain positive theoretical results, we impose more structure on preferences in this section.

Definition 3.1 *An agent* i*'s preference relation* $\succsim_i$ *is* ***guest-diseparable*** *if the set* I *of agents can be partitioned into the subsets* $I^{i,u}$ *of "unacceptable" guests and* $I^{i,a}$ *of "acceptable" guests such that constraints* ***C0***, ***C1*** *and* ***C2*** *below are satisfied. If in addition* $\succsim_i$ *also satisfies constraint* ***C3*** *below, then it is* ***guest-dichotomous***.

C0: $i \in I^{i,a}$

C1: *for any* $(h,j),(h',j') \in X^i$

$$(h,j) \succ_i (h',j') \text{ if } j \in I^{i,a} \text{ and } j' \in I^{i,u}$$

C2: *i strictly ranks houses such that for any* $(h,j),(h',j') \in X^i$

$$(h,j) \succ_i (h',j') \text{ if } j,j' \in I^{i,a} \text{ and } i \text{ ranks } h \text{ higher than } h'.$$

C3: *for any* $(h,j),(h,j') \in X^i$

$$(h,j) \sim_i (h,j') \text{ if } j,j' \in I^{i,a}$$

In simple words, under a guest-diseparable preference relation $\succsim_i$, i deems herself as an acceptable guest (**C0**), which is quite natural; her first priority is her house to be received by an acceptable guest (**C1**); and her ranking of pairs with acceptable guests is based upon her strict ranking of houses (**C2**). Under guest-dichotomy, additionally, i is indifferent between pairs with different acceptable guests but the same house (**C3**).

It is not hard to see that a guest-diseparable preference relation $\succsim_i$ is additively separable. We can construct an additively-separable utility function u_i consistent with $\succsim_i$ as follows: Relative to differences in utility levels across houses, utility levels for acceptable guests should be set (1) sufficiently close, i.e. for any $j,j' \in I^{i,a}$, $h,h' \in H$, $h \neq h'$

$$|u_i(j) - u_i(j')| < |u_i(h) - u_i(h')|$$

(2) and sufficiently above the utility levels for unacceptable guests, i.e. for any $j \in I^{i,a}, j' \in I^{i,u}$, $h,h' \in H$

$$|u_i(h) - u_i(h')| < u_i(j) - u_i(j').$$

If $\succsim_i$ is guest-dichotomous, the utility levels above for acceptable guests should be set the same.

In a house-swapping market and under guest-diseparable preference relations, we consider the following *acceptable top trading cycles mechanism* (ATTC), adapted from TTC: Imagine a diagram consisting of agents and houses.

Step 1: Let each agent "point" to her highest-ranked house among those houses owned by agents who deem her as an acceptable guest. Let each house point to its owner. There exists at least one cycle. Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram.

Step $t > 1$: Let each remaining agent "point" to her highest-ranked house among those remaining houses owned by agents who deem her as an acceptable guest. Let each remaining house point to its owner. There exists at least one cycle. Assign agents in cycles to the houses they point. Then remove these agents and houses from the diagram.

The algorithm terminates when every agent is assigned a house. Notice that ATTC is well-defined because every agent deems herself as an acceptable guest.

Theorem 3.1 *In a house-swapping market and under guest-diseparable preferences, the allocation induced by ATTC is a core allocation.*

Proof. See Appendix. ■

As stated in Theorem 3.1, just as TTC induces a core allocation in a housing market under strict preferences, ATTC induces a core allocation in a house-swapping market under guest-diseparable preferences. However, unlike in that setting, here the core allocation need not be unique. Further, unlike TTC in that setting, here ATTC can potentially be manipulated.

Proposition 3.4 *In a house-swapping market and under guest-diseparable preferences, the core allocation is not necessarily unique and ATTC is not strategy-proof.*

Proof. See Example 3.4. ■

Example 3.4 *Let $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3\}$, and the additively-separable utility profile u of agents be as given in the following table:*

u_1		u_2		u_3	
$u_1(h_2) = 10$	$u_1(1) = 13$	$u_2(h_3) = 10$	$u_2(2) = 13$	$u_3(h_1) = 10$	$u_3(3) = 13$
$u_1(h_3) = 5$	$u_1(2) = 12$	$u_2(h_1) = 5$	$u_2(1) = 12$	$u_3(h_3) = 5$	$u_3(1) = 12$
$u_1(h_1) = 0$	$u_1(3) = 11$	$u_2(h_2) = 0$	$u_2(3) = 11$	$u_3(h_2) = 0$	$u_3(2) = 11$

The preference relations of agents derived from u *are given in the table below. The table also presents in parentheses the derived utilities of agents from feasible (house,guest) pairs. It can be verified that the preference relations below are guest-diseparable such that every agent deems every agent as an acceptable guest.*

$\succsim_1$	$\succsim_2$	$\succsim_3$
$(h_2, 2)$ (22)	$(h_3, 1)$ (22)	$(h_1, 1)$ (22)
$(h_2, 3)$ (21)	$(h_3, 3)$ (21)	$(h_1, 2)$ (21)
$(h_3, 2)$ (17)	$(h_1, 1)$ (17)	$(h_3, 3)$ (18)
$(h_3, 3)$ (16)	$(h_1, 3)$ (16)	$(h_2, 1)$ (12)
$(h_1, 1)$ (13)	$(h_2, 2)$ (13)	$(h_2, 2)$ (11)

When ATTC is executed under the preference profile $\succsim$*, at Step 1 agent* 1 *points to her top-ranked house* h_2 *which points to agent* 2*; agent* 2 *points to her top-ranked house* h_3 *which points to agent* 3*; and agent* 3 *points to her top-ranked house* h_1 *which points to agent* 1*. A cycle forms and the associated assignments are made. The following core allocation is induced:*

$$\mu : \mu(1) = (h_2, 3), \mu(2) = (h_3, 1), \mu(3) = (h_1, 2).$$

Suppose that agent 1 *misstates her preferences and submits the manipulated preference relation* $\succsim_1'$ *given below. It can be verified that* $\succsim_1'$ *is also guest-diseparable such that agent* 1 *deems agent* 1 *and agent* 2 *as acceptable guests and agent* 3 *as an unacceptable guest. Below we also provided an additively-separable utility function* u_1' *consistent with* $\succsim_1'$*.*

$\succsim_1'$
$(h_2, 2)$ (22)
$(h_3, 2)$ (17)
$(h_1, 1)$ (13)
$(h_2, 3)$ (11)
$(h_3, 3)$ (6)

u_1'	
$u_1(h_2) = 10$	$u_1(1) = 13$
$u_1(h_3) = 5$	$u_1(2) = 12$
$u_1(h_1) = 0$	$u_1(3) = 1$

When ATTC is executed under the preference profile $(\succsim_1', \succsim_{-1})$, at Step 1 agent 1 points to her top-ranked house h_2 which points to agent 2; agent 2 points to her top-ranked house h_3 which points to agent 3; and agent 3 points to h_3 which points to herself (agent 3 now cannot point to her top-ranked house h_1 because under $\succsim_1'$ she is not an acceptable guest for agent 1). Agent 3 forms a cycle alone; she receives h_3. At Step 2 agent 1 points to h_2 which points to agent 2 and agent 2 points to h_1 which points to agent 1. A cycle forms and the associated assignments are made. The allocation induced is

$$\pi : \pi(1) = (h_2, 2), \pi(2) = (h_1, 1), \pi(3) = (h_3, 3).$$

Agent 1 is better off under π than under μ and so the manipulation benefitted her. Also note that in this example the core allocation is not unique as both π and μ are core allocations. $\diamondsuit$

As Theorem 3.2 implies, the multiplicity of core allocations and the susceptibility to manipulation of ATTC mentioned in Proposition 3.4 vanish if agents are exactly indifferent across acceptable guests.

Theorem 3.2 *In a house-swapping market and under guest-dichotomous preferences, there exists a unique core allocation and ATTC is strategy-proof.*

Proof. See Appendix. ■

Appendix

Proof of Theorem 3.1. Consider a house-swapping market $\langle I, H, \succsim \rangle$ where the preference profile $\succsim$ is guest-diseparable. Let μ^{ATTC} denote the allocation that ATTC induces under $\succsim$. Let I_t and H_t respectively be the set of agents and the set of houses that ATTC assigns at Step t. Let I^t and H^t respectively be the set of agents and the set of houses that ATTC assigns at Step t or after. Also, for $i \in I$ let i^H be the set of houses owned by agents who deem i as an acceptable guest.

Suppose μ^{ATTC} is not a core allocation. Let $\langle C, H^C, bl \rangle$ be a blocking triplet at μ^{ATTC} where the cardinality $|C|$ takes the smallest possible value. By design of ATTC, the guest of every agent is acceptable for her at μ^{ATTC}. Then for any $i \in C$, we get $bl(i) \in i^H$, because otherwise the agent who owns $bl(i)$ would deem i as an unacceptable guest and therefore she would be worse off under bl than under μ^{ATTC}. Let

$$C_t = C \cap I_t \text{ and } H_t^C = H^C \cap H_t,$$

i.e., C_t and H_t^C are the sets of coalition members and their houses that ATTC assigns at Step t. Note that the set of houses owned by agents in C_t is exactly H_t^C because under ATTC a house is assigned at the same step as its owner.

Let $\underline{t}$ be the earliest step such that $C_{\underline{t}} \neq \emptyset$. Consider an agent $i \in C_{\underline{t}}$. By design of ATTC, $\mu^{ATTC}(i)$ is the house in $H^{\underline{t}} \cap i^H$ that i ranks highest. Since i is weakly better off under bl than under μ^{ATTC}, either $bl(i) = \mu^{ATTC}(i)$ or $bl(i) \in i^H$ is a house that i ranks even higher than $\mu^{ATTC}(i)$, in which case $bl(i) \notin H^{\underline{t}}$. But then the agent in C who owns $bl(i)$ must have been assigned by ATTC before Step $\underline{t}$, which contradicts our choice of $\underline{t}$. So $bl(i) = \mu^{ATTC}(i)$. But the same arguments hold for any $i \in C_{\underline{t}}$ and so the houses in $H_{\underline{t}}^C$ are assigned to the agents in $C_{\underline{t}}$ in exactly the same way under μ^{ATTC} and under bl. Thus the agents in $C_{\underline{t}}$ are equally better off under bl and under μ^{ATTC} and there exists an agent in $C \setminus C_{\underline{t}}$ who is better off under bl than under μ^{ATTC}. But then $\left\langle C \setminus C_{\underline{t}}, H^C \setminus H_{\underline{t}}, \widetilde{bl} \right\rangle$ is also a blocking triplet where $\widetilde{bl}(i) = bl(i)$ for every $i \in C \setminus C_t$, which contradicts our initial choice of C as the blocking coalition with minimum cardinality. ■

Proof of Theorem 3.2. Consider a house-swapping market $\langle I, H, \succsim \rangle$ where the preference profile $\succsim$ is guest-dichotomous. Let μ^{ATTC} denote the allocation that ATTC induces under $\succsim$. Let I_t and H_t respectively be the set of agents and the set of houses that ATTC assigns at Step t. Let I^t and H^t respectively be the set of agents and the set of houses that ATTC assigns at Step t or after. Also, for $i \in I$ let i^H be the set of houses owned by agents who deem i as an acceptable guest. Our proof is in two parts.

Part 1, the uniqueness of a core allocation:

By Theorem 3.1 μ^{ATTC} is a core allocation. Suppose $\pi \neq \mu^{ATTC}$ is also a core allocation. By design of ATTC, the guest of every agent is acceptable for her at μ^{ATTC}. Then for any $i \in C$, we get $\pi(i) \in i^H$, because otherwise the agent who owns $\pi(i)$ would block π (she deems i as an unacceptable guest; if she is assigned her own house, however, her guest will be acceptable).

Consider the sets I_1 and H_1. By design of ATTC, for any agent $i \in I_1$, $\mu^{ATTC}(i)$ is the house that she ranks highest in i^H. But then, given that $\pi(i) \in i^H$ for every $i \in I$, if $\pi(i) \neq \mu^{ATTC}(i)$ for any $i \in I_1$, $\langle I_1, H_1, bl \rangle$ would be a blocking triplet at π where $bl(i) = \mu^{ATTC}(i)$ for every $i \in I_1$. That would contradict that π is a core allocation, so $\pi(i) = \mu^{ATTC}(i)$ for every $i \in I_1$.

Consider now the sets I_2 and H_2. By design of ATTC, for any agent $i \in I_2$, $\mu^{ATTC}(i)$ is the house that she ranks highest in $H^2 \cap i^H$. But then, given that $\pi(i) \in i^H$ and $\pi(i) \notin (H_1 \cap i^H)$ for every $i \in I$, if $\pi(i) \neq \mu^{ATTC}(i)$ for any $i \in I_2$, $\langle I_2, H_2, bl \rangle$ would be a blocking triplet at π where $bl(i) = \mu^{ATTC}(i)$ for every $i \in I_2$. That would contradict that π is a core allocation, so $\pi(i) = \mu^{ATTC}(i)$ for every $i \in I_2$.

But if we keep iterating the above arguments for I_3 and H_3, I_4 and H_4, and so on, we conclude that $\pi = \mu^{ATTC}$, which is a contradiction.

Part 2, the strategy-proofness of ATTC:

Consider any $i \in I$ and any misstatement $\succsim'_i \in P_i$ where $\succsim'_i$ is guest-dichotomous. We prove the strategy-proofness of ATTC by showing that i prefers the (house, guest) pair that she receives under the preference profile $\succsim$ to the pair that she receives under the preference profile $(\succsim'_i, \succsim_{-i})$.

Let Step T be when i joins a cycle when ATTC is executed under $(\succsim_i', \succsim_{-i})$. Let $i_1, i_2, \cdots, i_k, i$ be the ordered list that characterizes the cycle that i joins (i.e., i_1 points to h_{i_2} which points to i_2, i_2 points to h_{i_3} which points to i_3, $\cdots$, i_k points to h_i which points to i, and i points to h_{i_1} which points to i_1). The associated assignments are made and hence under $(\succsim_i', \succsim_{-i})$ i is assigned the pair (h_{i_1}, i_k). Note that $h_{i_1} \in i^H$ because otherwise i would not be allowed to point to h_{i_1}.

By its design, ATTC assigns i an acceptable guest if she submits her true preference relation. Then i is better off under $\succsim$ than under $(\succsim_i', \succsim_{-i})$ if i deems i_k as an unacceptable guest. So suppose i_k is an acceptable guest for i. Also, let Step T^* be when ATTC assigns i a house under $\succsim$. There are two possible cases:

Case 1: $T^* \geq T$

If $T^* \geq T$ then when ATTC is executed under $\succsim$ and under $(\succsim_i', \succsim_{-i})$, the same cycles form up to Step T (because the formation of these cycles depends only on $\succsim_{-i}$). Therefore, when ATTC is executed under $\succsim$, at Step T a chain forms, characterized by the ordered list $i_1, i_2, \cdots, i_k, i$ of agents, where h_{i_1} points to i_1, i_1 points to h_{i_2} which points to i_2, i_2 points to h_{i_3} which points to i_3,$\cdots$, i_k points to h_i which points to i. At Step T and after, this chain remains as it is without turning into a cycle as long as i remains in the diagram but not point to h_{i_1}. But then when ATTC is executed under $\succsim$, at Step T^* agent i points to either h_{i_1} or a house in i^H that she ranks even higher than h_{i_1}. Also, in either case her assigned guest is acceptable. But then i's assigned pair under $\succsim$ is not less preferable than (h_{i_1}, i_k).

Case 2: $T^* < T$

If $T^* < T$ then when ATTC is executed under $\succsim$ and $(\succsim_i', \succsim_{-i})$, the same cycles form up to Step T^* (because the formation of these cycles depends only on $\succsim_{-i}$). By design of ATTC, under $\succsim$, at Step T^*, i's assigned guest is acceptable for her, and her assigned house is the house that she ranks highest in $i^H \cap H^{T^*}$. Then the house that she receives under $\succsim$ is either h_{i_1} or a house that she ranks even higher. But then i's assigned pair under $\succsim$ is not less preferable than (h_{i_1}, i_k). ■

Bibliography

[1] A. Abdulkadiroğlu, Atila and T. Sönmez (1998), "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems," *Econometrica* 66: 689-701.

[2] A. Abdulkadiroğlu, Atila and T. Sönmez (1999), "House Allocation with Existing Tenants," *Journal of Economic Theory* 88:2 233-260.

[3] A. Abdulkadiroğlu, Atila and T. Sönmez (2003), "School Choice: A Mechanism Design Approach," *American Economic Review* 93-3: 729-747.

[4] R. Aumann (1964), "Markets with a Continuum of Traders," *Econometrica* 32: 39-50.

[5] A. Bogomolnaia and H. Moulin (2001), "A New Solution to the Random Assignment Problem," *Journal of Economic Theory* 100:2 295-328.

[6] M. L. Breton and A. Sen (1999), "Preferences, Strategyproofness, and Decomposability." *Econometrica* 67: 605-628.

[7] G. Carroll (2010), "A General Equivalence Theorem for Allocation of Indivisible Objects," MIT, working paper.

[8] A. Castaneda (2006-07-28), "Iraqis House-Swapping to Escape Violence." The Washington Post.

[9] Y. Chen and T. Sönmez (2002), "Improving efficiency of on-campus housing: An experimental study," *American Economic Review* 92-5: 1669-1686.

[10] F.Y. Edgeworth (1881), "Mathematical Psychics."

[11] L. Ehlers (2002) "Coalitional strategy-proof house allocation," *Journal of Economic Theory* 105:2 298-317.

[12] L. Ehlers and B. Klaus (2007) "Consistent house allocation," *Economic Theory* 30:3 561-574.

[13] L. Ehlers, B. Klaus and S. Pápai (2002) "Strategy-proofness and population-monotonicity in house allocation problems," *Journal of Mathematical Economics* 38:3 329-339.

[14] Ö. Ekici (2011), "Reclaim-proof Allocation of Indivisible Objects," Carnegie Mellon University, working paper.

[15] H. Ergin (2000) "Consistency in house allocation problems." *Journal of Mathematical Economics* 34:1 77-97.

[16] P. Fishburn (1970), "Utility Theory for Decision Making." New York: John Wiley.

[17] A. Hylland and R. Zeckhauser (1979), "The efficient allocations of individuals to positions," *Journal of Political Economy* 87:2 293-314.

[18] K. L. Jackson (2008-03-01), "On Holiday with Vacation Home Exchange." The Star-Ledger.

[19] D. E. Knuth (1996), "An Exact Analysis of Stable Allocation," *Journal of Algorithms* 20:2 431-444.

[20] J. Ma (1994), "Strategy-proofness and strict core in a market with indivisibilities," *International Journal of Game Theory* 23:1 75-83.

[21] Mas-Collel, Andreu (1991), "On the Uniqueness of Equilibrium Once Again," in W. A. Barnett, B. Cornet, C. d'Aspremont, J. Gabszewicz, and A. Mas-Colell (eds.) *Equilibrium Theory and Aplications*, Cambridge University Press, pp.275-296.

[22] E. Miyagawa (2002) "Strategy-proofness and the core in house allocation problems," *Games and Economic Behavior* 38:2 347-361.

[23] S. Pápai (2000) "Strategyproof assignment by hierarchical exchange," *Econometrica* 68:6 1403-1433.

[24] S. Pápai (2007) "Exchange in a general market with indivisible goods," *Journal of Economic Theory* 132:1 208-235.

[25] P. A. Pathak and J. Sethuraman (2011), "Loteries in Student Assignment: An Equivalence Result," *Theoretical Economics* 6:1 1-17.

[26] M. Pycia and M. U. Ünver (2011) "Incentive Compatible Allocation and Exchange of Discrete Resources," Boston College Working Papers in Economics, No: 715.

[27] S. Rosenbloom (2006-06-29), "At Home in the World". The New York Times.

[28] A. E. Roth (1982), "Incentive Compatibility in a market with indivisible goods," *Economics Letters* 9:2 127-132.

[29] A. E. Roth and A. Postlewaite (1977), "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economics* 4:2 131-137.

[30] A. E. Roth, T. Sönmez, and M.U. Ünver (2004), "Kidney Exchange," *Quarterly Journal of Economics* 119:2 457-488.

[31] A. E. Roth, T. Sönmez, and M.U. Ünver (2005), "Pairwise Kidney Exchange," *Journal of Economic Theory* 125:2 151-188.

[32] A. E. Roth, T. Sönmez, and M.U. Ünver (2005), "A Kidney Exchange Clearinghouse in New England," *American Economic Review Papers and Proceedings* 95:2 376-380.

[33] M.A. Satterthwaite, and H. Sonnenschein (1981) "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies* 48, 587-597.

[34] L. Shapley and H. Scarf (1974), "On cores and indivisibility." *Journal of Mathematical Economics* 1:1 23-28.

[35] T. Sönmez and M. U. Ünver (2005), "House allocation with existing tenants: An equivalence," *Games and Economic Behavior* 52:1 153-185.

[36] T. Sönmez and M. U. Ünver (2010), "House allocation with existing tenants: A Characterization," *Games and Economic Behavior* 69:2 425-445.

[37] L.G, Svensson (1994), "Queue allocation of indivisible goods," *Social Choice and Welfare* 11:4 323-330.

[38] L.G, Svensson (1999) "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare* 16:4 557-567.

[39] W. Thomson (2007), "Fair allocation rules," Rochester Center for economic research working paper no. 539.

[40] M. U. Ünver, "Dynamic Kidney Exchange," *Review of Economic Studies*, 77:1 372-414.

[41] Ö. Yılmaz (2010), "The probabilistic serial mechanism with private endowments," *Games and Economic Behavior* 69:2 475-491.