# Local Realizability Toposes and a Modal Logic for Computability

*S. Awodey, L. Birkedal, D. Scott*

April 15, 1999

Revised January 17, 2000

Technical Report No. CMU-PHIL-99

**Philosophy**

**Methodology**

**Logic**

**Carnegie Mellon**

**Pittsburgh, Pennsylvania 15213**

*Under consideration for publication in Math. Struct. in Comp. Science*

# Local Realizability Toposes and a Modal Logic for Computability

STEVEN AWODEY[2] and LARS BIRKEDAL[1] and DANA S. SCOTT[1]

[1] *School of Computer Science, Carnegie Mellon University, Pittsburgh, PA 15213, USA.*
[2] *Department of Philosophy, Carnegie Mellon University, Pittsburgh, PA 15213, USA.*

*Received 17 January 2000*

This work is a step toward developing a logic for types and computation that includes both the usual spaces of mathematics and constructions and spaces from logic and domain theory. Using realizability, we investigate a configuration of three toposes, which we regard as describing a notion of relative computability. Attention is focussed on a certain local map of toposes, which we study first axiomatically, and then by deriving a modal calculus as its internal logic. The resulting framework is intended as a setting for the logical and categorical study of relative computability.

## 1. Introduction

We report here on the current status of research on the Logic of Types and Computation at Carnegie Mellon University (Scott et al., 1998). The general goal of this research program is to develop a logical framework for the theories of types and computability that includes the standard mathematical spaces alongside the many constructions and spaces known from type theory and domain theory. One purpose of this goal is to facilitate the study of computable operations and maps on data that are not necessarily computable, such as the space of all real numbers.

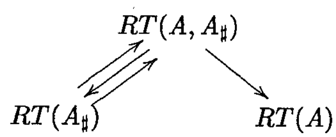
Concretely, in the research described here we use the realizability topos over the graph model $P\mathbb{N}$ of the (untyped) lambda calculus, together with the sub-graph model given by the recursively enumerable subsets, to represent the classical and computable worlds, respectively. There results a certain configuration of toposes that can be regarded as describing a notion of *relative computability*.† We study this configuration axiomatically, and derive a higher-order, modal logic in which to reason about it. The logic can then be applied to the original model to formalize reasoning about computability in that setting. Moreover, the resulting logical framework provides a general, categorical semantics and logical syntax for reasoning in a formal way about abstract computability, which it is hoped could also be useful for formally similar concepts, such as logical definability.

In somewhat more detail: Section 2 begins by recalling the standard realizability

† Not to be confused with the standard notion of computability relative to an oracle.

toposes $RT(A)$ and $RT(A_\sharp)$ resulting from a partial combinatory algebra $A$ and a subalgebra $A_\sharp$. We then identify a third category $RT(A, A_\sharp)$ which plays a key role; very roughly speaking, it represents the world of all (continuous) objects, but with only computable maps between them. This category $RT(A, A_\sharp)$ is a topos, the *relative realizability topos* on $A$ with respect to the subalgebra $A_\sharp$.

The toposes $RT(A)$ and $RT(A_\sharp)$ are not particularly well-related by themselves; the purpose of the relative realizability topos $RT(A, A_\sharp)$ is to remedy this defect. The *three* toposes are related to each other as indicated in the following diagram, in which the three functors on the left leg constitute a local geometric morphism, while the right leg is a logical morphism.




The local geometric morphism on the left is our chief concern and the focus of Section 3, which also mentions some examples and properties of these fairly well-understood maps of toposes. When we first encountered it, we were pleased to recognize our situation as an instance of one that F.W. Lawvere has already called attention to and dubbed an *adjoint cylinder* or, more colorfully, a *unity and identity of opposites* (Lawvere, 1991; Lawvere, 1989).

In Section 4 we present four sound and complete axioms for local maps of toposes. Actually, since the situation we are mainly interested in—*i.e.*, realizability—forces the local map to be localic, we give the axioms in a form that implies this condition. We simply mention here that a modification of axiom 2 about generators will accomodate all (bounded) local maps. More information concerning the abstract axiomatization of local maps (covering not only localic but also more general bounded local maps) can be found in (Awodey and Birkedal, 1999; Birkedal, 1999) — here we just recall the definitions and results needed in this paper. The axiomatization has been found useful in working with the particular situation we have in mind, but its general utility for local maps of toposes remains to be seen.

One application, of sorts, of the axioms for local maps is the investigation of their logical properties. These are given in Section 5 in the form of a logical calculus involving two propositional operations, written $\sharp\varphi$ and $\flat\varphi$, with $\sharp$ left adjoint to $\flat$. It turns out that $\sharp$ satisfies the S4 modal logic postulates for the box-operation. We here term the $\sharp$-calculus a *modal logic for computability*, since that is the interpretation we have in mind; but of course, this modal logic can be interpreted in any local topos. We intend to use it to investigate the logical relations that hold in the relative realizability topos; however, this aspect of our work is only just beginning.

In Section 6 we define and study *local triposes*, which are triposes that model the modal logic for computability. Any localic local map arises from a local tripos and any local tripos gives rise to a local map of toposes.

Note that any local map also induces a closely related pair of adjoint operations on *logical types* (objects), in addition to the ones on formulas (subobjects) studied here, relating our work to (Benaissa et al., 1999; Benton, 1995). The idea of a modal "computability" operator $\sharp$ is due to the senior author (January 1998) and was the original impetus for this work, parts of which are from the second author's doctoral thesis (Birkedal, 1999). The final brief section of the paper spells out the intended interpretation of the $\sharp$-calculus in the relative realizability topos $RT(A, A_\sharp)$.

## 2. Realizability toposes for computability

Let $(A, \cdot, \mathbf{K}, \mathbf{S})$ be a partial combinatory algebra (PCA); often we just denote it by its underlying set $A$. The binary operator $\cdot$ is the (partial) application and combinators $\mathbf{K}$ and $\mathbf{S}$ are taken to be part of the structure and not just required to exist.

Let $A_\sharp$ be a sub-PCA of $A$, that is $A_\sharp$ is a subset of $A$ containing $\mathbf{K}$ and $\mathbf{S}$ and closed under partial application. Intuitively, we are thinking of the realizers in $A$ as "continuous" realizers and of those in $A_\sharp$ as "computable" realizers. This intuition comes from the main example, where $A$ is $P\mathbb{N}$, the graph model on the powerset of the natural numbers, and $A_\sharp$ is $RE$, the recursively enumerable sub-graph-model. Note that the model $P\mathbb{N}$ has a continuum of (countable) sub-PCA's. As another example, one may consider Kleene's function realizability with $A = \mathbb{N}^\mathbb{N}$ and with $A_\sharp$ the set of total recursive functions. One may also consider van Oosten's combinatory algebra $\mathcal{B}$ for sequential computation and its effective subalgebra $\mathcal{B}_{\mathit{eff}}$, see (van Oosten, 1999; Longley, 1998).

The PCA's $A_\sharp$ and $A$ give rise to two realizability toposes $RT(A_\sharp)$ and $RT(A)$ in the standard way (Hyland et al., 1980). One may think of $RT(A)$ as a universe where all objects and all maps are realized by continuous realizers. Likewise, $RT(A_\sharp)$ may be thought of as a universe where all objects and all maps are realized by computable realizers. Unfortunately, these two toposes are not very well related; in particular, it is not clear how to talk about computable maps operating on continuous objects, which is what one would like to do for the purposes of, *e.g.*, computable analysis (Pour-El and Richards, 1989). Thus, one is led to introduce another realizability topos, $RT(A, A_\sharp)$, where, intuitively, equality on all objects is realized by continuous realizers and all maps are realized by computable realizers.[‡]

The topos $RT(A, A_\sharp)$ is constructed by modifying the underlying tripos for $RT(A)$ in the following way. The non-standard predicates $\varphi$, $\psi$ on a set $I$ are still functions $I \to PA$ into the powerset of $A$ and the Heyting pre-algebra operations are the same as in the tripos underlying $RT(A)$. The modification is in the definition of the entailment relation: we say $\varphi \vdash \psi$ over $I$ iff there is a realizer $a$ in $A_\sharp$ (not just in $A$) such that for all $i$ in $I$, all $b \in \varphi(i)$, $a \cdot b$ is defined and $a \cdot b \in \psi(i)$. In the terminology of Pitts (Pitts, 1981), we have changed the "designated truth values" to be those subsets of $A$ which have a non-empty intersection with $A_\sharp$. Denote this new tripos by $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$. Then $RT(A, A_\sharp)$ is the topos $\mathbf{Set}[P]$ represented by $P$.

Explicitly, objects of $RT(A, A_\sharp)$ are pairs $(X, \approx_X)$ with $X$ a set and $\approx_X : X \times X \to PA$

[‡] We first learned about the topos $RT(A, A_\sharp)$ from Thomas Streicher in February 1998, but the construction has actually been known for a long time; see (Pitts, 1981, Page 15, item (ii)).

a non-standard equality predicate with computable realizers for symmetry and transitivity. Morphisms from $(X, \approx_X)$ to $(Y, \approx_Y)$ are equivalence classes of functional relations $F\colon X \times Y \to PA$ with computable realizers proving that $F$ is a functional relation. Two such functional relations $F$ and $G$ are equivalent iff there are computable realizers showing them equivalent. We now see that intuitively, it makes sense to think of objects of $RT(A, A_\sharp)$ as objects with *continuous* realizers for existence and equality elements, and of morphisms $f = [F]$ as *computable* maps, since the realizers for the functionality of $F$ are required to be computable.

## 3. Geometry of the realizability toposes for computability

Let $Q\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ be the standard realizability tripos on $A_\sharp$, *i.e.*, the tripos underlying $RT(A_\sharp)$. We now define three **Set**-indexed functors between $Q$ and $P$:

$$\Delta\colon Q \to P \quad \text{and} \quad \Gamma\colon P \to Q \quad \text{and} \quad \nabla\colon Q \to P.$$

These are defined as follows. Over a set $I$, we have

$$\begin{array}{rcl} \Delta_I(\psi\colon I \to PA_\sharp)(i) & = & \psi(i) \\ \Gamma_I(\varphi\colon I \to PA)(i) & = & A_\sharp \cap \varphi(i) \\ \nabla_I(\psi\colon I \to PA_\sharp)(i) & = & \bigcup_{\varphi \in PA} \Big(\varphi \wedge \big(A_\sharp \cap \varphi \supset \psi(i)\big)\Big), \end{array}$$

where $\wedge$ and $\supset$ are calculated as in $P(1)$.

It is easy to see that $\Delta_I$ and $\Gamma_I$ are well-defined functors; to show this for $\nabla_I$ suppose that $\psi \vdash_I^Q \psi'$, where we write $\vdash_I^Q$ for the ordering in $Q(I)$. Then there is a realizer $c \in A_\sharp$ such that

$$c \in \bigcap_{i \in I} \big(\psi(i) \supset \psi'(i)\big).$$

Let

$$d = \lambda x.\, \langle \pi x, \lambda y.\, c(\pi'(x)(y)) \rangle.$$

Then $d \in A_\sharp$ (since $c \in A_\sharp$) and it is easy to verify that

$$d \in \bigcap_{i \in I} \big(\nabla_I(\psi)(i) \supset \nabla_I(\psi')(i),$$

as required.

Recall from (Hyland et al., 1980; Pitts, 1981) that a geometric morphism $f$ from a tripos $P$ to a tripos $Q$ is a pair of indexed functors $(f^*, f_*)$ with $f^*$ indexed left adjoint to $f_*$ such that with $f_I^*$ is left exact, for all $I$ in the base category.

**Theorem 3.1.** Under the foregoing definitions it follows that

— $(\Delta, \Gamma)$ is a geometric morphism of triposes from $P$ to $Q$.
— $(\Gamma, \nabla)$ is a geometric morphism of triposes from $Q$ to $P$.
— For all $I \in \mathbf{Set}$, $\Delta_I$ and $\nabla_I$ are both full and faithful.

*Proof.* It is easy to see that $\Delta$ is (indexed) left adjoint to $\Gamma$ using that $A_\sharp$ is closed under the partial application of $A$. Further, it is clear that $\Delta$ preserves finite limits and is full and faithful since the ordering in $P$ and $Q$ are defined in the same way (requiring computable realizers).

It remains to show that

$$\varphi \vdash_I^P \nabla\psi \iff \Gamma\varphi \vdash_I^Q \psi,$$

for all $\varphi \in P(I)$ and all $\psi \in Q(I)$ (where, of course, $\vdash_I^P$ is the ordering in $P(I)$ and $\vdash_I^Q$ is the ordering in $Q(I)$). To this end, suppose $\varphi \vdash_I^P \nabla\psi$, via a realizer $c \in A_\sharp$. Let

$$d = \lambda x.\, \pi(c(x))(\pi'(c(x))) \in A_\sharp.$$

It is easy to verify that $d$ is a a realizer for $\Gamma\varphi \vdash_I^Q \psi$.

For the other direction, suppose $d \in A_\sharp$ is a realizer for $\Gamma\varphi \vdash_I^Q \psi$. Then

$$c = \lambda x.\, \langle x, \lambda y.\, d(y)\rangle \in A_\sharp$$

is a realizer for $\varphi \vdash_I^P \nabla\psi$.

Since $\Delta$ is full and faithful and since $\Delta \dashv \Gamma \dashv \nabla$, also $\nabla$ is full and faithful, completing the proof of the theorem. □

By Proposition 4.7 in (Pitts, 1981), these geometric morphisms lift to two geometric morphisms between the toposes, as in

$$RT(A_\sharp) \underset{\nabla}{\overset{\Delta}{\rightleftarrows}} RT(A, A_\sharp) \quad (\Gamma: RT(A, A_\sharp) \to RT(A_\sharp)), \qquad \Delta \dashv \Gamma \dashv \nabla.$$

(Here we do not distinguish notationally between the functors at the tripos level and at the topos level). In particular, $\Delta$ preserves finite limits. Moreover, $\Delta, \nabla\colon RT(A_\sharp) \to RT(A, A_\sharp)$ are easily shown to also be both full and faithful. The resulting geometric morphism $(\Delta, \Gamma)\colon RT(A, A_\sharp) \to RT(A)$ is therefore a (connected) surjection, while the one given by $(\Gamma, \nabla)\colon RT(A_\sharp) \to RT(A, A_\sharp)$ is an embedding. Note that $\Gamma \circ \nabla \cong 1 \cong \Gamma \circ \Delta$. It then follows by standard results that there is a Lawvere-Tierney topology $j$ in $RT(A, A_\sharp)$ such that $RT(A_\sharp)$ is equivalent to the category $\mathrm{Sh}_j(RT(A, A_\sharp))$ of $j$-sheaves. We remark that one can show that in general the geometric morphism $(\Delta, \Gamma)\colon RT(A, A_\sharp) \to RT(A_\sharp)$ is *not* open (Birkedal, 1999).

The following theorem was known to Martin Hyland but apparently has never been published. We include a proof here.

**Theorem 3.2.** Let $\mathbb{C}$ be a finitely complete category and let $P$ and $Q$ be $\mathbb{C}$-triposes. Suppose $f = (f^*, f_*) : P \to Q$ is a geometric morphism of triposes. Then $\mathbb{C}[P]$ is localic over $\mathbb{C}[Q]$ via the induced geometric morphism $f = (f^*, f_*)\colon \mathbb{C}[P] \to \mathbb{C}[Q]$.

*Proof.* We want to prove that $\mathbb{C}[P]$ is equivalent to the category of $\mathbb{C}[Q]$-valued sheaves on the internal locale $f_*(\Omega_{\mathbb{C}[P]})$ in $\mathbb{C}[Q]$, where $\Omega_{\mathbb{C}[P]}$ is the subobject classifier in $\mathbb{C}[P]$. As usual (Johnstone, 1977) it suffices to show that, for all $X \in \mathbb{C}[P]$, there exists a

$Y \in \mathbb{C}[Q]$ and a diagram

$$\begin{array}{ccc} S & \rightarrowtail & f^*Y \\ \downarrow & & \\ X & & \end{array}$$

in $\mathbb{C}[P]$ presenting $X$ as a subquotient of $f^*Y$ for $Y$ an object of $\mathbb{C}[Q]$. Write $\nabla_P \colon \mathbb{C} \to \mathbb{C}[P]$ for the functor $I \mapsto (I, \exists_{\delta_I}(T))$, where $\delta_I \colon I \mapsto I \times I$ is the diagonal map (the "constant objects functor" (Pitts, 1981)).

By a familiar property of realizability toposes, we have that for all $X \in \mathbb{C}[P]$, there exists an object $I \in \mathbb{C}$ and a diagram

$$\begin{array}{ccc} S & \rightarrowtail & \nabla_P(I) = (I, \exists_{\delta_I}(T)) \\ \downarrow & & \\ X & & \end{array} \tag{1}$$

in $\mathbb{C}[P]$ presenting $X$ as a subquotient of a constant object $\nabla_P(I)$. Now since $f^*$ is the inverse image of a geometric morphism of triposes, $f^*$ preserves existential quantification (as an indexed left adjoint), so $f^*(\nabla_Q(I)) \cong \nabla_P(I)$, and the diagram in (1) is the required diagram. □

We summarize the consequences of Theorems 3.1 and 3.2 in the following:

**Corollary 3.3.** $RT(A, A_\sharp)$ is localic over $RT(A_\sharp)$ via the geometric morphism $(\Delta, \Gamma)$ : $RT(A, A_\sharp) \to RT(A_\sharp)$, which is a *localic local map of toposes*, since $\Gamma$ has a right adjoint $\nabla$, for which $\Gamma \circ \nabla \cong 1$.

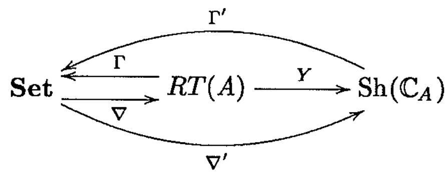
Local maps have been studied by (Johnstone and Moerdijk, 1989), and provide an instance of what Lawvere (Lawvere, 1991; Lawvere, 1989) has called *unity and identity of opposites.* The idea is that the full subcategories $\Delta(RT(A_\sharp))$ and $\nabla(RT(A_\sharp))$ are each equivalent to $RT(A_\sharp)$, and yet are "opposite" in the sense of being coreflective and reflective, respectively, in $RT(A, A_\sharp)$. We think of the objects in $\nabla(RT(A_\sharp))$ as sheaves, and here we think of those in $\Delta(RT(A_\sharp))$ as "computable".

Examples of local maps in addition to the basic ones mentioned in (Johnstone and Moerdijk, 1989) include the following:

(1) Let $RT(A)$ be a realizability topos, and let $i \colon \mathbb{C}_A \to RT(A)$ be a full subcategory of partitioned assemblies of suitably large, bounded cardinality, so that $\mathbb{C}_A$ is a small generating subcategory of projectives. The covering families in $\mathbb{C}_A$ are to be those which are epimorphic in $RT(A)$. Then the Grothendieck topos $\mathrm{Sh}(\mathbb{C}_A)$ is local; let us write its structure maps as $\Delta' \dashv \Gamma' \dashv \nabla' \colon \mathbf{Set} \to \mathrm{Sh}(\mathbb{C}_A)$. There is a restricted Yoneda embedding,

$$Y = RT(A)(i(\cdot), -) \colon \; RT(A) \to \mathrm{Sh}(\mathbb{C}_A),$$

for which the diagram below commutes, in the sense that $\Gamma \cong \Gamma' \circ Y$, $\nabla' \cong Y \circ \nabla$.




Thus we can regard $\Gamma\colon RT(A) \to \mathbf{Set}$ as what *would be* the direct image of a local map, if only $RT(A)$ had enough colimits.

(2) Let $\mathbb{C}$ be a small category with finite limits and $i\colon \mathbb{D} \hookrightarrow \mathbb{C}$ a full subcategory, closed under the same. The geometric morphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ with direct image the restriction $i^*$ along $i$ is then a local map. The image of $\widehat{\mathbb{D}}$ under the full embedding $i_!$ (where $i_! \dashv i^*$) then consists of those presheaves $P$ on $\mathbb{C}$ for which

$$PC \cong \operatorname*{colim}_{D,h:C\to D} PD.$$

These are the objects that we are interested in as candidates for being "computable", when $\mathbb{D}$ represents the computable subcategory. They are the ones termed "discrete" in the sequel.

Regarding the choice of terminology, we use the term "discrete" by analogy to topological examples. We would have liked to call these objects "cosheaves" since they are the objects that are *co*orthogonal to the morphisms inverted by **a**, and sheaves are those that are orthogonal. However, "cosheaf" has already been used to describe something else, namely a "covariant sheaf".

## 4. Axioms for localic local maps

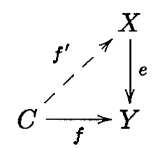
In this section we present a set of axioms for localic local maps. The axioms are sound and complete, in the sense that whenever a given topos satisfies the axioms then it gives rise to a localic local map of toposes and, moreover, any localic local map of toposes satisfies the axioms. See (Awodey and Birkedal, 1999; Birkedal, 1999) for more information about the abstract axiomatization of (not only localic but also more general bounded) local maps. Here we just recall the definitions and results needed in the following to describe a *modal logic for computability.* First we need a couple of definitions.

For the remainder of this section let $\mathcal{E}$ be an elementary topos and $j$ a Lawvere-Tierney topology in $\mathcal{E}$. We write $V \mapsto \overline{V}$ for the associated closure operation on subobjects $V \rightarrowtail X$. We say that $j$ is **principal** if, for all $X \in \mathcal{E}$, the closure operation on $Sub(X)$ has a left adjoint $U \mapsto \mathring{U}$, called **interior**, that is,

$$\mathring{U} \leq V \iff U \leq \overline{V} \qquad \text{in } Sub(X).$$

The interior operation is *not* assumed to be natural; that is, it is not assumed to commute with pullbacks. It follows that in general the interior operation is not induced by an internal map on the subobject classifier $\Omega$ in the $\mathcal{E}$, and in that sense is not a *logical operation* (in the internal logic of $\mathcal{E}$).

The interior operation extends to a functor $\mathcal{E} \to \mathcal{E}$, since, whenever $f\colon X \to Y$, we have $\overset{\circ}{X} \leq f^*(\overset{\circ}{Y})$. We say that an object $X \in \mathcal{E}$ is **open** if $\overset{\circ}{X} \cong X$. An object is open iff the interior of its diagonal equals its diagonal. An object $C \in \mathcal{E}$ is called **discrete** if it is **coorthogonal** to all morphisms inverted by the associated sheaf functor $\mathbf{a}$; that is, $C$ is discrete if for all $e\colon X \to Y$ such that $\mathbf{a}(e)$ is an isomorphism, for all $f\colon C \to Y$, there exists a unique $f'\colon C \to X$ such that



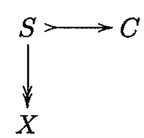
commutes.

Recall, *e.g.*, from (Johnstone, 1977), that a sheaf can be characterized as an object which is orthogonal to all morphisms inverted by $\mathbf{a}$, and that it suffices to test orthogonality just with respect to the dense monomorphisms. For discrete objects there is a similar simplification: an object is discrete iff it is coorthogonal to all **codense epimorphisms**, where an epimorphism $e\colon X \twoheadrightarrow Y$ is codense iff the interior of its kernel is included in the diagonal of $X$. We write $\mathrm{D}_j\mathcal{E}$ for the full subcategory of $\mathcal{E}$ on the discrete objects.

Now we propose the following **axioms for a localic local map** on a topos $\mathcal{E}$ with topology $j$.

**Axiom 1.** The topology $j$ is principal.

**Axiom 2.** For all $X \in \mathcal{E}$, there exists a discrete object $C$ and a diagram



in $\mathcal{E}$, presenting $X$ as a subquotient of $C$.

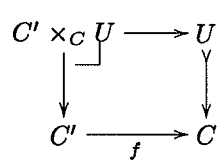
**Axiom 3.** For all discrete $C \in \mathcal{E}$, if $X \rightarrowtail C$ is open, then $X$ is also discrete.

**Axiom 4.** For all discrete $C, C' \in \mathcal{E}$, the product $C \times C'$ is again discrete.

Let $\mathcal{E}$ be a topos with a topology $j$ satisfying the Axioms 1–4 for localic local maps. We can then prove (Awodey and Birkedal, 1999; Birkedal, 1999):

**Theorem 4.1.** The category of discrete objects $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$, that is, the inclusion $\Delta\colon \mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ has a right adjoint. Moreover, $\mathrm{D}_j\mathcal{E}$ is a topos, $\Delta$ is left exact, and $(\Delta, \Gamma)\colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ is a localic local map.

**Corollary 4.2.** For any discrete $C, C' \in \mathcal{E}$ and any $f\colon C' \to C$, and all open subobjects $U \rightarrowtail C$, the pullback $C' \times_C U$ indicated in:



is open.

**Theorem 4.3.** Every localic local map of toposes satisfies Axioms 1-4 for localic local maps.

*Proof.* See (Awodey and Birkedal, 1999; Birkedal, 1999). □

## 5. A modal logic for computability

Let $\mathcal{E}$ be a topos with a topology $j$ satisfying the axioms set out in the previous section. In this section our goal is to describe a *logic* with which one can reason about *both* of the two toposes $\mathcal{E}$ and $\mathrm{D}_j\mathcal{E}$. This will then apply to $RT(A, A_\sharp)$ and $RT(A_\sharp)$, see Section 7.

As a first attempt, one may consider the ordinary internal logic of $\mathcal{E}$ extended with a closure operator induced by the topology $j$ and try to extend it further with a logical operator corresponding to the interior operation. But since interior does not commute with pullback in general, it is not a logical operation on all subobjects of objects of $\mathcal{E}$. However, since the interior of an object $X$ is the least dense subobject of $X$, one may instead add a new atomic predicate $U_X$ for each type $X$ and write down axioms expressing that it is the least dense subobject. This is straightforward. But, as yet, we do not have a convenient internal logical characterization in these terms of the discrete objects.

Instead we shall describe another approach where types and terms are objects and morphisms of $\mathrm{D}_j\mathcal{E}$ and predicates are all the predicates in $\mathcal{E}$ on objects from $\mathrm{D}_j\mathcal{E}$. More precisely, we consider the internal logic of the fibration $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$ obtained by change-of-base as indicated in:

$$\begin{array}{ccc} Pred & \longrightarrow & Sub(\mathcal{E}) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{D}_j\mathcal{E} & \xrightarrow[\Delta]{} & \mathcal{E}. \end{array}$$

Thus a predicate on an object $X \in \mathrm{D}_j\mathcal{E}$ is a subobject of $\Delta X$ in $\mathcal{E}$. Since

$$Sub_{\mathcal{E}}(\Delta X) \cong \mathcal{E}(\Delta X, \Omega_{\mathcal{E}}) \cong \mathrm{D}_j\mathcal{E}(X, \Gamma\Omega_{\mathcal{E}}), \tag{2}$$

the internal locale $\Gamma\Omega_{\mathcal{E}}$ is a generic object for the fibration $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$. Hence this internal logic is many-sorted higher-order intuitionistic logic.

Note that, since $\mathcal{E}$ is localic over $\mathrm{D}_j\mathcal{E}$, we can completely describe $\mathcal{E}$ in this internal logic in the standard way (Fourman and Scott, 1979) as partial equivalence relations and functional relations between such. In other words, applying the tripos-to-topos construction to the tripos $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$ results in a topos equivalent to $\mathcal{E}$.

By Corollary 4.2, the interior operation is a logical operation on predicates; we denote it by $\sharp$. Note that the ordinary logic of $\mathrm{D}_j\mathcal{E}$ is obtained by restricting attention to predicates of the form $\sharp\varphi$ — in fact it suffices to restrict attention to predicates with a $\sharp$ in front of every $\supset$ and $\forall$ subformula. The topology $j$ in $\mathcal{E}$ induces a closure operator, which we are pleased to denote $\flat$, on predicates in this internal logic.

We now describe how the $\sharp$ and $\flat$ operations can be axiomatized. Logical entailment is written $\Gamma \mid \varphi \vdash \psi$, where $\Gamma$ is a context of the form $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ giving types $\sigma_i$ to variables $x_i$, and where $\varphi$ and $\psi$ are formulas with free variables in $\Gamma$. There are the usual rules of many-sorted higher-order intuitionistic logic plus the following axioms and rules:

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi}\ (3) \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi}\ (4)$$

$$\frac{}{\Gamma \mid \top \vdash \sharp(\top)}\ (5) \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)}\ (6)$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\overline{\Gamma \mid \varphi \vdash \flat\psi}}\ (7) \qquad \frac{}{x : \sigma, y : \sigma \mid x = y \vdash \sharp(x = y)}\ (8)$$

One can then show that $\sharp$ has the formal properties of the box operator in the modal logic S4, *i.e.*, for formulas $\varphi$ and $\psi$ in context $\Gamma$:

$$\vdash \sharp\varphi \supset \varphi$$
$$\vdash \sharp(\varphi \supset \psi) \supset (\sharp\varphi \supset \sharp\psi)$$
$$\vdash \sharp\varphi \supset \sharp\sharp\varphi$$

and

$$\frac{\vdash \varphi}{\vdash \sharp\varphi}\ (9)$$

We therefore refer to this logic as a **modal logic for computability**.

We remark that the following principles of inference for quantifiers can be derived from (3)–(8):

$$\frac{\Gamma \mid \top \vdash \sharp\forall x{:}X.\, \varphi}{\overline{\Gamma \mid \top \vdash \forall x{:}X.\, \sharp\varphi}}\ (10)$$

$$\frac{}{\Gamma \mid \sharp\exists x{:}X.\, \varphi \dashv\vdash \exists x{:}X.\, \sharp\varphi}\ (11)$$

One can also show that, for any formula $\varphi$ in context $\Gamma$,

$$\sharp(\sharp\varphi \supset \psi) \dashv\vdash \sharp\varphi \supset \sharp\psi. \qquad (12)$$

Quite generally, the modal logic of any local map of toposes

$$\Gamma : \mathcal{E} \to \mathcal{F}, \qquad \Delta \dashv \Gamma \dashv \nabla$$

can be used to compare the internal logic of $\mathcal{E}$ with that of $\mathcal{F}$, since the types are then the discrete objects $E$ in $\mathcal{E}$, *i.e.*, those for which $E \cong \Delta\Gamma E$, and

$$Sub_{\mathcal{F}}(\Gamma E) \cong OpenSub_{\mathcal{E}}(E),$$

where $OpenSub_{\mathcal{E}}(E) \subseteq Sub_{\mathcal{E}}(E)$ is the subposet of open subobjects of $E$ in $\mathcal{E}$. Observe, *e.g.*, that the natural numbers object $N$ is among the discrete objects, and that the identity relation on any discrete object is open.

We now describe two example applications of the modal logic.

(1) As in (Awodey and Birkedal, 1999) call a formula $\vartheta$ **stable** if it is built up from atomic predicates (including equations) and first-order logic and if for every subformula of the form $\varphi \supset \psi$, the formula $\varphi$ has no $\forall$ or $\supset$. For any sentence $\vartheta$, we write $\mathcal{F} \models \vartheta$ to mean that $\vartheta$ holds in the standard internal logic of $\mathcal{F}$ with basic types $\sigma$ interpreted by objects $X_\sigma$ of $\mathcal{F}$ and atomic formulas $R$ on type $\sigma$ interpreted as subobjects $S_R \rightarrowtail X_\sigma$. We then write $\mathcal{E} \models \vartheta$ to mean that $\vartheta$ holds in the standard logic of $\mathcal{E}$ with basic types $\sigma$ interpreted by objects $\Delta X_\sigma$ and atomic formulas $R$ interpreted by $\Delta S_R \rightarrowtail \Delta X_\sigma$.

**Proposition 5.1.** For any stable sentence $\vartheta$,

$$\mathcal{F} \models \vartheta \qquad \text{iff} \qquad \mathcal{E} \models \vartheta.$$

*Proof Sketch* There are the interpretations $[\![-]\!]_{\mathcal{F}}$ and $[\![-]\!]_{\mathcal{E}}$ for which one shows by induction that for any stable formula $\vartheta$

$$\Delta[\![\vartheta]\!]_{\mathcal{F}} = [\![\sharp\vartheta]\!]_{\mathcal{E}},$$

using the fact that $\Gamma$ preserves $\forall$ along maps between discrete objects. Thus for any stable sentence $\vartheta$:

$$\begin{aligned}
\mathcal{F} \models \vartheta \qquad & \text{iff} \quad 1 = [\![\vartheta]\!]_{\mathcal{F}} \\
& \text{iff} \quad 1 = \Delta 1 = \Delta[\![\vartheta]\!]_{\mathcal{F}} = [\![\sharp\vartheta]\!]_{\mathcal{E}} \\
& \text{iff} \quad \mathcal{E} \models \sharp\vartheta \\
& \text{iff} \quad \mathcal{E} \models \vartheta.
\end{aligned}$$

□

The proposition can be used to show that, *e.g.*, if $\mathcal{F}$ has choice for functions from $N$ to $N$ in the external sense, then so does $\mathcal{E}$. Indeed, let $R$ by any relation (not necessarily open) on $N$ in $\mathcal{E}$ and suppose that

$$\mathcal{E} \models \forall n{:}N.\, \exists m{:}N.\, R(n,m).$$

Then we reason informally as follows

| | |
|---|---|
| $\mathcal{E} \models \forall n{:}N.\, \exists m{:}N.\, R(n,m)$ | |
| $\mathcal{E} \models \sharp\forall n{:}N.\, \exists m{:}N.\, R(n,m)$ | by (9) |
| $\mathcal{E} \models \forall n{:}N.\, \sharp\exists m{:}N.\, R(n,m)$ | by (10) |
| $\mathcal{E} \models \forall n{:}N.\, \exists m{:}N.\, \sharp R(n,m)$ | by (11) |
| $\mathcal{F} \models \forall n{:}N.\, \exists m{:}N.\, \sharp R(n,m)$ | by stability |
| $\mathcal{F} \models \forall n{:}N.\, \sharp R(n,f(n))$ | for some $f\colon N \to N$ by AC in $\mathcal{F}$ |
| $\mathcal{E} \models \forall n{:}N.\, \sharp R(n,f(n))$ | by stability |
| $\mathcal{E} \models \forall n{:}N.\, R(n,f(n))$ | by (3) |

(2) As in (Birkedal, 1999), we show that if the topos of discrete objects satisfies the arithmetic form of Church's Thesis (in the sense of, *e.g.*, (Troelstra and van Dalen, 1988; Troelstra, 1973)), then a $\sharp$'ed version is satisfied by $\mathcal{E}$.

Observe that $\mathcal{E}$ has a natural numbers object if and only if $\mathrm{D}_j\mathcal{E}$ does, because both $\Delta\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ and $\Gamma\colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ are inverse images of geometric morphisms, which preserve the natural numbers object (Johnstone, 1977, Proposition 6.12). Let $N$ be a natural numbers object in $\mathrm{D}_j\mathcal{E}$.

Recall from Kleene's Normal Form Theorem that the basic predicates of recursion theory can be defined from Kleene's $T$-predicate and output function $U\colon N \to N$, see, *e.g.*, (Troelstra and van Dalen, 1988). The predicates $T \rightarrowtail N \times N \times N$ and $U(-) = (-)$ are both primitive recursive. Hence their interpretation is preserved by the inclusion $\mathrm{D}_j\mathcal{E} \hookrightarrow \mathcal{E}$ (see (Birkedal, 1999) for a detailed argument).

Recall from (Troelstra and van Dalen, 1988, Section 4.3), that the **arithmetical form of Church's Thesis** is the schema

$$\mathrm{CT}_0 \qquad \forall n.\, \exists m.\, \varphi(n,m) \supset \exists k.\, \forall n.\, \exists m.\, \big(\varphi(n, Um) \wedge T(k,n,m)\big),$$

where all the variables range over the type of natural numbers $\mathsf{N}$ and where $\varphi$ is a formula.

For a first-order formula $\varphi$, we denote by $|\varphi|$ the formula which is like $\varphi$ except that there is a $\sharp$ in front of every $\forall$ and $\supset$ subformula.

**Proposition 5.2.** Let $\varphi$ be a formula and suppose that the $\varphi$ instance of $\mathrm{CT}_0$ holds in $\begin{smallmatrix} Sub\,\mathrm{D}_j\mathcal{E} \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$. Then the following formula holds in $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$:

$$\sharp(\forall n.\, \exists m.\, |\varphi|(n,m)) \supset \exists k.\, \forall n.\, \exists m.\, \big(|\varphi|(n, Um) \wedge T(k,n,m)\big).$$

*Proof.* By the fact that $\mathrm{D}_j\mathcal{E} \vDash \varphi$ iff $|\varphi|$ holds in $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$ we get that

$$\sharp(\forall n.\, \exists m.\, |\varphi|(n,m)) \supset \exists k.\, \sharp\forall n.\, \exists m.\, \big(|\varphi|(n, Um) \wedge T(k,n,m)\big)$$

holds in $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$. The required then follows by the modal logic using (11) and (12). □

Thus, in particular, if all geometric instances (*i.e.*, with $\varphi$ built up using only atomic relations, and $\top$, $\wedge$, $\bot$, $\vee$, $\exists$) of $\mathrm{CT}_0$ hold in $\mathrm{D}_j\mathcal{E}$, then all geometric instances of

$$\mathrm{CT}_0^\sharp \qquad \sharp(\forall n.\, \exists m.\, \varphi(n,m)) \supset \exists k.\, \forall n.\, \exists m.\, \big(\varphi(n, Um) \wedge T(k,n,m)\big)$$

hold in $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$.

## 6. Local Triposes

In this section we introduce the notion of a *local tripos*, assuming familiarity with the basic theory of triposes as set out in (Pitts, 1981).

The model of the modal logic for computability given by the fibration $\begin{smallmatrix} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{smallmatrix}$ defined above is in fact a tripos, namely the standard tripos on the internal locale $\Gamma\Omega_{\mathcal{E}}$, see (2) above. Indeed, we can give the following general definition.

**Definition 6.1.** Let $P = \mathcal{F}(-,\Sigma)$ be a canonically-presented tripos on an object $\Sigma$ in a topos $\mathcal{F}$. The tripos $P$ is said to be **local** if it comes together with maps $I, J\colon \Sigma \to \Sigma$ in $\mathcal{F}$ satisfying:

1 $p\colon \Sigma \mid Ip \vdash p$
2 $p\colon \Sigma \mid Ip \vdash IIp$
3 $\emptyset \mid \top \vdash I(\top)$
4 $p, q\colon \Sigma \mid Ip \wedge Iq \vdash I(p \wedge q)$
5

$$\frac{p, q\colon \Sigma \mid Ip \vdash q}{p, q\colon \Sigma \mid p \vdash Jq}$$

all in the logic of $P$.

The axioms and rules that have to hold for $I$ and $J$ are of course just as for $\sharp$ and $\flat$ in the modal logic for local maps in Section 5.§ In other words, a local tripos models the modal logic for local maps. The following proposition is clear:

**Proposition 6.2.** Let $P = \mathcal{F}(-,\Sigma)$ be a local tripos *qua* $I, J\colon \Sigma \to \Sigma$. Then $J$ is a Lawvere-Tierney topology on $P$ (as defined in (Pitts, 1981)).

Let $P = \mathcal{F}(-,\Sigma)$ be a local $\mathcal{F}$-tripos qua $I, J\colon \Sigma \to \Sigma$. We define a new canonically presented $\mathcal{F}$-tripos $P_I$ as follows (in the following we prove that $P_I$ so defined indeed is a tripos). Let $I\Sigma$ be the image of $I$ in $\mathcal{F}$. Tripos $P_I$ is canonically presented on $I\Sigma$. The ordering is defined as in $P$, that is, for $\varphi, \psi \in \mathcal{F}(X, I\Sigma)$ ($P_I$'s fibre over $X \in \mathcal{F}$), we have $\varphi \vdash^{P_I} \psi$ iff $\varphi \vdash^{P} \psi$.

Since $J$ is a topology by Proposition 6.2 we have a well-defined $\mathcal{F}$-tripos $P_J$ as in (Pitts, 1981, Chapter 5). We recall that $P_J$ is canonically presented on $\Sigma$ and that entailment is defined as $\varphi \vdash^{P_J} \psi$ iff $\varphi \vdash^{P} J\psi$.

It is easy to verify that composing with $I\colon \Sigma \to \Sigma$ gives an indexed functor, also denoted $I$, from $P_J$ to $P_I$ over $\mathcal{F}$. Likewise, composing with $J\colon \Sigma \to \Sigma$ gives a fibred functor, also denoted $J$, from $P_I$ to $P_J$ over $\mathcal{F}$.

**Lemma 6.3.** Functor $I$ is a fibred left adjoint to $J$ and the triposes $P_I$ and $P_J$ are equivalent, as indexed categories over $I$, via the functors $I$ and $J$.

*Proof.* Straightforward. □

By the lemma it follows that $P_I$ has all the first-order structure required in the definition of a tripos (since it is defined categorically and thus preserved by equivalence functors). It is clear that $id\colon I\Sigma \to I\Sigma$ is a generic object for $P_I$ and thus $P_I$ is indeed a tripos as claimed.

§ The only exception is that we in the definition of local tripos have left out the rule for equality – the rule for equality follows since equality in a tripos is given using existential quantification and truth $\top$ and $I$ commutes with existential quantification as a left adjoint and with $\top$ by item (3).

**Theorem 6.4.** Let $P = \mathcal{F}(-,\Sigma)$ be a local $\mathcal{F}$-tripos qua $I, J\colon \Sigma \to \Sigma$. Then $P$ gives rise to a localic local map of toposes from $\mathcal{F}[P]$ to $\mathcal{F}[P_I]$.

*Proof.* We define three indexed functors over $\mathcal{F}$, as in

$$P_I \underset{J}{\overset{\Delta}{\rightleftarrows}} P \quad (\Delta\colon P_I \to P,\ I\colon P \to P_I,\ J\colon P_I \to P) \qquad \Delta \dashv I \dashv J.$$

Functor $\Delta$ is simply the inclusion functor. Functor $I$ is induced by composing with $I\colon \Sigma \to \Sigma$ and functor $J$ is induced by composing with $J\colon \Sigma \to \Sigma$. It is easy to see that all three functors are indexed, since $P$ and $P_I$ are canonically presented, and that $\Delta$ is left adjoint to $I$ and that $I$ is left adjoint to $J$. The functor $\Delta$ is left exact since $I\Sigma$ is closed under finite limits in $\Sigma$ by items (3) and (4) in the definition of a local tripos. Hence $(\Delta, I)$ is a geometric morphism of triposes, as is also $(I, J)$. Functor $\Delta$ is clearly full and faithful and thus also $J$ is full and faithful (it is also straightforward to verify directly that $J$ is full and faithful).

It follows now, in the same way as in Section 3, that the geometric morphism from $\mathcal{F}[P]$ to $\mathcal{F}[Q]$ induced by $(\Delta, I)$ is a localic local map. □

Conversely, one can easily see:

**Theorem 6.5.** Every localic local map of toposes arises from a local tripos (in the way given by the proof of Theorem 6.4).

**Example 6.6.** The relative realizability tripos $P$ from Section 2 is a local tripos. The maps $I$ and $J$ are given by $\Delta\Gamma$ and $\nabla\Gamma$ respectively, see Section 3.

In this realizability example, the topos $\mathcal{E}$ is the topos $RT\,(A_\sharp)$ and the topos $\mathrm{D}_j\mathcal{E}$ is the topos $RT\,(A, A_\sharp)$. Note that if we were just given the topos $RT\,(A_\sharp)$ with the internal locale $\Gamma\Omega_{RT\,(A,A_\sharp)} = (PA, \approx)$ with $(p \approx q) = (p \supset\subset q) \cap A_\sharp$, it would be quite hard to recognize the canonical tripos on this locale as being local, because it is complicated to calculate with internal adjoints *etc.* in $RT\,(A_\sharp)$. That is one reason why it can be advantageous to describe a localic local map of toposes via a local tripos (rather than just in terms of internal locale theory).

**Example 6.7 (Extensional Realizability).** Let $A$ be a PCA and let $P$ be the standard **Set**-realizability tripos over $A$. Let $\mathbf{PER}(A)$ denote the category of partial equivalence relations over $A$. Define a tripos $R$ over **Set** by taking predicates over sets $I$ to be elements of $\mathbf{PER}(A)^I$, that is, $I$-indexed families of PER's. For two such families $\varphi$ and $\psi$, we define the ordering over $I$ to be $\varphi \vdash \psi$ iff there is an $a \in A$ such that, for all $i \in I$, $a$ is in the domain of the PER $\psi(i)^{\varphi(i)}$ (the exponential in the category of PER's). See (Pitts, 1981, Section 1.6) and (van Oosten, 1997) for more details. Then the tripos $R$ is local over $P$, since the forgetful functor mapping a PER to its domain has both left and right (fibred) full and faithful adjoints. Over 1, the left adjoint maps a subset of $A$ to the discrete PER on the subset and the right adjoint maps a subset of $A$ to the PER with only one equivalence class. See (Pitts, 1981, Example 4.9(iii)) and (van Oosten, 1997) for more details. We denote the topos resulting from the tripos $R$ by $\mathrm{Ext}(A)$.

This example is special in the sense that the inclusion of $RT(A)$ into $\mathrm{Ext}(A)$ is an *open* inclusion by Proposition 3.6 of (van Oosten, 1997). That means that the principal topology $j$ in $\mathrm{Ext}(A)$, for which $RT(A)$ is equivalent to the category of sheaves, is of the form $j = (u \supset -)$ for some $u\colon 1 \to \Omega$. As a consequence, $j$ has an internal left adjoint, namely $(u \times -)\colon \Omega \to \Omega$. This left adjoint induces the interior operation in $\mathrm{Ext}(A)$, so the interior operation *does* commute with pullback in this example.

## 7. Interpretation of the modal logic in $RT(A_\sharp)$ and $RT(A, A_\sharp)$

Finally, we briefly describe in concrete terms how the modal logic for computability is interpreted in $RT(A_\sharp)$ and $RT(A, A_\sharp)$.

Types and terms are interpreted by objects and morphisms of $RT(A_\sharp)$ in the standard way. A predicate $\varphi$ on an type $(X, \approx_X) \in RT(A_\sharp)$ is an equivalence class of a strict, extensional relation in $P(X \times X)$ (recall $P$ is the tripos underlying $RT(A, A_\sharp)$), that is, $\varphi$ is an equivalence class of set-theoretic functions $X \times X \to PA$ which are strict and extensional via computable realizers, two such functions being equivalent iff they are isomorphic as objects of $P(X \times X)$.

On such a predicate $\varphi$ on an object $(X, \approx)$, $\sharp\varphi$ is just $x \mapsto \varphi(x) \cap A_\sharp$ and $\flat\varphi$ is

$$x \mapsto \bigcup_{\psi \in PA} (\psi \wedge (\psi \cap A_\sharp \supset \varphi(x) \cap A_\sharp)).$$

Thus we can think of $\sharp\varphi$ as $\varphi$ being computably true, *i.e.*, realized via computable realizers.

Objects of $RT(A, A_\sharp)$ are then described as pairs $((X, \approx), \varphi)$ with $(X, \approx) \in RT(A_\sharp)$ and $\varphi$ a partial equivalence in *Pred* on $(X, \approx) \times (X, \approx)$. Likewise, morphisms are described as functional relations in the standard way.

In this realizability model, we have the following further principle for $\sharp$:

$$\Gamma \mid \neg\varphi \dashv\vdash \sharp\neg\varphi$$

because the types are the objects of $RT(A_\sharp)$. From this it follows that

$$\Gamma \mid \neg\neg\varphi \dashv\vdash \sharp\neg\neg\varphi$$

which accords with the intuition that double-negation closed formulas have no computational content.

Note also that since $\Gamma\colon RT(A_\sharp) \to \mathbf{Set}$ has a right adjoint, the same is true for the global sections functor $\Gamma\colon RT(A, A_\sharp) \to \mathbf{Set}$. Thus in $RT(A, A_\sharp)$ too, 1 is indecomposable and projective; so $RT(A, A_\sharp)$ has the logical disjunction and existence properties.

**Example 7.1.** Let $A = P\mathbb{N}$ be the graph model and let $A_\sharp = RE$ be the recursively enumerable sub-graph-model. The arithmetical form of Church's Thesis holds in $RT(RE)$, but not in $RT(P\mathbb{N})$ (since arithmetic in $RT(P\mathbb{N})$ is classical). Since there is a logical functor from $RT(A, A_\sharp) = RT(P\mathbb{N}, RE)$ to $RT(A) = RT(P\mathbb{N})$, see (Birkedal, 1999), Church's Thesis does not hold in $RT(A, A_\sharp) = RT(P\mathbb{N}, RE)$ either. However, using the modal logic for computability, we can express that a form of Church's Thesis does hold in $RT\,(P\mathbb{N}, RE)$: by Proposition 5.2, for every arithmetical formula $\varphi$,

$$\sharp(\forall n.\, \exists m.\, |\varphi|(n,m)) \supset \exists k.\, \forall n.\, \exists m.\, \big(|\varphi|(n, Um) \wedge T(k,n,m)\big)$$

holds in the modal logic.

More information about the relative realizability topos $RT\,(A, A_\sharp)$ can be found in the second author's PhD-thesis (Birkedal, 1999).

## Acknowledgements

We gratefully acknowledge very useful discussions with Andrej Bauer, Martin Hyland, Peter Johnstone, John Longley, Jaap van Oosten, Pino Rosolini, Alex Simpson, and Thomas Streicher.

## References

Awodey, S. and Birkedal, L. (1999). Elementary axioms for local maps of toposes. Technical Report CMU–PHIL–103, Department of Philosophy, Carnegie Mellon University. Submitted for publication.

Benaissa, Z.-L., Moggi, E., Taha, W., and Sheard, T. (1999). A categorical analysis of multi-level languages. Manuscript.

Benton, P. (1995). A mixed linear and non-linear logic: Proofs, terms and models (preliminary report). Technical report, University of Cambridge.

Birkedal, L. (1999). *Developing Theories of Types and Computability.* PhD thesis, School of Computer Science, Carnegie Mellon University.

Fourman, M. and Scott, D. (1979). Sheaves and logic. In Fourman, M., Mulvey, C., and Scott, D., editors, *Applications of Sheaves*, pages 302–401. Springer-Verlag.

Hyland, J., Johnstone, P., and Pitts, A. (1980). Tripos theory. *Math. Proc. Camb. Phil. Soc.*, 88:205–232.

Johnstone, P. (1977). *Topos Theory.* Number 10 in LMS Mathematical Monographs. Academic Press, London.

Johnstone, P. and Moerdijk, I. (1989). Local maps of toposes. *Proc. London Math. Soc.*, 3(58):281–305.

Lawvere, F. (1989). Toposes generated by codiscrete objects in combinatorial topology and functional analysis. Notes for Colloquium lectures given at North Ryde, New South Wales, Australia on April 18, 1989 and at Madison, USA, on December 1, 1989.

Lawvere, F. (1991). Some thoughts on the future of category theory. In Carboni, A., Pedicchio, M., and Rosolini, G., editors, *Category Theory. Proceedings of the International Conference held in Como, Italy, July 22–28, 1990*, volume 1488 of *Lecture Notes in Mathematics*, pages 1–13. Springer-Verlag.

Longley, J. (1998). The sequentially realizable functionals. Technical Report ECS–LFCS–98–402, University of Edinburgh.

Pitts, A. (1981). *The Theory of Triposes.* PhD thesis, Cambridge University.

Pour-El, M. and Richards, J. (1989). *Computability in Analysis and Physics.* Springer-Verlag.

Scott, D., Awodey, S., Bauer, A., Birkedal, L., and Hughes, J. (1998). Logics of Types and Computation at Carnegie Mellon University. `http://www.cs.cmu.edu/Groups/LTC/`.

Troelstra, A., editor (1973). *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. Springer. With contributions by A.S. Troelstra, C.A. Smoryński, J.I. Zucker and W.A. Howard.

Troelstra, A. and van Dalen, D. (1988). *Constructivism in Mathematics*. North-Holland. 2 volumes.

van Oosten, J. (1997). Extensional realizability. *Annals of Pure and Applied Logic*, 84:317–349.

van Oosten, J. (1999). A combinatory algebra for sequential functionals of finite type. In Cooper, S. and Truss, J., editors, *Models and Computability*, pages 389–406. Cambridge University Press.